

Geometric structure of the equivalence classes of a controllable pair *

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Abstract-Given a pair of matrices representing a controllable linear system, we study its equivalence classes by the single or combined action of feedbacks and change of state and input variables, as well as their intersections. In particular, we prove that they are differentiable manifolds and we compute their dimensions. Some remarks concerning the effect of different kinds of feedbacks are derived.

Keywords-Controllable pairs, Linear systems, Orbits by feedback, Orbits by variables change, System perturbations.

I. INTRODUCTION

Several equivalence relations between pairs of matrices representing linear systems are considered in the literature. For example, the one corresponding to change of basis in the state variables, or the so-called block similarity, which also involves changes in the input space and feedbacks. It seems natural to consider the equivalence classes related to each one of these transformations and to two of them. We ask for the geometric relations between the different equivalence classes, such as the relative codimension and specially the study of their non trivial intersections.

Partial approaches to this subject appear in the literature. Here we tackle an unified treatment in order to simplify the proofs and to present a full panorama of the geometric hierarchy of these equivalence classes. Some non obvious remarks will be derived concerning the effect of the feedbacks.

The starting point is the differentiable structure of each equivalence class, which follows from the Closed Orbit Lemma. The computation of their dimension is based on Arnold's technique of versal deformations, that is to say, transversal manifolds to the considered classes (or orbits) in some other coarser one. In fact, we use the results in [3] and [14] to obtain "adapted" deformations having similar patterns, in such a way that different families of parameters are responsible for the corresponding deformation. Moreover, it gives a local adapted parameterization of the different equivalence classes.

Concerning the intersections, in general they must not be an orbit, even not a differentiable manifold. In our case, it is so due to the transversality conditions hold and it is possible in each case a particular description as orbit with regard to suitable subgroups. Even more, in some cases this subgroup is just the intersection of those generating the intersection orbits.

The study of the differentiable equivalence classes is tackled in Sections 3 (simple actions) and 4 (multiple actions), whereas Section 5 is devoted to their intersections. Previously we introduce some definitions and notation in Section 2.

II. PRELIMINARIES

Let $\mathcal{M} = \mathbb{C}^{n^2} \times \mathbb{C}^{n \times m}$ be the differentiable manifold of pairs of matrices $\mathcal{M} = \{(A, B) : A \in \mathbb{C}^{n^2}, B \in \mathbb{C}^{n \times m}\}$ and \mathcal{M}^* the open dense subset of \mathcal{M} formed by the controllable pairs with rank $B = m$, that is to say, the *full rank controllable pairs*.

The usual *block similarity* (or *BK-equivalence*) is induced by the group action:

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$$\mathcal{G} = \{g = \begin{pmatrix} S & 0 \\ R & T \end{pmatrix} : S \in Gl_n, T \in Gl_m, R \in \mathbb{C}^{m \times n}\}.$$

$$g * (A, B) = S^{-1}(A, B) \begin{pmatrix} S & 0 \\ R & T \end{pmatrix} = (S^{-1}AS + S^{-1}BR, S^{-1}BT),$$

so that the *BK-equivalence class* of a pair (A, B) is the orbit

$$O_{BK}(A, B) = \{g * (A, B) : g \in \mathcal{G}\}.$$

The actions of S, T, R are called a change of state variables, a change of input variables and a feedback, respectively. In a natural way, we can also consider the subgroups relative to only some of these actions and their corresponding orbits.

Definition 0.1 Let $(A, B) \in \mathcal{M}$. We consider the following suborbits of $O_{BK}(A, B)$ defined by

1. $O_{ST}(A, B), O_{SR}(A, B), O_{TR}(A, B)$ when $R = 0, T = I_m, S = I_n$, respectively.
2. $O_S(A, B), O_T(A, B), O_R(A, B)$ when $R = 0$ and $T = I_m, R = 0$ and $S = I_n, S = I_n$ and $T = I_m$, respectively.

Our aim is to study these orbits and their intersections. It follows directly from the Closed Orbit Lemma (see for example [10]) that all the above orbits are differentiable manifolds. Their dimensions will be computed in sections 3 and 4. Concerning their intersections, we notice that in general the intersection of two differentiable manifolds must not be so even if they are group orbits. However, in Section 5 we will see that in our case it follows from transversality conditions.

As we have pointed out, we restrict ourselves to the generic case of full rank controllable pairs. Several canonical forms and complete invariants are well-known with regard to block similarity (for a survey, see [12]). We will use the following *BK-form*, defined by means of the controllability indices. As notation, we write $E_q = (0 \dots 010 \dots 0)^t$, where the 1-valued entry is in the q -position and the size corresponds to the context, and $N_p = (0, E_1, \dots, E_{p-1})$ is the upper nilpotent p -block.

Definition 0.2 Given a full rank controllable pair of matrices $(A, B) \in \mathcal{M}^*$ and $k_1 \geq k_2 \geq \dots \geq k_m > 0$ its controllability indices, it is known that there is another pair (A_c, B_c) in its *BK-orbit* such that

$$(A_c, B_c) = (\text{diag}(N_{k_1}, N_{k_2}, \dots, N_{k_m}), (E_{l_1}, E_{l_2}, \dots, E_{l_m})),$$

$$\text{where } l_i = \sum_{j=1}^i k_j.$$

It is said that (A_c, B_c) is the *Brunovsky canonical form* of (A, B) or that (A_c, B_c) is a *BK-pair*.

We express the pairs $(C, D) \in \mathcal{M}$ linked to a full rank controllable pair (A, B) with controllability indices $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ divided into blocks: $C = (C_{i,j})_{1 \leq i,j \leq m}, C_{i,j} \in \mathbb{C}^{k_i \times k_j}, D = (D_1, D_2, \dots, D_m), D_i \in \mathbb{C}^n$.

When 0 appears in a block matrix, it will be a null block of the suitable size (it could be empty).

The following families of parameterized matrices divided into the above blocks will be widely used in the next sections:

Definition 0.3 1. Given $A_\alpha, A_\beta \in \mathbb{C}^{n \times n}$ and $B_\gamma, B_\delta \in \mathbb{C}^{n \times m}$ with $\alpha_{i,j,p}, \beta_{i,j,p}, \gamma_{i,j}, \delta_{i,j,l} \in \mathbb{C}$ such that

$$A_{\alpha_{i,j}} = E_{k_i} \cdot (\alpha_{i,j,1}, \alpha_{i,j,2}, \dots, \alpha_{i,j, \min\{k_i, k_j\}}, 0).$$

$$A_{\beta_{i,j}} = E_{k_i} \cdot (0, \beta_{i,j, k_j - k_i + 1}, \dots, \beta_{i,j, k_j}) \text{ if } k_i < k_j, \text{ and } A_{\beta_{i,j}} = 0 \text{ otherwise.}$$

$$B_{\gamma_{i,j}} = \sum_{k_i > k_j} \gamma_{i,j} E_{k_i}.$$

$$B_{\delta_{i,j}} = \sum_{1 \leq i < j \leq m} \sum_{1 \leq l \leq k_{i,j}} \delta_{i,j,l} E_{k_i - l}, \text{ where } k_{i,j} = \max\{0, k_i - k_j - 1\}.$$

2. We denote the number of parameters of $A_\alpha, A_\beta, B_\gamma, B_\delta$ by

$$n_\alpha = \sum_{i,j=1}^m \min\{k_i, k_j\} = \sum_{i=1}^m (2i - 1)k_i.$$

$$n_\beta = \sum_{i,j=1}^m \max\{0, k_i - k_j\}.$$

$$n_\gamma = \sum_{i,j=1}^m \Gamma_{i,j}, \text{ where } \Gamma_{i,j} = 1 \text{ if } k_i > k_j, \text{ and } 0 \text{ otherwise.}$$

$$n_\delta = \sum_{i,j=1}^m \max\{0, k_i - k_j - 1\}.$$

Notice that $n_\alpha + n_\beta = nm, n_\gamma + n_\delta = n_\beta, n_\alpha + n_\gamma + n_\delta = nm$.

Example 0.4 If $k_1 = 6, k_2 = 3, k_3 = k_4 = 2$, and the pair (A_c, B_c) is the corresponding Brunovsky canonical form, then $A_c + A_\alpha + A_\beta$ and $B_c + B_\gamma + B_\delta$ are, respectively,

0	1	0	0	0	0	0	0	...
0	0	1	0	0	0	0	0	...
0	0	0	1	0	0	0	0	...
0	0	0	0	1	0	0	0	...
0	0	0	0	0	1	0	0	...
0	0	0	0	0	0	1	0	...
$\alpha_{1,1,1}$	$\alpha_{1,1,2}$	$\alpha_{1,1,3}$	$\alpha_{1,1,4}$	$\alpha_{1,1,5}$	$\alpha_{1,1,6}$	$\alpha_{1,2,1}$...	
0	0	0	0	0	0	0	0	...
0	0	0	0	0	0	0	0	...
$\alpha_{2,1,1}$	$\alpha_{2,1,2}$	$\alpha_{2,1,3}$	$\beta_{2,1,4}$	$\beta_{2,1,5}$	$\beta_{2,1,6}$	$\alpha_{2,2,1}$...	
0	0	0	0	0	0	0	0	...
$\alpha_{3,1,1}$	$\alpha_{3,1,2}$	$\beta_{3,1,3}$	$\beta_{3,1,4}$	$\beta_{3,1,5}$	$\beta_{3,1,6}$	$\alpha_{3,2,1}$...	
0	0	0	0	0	0	0	0	...
$\alpha_{4,1,1}$	$\alpha_{4,1,2}$	$\beta_{4,1,3}$	$\beta_{4,1,4}$	$\beta_{4,1,5}$	$\beta_{4,1,6}$	$\alpha_{4,2,1}$...	

...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	$\alpha_{1,2,2}$	$\alpha_{1,2,3}$	$\alpha_{1,3,1}$	$\alpha_{1,3,2}$	$\alpha_{1,4,1}$	$\alpha_{1,4,2}$	
...	1	0	0	0	0	0	
...	0	1	0	0	0	0	
...	$\alpha_{2,2,2}$	$\alpha_{2,2,3}$	$\alpha_{2,3,1}$	$\alpha_{2,3,2}$	$\alpha_{2,4,1}$	$\alpha_{2,4,2}$	
...	0	0	0	1	0	0	
...	$\alpha_{3,2,2}$	$\beta_{3,2,3}$	$\alpha_{3,3,1}$	$\alpha_{3,3,2}$	$\alpha_{3,4,1}$	$\alpha_{3,4,2}$	
...	0	0	0	0	0	1	
...	$\alpha_{4,2,2}$	$\beta_{4,2,3}$	$\alpha_{4,3,1}$	$\alpha_{4,3,2}$	$\alpha_{4,4,1}$	$\alpha_{4,4,2}$	

0	0	0	0
0	0	0	0
0	0	$\delta_{1,3,3}$	$\delta_{1,4,3}$
0	$\delta_{1,2,2}$	$\delta_{1,3,2}$	$\delta_{1,4,2}$
0	$\delta_{1,2,1}$	$\delta_{1,3,1}$	$\delta_{1,4,1}$
1	$\gamma_{1,2}$	$\gamma_{1,3}$	$\gamma_{1,4}$
0	0	0	0
0	0	0	0
0	1	$\gamma_{2,3}$	$\gamma_{2,4}$
0	0	0	0
0	0	1	0
0	0	0	0
0	0	0	1

III. THE ORBITS O_S, O_T AND O_R

It is easy to prove that, for full rank controllable pairs, the single action of the matrices of the form S, T and R is simple, generating orbits of constant dimension.

Proposition 0.5 *Let $(A, B) \in \mathcal{M}$; then*

1. *If (A, B) is controllable, $\dim O_S(A, B) = n^2$.*
2. *If B has full rank, $\dim O_R(A, B) = nm$.*
3. *If B has full rank, $\dim O_T(A, B) = m^2$.*

As we have pointed out, we are interested in the intersections of the above orbits and the mixed action of the matrices of the form S, T and R . Actually, we focus on the cases when the S -action is involved because of the trivial fact that:

Proposition 0.6 *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair of matrices; then*

1. $O_T(A, B) \cap O_R(A, B) = \{(A, B)\}$.
2. $\dim O_{TR}(A, B) = \dim O_T(A, B) + \dim O_R(A, B) = m^2 + nm$.

IV. THE ORBITS O_{BK}, O_{ST} AND O_{SR}

We recall that the triple action of S, T and R corresponds to the usual block similarity. The geometric structure of the BK-orbits has been studied in [3] and [4]. In particular, for a full rank controllable pair, we have

Theorem 0.7 [3] *Given a full rank controllable pair $(A, B) \in \mathcal{M}^*$ with controllability indices $k_1 \geq k_2 \geq \dots \geq k_m > 0$, then*

$$\dim O_{BK}(A, B) = n^2 + nm - n_\delta = n^2 + n_\alpha + n_\gamma.$$

If (A_c, B_c) is its Brunovsky canonical form, a BK-miniversal deformation of (A_c, B_c) in \mathcal{M}^ is the n_δ -dimensional linear manifold $(A_c, B_c) + \{(0, B_\delta)\}_\delta$.*

Canonical forms with regard to the change of states have been obtained for controllable pairs by several authors ([1],[7],[9],[11],[13],[14]). In fact, we will base the study of the orbits O_{ST} and O_{SR} on the following result, which is a direct consequence of Theorem (2.2) and (2.3) in [14], jointly with the above Theorem 0.7 and (1) of Proposition 0.5:

Theorem 0.8 *Let $(A, B) \in \mathcal{M}^*$ a full rank controllable pair with controllability indices $k_1 \geq k_2 \geq \dots \geq k_m > 0$, and (A_c, B_c) its Brunovsky canonical form. Then:*

An S -miniversal deformation of (A_c, B_c) in its BK-orbit is given by the $(n_\alpha + n_\gamma)$ -dimensional linear manifold $(A_c, B_c) + \{(A_\alpha, B_\gamma)\}_{\alpha, \gamma}$.

This S -miniversal deformation of (A_c, B_c) has the following quite singular property: the γ -parameters, and only them, can be eliminated by the T -action; and analogously, the α ones, and only them, by the R -action. Therefore:

Proposition 0.9 *In the conditions of (0.8):*

1. The n_α -dimensional linear manifold $(A_c, B_c) + \{(A_\alpha, 0)\}_\alpha \subset O_{SR}(A_c, B_c)$ is an ST-miniversal deformation of (A_c, B_c) in its BK-orbit.
2. The n_γ -dimensional linear manifold $(A_c, B_c) + \{(0, B_\gamma)\}_\gamma \subset O_{ST}(A_c, B_c)$ is an SR-miniversal deformation of (A_c, B_c) in its BK-orbit.

$$\left(\left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\delta_2 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline -\delta_1/\delta_2 & 1 \\ 0 & 0 \\ 1 & 0 \end{array} \right) \right) \in$$

 $O_{BK}(3, 2).$

As a first direct consequence of this result, we have:

- If $\delta_2 = 0, \delta_1 \neq 0$, $(A_c, B_c(\delta_1, \delta_2, 0))$ is S-equivalent to

Corollary 0.10 Given a full rank controllable BK-pair $(A_c, B_c) \in \mathcal{M}^*$, we have

$$\dim O_{ST}(A_c, B_c) = n^2 + n_\gamma,$$

$$\dim O_{SR}(A_c, B_c) = n^2 + n_\alpha.$$

$$\left(\left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1/\delta_1 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} \right) \right) \in$$

 $O_{BK}(3, 2).$

Example 0.11 Let us consider $n = 5$ and $m = 2$. Then, there are only two kinds of orbits, according to the controllability indices being $(4, 1)$ or $(3, 2)$. Let us obtain the bifurcation diagram in Figure 1.

An SR-miniversal deformation in \mathcal{M}^* of the pair

$$(A_c, B_c) = \left(\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right) \right) \in$$

 $O_{BK}(4, 1)$

is the 3-dimensional linear manifold formed by the pairs

$$(A_c, B_c(\delta_1, \delta_2, \gamma)) = \left(\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \left(\begin{array}{c|c} 0 & 0 \\ 0 & \delta_1 \\ 0 & \delta_2 \\ \hline 1 & \gamma \\ 0 & 1 \end{array} \right) \right).$$

Here, the orbit $O_{BK}(4, 1)$ appears as the γ -axis, formed by pairs belonging to different SR-orbits of type $k = (4, 1)$, parameterized by γ .

The remaining points correspond to the orbit $O_{BK}(3, 2)$. In particular, the points in the plane (δ_1, δ_2) , or equivalently $\gamma = 0$, with $(\delta_1, \delta_2) \neq (0, 0)$ can be S-transformed according to 0.8. A quite laborious computation shows that

- If $\delta_2 \neq 0$, $(A_c, B_c(\delta_1, \delta_2, 0))$ is S-equivalent to

Each one lies in a different ST-orbit, because they correspond to different values of the α -parameters.

Since the SR-orbits correspond to the γ -parameters, they are the axes $\delta_2 \neq 0, \delta_1 = 0$ and $\delta_1 \neq 0, \delta_2 = 0$, and the parabolas $\frac{\delta_1}{\delta_2} = \text{constant}, \delta_2 \neq 0$.

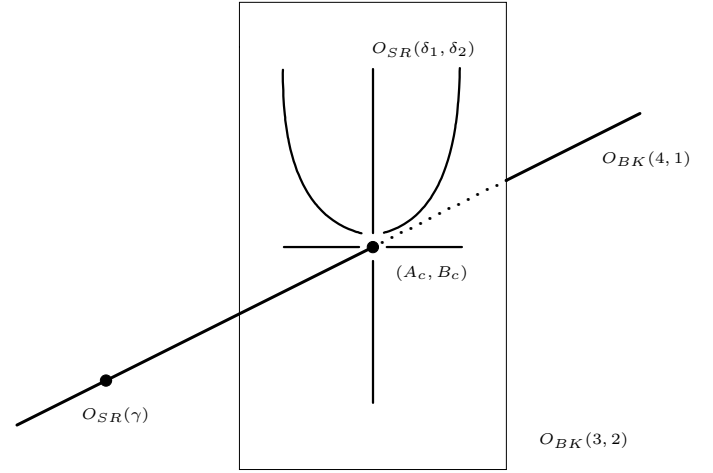


Figure 1: The orbits O_{BK} , O_{ST} and O_{SR}

V. THE INTERSECTIONS $O_S \cap O_T$, $O_R \cap O_S$ AND $O_{SR} \cap O_{ST}$

Next we study the intersection of the orbits in 3. In general, an orbit intersection must not be an orbit, not even a differentiable manifold. However, that it is so in our case because the transversality conditions hold:

Proposition 0.12 Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair with controllability indices $k_1 \geq \dots \geq k_m > 0$. Then the following orbit intersections are differentiable submanifolds of $O_{BK}(A, B)$ and:

$$\dim(O_S(A, B) \cap O_T(A, B)) = n^2 + m^2 - \dim O_{ST}(A, B),$$

$$\dim(O_S(A, B) \cap O_R(A, B)) = n^2 + nm - \dim O_{SR}(A, B).$$

If (A, B) is a BK-pair, we have $m^2 - n_\gamma$ and $nm - n_\alpha$, respectively.

Clearly, given two general subgroups \mathcal{G}_1 and \mathcal{G}_2 , the intersection of their orbits is not the one with regard to $\mathcal{G}_1 \cap \mathcal{G}_2$. For example, the intersection of the S-subgroup and the T-subgroup is simply the identity matrix. In fact, it must not be an orbit. However, let us see that the intersections in (0.12) are actually orbits with regard to the action of suitable subgroups which will depend on the pair (A, B) . We will describe it explicitly when (A, B) is a BK-pair.

For the first intersection we will do it by means of Toeplitz matrices. Let us recall the class of them which will be used:

Definition 0.13 (1) $X \in \mathbb{C}^{p \times q}$ is called a upper triangular Toeplitz matrix if

(i) It is constant along the diagonals $x_{i,j} = x_{i+1,j+1}$.

(ii) $x_{i,1} = 0$ if $i > 1$.

(iii) $x_{1,i} = 0$ if $i \leq q - p$.

(2) If k_1, \dots, k_m is a partition of n (that is, $k_1 + \dots + k_m = n$), $X \in \mathbb{C}^{n \times n}$ is called a block upper triangular Toeplitz matrix if

$$X = (X_{i,j})_{1 \leq i, j \leq m}, \quad X_{i,j} \in \mathbb{C}^{k_i \times k_j}$$

and each $X_{i,j}$ is an upper triangular Toeplitz matrix. We will denote the set of these matrices by $UTT(k_1, \dots, k_m)$.

We recall also that if A is a block diagonal nilpotent matrix

$$A = \text{diag}(N_{k_1}, \dots, N_{k_m})$$

then a non-singular matrix $S \in Gl_n$ belongs to its centralizer (that is: $S^{-1}AS = A$) if and only if $S \in UTT(k_1, \dots, k_m)$.

With this notation, we have:

Proposition 0.14 Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.

(1) The submanifold $O_S(A, B) \cap O_T(A, B)$ is the orbit of (A, B) with regard to the action of $\mathcal{G}_{ST}(A, B)$ formed by the $S \in Gl_n$ such that $AS = SA$, and there is $T \in Gl_m$ such that: $SB = BT$.

(2) In particular, if $(A_c, B_c) \in \mathcal{M}^*$ is a BK-pair with controllability indices $k_1 \geq \dots \geq k_m > 0$, then:

$$\mathcal{G}_{ST}(A_c, B_c) = \{S \in Gl_n \cap UTT(k_1, \dots, k_m) :$$

$$S_{i,j} = 0 \quad \text{if } k_i > k_j, \quad \text{and } S_{i,j} = (0, s_{i,j}I_{k_i}) \\ \text{if } k_i \leq k_j\}.$$

Proof.

(1) It is obvious that $O_T(A, B) \cap O_S(A, B) = \{(A, SB) : S \in \mathcal{G}_{ST}\}$. Hence, it is sufficient to check that \mathcal{G}_{ST} is a subgroup of the centralizer of A : if for $i = 1, 2$, $S_i \in \mathcal{G}_{ST}$ and $T_i \in Gl_m$ are such that $S_i B = B T_i$, then left multiplying by S_i^{-1} and right multiplying by T_i^{-1} we have $S_i^{-1} B = B T_i^{-1}$ and $S_1 S_2^{-1} B = S_1 B T_2^{-1} = B T_1 T_2^{-1}$.

(2) We have noticed that the non-singular matrices S such that $AS = SA$ are just those in $UTT(k_1, \dots, k_m)$.

In the other hand, if $B_c = (E_{l_1}, E_{l_2}, \dots, E_{l_m})$, the columns of $S B_c$ will be the columns l_1, l_2, \dots, l_m of S . Moreover, because $S B_c = B_c T$, these columns must be linear combinations of the columns of B_c . Hence, $(S_{i,j})_{t,k_j} = 0$ if $1 \leq t < k_i$, and the proposition is proved. ■

Concerning the second intersection in (0.12), we have again a general description as an orbit with regard to a group depending on the pair (A, B) , and an explicit description for BK-pairs (A_c, B_c) :

Theorem 0.15 Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.

(1) The submanifold $O_S(A, B) \cap O_R(A, B)$ is the orbit of (A, B) with regard to the action of $\mathcal{G}_{SR}(A, B)$ formed by the $S \in Gl_n$ such that $S^{-1}B = B$, and there is $R \in \mathbb{C}^{m \times n}$ such that: $S^{-1}AS = A + BR$.

- (2) In particular, if $(A_c, B_c) \in \mathcal{M}^*$ is a BK-pair with controllability indices $k_1 \geq \dots \geq k_m > 0$, it is a n_β -dimensional linear manifold:

$$O_S(A_c, B_c) \cap O_R(A_c, B_c) = (A_c, B_c) + \{(A_\beta, 0)\}_\beta.$$

Proof.

- (1) It is obvious that:

$$O_S(A, B) \cap O_R(A, B) = \{(S^{-1}AS, B) :$$

$$S \in \mathcal{G}_{SR}(A, B)\}.$$

Hence, it is sufficient to check that \mathcal{G}_{SR} is a subgroup of Gl_n : if for $i = 1, 2$ one has $S_i^{-1}AS_i = A + BR_i$ and $S_i^{-1}B = B$, then

$$(S_1S_2^{-1})^{-1}B = S_2S_1^{-1}B = S_2B = B$$

$$(S_1S_2^{-1})^{-1}A(S_1S_2^{-1}) = S_2(A + BR_1)S_2^{-1} =$$

$$A + BR_2 + S_2BR_1S_2^{-1} = A + B(R_2 + R_1S_2^{-1})$$

- (2) For simplicity we will refer to the given pair as (A, B) , to its orbits as O_R and O_S and to the n_β -linear manifold as L_β .

Firstly, we have that $L_\beta \subset O_R$ because taking the rows $R^j = A_\beta^{lj}$ and bearing in mind that $B^{lj} = E_j^T$ and $B^i = 0$ otherwise, it is easy to see that $A_\beta = BR$.

Secondly, to prove that $L_\beta \subset O_S$ we must see that there is $X \in Gl_n(\mathbb{C})$ such that $XAX^{-1} = A + A_\beta$ and $XB = B$ or, equivalently, $XA = AX + A_\beta X$ and $XB = B$.

Expressing these conditions in blocks, we have

$$(a) \quad X_{i,j}N_{k_j} = N_{k_i}X_{i,j} + \sum_{p=1}^{i-1} A_{\beta,i,p}X_{p,j},$$

$$(b) \quad (X_{i,j})_{k_j} = \delta_{i,j}E_{k_i}.$$

We define

$$(a) \quad S \in Gl_n(\mathbb{C}) \text{ such that } S_{i,i} = I_{k_i}, \\ S_{i,j} = 0 \text{ if } k_i \geq k_j \text{ and } (S_{i,j})^p = \\ \sum_{q=1}^{k_j-k_i} \beta_{i,j,q} E_{p+q-1}^T \text{ if } k_i < k_j.$$

$$(b) \quad X \in Gl_n(\mathbb{C}) \text{ such that } X_{i,i} = I_{k_i}, X_{i,j} = \\ 0 \text{ if } i < j, X_{i,i-1} = S_{i,i-1} \text{ and } X_{i,j} = \\ \sum_{p=i-1}^j S_{i,p}X_{p,j} \text{ if } i > j.$$

In example (0.4), the matrix S is

1	0	0	0	0	0	0	0	...
0	0	1	0	0	0	0	0	...
0	0	0	1	0	0	0	0	...
0	0	0	0	1	0	0	0	...
0	0	0	0	0	1	0	0	...
0	0	0	0	0	0	1	0	...
$\beta_{2,1,4}$	$\beta_{2,1,5}$	$\beta_{2,1,6}$	0	0	0	1	...	
0	$\beta_{2,1,4}$	$\beta_{2,1,5}$	$\beta_{2,1,6}$	0	0	0	...	
0	0	$\beta_{2,1,4}$	$\beta_{2,1,5}$	$\beta_{2,1,6}$	0	0	...	
$\beta_{3,1,3}$	$\beta_{3,1,4}$	$\beta_{3,1,5}$	$\beta_{3,1,6}$	0	0	$\beta_{3,2,3}$...	
0	$\beta_{3,1,3}$	$\beta_{3,1,4}$	$\beta_{3,1,5}$	$\beta_{3,1,6}$	0	0	...	
$\beta_{4,1,3}$	$\beta_{4,1,4}$	$\beta_{4,1,5}$	$\beta_{4,1,6}$	0	0	$\beta_{4,2,3}$...	
0	$\beta_{4,1,3}$	$\beta_{4,1,4}$	$\beta_{4,1,5}$	$\beta_{4,1,6}$	0	0	...	

...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	0	0	0	0	0	0	0
...	1	0	0	0	0	0	0
...	0	1	0	0	0	0	0
...	0	0	1	0	0	0	0
...	0	0	0	1	0	0	0
...	$\beta_{3,2,3}$	0	0	1	0	0	0
...	0	0	0	0	1	0	0
...	$\beta_{4,2,3}$	0	0	0	0	1	0

Notice that obviously $SB = B$. In addition, if $i > j$, we have

- $S_{i,j}A_{\beta,j,p} = 0$.
- $S_{i,j}N_{k_j} = N_{k_i}S_{i,j} + A_{\beta,i,j}$.
- $X_{i,j}N_{k_j} = \sum_{p=i-1}^j S_{i,p}X_{p,j}N_{k_j} = \sum_{p=i-1}^j S_{i,p}N_{k_p}X_{p,j} = \sum_{p=i-1}^j (N_{k_i}S_{i,p} + A_{\beta,i,p})X_{p,j} = N_{k_i}X_{i,j} + \sum_{p=i-1}^j A_{\beta,i,p}X_{p,j} = N_{k_i}X_{i,j} + \sum_{p=1}^{i-1} A_{\beta,i,p}X_{p,j}$.
- $(X_{i,i})_{k_i} = E_{k_i}$, $(X_{i,i-1})_{k_{i-1}} = (S_{i,i-1})_{k_{i-1}} = 0$ and by recurrence $(X_{i,j})_{k_j} = \delta_{i,j}E_{k_i}$.

The last equalities prove that $L_\beta \subset O_S$. Then, $L_\beta \subset O_S \cap O_R$. The two manifolds have the same dimension n_β but it does not imply the equality. However it is a straightforward computation that if a pair $(A_0, B_0) = (A, B) + (A_\alpha, 0) + (A_\beta, 0) \in O_R(A, B)$ belongs to $O_S(A, B)$ then $\alpha = 0$, and hence it lies in L_β : the condition $rank A_0 = rank A$, implies that the α -parameters in the first column of each block (i.e., in the columns 1, $q_1 + 1$, $q_2 + 1, \dots$) must be zero; then, the condition $rank A_0^2 = rank A_c^2$ implies that the α -parameters in the second column of each block (i.e., in the columns 2, $q_1 + 2, \dots$) must be zero; and so on. ■

Remark 0.16 1. It is well known that, if (A, B) is controllable, the eigenvalues of A can be

shifted by means of suitable feedbacks. More in general, the Rosenbrock's Theorem details the effects of feedbacks on the Jordan form of A . Last assertion in theorem 0.15 shows that in fact the β -feedbacks do not change the Jordan invariants of A_c . On the other hand, the Arnold's theory gives that $A_c + \{A_\alpha\}_\alpha$ is a Jordan-miniversal deformation of A_c , so that any non-zero α -feedback of (A_c, B_c) modifies the Jordan invariants of A_c .

2. More explicitly, in the above proof one has described, for each β -feedback, the change of basis $S_\beta \in Gl_n$ such that

$$\begin{aligned} S_\beta^{-1}A_cS_\beta &= A_c + A_\beta = A_c + B_cR_\beta, \\ S_\beta^{-1}B_c &= B_c. \end{aligned}$$

It gives an alternative explicit description of $O_S \cap O_R$:

$$\begin{aligned} O_S(A_c, B_c) \cap O_R(A_c, B_c) &= (A_c, B_c) + \{(A_\beta, 0)\}_\beta \\ &= \{(S_\beta^{-1}A_cS_\beta, B_c)\}_\beta. \end{aligned}$$

As above, let us see that $O_{SR}(A, B) \cap O_{ST}(A, B)$ is a differentiable submanifold of $O_{BK}(A, B)$. Notice that obviously $O_S(A, B) \subset O_{SR}(A, B) \cap O_{ST}(A, B)$. We will see that in fact the converse is also true if (A, B) is a BK-pair. In order to that, we proof previously a similar result concerning $O_R(A_c, B_c) \cap O_{ST}(A_c, B_c)$:

Lemma 0.17 *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.*

- (1) *The intersection $O_{ST}(A, B) \cap O_R(A, B)$ is a submanifold of $O_{BK}(A, B)$.*

- (2) *In particular, if $(A_c, B_c) \in \mathcal{M}^*$ is a BK-pair,*

$$\begin{aligned} (A_c, B_c) + \{(A_\beta, 0)\}_\beta &= O_S(A_c, B_c) \cap O_R(A_c, B_c) \\ &= O_{ST}(A_c, B_c) \cap O_R(A_c, B_c). \end{aligned}$$

Proof.

- (1) As in (0.12).
 (2) With the notation in the proof of (5.4(2)), we have shown then that:

$$L_\beta = O_S \cap O_R(A_c, B_c) \subset O_{ST}(A_c, B_c) \cap O_R(A_c, B_c).$$

If $(A', B') \in O_{ST}(A_c, B_c) \cap O_R(A_c, B_c)$, then:

$$A' = S^{-1}AS = A + A_\alpha + A_\beta, \quad B' = S^{-1}BT = B, O_{SR}.$$

for some S, T, α, β . We have also proved that the first relation implies $\alpha = 0$, and that then (see Remark (0.16)):

$$A + A_\beta = S_\beta^{-1}AS_\beta, \quad B = S_\beta^{-1}B.$$

Hence, $(A', B') \in O_S \cap O_R$. ■

Theorem 0.18 *Let $(A, B) \in \mathcal{M}^*$ be a full rank controllable pair.*

- (1) *The intersection $O_{SR}(A, B) \cap O_{ST}(A, B)$ is a differentiable submanifold of $O_{BK}(A, B)$ and*

$$n^2 \leq \dim(O_{SR}(A, B) \cap O_{ST}(A, B)) =$$

$$\dim O_{SR}(A, B) + \dim O_{ST}(A, B) - (n^2 + n_\alpha + n_\gamma).$$

- (2) *In particular, if $(A_c, B_c) \in \mathcal{M}^*$ is a BK-pair, then*

$$O_{SR}(A_c, B_c) \cap O_{ST}(A_c, B_c) = O_S(A_c, B_c).$$

Proof.

1. As in (0.12).

2. For BK-pairs, we have $\dim((O_{SR}(A, B) \cap O_{ST}(A, B))) = n^2 = \dim O_S(A, B)$.

Again we refer to the given pair simply as (A, B) . If (A', B') lies in the intersection, there will be S_1, S_2, R and T such that:

$$A' = S_1^{-1}AS_1 = S_2^{-1}(A + BR)S_2,$$

$$B' = S_1^{-1}BT = S_2^{-1}B.$$

Let $A'' = S_2A'S_2^{-1} = S_3^{-1}AS_3 = A + BR$, where $S_3 = S_1S_2^{-1}$. Then, $B = S_2B' = S_3^{-1}BT$. Clearly

$$(A'', B) \in O_S(A', B'),$$

$$(A'', B) \in O_{ST}(A, B) \cap O_R(A, B) \supset$$

$$O_S(A, B) \cap O_R(A, B).$$

But (0.17) ensures that the last inclusion is in fact an equality. Hence $(A'', B) \in O_S(A, B)$. From it and $(A'', B) \in O_S(A', B')$ one has $(A', B') \in O_S(A, B)$. ■

Remark 0.19 *Last assertion in theorem 0.18 shows that, in this case, the intersection $O_{ST} \cap O_{SR}$ is just the orbit generated by the action of the intersection Lie subgroup of those generating O_{ST} and*

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