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PHD THESIS:

**On some Spectral and Combinatorial Properties of
Distance-Regular Graphs and their Generalizations.**

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1 Introduction and preliminaries

In this work we present results and contributions in some different areas of Graph Theory. We contribute to the study of generalizations of distance-regular graphs by describing algebraic tools and properties that we can apply to any graph. By the study of these algebraic tools, we can characterize combinatorial properties and structures of the graph.

One of our main purposes for this work is to study the different fonts of information (of an algebraic and combinatorial nature) we can extract from a graph, and to study the relationships between them. We describe some procedures and methods from which we can obtain any piece of information in terms of any other. We conclude that there exists an equivalence between the algebraic and combinatorial properties. Moreover, these equivalences and procedures constitute a tool from which we can obtain more properties and characterizations of any graph.

Along this work, we also define a new family of graphs, the *distance mean-regular graphs*, which are a generalization of the distance-regular graphs. We use different tools and procedures to study this kind of graphs. We use some combinatorial techniques to study the first properties of these graphs. Also, we define some algebraic structures of which we study not only its properties but also the existent relationships between them. This algebraic study allows us to give more properties and characterizations of the graphs.

In the last chapter of this work, we focus our attention to a particular problem, the *vertex-isoperimetric problem* in Johnson graphs, which constitute a family of distance-regular graphs. By using different combinatorial techniques, we are able to solve it in some cases, and to describe properties about the behaviour of the problem in the other cases.

As usual, we are going to consider a graph $\Gamma = (V, E)$ with a non empty vertex set $V = \{v_1, \dots, v_n\}$ and with E , a set of edges $\{v_i, v_j\}$ with $v_i, v_j \in V$. Given the vertices v_i and v_j in V , we say they are adjacent if $\{v_i, v_j\} \in E$. Let \mathbf{A} be the adjacency matrix that has dimension $|V| \times |V|$ and its entries are defined by:

$$(\mathbf{A})_{i,j} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \\ 0, & \text{if } \{v_i, v_j\} \notin E \end{cases}$$

The spectrum of Γ is the set of eigenvalues of the adjacency matrix together with their multiplicities, $\text{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \dots, \lambda_d^{m(\lambda_d)}\}$. The spectrum of the graph is the first piece of information that we study, and it is the one from where we obtain some of the other pieces.

An important family of graphs is constituted by the *distance-regular graphs*. They have been well studied by several authors like Biggs in [29], Brouwer, Cohen and Neumaier in [26], Cvetković, Doob and Sachs in [7] and Van Dam, Koolen and Tanaka in [14], for example. A lot of literature has also been written about characterizations of these graphs and their algebraic properties, as Brouwer and Haemers in [27] with *Spectra of Graphs*; or Van Dam [10] and Fiol, Gago and Garriga [17] with the *Spectral excess theorem*, among

others.

A graph Γ is distance-regular if for any integers i, j and h , and given vertices u and v at distance h , the number of vertices at exactly distance i from u and at distance j from v is independent of the chosen vertices u and v . That is, the so-called *intersection parameter*

$$p_{i,j}^h = |\Gamma_i(u) \cap \Gamma_j(v)|$$

depends only on the parameters i, j and h for $0 \leq i, j, h \leq D$, with D being the diameter of the graph; and where $\Gamma_i(u)$ is the set of vertices in Γ at distance i from the vertex u .

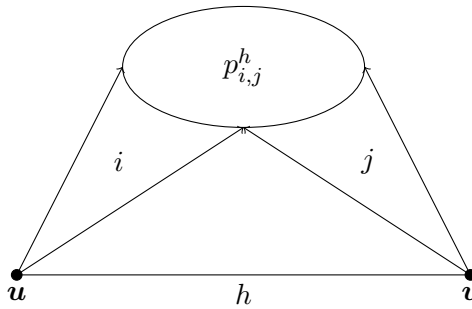


Figure 1: Scheme of the $p_{i,j}^h$ parameters.

Taking in $p_{i,j}^h$ the values $i = h, h + 1, h - 1$ and $j = 1$, we obtain the *intersection numbers* (or *intersection parameters*) a_i, b_i and c_i , respectively, of the graph. For a given vertex u taken as origin vertex, we consider a vertex v in $\Gamma_i(u)$. The parameters a_i represent the number of adjacent vertices of the vertex v in the subgraph $\Gamma_i(u)$; the parameters b_i represent the number of adjacent vertices of v in $\Gamma_{i+1}(u)$; and the parameters c_i represent the number of adjacent vertices of v in the subgraph $\Gamma_{i-1}(u)$. See Fig. 2 for the case $j = 2$.

Then, the graph is distance-regular if and only if the parameters a_i, b_i and c_i do not depend neither on the vertex v nor on the origin vertex u . In this case, we say that they are well defined. One of the main properties of the intersection numbers is that all the parameters $p_{i,j}^h$ are uniquely determined by them. This allows us to obtain a lot of combinatorial properties and the behaviour of the graph in terms of the intersection numbers. However, these parameters are not enough to determine the graph. There exist non-isomorphic graph with the same spectrum, and also, there exist non-isomorphic distance-regular graphs with the same intersection numbers (see [20]).

Some basic examples of the properties of intersection numbers are that the degree is calculated as $a_i + b_i + c_i = k$ for every i , or that a distance-regular graph is bipartite if and only if $a_i = 0$ for all i . Some other combinatorial properties like the girth can be also determined by them. The behaviour of this parameters have also some monotony properties: it holds that $k = b_0 \geq b_1 \geq \dots \geq b_{d-1} > 0$ or $1 = c_1 \leq c_2 \leq \dots \leq c_d$.

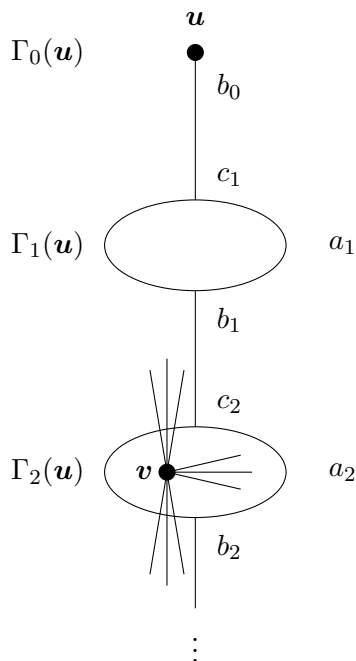


Figure 2: Intersection parameters diagram.

In general, the spectrum of a graph is a very useful tool to study some of its properties. In the case of the distance-regular graphs this tool is specially suitable. The intersection numbers of a distance-regular graph can be deduced from the spectrum, and vice versa.

In this work we introduce a natural generalization of the intersection parameters. If we consider any graph Γ and its spectrum and apply the same procedures as if it were distance-regular, we obtain, from the spectrum, the parameters that we call the *preintersection numbers* α_i , β_i , γ_i and $\xi_{i,j}^h$ (which generalize a_i , b_i , c_i and $p_{i,j}^h$ respectively). These parameters, introduced by Abiad, van Dam and Fiol in [1], can be defined for any graph and they keep a lot of combinatorial properties that have the intersection numbers in a distance-regular graph.

Obviously, these parameters α_i , β_i and γ_i do not give, in general, the degree of a vertex in $\Gamma_{i-1}(u)$, $\Gamma_i(u)$ or $\Gamma_{i+1}(u)$ respectively. They do it only in the case that the graph is distance-regular. However, if the graph is regular of degree k , then $\alpha_i + \beta_i + \gamma_i = k$ for every i . Also, the preintersection numbers can determine combinatorial properties, like the girth or characterize when the graph is bipartite.

The preintersection numbers are another important piece of information that we study in this work. In the same way that, in a distance-regular graph, the parameters $p_{i,j}^h$ and the spectrum can be determined from the parameters a_i , b_i and c_i ; in any graph, the parameters $\xi_{i,j}^h$ and the spectrum can be determined from the parameters α_i , β_i and γ_i .

In a distance-regular graph we can define also a family of polynomials called the *distance polynomials*. These polynomials and its properties have been studied by Fiol and Garriga

in [32], among others. In [21], Hoffman introduced the Hoffman polynomial H_Γ of a graph Γ , which characterizes the regularity of Γ . Namely, it holds that $H_\Gamma(\mathbf{A}) = \mathbf{J}$ if and only if Γ is regular, where \mathbf{A} is the adjacency matrix and \mathbf{J} the all-ones matrix. This polynomial is closely related to the spectrum of the graph, because the roots of H_Γ are the eigenvalues of Γ , except the first eigenvalue λ_0 .

The distance polynomials p_0, p_1, \dots, p_d of a distance-regular graph Γ can be seen as a decomposition of the Hoffman polynomial. That is, $p_0 + p_1 + \dots + p_d = H_\Gamma$. This polynomials satisfies that $p_i(\mathbf{A}) = \mathbf{A}_i$ where \mathbf{A}_i is the i -distance matrix for each $0 \leq i \leq d$. But they also have some particular combinatorial properties which help us in the study of the graph. For example, they determine the number of vertices at any distance i from any origin vertex: taking k as the degree of the graph, we have that $p_i(k) = |\Gamma_i|$ for every i .

These polynomials have also a lot of combinatorial properties which relate them to the intersection numbers. The principal equality is given by the expression (the *three term recurrence equality*)

$$xp_i = b_{i-1}p_{i-1} + a_i p_i + c_{i+1}p_{i+1}.$$

This equality is going to be the fundamental step for obtaining relationships between the intersection numbers and the distance polynomials. Also, the distance polynomials are well related with the spectrum of the graph, because they are an orthogonal family of polynomials in the space defined by the scalar product $\langle \cdot, \cdot \rangle_\Gamma$ defined as:

$$\langle f, g \rangle_\Gamma = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i)g(\lambda_i),$$

where \mathbf{A} is the adjacency matrix of the graph Γ , and λ_i are the eigenvalues of Γ with multiplicity $m_i = m(\lambda_i)$. This scalar product is going to be the principal relation between the polynomials and the spectrum. But also, from it we can obtain the intersection numbers as Fourier coefficients of the distance polynomials in the Hilbert space with $\langle \cdot, \cdot \rangle_\Gamma$:

$$p_{ij}^h = \frac{\langle p_i p_j, p_h \rangle_\Gamma}{\|p_h\|_\Gamma^2} = \frac{1}{np_h(\lambda_0)} \sum_{r=0}^d m(\lambda_r) p_i(\lambda_r) p_j(\lambda_r) p_h(\lambda_r),$$

and $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$ and $c_i = p_{1,i-1}^i$.

In the same way we extended the intersection numbers of the distance-regular graphs to the preintersection numbers for any graph, we can extend the distance polynomials to the *predistance polynomials*. They constitute the third principal piece of information of any graph.

As in the case of the preintersection numbers, they preserve properties from the distance polynomials in the distance-regular graphs. The three terms equality we used to relate the intersection numbers with the distance polynomials holds in the case of preintersection numbers and predistance polynomials:

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}.$$

This is the natural and principal way in which the predistance polynomials work together with the preintersection numbers. Recall that in a general graph Γ , the values $|\Gamma_i|$ are in general not well defined. That is, $|\Gamma_i(u)|$ depends of the vertex u . However, we can use the value of $p_i(\lambda_0)$ in the same way we used $|\Gamma_i(u)|$ in the distance-regular graphs. We can write $p_{i-1}(\lambda_0)\beta_{i-1} = p_i(\lambda_0)\gamma_i$. This is an example of how the predistance polynomials are a useful tool with which we can extend results we could not obtain with the distance polynomials.

The family of predistance polynomials has the same relationship with the spectrum. The scalar product defined before is the link with which they can work together. It holds that the predistance polynomials are an orthogonal family of polynomials in the Hilbert space $(\mathbb{R}^{\mathbb{R}}, \langle \cdot, \cdot \rangle_{\Gamma})$:

$$\langle f, g \rangle_{\Gamma} = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i)g(\lambda_i),$$

for every pair of functions f and g . That is,

$$\langle p_i, p_j \rangle_{\Gamma} = 0$$

for every p_i and p_j with $i \neq j$. Preintersection numbers can be also obtained as Fourier coefficients of the predistance polynomials:

$$\xi_{ij}^h = \frac{\langle p_i p_j, p_h \rangle_{\Gamma}}{\|p_h\|_{\Gamma}^2} = \frac{1}{np_h(\lambda_0)} \sum_{r=0}^d m(\lambda_r) p_i(\lambda_r) p_j(\lambda_r) p_h(\lambda_r),$$

being $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$ and $\gamma_i = \xi_{1,i-1}^i$.

As in the case of the distance-regular graphs, the sum of all the predistance polynomials is the Hoffman polynomial H_{Γ} of the graph. This polynomial, in general, does not satisfy $H_{\Gamma}(\mathbf{A}) = \mathbf{J}$. That holds only when Γ is regular. However, it has the same algebraic properties that in the case of the distance-regular graph. Its roots are the distinct eigenvalues but the first one of the spectrum, and some other properties that we can see in Section 2.2.

Another piece of information we use in this work is the average number of closed walks of length ℓ that there exist in Γ . In general, for any pair of vertices u and v and any value ℓ , the number of walks joining u with v of length exactly ℓ can be calculated as $(\mathbf{A}^{\ell})_{uv}$. Thus the average number of closed walks of length ℓ in a graph, can be expressed as:

$$\bar{c}(\ell) = \frac{1}{n} \text{tr}(\mathbf{A}^{\ell}).$$

The trace of a matrix is a value closely related with its spectrum. We have that $\text{tr}(\mathbf{A}^{\ell}) = \sum_i \lambda_i^{\ell} m(\lambda_i)$ where λ_i is an eigenvalue of the graph and $m(\lambda_i)$ its multiplicity. Thus, the average number of closed walks of any fixed length ℓ can be calculated directly from the spectrum of the graph.

Note that the scalar product of the canonical basic polynomials $\langle x^i, x^j \rangle_\Gamma$ for any pair i, j such that $i + j = \ell$ denote also the number of closed walks in the graph. This is the way in which this parameter is also related with the predistance polynomials of the graph.

We not only use the powers of the adjacency matrix to obtain combinatorial properties; we study also the relationship between the different powers of the adjacency matrix. We may consider the algebra defined as $\mathcal{A} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d)$ where the graph Γ has $d + 1$ distinct eigenvalues. In the case of the distance-regular graphs, this algebra is called the Bose-Mesner algebra (see [27] or [26], for example).

In general, any algebra of dimension n is defined by its basic elements and n^3 parameters. They describe how the basic elements interact between them. That is, given the n basic elements $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and n^3 parameters $c_{h,i,j}$, the multiplication operation is defined as:

$$\mathbf{e}_i \star \mathbf{e}_j = \sum_{h=1}^n c_{h,i,j} \mathbf{e}_h.$$

In the case of the Bose-Mesner algebra, these parameters are given by the intersection parameters p_{ij}^h . We use the preintersection numbers and define an algebra in the same terms. We define the *adjacency algebra* as:

$$\mathbf{A}_i \mathbf{A}_j = \sum_{h=0}^d \xi_{ij}^h \mathbf{A}_h,$$

where \mathbf{A}_i represents the i -distance matrix of the graph. This algebraic structure constitute another way to study the preintersection numbers. The study of this structure (and its relation with in the Bose-Mesner algebra) will give us some characterizations of the distance-regularity and other properties of the graph (see Subsection 5.8).

A large part of this work deals not only with the properties of the pieces of information we have been discussing, but also with the relationship between them. In Chapter 2 we specifically discuss these pieces and study some of their first properties. Moreover, in Chapter 3 we present specific formulas and procedures to obtain each one of the pieces of information in terms of the others. It is worth noting that this chapter shows that the characterizations, properties, or information we obtain for any graph can be studied in many different ways.

This part of the work has two consequences. The first one is that the different pieces of information are equivalent, even if they have combinatorial or algebraic nature. The second one is that the tools we introduce can be used to prove new properties and results in Spectral Graph Theory.

In Subsection 3.7 we illustrate with an example all these equivalences and procedures. We calculate all the pieces of information (and properties) in terms of each other, for a graph which is not regular.

The equivalences and procedures explained in Chapter 3 allow us to prove combinatorial and algebraic properties of the graphs (see Chapter 4).

Chapter 5 is devoted to introduce a new family of graphs, the *distance mean-regular graphs*. We define this kind of graphs as a generalization of vertex-transitive graphs or distance-regular graphs. That is why the techniques and procedures we study in this work can be applied to many graphs.

A graph is *distance mean-regular* if for a given vertex u , the averages of the intersection numbers $p_{ij}^h(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|$ (number of vertices at distance i from u and distance j from v) computed over all vertices v at a given distance $h \in \{0, 1, \dots, D\}$ from u , do not depend on u . In other words, we can fix an origin vertex u , as we did in the case of distance-regular graphs, and calculate the average degree of a vertex at distance i from u , by taking in to account the decomposition defined by the adjacent vertices at distances $i-1$, i and $i+1$ from u . The graph is distance mean-regular if these average values are not dependent on the chosen origin vertex u , and we call these values \bar{a}_i , \bar{b}_i and \bar{c}_i respectively. These parameters are called *intersection mean-parameters*, and we can write them in the *mean-matrix* $\bar{\mathbf{B}}$:

$$\bar{\mathbf{B}} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & & & & & & \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 & & & & & \\ & \bar{c}_2 & \bar{a}_2 & \bar{b}_2 & & & & \\ & & & \ddots & & & & \\ & & & & \bar{c}_{d-1} & \bar{a}_{d-1} & \bar{b}_{d-1} & \\ & & & & & \bar{c}_d & \bar{a}_d & \end{pmatrix}.$$

We first remark that the parameters \bar{a}_i , \bar{b}_i and \bar{c}_i are, as α_i , β_i and γ_i , generalizations of the intersection numbers of a distance-regular graph. However there exist several differences between them. The most important difference we have to remark is that the preintersection numbers are defined for any graph. We extend the techniques used in distance-regular graphs to any other graph, even if it is not regular. However, the parameters \bar{a}_i , \bar{b}_i and \bar{c}_i are only defined in the case in which the graph is distance mean-regular, and its values do not coincide in general with the preintersection numbers.

These parameters have important combinatorial properties in the graph (see Subsections 5.4 and 5.3, for example). But they are not the unique tool we use for the study of this kind of graphs. Taking the intersection mean-parameters as base, we define the *distance mean-polynomials* $\bar{p}_0, \bar{p}_1, \dots, \bar{p}_D$ with the equivalence formula:

$$x\bar{p}_i(x) = \bar{b}_{i-1}\bar{p}_{i-1}(x) + \bar{a}_i\bar{p}_i(x) + \bar{c}_{i+1}\bar{p}_{i+1}(x).$$

We can see that the family of predistance polynomials are $d+1$ different polynomials, where the graph has $d+1$ different eigenvalues. However, this family of polynomials has $D+1$ polynomials, being D the diameter of the graph. We do not only define this polynomials, but we define also the *pseudo-spectrum* of the graph. In a distance regular graph, the eigenvalues of the matrix \mathbf{B} are the distinct eigenvalues of the graph without their multiplicities. In this case, we take the spectrum of $\bar{\mathbf{B}}$, which has $D+1$ distinct eigenvalues, as the distinct eigenvalues of the pseudo-spectrum of a distance mean-regular graph. We calculate the pseudo-multiplicities with the same procedures we use for any

other graph (see Subsection 3.3). In this way we obtain $\{\mu_0^{m(\mu_0)}, \dots, \mu_D^{m(\mu_D)}\}$ as a spectrum that has n eigenvalues (the number of vertices of the graph) and $D+1$ distinct eigenvalues. We can look at the pseudo-spectrum as the spectrum the graph would have if it was distance-regular.

The distance mean-polynomials constitute an orthogonal family of polynomials with respect to the scalar product defined over the pseudo-spectrum of the graph:

$$\langle f, g \rangle_\star = \frac{1}{n} \sum_{i=0}^D m(\mu_i) f(\mu_i) g(\mu_i),$$

and the intersection mean-numbers can be also obtained as Fourier coefficients of the polynomials:

$$\bar{a}_i = \frac{\langle x\bar{p}_i, \bar{p}_i \rangle_\star}{\|\bar{p}_i\|_\star^2}, \quad \bar{b}_i = \frac{\langle x\bar{p}_{i+1}, \bar{p}_i \rangle_\star}{\|\bar{p}_i\|_\star^2}, \quad \bar{c}_i = \frac{\langle x\bar{p}_{i-1}, \bar{p}_i \rangle_\star}{\|\bar{p}_i\|_\star^2}.$$

That is, these three tools (the intersection mean-numbers, the distance mean-polynomials and the pseudo-spectrum) of a graph work well together. However, these are not the unique tools we use in order to study this kind of graphs. With the help of the distance mean-polynomials, the adjacency matrix and the mean-matrix, we define four matrix algebras:

- $\mathcal{A} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d)$, where \mathbf{A} is the adjacency matrix and $d+1$ is the number of distinct eigenvalues;
- $\mathcal{D} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_D)$, where \mathbf{A}_i is the i -distance matrix;
- $\bar{\mathcal{D}} = \text{span}(\mathbf{I}, \mathbf{A}, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_D)$, where $\bar{\mathbf{A}}_i = \bar{p}_i(\mathbf{A})$;
- $\bar{\mathcal{B}} = \text{span}(\mathbf{I}, \bar{\mathbf{B}}, \bar{\mathbf{B}}_2, \dots, \bar{\mathbf{B}}_D)$, where $\bar{\mathbf{B}}_i = \bar{p}_i(\mathbf{B})$.

The study of this algebras and the relationship between them give us some important properties and characterizations of distance mean-regular graphs (see Subsection 5.8).

Finally, in Chapter 6, we study a particular problem in a family of distance-regular graphs: The *vertex-isoperimetric problem* in the Johnson graphs. The Johnson graph $J(n, m)$ has as vertices the m -subsets of the set $\{1, \dots, n\}$, and two of them are adjacent if and only if they share $m-1$ of their elements. For example, the Johnson graph $J(5, 2)$ is shown in Fig. 3. This kind of graphs constitute an important example of distance-regular and distance-transitive graphs.

The isoperimetric problem consists in, for any cardinality k , determine the set S such that $|S| = k$ with the minimum cardinality of the boundary $|\partial S|$ amount all the sets of cardinality k . In this case, S is an optimal set. If we consider the boundary of S as the edge-boundary, that is, as the set of edges joining a vertex of S with a vertex outside of S , then we are studying the edge-isoperimetric problem. The edge-isoperimetric problem has

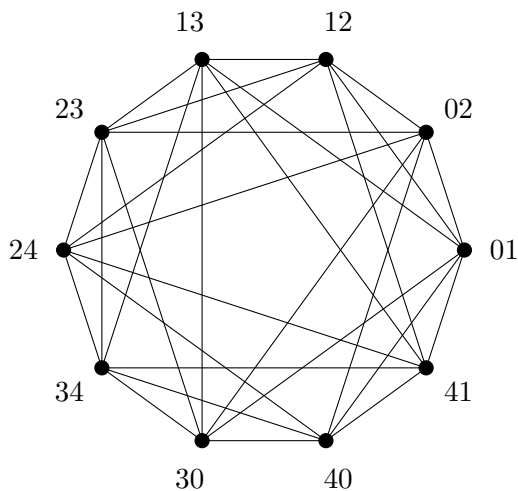


Figure 3: The Johnson graph $J(5, 2)$.

been studied by many different authors like Ahlswede and Katona in [41] or Bey in [42]. If we consider the boundary of S as the vertex boundary, that is, the number of vertices not in S but adjacent to a vertex of S , then we are on the vertex-isoperimetric problem.

The main purpose of this chapter is to contribute to the study of the vertex-isoperimetric problem for the Johnson graph. We solve explicitly the problem for some particular cases of the graph $J(n, m)$. We give the family of optimal sets of every cardinality k for the graph $J(n, 2)$ (see Subsection 6.2) and $J(2m - 2, m)$ (see subsection 6.3). In both cases, the optimal sets are the initial segments of the colexicographic order. The Johnson graph $J(2m - 2, m)$ is relevant, because it is the base case in the induction proof of Theorem 6.4. In this theorem it is proved that the colexicographic order is also optimal for sets of small cardinality for any Johnson graph. In Subsection 6.4 we also study the behaviour of the graph $J(n, m)$ for larger values of n and we do an asymptotic study of the graph. In this case, the colexicographic order is again the optimal order.

However, the colexicographic order is not optimal in all the cases. An important subsection of this chapter is devoted to present an example of that. In Subsection 6.5 we give a family of cardinals whose initial segment of the colexicographic order is not optimal. This is an important result because we show when this order is the optimal solution and where is not, proving that this is an interesting problem that can be studied in future works.

2 Different pieces of information

Distance-regular graphs are a key concept in combinatorics, because of their rich structure and multiple applications. Indeed, they have important connections with other branches of mathematics, as well as other areas of graph theory. For background on distance-regular graphs, we refer the reader to Bannai and Ito [24], Biggs [29], Brouwer, Cohen, and Neumaier [26], Brouwer and Haemers [27], Godsil [35], and Van Dam, Koolen and Tanaka [31].

The distance-regular graphs are a well known kind of graph because their algebraic and combinatorial properties. The relationship between these algebraic and combinatorial concepts have been studied by many authors (see Brouwer and Haemers at [27], van Dam, Koolen and Tanaka and [14], Cámara, Fàbrega, Fiol, and Garriga at [30],...). These results give us interesting properties about the internal structure of the graph, but also we have interesting and useful characterizations of the distance regularity.

Two main algebraic concepts are the intersection parameters and the distance polynomials of the graph. Both are well determined by the spectrum of the graph, but is well-known that many properties can be independently derived from each one of these concepts. In this section we extend these basic concepts to any (not necessarily distance-regular) other graph, introducing what we call the preintersection parameters and the predistance polynomials.

First we study independently these concepts along with others like the spectrum or the intersection numbers. We see how each one of these pieces of information of the graph is enough to give some properties of the graph in the same way that the intersection numbers give us combinatorial properties about the distribution of the adjacencies of each vertex, or in the way in which the distance polynomials give us the distribution of the vertices. In future sections we will study the relationships between the different pieces of information, and the way in which we can obtain one in terms of any each other.

We first recall some basic concepts, notation, and results on which our study is based. For more background on spectra of graphs, distance-regular graphs, and their characterizations, see [29, 26, 27, 7, 14, 16, 35]. Throughout this work, $\Gamma = (V, E)$ stands for a (simple and finite) connected graph with vertex set V and edge set E . We denote by n the number of vertices and by e the number of edges. Adjacency between vertices u and v ($uv \in E$) will be denoted by $u \sim v$. The *adjacency matrix* \mathbf{A} of Γ is the 01-matrix, with rows and columns indexed by the vertices, such that $(\mathbf{A})_{uv} = 1$ if and only if $u \sim v$.

2.1 The spectrum

One of the most important tools in the study of the algebraic properties of a graph Γ is its spectrum. The *spectrum* of Γ is the set of eigenvalues of its adjacency matrix \mathbf{A} together with their multiplicities:

$$\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}, \quad (1)$$

where $\lambda_0 > \lambda_1 > \dots > \lambda_d$, and the superscript m_i stand for the multiplicity of the eigenvalue λ_i , for $i = 0, \dots, d$. Notice that, since Γ is connected, $m_0 = 1$, and if Γ is k -regular, then $\lambda_0 = k$. Throughout the paper, d will always denote the number of distinct eigenvalues minus one.

We also use the spectrum as base for build a scalar product of two functions with respect to the spectrum of a graph $\text{sp } \Gamma$. We define it as:

$$\langle f, g \rangle_{\Gamma} = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i)g(\lambda_i), \quad (2)$$

for any two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$. This product is the key to determine the relation between the different elements of a unique piece of information, as well as the relation between some different pieces of information.

2.2 The predistance polynomials

Given a graph Γ with spectrum as above, the *predistance polynomials* p_0, \dots, p_d , introduced by Fiol and Garriga in [32], are polynomials in $\mathbb{R}_d[x]$, with degree $p_i = i$, which are orthogonal with respect to the scalar product given in (2) and normalized in such a way that $\|p_i\|_{\Gamma}^2 = p_i(\lambda_0)$ (this always makes sense since it is known that $p_i(\lambda_0) > 0$ for every $i = 0, \dots, d$). Some basic properties of these polynomials, which can be seen as a generalization of the distance polynomials of a distance-regular graph, are given in the following lemma, see [30].

Lemma 2.1. *Let Γ be a graph with average degree $\bar{k} = 2e/n$, predistance polynomials p_i , and consider their sums $q_i = p_0 + \dots + p_i$, for $i = 0, \dots, d$. Then,*

(a) $p_0 = 1$, $p_1 = (\lambda_0/\bar{k})x$, and the constants of the three-term recurrence

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \quad (3)$$

where $\beta_{-1} = \gamma_{d+1} = 0$, satisfy:

(a1) $\alpha_i + \beta_i + \gamma_i = \lambda_0$, for $i = 0, \dots, d$;

(a2) $p_{i-1}(\lambda_0)\beta_{i-1} = p_i(\lambda_0)\gamma_i$, for $i = 1, \dots, d$.

(b) $p_d(\lambda_0) = n \left(\sum_{i=0}^d \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}$, where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$, for $i = 0, \dots, d$.

(c) $1 = q_0(\lambda_0) < q_1(\lambda_0) < \dots < q_d(\lambda_0) = n$, and $q_d(\lambda_i) = 0$ for every $i \neq 0$. Thus, $q_d = H$ is the Hoffman polynomial characterizing the regularity of Γ by the condition $H(\mathbf{A}) = \mathbf{J}$, where \mathbf{J} stands for the all-1 matrix (see Hoffman [21]).

(d) The three-term recurrence (3) can be represented through a tridiagonal $(d+1) \times (d+1)$ matrix \mathbf{R} such that, in the quotient ring $\mathbb{R}[x]/(m)$, where (m) is the ideal generated by the minimal polynomial $m = \prod_{i=0}^d (x - \lambda_i)$ of \mathbf{A} , it satisfies

$$x\mathbf{p} = x \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & & \\ & & & \ddots & \gamma_d \\ & & & \beta_{d-1} & \alpha_d \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} = \mathbf{R}\mathbf{p}. \quad (4)$$

2.3 The preintersection numbers

The intersection numbers are a very well studied family of parameters which determines a lot of combinatorial properties of a distance-regular graph. Chosen any origin vertex u in a distance-regular graph, the parameters a_i , b_i and c_i represent the adjacencies of a vertex at distance i , $i+1$ and $i-1$ respectively from u .

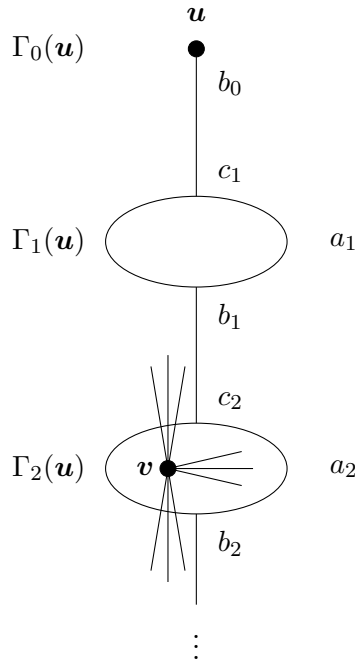


Figure 4: a_i , b_i and c_i parameters diagram.

This combinatorial properties can be calculated also by algebraic techniques. By extending this algebraic procedures to any graph, we can define the *preintersection numbers* of any graph as a generalization of the intersection numbers of a distance-regular graph, which

are closely related to its combinatorial properties (see e.g. Biggs [29]). In the more general context of any graph, the preintersection numbers give us an algebraic information on the graph, which is of the same nature as the spectrum of its adjacency matrix. More precisely, the *preintersection numbers* ξ_{ij}^h , $i, j, h \in \{0, \dots, d\}$, are the Fourier coefficients of $p_i p_j$ in terms of the basis $\{p_h\}_{0 \leq h \leq d}$, that is,

$$\xi_{ij}^h = \frac{\langle p_i p_j, p_h \rangle_\Gamma}{\|p_h\|_\Gamma^2} = \frac{1}{np_h(\lambda_0)} \sum_{r=0}^d m(\lambda_r) p_i(\lambda_r) p_j(\lambda_r) p_h(\lambda_r). \quad (5)$$

Notice that, in particular, the coefficients of the three-term recurrence (3) are $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$, and $\gamma_i = \xi_{1,i-1}^i$. In fact, from our derivations will be clear that such coefficients determine all the other preintersection numbers.

2.4 Bose-Mesner algebras and intersection numbers

Bose-Mesner algebras and Krein parameters are concepts originally defined in terms of the association scheme, but they can also be considered in the context of distance-regular graphs. In this work we generalize these concepts to extend them to every graph. For a graph of $d + 1$ distinct eigenvalues, we consider the algebra $\mathcal{A} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d)$, where \mathbf{A} is the adjacency matrix of the graph (see Subsection 5.8). In general, we are not interested in the structure of this algebra but in the relation between it and other algebras. Then, our aim is to study properties and characterization of some kind of graphs (for instance Theorem 5.13 or Proposition 5.14).

Every algebra is determined by the parameters p_{ij}^h which determine the relation between the elements in the base:

$$\mathbf{A}_i \mathbf{A}_j = \sum_{h=0}^d p_{ij}^h \mathbf{A}_h, \quad h, i, j = 0, 1, \dots, d,$$

where d is the dimension of the algebra and each A_i the i -distance matrix of the graph. In the case when the algebra studied is the Bose-Mesner algebra, these parameters are called the intersection numbers of the graph. In the same way we extend the Bose-Mesner algebra to any other non-distance-regular graph, we extend the concept of intersection parameters to any graph. This generalized parameters have a well-known combinatorial meaning in the case in the distance-regular graphs, as well as an algebraic meaning. As mentioned above, they are some Fourier coefficient given in terms of the predistance polynomials of the graph (we will study them in Chapter 3 or in Subsection 5.8).

2.5 Closed walks

The average number of closed walks of length ℓ are combinatorial parameters which can be computed either directly from the graph, or with algebraic methods. The closed walks

are related with the scalar product defined in (2) because we can write for each $\bar{c}(\ell)$ in the graph Γ : $\bar{c}(\ell) = \langle 1, x^\ell \rangle_\Gamma$. Each parameter $\bar{c}(\ell)$ can be expressed as

$$\bar{c}(\ell) = \frac{1}{n} \text{tr}(\mathbf{A}^\ell) = \frac{1}{n} \sum_{i=0}^d m_i \lambda_i^\ell$$

for $\ell = 0, 1, 2, \dots$

Thus, they are related with the predistance polynomials of the graph, and it can be seen that the information given by the parameters $\bar{c}(\ell)$ for $\ell = 0, \dots, d$ is enough to complete the information given by the $d+1$ distinct eigenvalues, or the preintersection numbers. In the case that Γ is a distance regular graph, these parameters match with the exact values of the numbers of closed walks in each vertex.

We also define a generalization of this parameter. Let $\bar{c}(\ell)_k$ be the mean number of different walks between vertices at distance k . In particular, $\bar{c}(\ell) = \bar{c}(\ell)_0$. And let be simply $c(\ell)_k$ in the case in which it is well defined. In Chapter 4, we use the parameters $\bar{c}(\ell)_k$ and its relationship with in the preintersection parameters to give results about the characterization of distance-regularity of a graph.

3 Formulas and procedures for equivalence

In this section we study the equivalence between the pieces of information described in Section 2. We give explicit formulas and procedures to obtain each one of the pieces of information from each other.

These procedures are very useful by themselves, but the importance of this section is also to prove that the same information involved in the spectrum of any graph, is also contained in the other pieces of information (preintersection numbers or the predistance polynomials).

3.1 From the spectrum to the predistance polynomials

As mentioned above, the spectrum of a graph plays a central role in the study of its algebraic and combinatorial properties. To obtain the predistance polynomials introduced in Subsection 2.2 we consider the scalar product defined in (2), in the introduction of the Chapter 2, and apply the Gram-Schmidt orthogonalization method from the basis $\{1, x, \dots, x^d\}$, normalizing the obtained sequence of orthogonal polynomials in such a way that $\|p_i\|^2 = p_i(\lambda_0)$. (This makes sense since, from the theory of orthogonal polynomials, it is known that $p_i(\lambda_0) > 0$ for any $i = 0, \dots, d$.)

This orthogonalization can be implemented by using the trace of the matrices \mathbf{A}^k . These values can be easily calculated as $\text{tr}(\mathbf{A}^k) = \sum_{i=0}^d m_i \lambda_i$. A general step of the Gram-Schmidt orthogonalization is:

$$r_i = x^i - \sum_{j=0}^{i-1} \frac{\langle x^i, r_j \rangle_{\Gamma}}{\langle r_j, r_j \rangle_{\Gamma}} r_j.$$

We can express each scalar product as:

$$\langle f, g \rangle_{\Gamma} = \frac{1}{n} \text{tr}((fg)(\mathbf{A})) = \frac{1}{n} [z_0 \text{tr}(\mathbf{I}) + z_1 \text{tr}(\mathbf{A}) + z_2 \text{tr}(\mathbf{A}^2) + \dots],$$

where $(fg)(x) = z_0 + z_1 x + z_2 x^2 + \dots$. In the particular cases in which we calculate the scalar product of two polynomials of the canonical base, we have that $\langle x^i, x^j \rangle_{\Gamma} = \frac{1}{n} \text{tr}(\mathbf{A}^{i+j})$.

An example of this method can be seen at Subsection 3.7.

As mentioned in Lemma 2.1, $H = p_0 + \dots + p_d$ is the Hoffman polynomial satisfying $H(\lambda_i) = 0$ for $i > 0$, $H(\lambda_0) = n$, and characterizing the regularity of the graph by the condition $H(\mathbf{A}) = \mathbf{J}$.

3.2 From the predistance polynomials to the spectrum

In this subsection we show how the spectrum of a graph Γ can be obtained from its predistance polynomials.

Proposition 3.1. *Let p_0, p_1, \dots, p_d be the predistance polynomials of a graph Γ , with ω_i^j being the coefficient of x^j in p_i . Then,*

- (a) *The different eigenvalues $\lambda_i \neq \lambda_0$ of Γ are the d distinct zeros of the Hoffman polynomial $H = p_0 + p_1 + \dots + p_d$.*
- (b) *The largest eigenvalue (spectral radius) is*

$$\lambda_0 = -\frac{\omega_1^1 \omega_2^0}{\omega_2^2}. \quad (6)$$

- (c) *The multiplicity of each eigenvalue λ_i , for $i = 0, \dots, d$, is*

$$m_i = n \left(\sum_{j=0}^d \frac{p_j(\lambda_i)^2}{p_j(\lambda_0)} \right)^{-1}. \quad (7)$$

Proof. (a) As mentioned in Lemma 2.1, $H = p_0 + \dots + p_d$ is the Hoffman polynomial satisfying $H(\lambda_i) = 0$ for $i = 1, \dots, d$.

(b) The expressions for p_0 and p_1 (see Lemma 2.1(a)) imply that $\omega_0^0 = 1$ and $\omega_1^0 = 0$. Then,

$$\alpha_0 = 0, \quad \alpha_1 = -\frac{\omega_2^1}{\omega_2^2}, \quad \text{and} \quad \beta_0 = -\frac{\omega_1^1 \omega_2^0}{\omega_2^2}, \quad (8)$$

and (6) follows from $\lambda_0 = \alpha_0 + \beta_0$.

(c) See the proof of Proposition 3.7. This is a result from [16].

□

The spectral radius can also be determined as the largest root of the polynomial

$$h = \left(\sum_{i=1}^d \frac{\lambda_i}{p_d(\lambda_i)} \prod_{\substack{j=1 \\ j \neq i}}^d \frac{x - \lambda_j}{\lambda_i - \lambda_j} \right) p_d(x) - x. \quad (9)$$

This comes from the combination of the following two facts: the multiplicity of each eigenvalue can be also calculated as

$$m_i = \frac{\phi_0 p_d(\lambda_0)}{\phi_i p_d(\lambda_i)}, \quad \text{for } i = 0, \dots, d, \quad (10)$$

where $\phi_i = \prod_{j=0, j \neq i}^d (\lambda_0 - \lambda_j)$, see [16], and the sum of all the eigenvalues has to be zero, $\sum_{i=0}^d m_i \lambda_i = \text{tr } \mathbf{A} = 0$. Note that, in fact, the polynomial h has also the roots $\lambda_1, \dots, \lambda_d$.

Another approach is to notice that each coefficient of $H(x) = \sum_{j=0}^d h_j x^j$ can be written as $h_j = w_j^j + w_{j+1}^j + \cdots + w_d^j$, where w_i^j is the coefficient of degree j of the polynomial p_i . In particular, if Γ is regular, then

$$H(x) = \frac{n}{\pi_0} \prod_{i=1}^d (x - \lambda_i) = \frac{n}{\pi_0} \sum_{C \subset [d]} (-1)^{|C|} x^{d-|C|} \left(\prod_{j \in C} \lambda_j \right),$$

where $[d] = \{0, 1, \dots, d\}$, $\pi_0 = \prod_{j=1}^d |\lambda_0 - \lambda_j|$, and, hence, we have the system of d equations

$$h_j = w_j^j + w_{j+1}^j + \cdots + w_d^j = \frac{n}{\pi_0} \sum_{|C|=d-j} (-1)^{d-j} \left(\prod_{i \in C} \lambda_i \right) \quad j = 0, \dots, d-1,$$

with unknowns $\lambda_1, \dots, \lambda_d$.

3.3 From the predistance polynomials to the preintersection numbers

In this subsection we assume that the predistance polynomials of a graph Γ are given and, from them, we want to obtain its preintersection numbers. Of course, we could do so by applying (5), but this requires to know the spectrum of Γ , which requires an intermediate computation (as shown in Subsection 3.2). Consequently, we want to relate directly the preintersection numbers to (the coefficients of) the predistance polynomials. With this aim, we use both the three-term recurrence (3) and the generic expression of each polynomial p_i as above. This leads to the following result.

Proposition 3.2. *Given the predistance polynomials of a graph Γ , $p_i = \sum_{j=0}^i \omega_i^j x^j$, its preintersection numbers are:*

- (a) $\alpha_0 = -\frac{\omega_1^0}{\omega_1^1}, \quad \alpha_i = \frac{\omega_i^{i-1}}{\omega_i^i} - \frac{\omega_{i+1}^i}{\omega_{i+1}^{i+1}} \quad (1 \leq i \leq d-1);$
- (b) $\beta_i = \frac{\omega_{i+1}^{i-1}}{\omega_i^i} - \frac{\omega_{i+1}^i}{\omega_i^i} \left(\frac{\omega_{i+1}^i}{\omega_{i+1}^{i+1}} - \frac{\omega_{i+2}^{i+1}}{\omega_{i+2}^{i+2}} \right) - \frac{\omega_{i+1}^{i+1}}{\omega_{i+2}^{i+2}} \frac{\omega_{i+2}^i}{\omega_i^i} \quad (0 \leq i \leq d-2);$
- (c) $\gamma_i = \frac{\omega_{i-1}^{i-1}}{\omega_i^i} \quad (1 \leq i \leq d).$

Proof. By using the expressions of p_{i-1} , p_i , and p_{i+1} in (3), and considering the terms of degree $i+1$, we get

$$\omega_i^i = \gamma_{i+1} \omega_{i+1}^{i+1}, \quad i = 0, \dots, d-1,$$

giving (c).

Analogously, from the term of degree i , we have

$$\omega_i^{i-1} = \alpha_i \omega_i^i + \gamma_{i+1} \omega_{i+1}^i$$

whence, by using the value of γ_{i+1} , we obtain

$$\omega_i^{i-1} = \alpha_i \omega_i^i + \frac{\omega_i^i}{\omega_{i+1}^{i+1}} \omega_{i+1}^i$$

giving (a) for $1 \leq i \leq d-1$. The value of α_0 is obtained from (3) with $i=0$ and the value of γ_1 .

Finally, looking at the terms of degree $i-1$:

$$\omega_i^{i-2} = \beta_{i-1} \omega_{i-1}^{i-1} + \alpha_i \omega_i^{i-1} + \gamma_{i+1} \omega_{i+1}^{i-1},$$

and using the values for α_i and γ_{i+1} , we have

$$\omega_i^{i-2} = \beta_{i-1} \omega_{i-1}^{i-1} + \left(\frac{\omega_i^{i-1}}{\omega_i^i} - \frac{\omega_{i+1}^i}{\omega_{i+1}^{i+1}} \right) \omega_i^{i-1} + \frac{\omega_{i-1}^{i-1}}{\omega_i^i} \omega_{i+1}^{i-1}.$$

This yields the value of β_i for $1 \leq i \leq d-2$. The value of β_0 is obtained from (3) with $i=1$, and the values of α_1 and γ_2 . This also yields (b) with $i=0$, by setting $\omega_1^{-1} = 0$. \square

Note that, in the above result, α_d and β_{d-1} do not need to be mentioned, since they are computed by using Lemma 2.1(a1) with $\lambda_0 = \alpha_0 + \beta_0$.

A matrix approach

The above computation can be also carried out by using a matrix approach. To this end, let us consider the given matrices

$$\mathbf{\Omega} = \begin{pmatrix} \omega_0^0 & & & & & \\ \omega_1^0 & \omega_1^1 & & & & \\ \omega_2^0 & \omega_2^0 & \omega_2^2 & & & \\ \vdots & & & \ddots & & \\ \omega_d^0 & & \dots & & \omega_d^d & \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (11)$$

where, as above, the ω_i^j , $i, j = 0, \dots, d$ stand for the coefficients of the predistance polynomials; and the \mathbf{U} matrix is a $(d+1) \times (d+1)$ matrix with ones in the upper diagonal. From them, we want to find the tridiagonal matrix of the preintersection numbers of Γ :

$$\mathbf{R} = \begin{pmatrix} \alpha_0 & \gamma_1 & & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & & \\ & \beta_1 & \alpha_2 & & & \\ & & & \ddots & & \gamma_d \\ & & & & \beta_{d-1} & \alpha_d \end{pmatrix}.$$

Then, we have the following result.

Proposition 3.3. *Let Γ be a graph with predistance polynomials p_0, \dots, p_d , and coefficient matrix Ω . Let Ω' and \mathbf{R}' be the matrices obtained, respectively, from Ω and \mathbf{R} by removing its last row. Then,*

$$\mathbf{R}' = \Omega' \mathbf{U} \Omega^{-1},$$

Proof. By using the (column) vectors $\mathbf{p} = (p_0, p_1, \dots, p_d)^\top$ and $\mathbf{x} = (1, x, \dots, x^d)^\top$, and \mathbf{p}' and \mathbf{x}' obtained from \mathbf{p} and \mathbf{x} by deleting the last entry, we have $\mathbf{p} = \Omega \mathbf{x}$, $\mathbf{p}' = \Omega' \mathbf{x}'$, and $x \mathbf{x}' = \mathbf{U} \mathbf{x}$. Moreover, the first d equations in (4) are $x \mathbf{p}' = \mathbf{R}' \mathbf{p}$. Then, all together yields

$$x \Omega' \mathbf{x}' = \Omega' \mathbf{U} \mathbf{x} = \mathbf{R}' \Omega \mathbf{x},$$

so that $(\Omega' \mathbf{U} - \mathbf{R}' \Omega) \mathbf{x} = \mathbf{0}$ and, then, it must be $\Omega' \mathbf{U} = \mathbf{R}' \Omega$, whence the result follows. \square

Finally, the last row of \mathbf{R} is computed by using Lemma 2.1(a1).

3.4 From the preintersection numbers to the predistance polynomials

To obtain the predistance polynomials from the preintersection numbers of a graph Γ , we just need to apply the three-term recurrence (3) which, initialized by $p_0 = 1$, yields:

$$p_i = \frac{1}{\gamma_i} [(x - \alpha_{i-1}) p_{i-1} - \beta_{i-2} p_{i-2}], \quad i = 1, \dots, d. \quad (12)$$

In particular, as stated in Lemma 2.1(a), we get $p_1 = (\lambda_0/\bar{k})x$, so that Γ is regular if and only if $p_1 = x$.

Alternatively, we can also compute p_i directly by using the principal submatrix of the recurrence matrix \mathbf{R} in (4). Namely,

$$\mathbf{R}_i = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & & \\ & & & \ddots & \gamma_i \\ & & & \beta_{i-1} & \alpha_i \end{pmatrix}, \quad i = 0, 1, \dots, d.$$

Proposition 3.4. *The predistance polynomial p_i associated to the recurrence matrix \mathbf{R} is*

$$p_i = \frac{1}{\gamma_0 \cdots \gamma_i} p_c(\mathbf{R}_{i-1}), \quad i = 1, \dots, d, \quad (13)$$

where $p_c(\mathbf{R}_{i-1})$ stands for the characteristic polynomial of \mathbf{R}_{i-1} .

Proof. By induction. The result holds for $i = 1, 2$ since, by (12), we get

$$p_1 = \frac{1}{\gamma_1} (x - \alpha_0) = \frac{1}{\gamma_1} p_c(\mathbf{R}_0), \quad p_2 = \frac{1}{\gamma_1 \gamma_2} [(x - \alpha_0)(x - \alpha_1) - \beta_0 \gamma_1] = \frac{1}{\gamma_1 \gamma_2} p_c(\mathbf{R}_1).$$

Then, we assume that the result holds for all values smaller than $i(\geq 3)$ and prove that $\det(x\mathbf{I} - \mathbf{R}_{i-1}) = p_i$ developing by the last column. \square

Also, we can obtain explicit formulas for the coefficients of the polynomials in terms of the preintersection numbers.

Lemma 3.5. *Given the preintersection numbers α_i , β_i , and γ_i of a graph Γ , the three coefficients of the higher degree terms of its predistance polynomial $p_i = \omega_i^i x^i + \omega_i^{i-1} x^{i-1} + \dots + \omega_i^0$, are:*

$$\begin{aligned} (i) \quad \omega_i^i &= \frac{1}{\gamma_1 \gamma_2 \dots \gamma_i}; \\ (ii) \quad \omega_i^{i-1} &= -\frac{\alpha_0 + \dots + \alpha_{i-1}}{\gamma_1 \gamma_2 \dots \gamma_i}; \\ (iii) \quad \omega_i^{i-2} &= \frac{\sum_{0 \leq r < s \leq i-1} \alpha_r \alpha_s - \sum_{r=0}^{i-2} \beta_r \gamma_{r+1}}{\gamma_1 \gamma_2 \dots \gamma_i}. \end{aligned}$$

Proof. By using induction with the three-term recurrence (3), we get:

- (i) The principal coefficient of the polynomial $p_i = \frac{1}{\gamma_i} [(x - \alpha_{i-1})p_{i-1} - \beta_{i-2}p_{i-2}]$ is the principal coefficient of $\frac{1}{\gamma_i} p_{i-1}$, that is, $\frac{1}{\gamma_i} \omega_{i-1}^{i-1}$.
- (ii) The second coefficient of p_i can be expressed in terms of the previous coefficients as:

$$\omega_i^{i-1} = \frac{\omega_{i-1}^{i-2} - \alpha_{i-1} \omega_{i-1}^{i-1}}{\gamma_i},$$

and using the first statement we have:

$$\omega_i^{i-1} = \frac{\omega_{i-1}^{i-2}}{\gamma_i} - \alpha_{i-1} \frac{1}{\gamma_2 \dots \gamma_i}.$$

- (iii) For the coefficient of the third highest degree term, we get:

$$\omega_i^{i-2} = \frac{\omega_{i-1}^{i-3} - \alpha_{i-1} \omega_{i-1}^{i-2} - \beta_{i-2} \omega_{i-2}^{i-2}}{\gamma_i},$$

which, in addition with the previous results, it can be expressed as:

$$\omega_i^{i-2} = \frac{\omega_{i-1}^{i-3}}{\gamma_i} - \frac{\alpha_1 \alpha_{i-1} + \dots + \alpha_{i-2} \alpha_{i-1}}{\gamma_2 \dots \gamma_i} - \frac{\beta_{i-2} \gamma_{i-1}}{\gamma_2 \dots \gamma_i}.$$

\square

Of course, this procedure can be carried on by calculating each ω_i^j from the three-term recurrence and using the expressions of the previously computed $\omega_i^i, \dots, \omega_i^{j+1}$.

A linear system approach

The above computations can be also carried out by using a matrix approach. Indeed, they can be set as a linear system by using the matrix approach in Proposition 3.3 of the previous subsection.

Considering the preintersection parameters as fixed scalars and ω_i^j as variables in the equality $\mathbf{R}'\boldsymbol{\Omega} = \boldsymbol{\Omega}'\mathbf{U}$, we have a set of linear equations:

$$\{\beta_{i-1}\omega_{i-1}^j + \alpha_i\omega_i^j + \gamma_{i+1}\omega_{i+1}^j = \omega_i^{j-1}\}_{0 \leq i \leq d-1, 0 \leq j \leq i+1 \leq d}.$$

This is an homogeneous linear equation system of order $\frac{d(d+3)}{2}$ with $\frac{(d+1)(d+2)}{2}$ variables.

Finally, we add the equation $\omega_0^0 = 1$ and obtain a unique solution of the system with the predistance polynomial coefficients.

This linear system only depends on the preintersection numbers α_i , β_i and γ_i . Thus, all the $\frac{(d+1)(d+2)}{2}$ polynomial coefficients depend only on the $2d$ intersection parameters (recall that the third parameter is determined by the two others because $\alpha_i + \beta_i + \gamma_i = \beta_0$).

Proposition 3.6. *The linear system given by the equations*

$$\begin{aligned} \omega_0^0 &= 1 \\ \{\beta_{i-1}\omega_{i-1}^j + \alpha_i\omega_i^j + \gamma_{i+1}\omega_{i+1}^j &= \omega_i^{j-1}\}_{0 \leq i \leq d-1, 0 \leq j \leq i+1 \leq d} \end{aligned}$$

has the (unique) solution

$$\mathbf{X}_\omega = \begin{pmatrix} \omega_0^0 \\ \vdots \\ \omega_d^0 \\ \vdots \\ \omega_d^d \end{pmatrix}.$$

Proof. By construction, the column vector of the polynomial coefficients \mathbf{X}_ω is a solution of the system. To show unicity, we have to see that the coefficient matrix

$$M = \begin{pmatrix} 1, 0, \dots, 0 \\ \mathbf{R}' \\ -I & \mathbf{R}' & & \\ & -I & \mathbf{R}' & \\ & & \dots & \\ & 1, 0, \dots, 0 & & \\ & & 1, 0, 0, \dots, 0 & \\ & & 0, 1, 0, \dots, 0 & \dots \end{pmatrix}$$

has rank $(d+1)^2$. To this end, we reorder its rows to get a triangular block matrix:

$$M^* = \begin{pmatrix} 1, 0, \dots, 0 \\ \mathbf{R}' \\ -I & 1, 0, \dots, 0 \\ & \mathbf{R}' \\ & -I & 1, 0, \dots, 0 \\ & & \mathbf{R}' \\ & & -I & \ddots \\ & & \dots & \dots \end{pmatrix}. \quad (14)$$

Each one of the $d+1$ square blocks in the diagonal is formed by the matrix \mathbf{R}' with the first canonical vector added as the first row. This gives a triangular matrix with nonnull term in the diagonal. The determinant of the submatrix of M^* of its first $(d+1)^2$ rows is $(\gamma_1 \gamma_2 \dots \gamma_d)^{d+1} \neq 0$. Finally, we can avoid the rows and columns corresponding to the indexes i, j with $i < j$, because $\omega_i^j = 0$ for every $i < j$. \square

We can see an explicit example in Subsection 3.7.

3.5 From the preintersection numbers to the spectrum

Let us now see how the preintersections numbers of a graph allow us to compute its spectrum.

Proposition 3.7. *Given a graph Γ with $d+1$ distinct eigenvalues and matrix \mathbf{R} of preintersection numbers, its spectrum $\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ can be computed in the following way:*

- (a) *The different eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$ of Γ are the eigenvalues of \mathbf{R} , that is the (distinct) zeros of its characteristic polynomial $p_c(\mathbf{R}) = \det(x\mathbf{I} - \mathbf{R})$.*
- (b) *Let \mathbf{u}_i and \mathbf{v}_i be the standard (with first component 1) left and right eigenvectors corresponding to λ_i . Then, the multiplicities are given by the formulas*

$$m_i = \frac{n}{\langle \mathbf{u}_i, \mathbf{v}_i \rangle} \quad i = 0, \dots, d, \quad (15)$$

where $n = \langle \mathbf{u}_0, \mathbf{v}_0 \rangle$ is the number of vertices of Γ .

Proof. Let \mathbf{P} be the matrix indexed with $0, \dots, d$, and with entries $\mathbf{P}_{ij} = p_i(\lambda_j)$. Then, because of (4), its i -th column \mathbf{v}_i is a right λ_i -eigenvector of the recurrence matrix \mathbf{R} : $\mathbf{R}\mathbf{P} = \mathbf{P}\mathbf{D}$, where $\mathbf{D} = \text{diag}(\lambda_0, \dots, \lambda_d)$. Then, as $\mathbf{P}^{-1}\mathbf{R} = \mathbf{D}\mathbf{P}^{-1}$, the i -th row \mathbf{u}_i of \mathbf{P}^{-1} is a left λ_i -eigenvector of \mathbf{R} . Moreover, because of the orthogonal property of the

predistance polynomials with respect to the scalar product (2), the inverse of the matrix \mathbf{P} is

$$\mathbf{P}^{-1} = \frac{1}{n} \begin{pmatrix} m_0 \frac{p_0(\lambda_0)}{n_0} & m_0 \frac{p_1(\lambda_0)}{n_1} & \dots & m_0 \frac{p_d(\lambda_0)}{n_d} \\ m_1 \frac{p_0(\lambda_1)}{n_0} & m_1 \frac{p_1(\lambda_1)}{n_1} & \dots & m_1 \frac{p_d(\lambda_1)}{n_d} \\ \vdots & \vdots & & \vdots \\ m_d \frac{p_0(\lambda_d)}{n_0} & m_d \frac{p_1(\lambda_d)}{n_1} & \dots & m_d \frac{p_d(\lambda_d)}{n_d} \end{pmatrix}$$

where $n_i = p_i(\lambda_0)$. Then, from $(\mathbf{P}^{-1}\mathbf{P})_{ii} = 1$, $0 \leq i \leq d$, we get

$$m_i = n \left(\sum_{j=0}^d \frac{p_j(\lambda_i)^2}{p_j(\lambda_0)} \right)^{-1} = \frac{n}{\langle \mathbf{u}_i, \mathbf{v}_i \rangle} \quad i = 0, \dots, d, \quad (16)$$

as claimed. Finally, notice that, as $m_0 = 1$, $n = \langle \mathbf{u}_0, \mathbf{v}_0 \rangle$. \square

Note also that, in (16), the right and left eigenvectors are, respectively, $\mathbf{v}_i = (p_0(\lambda_i), p_1(\lambda_i), \dots, p_d(\lambda_i))^\top$, and $\mathbf{u}_i = \left(\frac{p_0(\lambda_i)}{p_0(\lambda_0)}, \frac{p_1(\lambda_i)}{p_1(\lambda_0)}, \dots, \frac{p_d(\lambda_i)}{p_d(\lambda_0)} \right)$. In the particular case when Γ is a distance-regular graph, an alternative proof of (15) without using the orthogonal polynomials was given by Biggs [29].

3.6 From the spectrum to the preintersection numbers

As far as we know, in the case of distance-regular graphs there were no formulas relating directly the preintersection numbers to the eigenvalues and multiplicities of a graph. Within this context, in the Appendix of the paper by Van Dam and Haemers [12], the authors wrote the following: “In this appendix we sketch a proof of the following result: for a distance-regular graph the spectrum determines the intersection array. This less-known but relevant result (mentioned in the introduction) has been observed before, but it does not seem to be readily available in the literature.” Their method consists of three steps: first, use the scalar product (2) to find the predistance polynomials, as explained in Subsection 3.1 (apply Gram-Schmidt orthogonalisation and the normalization condition); second, compute the distance matrices of the graph by applying the distance polynomials to its adjacency matrix; and third, calculate the intersection parameters from the distance matrices.

However, in our context of a general graph, this method does not apply, since neither the distance matrices can be obtained from the predistance polynomials, nor the preintersection numbers are related to such matrices. Instead, an alternative would be to compute the predistance polynomials as in Subsection 3.1, and then calculate the preintersection numbers by applying the results of Subsection 3.4. Let us see that, if we follow properly this procedure, we can obtain explicit formulas for the preintersection numbers in terms only of the information given by the spectrum. To this end, we call into play the average numbers of closed walks as a new piece of information. In fact, these averages also determine univocally the spectrum, in the same way as the predistance polynomials and

the preintersection numbers do. These averages can be seen as a generalization of the numbers of closed ℓ -walks in a distance-regular graph, where, for any fixed length ℓ , they do not depend on the root vertex.

A Fourier coefficient approach on closed walks

We give a procedure to obtain the preintersection parameters in terms of the spectrum, by considering them as Fourier coefficients of the predistance polynomials. This procedure, also involve the relation between the predistance polynomials and the closed walks.

The average closed walks can be seen as procedure in instance to obtain the classic pieces of information. Also, they can be seen as an independent combinatorial piece of information from whose we can obtain algebraic and combinatorial properties of the graph.

Proposition 3.8. *Let Γ be a graph. Then, its preintersection numbers can be computed directly from its spectrum $\text{sp}\Gamma$ by using the average number of closed walks of length ℓ , that is,*

$$\bar{c}(\ell) = \frac{1}{n} \text{tr}(\mathbf{A}^\ell) = \frac{1}{n} \sum_{i=0}^d m_i \lambda_i^\ell$$

for $\ell = 0, 1, 2, \dots$, and their first values are:

$$\alpha_0 = 0, \quad \beta_0 = \lambda_0, \tag{17}$$

$$\gamma_1 = \frac{\bar{c}(2)}{\lambda_0}, \quad \alpha_1 = \frac{\bar{c}(3)}{\bar{c}(2)}, \quad \beta_1 = \lambda_0 - \alpha_1 - \gamma_1, \tag{18}$$

$$\gamma_2 = \frac{\lambda_0[\bar{c}(2)\bar{c}(4) - \bar{c}(3)^2 - \bar{c}(2)^3]}{\bar{c}(2)[\lambda_0^2\bar{c}(2) - \bar{c}(3)\lambda_0 - \bar{c}(2)^2]}, \quad \alpha_2 = \frac{\bar{c}(2)^2\bar{c}(5) - 2\bar{c}(2)\bar{c}(3)\bar{c}(4) - \bar{c}(3)^3}{\bar{c}(2)[\bar{c}(2)\bar{c}(4) - \bar{c}(3)^2 - \bar{c}(2)^3]}, \quad \beta_2 = \dots \tag{19}$$

Proof. The proof is by induction. We know that, giving the predistance polynomials p_0, \dots, p_{i-1} , $i \geq 1$, the Gram-Schmidt method yields

$$p_i = \frac{r_i(\lambda_0)}{\|r_i\|^2} r_i \tag{20}$$

where

$$r_i = x^i - \sum_{j=0}^{i-1} \frac{\langle x^i, p_j \rangle}{\|p_j\|^2} p_j = x^i - \sum_{j=0}^{i-1} \frac{\sum_{h=0}^d m(\lambda_h) \lambda_h^i p_j(\lambda_h)}{\sum_{h=0}^d m(\lambda_h) p_j^2(\lambda_h)} p_j.$$

Then, from $p_0 = 1$, we obtain that $p_1 = \frac{\lambda_0}{\bar{c}(2)}x$, whence, applying the formulas

$$\gamma_i = \frac{\langle xp_{i-1}, p_i \rangle}{\|p_i\|^2} = \frac{1}{p_i(\lambda_0)} \frac{1}{n} \sum_{j=0}^d m(\lambda_j) \lambda_j p_i(\lambda_j) p_{i-1}(\lambda_j) = \frac{\text{tr}(\mathbf{A}(p_{i-1}p_i)(\mathbf{A}))}{\text{tr}(p_i^2(\mathbf{A}))}, \quad (21)$$

$$\alpha_i = \frac{\langle xp_i, p_i \rangle}{\|p_i\|^2} = \frac{1}{p_i(\lambda_0)} \frac{1}{n} \sum_{j=0}^d m(\lambda_j) \lambda_j p_i^2(\lambda_j) = \frac{\text{tr}(\mathbf{A}p_i^2(\mathbf{A}))}{\text{tr}(p_i^2(\mathbf{A}))}, \quad (22)$$

$$\beta_i = \lambda_0 - \alpha_i - \gamma_i, \quad (23)$$

with $i = 0, 1$ we get (17) and (18). In general, if all the coefficients of the predistance polynomials p_0, \dots, p_{i-1} , $i \geq 1$, are given in terms of the numbers $\bar{c}(\ell)$'s, we proceed in the same way by first calculating p_i and then applying the formulas (21)-(22). This assures that the obtained preintersections parameters α_i , β_i , and γ_i will be expressed also in terms of the $\bar{c}(\ell)$'s. For instance, the computation for $i = 2$ give the results in (19). \square

3.7 An example

In this subsection, we illustrate the previous results given in this section with one example. Let Γ be the graph 4.47 of Table 4 in the textbook of Cvetković, Doob, and Sachs [7], shown in Fig. 5, which has $n = 9$ vertices, and spectrum

$$\text{sp } \Gamma = \left\{ 3^1, \left(\frac{-1 + \sqrt{13}}{2} \right)^2, 0^3, (-1)^1, \left(\frac{-1 - \sqrt{13}}{2} \right)^2 \right\}.$$

Thus, Γ has $d + 1 = 5$ distinct eigenvalues. We are going to apply the procedures and the equivalence results in order to obtain all the other pieces of information.

From the spectrum to the predistance polynomials

As mentioned in Subsection 3.1, the sequence of predistance polynomials p_0, p_1, p_2, p_3, p_4 are orthogonal with respect to the scalar product

$$\langle f, g \rangle_{\Gamma} = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i) g(\lambda_i),$$

and normalized in such a way that $\|p_i\|_{\Gamma}^2 = p_i(\lambda_0)$. Then, we can obtain them by applying the Gram-Schmidt method, starting from the sequence $1, x, x^2, x^3, x^4$, and using the values:

$$\text{tr}(\mathbf{A}^0) = 9, \quad \text{tr}(\mathbf{A}^1) = 0, \quad \text{tr}(\mathbf{A}^2) = 24, \quad \text{tr}(\mathbf{A}^3) = 6, \quad \text{tr}(\mathbf{A}^4) = 144, \dots$$

we can calculate for the first polynomial:

$$r_1 = x - \frac{\langle x, 1 \rangle_{\Gamma}}{\langle 1, 1 \rangle_{\Gamma}} 1 = x - \frac{\text{tr}(\mathbf{A})}{\text{tr}(\mathbf{A}^0)} = x,$$

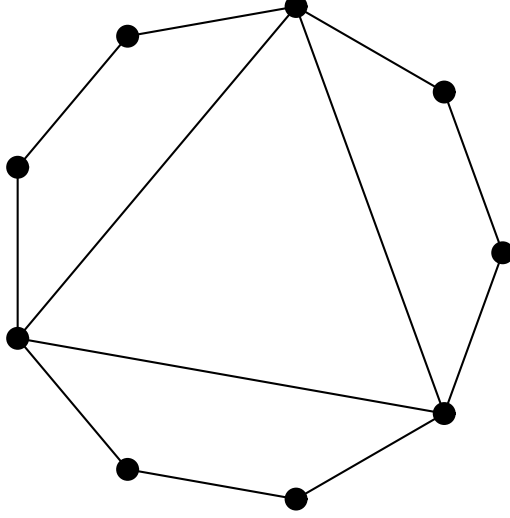


Figure 5: The graph 4.47 in Table 4 of [7].

and normalize with:

$$p_1 = \frac{r_1(\lambda_0)}{\langle r_1, r_1 \rangle_\Gamma} r_1 = \frac{3}{\frac{1}{9} \text{tr}(\mathbf{A}^2)} r_1 = \frac{9}{8} x.$$

For the second predistance polynomial, we have:

$$r_2 = x^2 - \frac{\langle x^2, 1 \rangle_\Gamma}{\langle 1, 1 \rangle_\Gamma} 1 - \frac{\langle x^2, p_1 \rangle_\Gamma}{\langle p_1, p_1 \rangle_\Gamma} p_1 = x^2 - \frac{\text{tr}(\mathbf{A}^2)}{\text{tr}(\mathbf{A}^0)} - \frac{\text{tr}(\mathbf{A}^2 p_1(\mathbf{A}))}{\text{tr}(p_1^2(\mathbf{A}))} p_1 = x^2 - \frac{x}{4} - \frac{8}{3}.$$

We normalize this polynomial with the values $r_2(\lambda_0) = 3^2 - \frac{3}{4} - \frac{8}{3} = \frac{67}{12}$ and

$$\begin{aligned} \langle r_2, r_2 \rangle_\Gamma &= \frac{1}{9} \text{tr} \left[\left(\mathbf{A}^2 - \frac{1}{4} \mathbf{A} - \frac{8}{3} \mathbf{I} \right)^2 \right] \\ &= \frac{1}{9} \left[\text{tr}(\mathbf{A}^4) - \frac{1}{2} \text{tr}(\mathbf{A}^3) - \frac{253}{48} \text{tr}(\mathbf{A}^2) + \frac{4}{3} \text{tr}(\mathbf{A}^1) + \frac{64}{9} \text{tr}(\mathbf{A}^0) \right] = \frac{157}{18}. \end{aligned}$$

We obtain:

$$p_2 = \frac{r_2(\lambda_0)}{\langle r_2, r_2 \rangle_\Gamma} r_2 = \frac{201}{314} r_2 = -\frac{268}{157} - \frac{201}{1256} x + \frac{201}{314} x^2.$$

Going on with that method we obtain the predistance polynomials of the graph:

$$\begin{aligned}
p_0(x) &= 1, \\
p_1(x) &= \frac{9}{8}x, \\
p_2(x) &= -\frac{268}{157} - \frac{201}{1256}x + \frac{201}{314}x^2, \\
p_3(x) &= \frac{23607}{50711} - \frac{83082}{50711}x - \frac{732}{2983}x^2 + \frac{183}{646}x^3, \\
p_4(x) &= \frac{78}{323} + \frac{547}{1292}x - \frac{32}{57}x^2 - \frac{113}{969}x^3 + \frac{1}{12}x^4.
\end{aligned}$$

From the predistance polynomials to the spectrum

To obtain the spectrum from the predistance polynomials, we can use the results in Proposition 3.1. So, the Hoffman polynomial $H = p_0 + p_1 + p_2 + p_3 + p_4$ is

$$H(x) = -\frac{1}{4}x - \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4,$$

with zeros being the distinct eigenvalues different from λ_0 :

$$\lambda_1 = \frac{-1 + \sqrt{13}}{2}, \quad \lambda_2 = 0, \quad \lambda_3 = -1, \quad \lambda_4 = \frac{-1 - \sqrt{13}}{2}.$$

Moreover, the largest root of the polynomial given by (9) is $\lambda_0 = 3$. Alternatively, by (6),

$$\lambda_0 = -\frac{\omega_1^1 \omega_2^0}{\omega_2^2} = -(9/8)(-268/157)/(201/314) = 3.$$

Moreover, the values of the parameters ϕ_i and $p_d(\lambda_i)$ are:

$$\phi_0 = 108, \quad \phi_1 = \frac{3}{2}[13 - 7\sqrt{13}], \quad \phi_2 = 9, \quad \phi_3 = -12, \quad \phi_4 = \frac{3}{2}[13 + 7\sqrt{13}],$$

and

$$\begin{aligned}
p_4(\lambda_0) &= \frac{39}{646}, \quad p_4(\lambda_1) = -\frac{1}{646}[39 + 21\sqrt{13}], \quad p_4(\lambda_2) = \frac{78}{323}, \\
p_4(\lambda_3) &= -\frac{351}{646}, \quad p_4(\lambda_4) = \frac{1}{646}[-39 + 21\sqrt{13}].
\end{aligned}$$

Thus, by applying (10), we get the multiplicities:

$$m_0 = 1, \quad m_1 = 2, \quad m_2 = 3, \quad m_3 = 1, \quad m_4 = 2.$$

From the predistance polynomials to the preintersection numbers

To obtain the preintersection numbers by using the predistance polynomials, we apply Proposition 3.3 giving the relationship between the preintersection matrix and the polynomial coefficient matrix of the graph.

The matrix Ω containing the polynomial coefficients of the graph Γ is:

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 9/8 & 0 & 0 & 0 \\ -268/157 & -201/1256 & 201/314 & 0 & 0 \\ 23607/50711 & -83082/50711 & -732/2983 & 183/646 & 0 \\ 78/323 & 547/1292 & -32/57 & -113/969 & 1/12 \end{pmatrix}.$$

Then, with U given in (11), Proposition 3.3 yields:

$$\mathbf{R}' = \Omega' U \Omega^{-1} = \begin{pmatrix} 0 & 8/9 & 0 & 0 & 0 \\ 3 & 1/4 & 471/268 & 0 & 0 \\ 0 & 67/36 & 387/628 & 21641/9577 & 0 \\ 0 & 0 & 6588/10519 & 27036/50711 & 1098/323 \end{pmatrix},$$

and finally, we add the last row of the matrix \mathbf{R} of preintersection numbers by using the equality $\alpha_i + \beta_i + \gamma_i = \beta_0 = 3$:

$$\begin{pmatrix} \alpha_0 & \gamma_1 & & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & & \\ & \beta_1 & \alpha_2 & \gamma_3 & & \\ & & \beta_2 & \alpha_3 & \gamma_4 & \\ & & & \beta_3 & \alpha_4 & \end{pmatrix} = \begin{pmatrix} 0 & 8/9 & & & & \\ 3 & 1/4 & 471/268 & & & \\ & 67/36 & 387/628 & 21641/9577 & & \\ & & 6588/10519 & 27036/50711 & 1098/323 & \\ & & & 4082/19703 & -129/323 & \end{pmatrix}.$$

From the preintersection numbers to the predistance polynomials

In order to obtain the predistance polynomials from the preintersection numbers of Γ , we apply the three-term recurrence (12) which, initialized with $p_0 = 1$, yields:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= \frac{(x - \alpha_0)p_0 - \beta_{(-1)}p_{(-1)}}{\gamma_1} = \frac{9}{8}x, \\ p_2(x) &= \frac{(x - \frac{1}{4})p_1 - 3}{\frac{471}{268}} = -\frac{268}{157} - \frac{201}{1256}x + \frac{201}{314}x^2, \\ p_3(x) &= \frac{(x - \frac{387}{628})p_2 - \frac{67}{36}p_1}{\frac{21641}{9577}} = \frac{23607}{50711} - \frac{83082}{50711}x - \frac{732}{2983}x^2 + \frac{183}{646}x^3, \\ p_4(x) &= \frac{(x - \frac{27036}{50711})p_3 - \frac{6588}{10519}p_2}{\frac{1098}{323}} = \frac{78}{323} + \frac{547}{1292}x - \frac{32}{57}x^2 - \frac{113}{969}x^3 + \frac{1}{12}x^4. \end{aligned}$$

Alternatively, we can compute the characteristic polynomial of each submatrix \mathbf{R}_{i-1} for $i = 1, \dots, d$. Then, we get:

$$\begin{aligned}
p_1(x) &= \frac{9}{8} \det(x\mathbf{I} - \begin{pmatrix} 0 \end{pmatrix}) = \frac{9}{8}x, \\
p_2(x) &= \frac{9}{8} \cdot \frac{268}{471} \det\left(x\mathbf{I} - \begin{pmatrix} 0 & 8/9 \\ 3 & 1/4 \end{pmatrix}\right) = -\frac{268}{157} - \frac{201}{1256}x + \frac{201}{314}x^2, \\
p_3(x) &= \frac{9}{8} \cdot \frac{268}{471} \cdot \frac{9577}{21641} \det\left(x\mathbf{I} - \begin{pmatrix} 0 & 8/9 & \\ 3 & 1/4 & 471/268 \\ & 67/36 & 387/628 \end{pmatrix}\right) \\
&= \frac{23607}{50711} - \frac{83082}{50711}x - \frac{732}{2983}x^2 + \frac{183}{646}x^3, \\
p_4(x) &= \frac{9}{8} \cdot \frac{268}{471} \cdot \frac{9577}{21641} \cdot \frac{323}{1098} \det\left(x\mathbf{I} - \begin{pmatrix} 0 & 8/9 & & \\ 3 & 1/4 & 471/268 & \\ & 67/36 & 387/628 & 21641/9577 \\ & & 6588/10519 & 27036/50711 \end{pmatrix}\right) \\
&= \frac{78}{323} + \frac{547}{1292}x - \frac{32}{57}x^2 - \frac{113}{969}x^3 + \frac{1}{12}x^4.
\end{aligned}$$

We can also check that the principal coefficient of each predistance polynomials is easily determined by the parameters γ_i 's:

$$\begin{aligned}
\omega_1^1 &= \frac{1}{\gamma_1} = \frac{9}{8}, \\
\omega_2^2 &= \frac{1}{\gamma_1\gamma_2} = \frac{9}{8} \cdot \frac{268}{471} = \frac{201}{314}, \\
\omega_3^3 &= \frac{1}{\gamma_1\gamma_2\gamma_3} = \frac{9}{8} \cdot \frac{268}{471} \cdot \frac{9577}{21641} = \frac{183}{646}, \\
\omega_4^4 &= \frac{1}{\gamma_1\gamma_2\gamma_3\gamma_4} = \frac{9}{8} \cdot \frac{268}{471} \cdot \frac{9577}{21641} \cdot \frac{323}{1098} = \frac{1}{12}.
\end{aligned}$$

With a linear system approach

Let \mathbf{C} be the matrix of dimension 15 of the linear system described in Proposition 3.6 and \mathbf{e}_1 the first canonical vector of dimension 15.

so giving $n = \langle \mathbf{u}_0, \mathbf{v}_0 \rangle = 9$. Now, let us consider, for example, the eigenvalue $\lambda_2 = 0$. Then, the corresponding left and right normalized eigenvectors of \mathbf{R} are

$$\mathbf{u}_2 = \left(1, 0, -\frac{32}{67}, 0, \frac{86}{183}, 4 \right) \quad \text{and} \quad \mathbf{v}_2 = \left(1, 0, -\frac{286}{157}, \frac{23607}{50711}, \frac{78}{323} \right).$$

Then, we get

$$m_2 = \frac{n}{\langle \mathbf{u}_2, \mathbf{v}_2 \rangle} = 3,$$

and similar computations give $\langle \mathbf{u}_1, \mathbf{v}_1 \rangle = \frac{9}{2}$, $\langle \mathbf{u}_3, \mathbf{v}_3 \rangle = 9$, and $\langle \mathbf{u}_4, \mathbf{v}_4 \rangle = \frac{9}{2}$ so giving the other multiplicities $m_1 = 2$, $m_3 = 1$, and $m_4 = 2$.

From the spectrum to the preintersection number. A Fourier coefficient approach on closed walks

In our case, the average numbers of walks of length $\ell = 0, 1, \dots, 5$ turn out to be

$$\bar{c}(0) = 1, \quad \bar{c}(1) = 0, \quad \bar{c}(2) = \frac{8}{3}, \quad \bar{c}(3) = \frac{2}{3}, \quad \bar{c}(4) = 16, \quad \bar{c}(5) = \frac{40}{3},$$

and, then, Proposition 3.8 gives:

$$\alpha_0 = 0, \quad \beta_0 = 3, \quad \gamma_1 = \frac{8}{9}, \quad \alpha_1 = \frac{1}{4}, \quad \beta_1 = \frac{67}{36}, \quad \gamma_2 = \frac{471}{268}, \quad \alpha_2 = \frac{387}{628}, \quad \beta_2 = \frac{6588}{10519}, \dots$$

and we can keep applying the method to obtain the remaining preintersection numbers.

4 Characterizations of distance-regularity in terms of the pieces of information

In this Chapter we present some applications of the information given by the spectrum, the predistance polynomials, and the preintersection numbers of a given graph. These pieces of information have some combinatorial and structural implications. Moreover we show how the equivalences of these informations allows us to rewrite some of the properties and/or conditions.

Some of these consequences can be derived from the preintersection numbers, which are the generalization of the intersection numbers in a distance regular graph, that has combinatorial meaning. However, the preintersection numbers are defined and calculated as a completely algebraic piece of information. And even being an algebraic family of parameters, can be interpreted some combinatorial properties of the graph in the same way that intersection parameters do in a distance regular. Moreover, other properties can be deduced from pure algebraic pieces of information like the spectrum or the predistance polynomials.

4.1 Characterizations of distance-regularity

We can start giving some characterizations of distance-regularity in graphs, which are given in terms of the different studied informations. We begin with the so-called ‘spectral excess theorem’ (see Fiol and Garriga [32]), which can be seen as a quasi-spectral characterization of a distance-regular graph.

Theorem 4.1. (The spectral excess theorem) *Let $\Gamma = (V, E)$ be a regular graph with spectrum/predistance polynomials/preintersection numbers as above. Then Γ is distance-regular if and only if its spectral excess*

$$p_d(\lambda_0) = \frac{\beta_0 \beta_1 \cdots \beta_{d-1}}{\gamma_1 \gamma_2 \cdots \gamma_d} = n \left(\sum_{i=0}^d \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1},$$

(where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$, for $i = 0, \dots, d$) equals the average excess

$$\bar{k}_d = \frac{1}{n} \sum_{u \in V} |\Gamma_d(u)|.$$

The following two theorems proved in different moments are easily related between them by the results in Subsections 3.3 and 3.4, in which we explain the relation between the preintersection parameters and the predistance polynomials in a graph. More in particular, in Lemma 3.5 we express the principal coefficient ω_i^i of the predistance polynomial p_i as:

$$\omega_i^i = \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_i}.$$

That is, the first i predistance polynomials of any graph are monic polynomials if and only if $\gamma_1 = \dots = \gamma_i = 1$.

Theorem 4.2. (Abiad, Van Dam, Fiol [1]) *Let Γ be a graph with $d+1$ distinct eigenvalues and preintersection numbers γ_i , $i = 1, \dots, d$.*

- (a) *If $\gamma_1 = \dots = \gamma_{d-1} = 1$, then Γ is distance-regular.*
- (b) *If Γ is bipartite and $\gamma_1 = \dots = \gamma_{d-2} = 1$, then Γ is distance-regular.*

Theorem 4.3. (Abiad, Van Dam, Fiol, 2016) *Let Γ be a graph with $d+1$ distinct eigenvalues and predistance polynomials p_i , $i = 0, 1, \dots, d$.*

- (a) *If all the p_i 's, are monic for $i = 1, \dots, d-1$, then Γ is distance-regular.*
- (b) *If Γ is bipartite and all the p_i 's, are monic for $i = 1, \dots, d-2$, then Γ is distance-regular.*

This two equivalent results show how the pieces of information can give us properties of the graphs as relevant as their distance-regularity. But also the possibility of view similar results in different ways.

4.2 Combinatorial properties of the pieces of information

We can see also combinatorial properties: we can easily check if the graph is bipartite or how large is its odd girth with simply checking at the coefficients of its predistance polynomials.

Proposition 4.4. *Let Γ be a graph with $d+1$ distinct eigenvalues. Then,*

- (a) *Γ is bipartite if and only if $\alpha_0 = \dots = \alpha_d = 0$.*
- (b) *If Γ is not bipartite, then it has odd girth $2m+1$ if and only if $\alpha_0 = \dots = \alpha_{m-1} = 0$ and $\alpha_m > 0$.*

Proof. Let us first prove necessity. If Γ has odd girth $2m+1$, then $\text{tr } \mathbf{A}^{2i+1} = 0$ for $i = 0, \dots, m-1$ and $\text{tr } \mathbf{A}^{2m+1} \neq 0$. Using this, it can be shown by induction (like in [13]) that $\alpha_i = 0$ for $i < m$ and that the predistance polynomials p_i are odd or even depending on the parity of i , for $i \leq m$. Moreover,

$$\alpha_m = \frac{1}{p_m(\lambda_0)} \langle xp_m, p_m \rangle_G = \frac{1}{np_m(\lambda_0)} \text{tr}(\mathbf{A}p_m^2(\mathbf{A})) \neq 0,$$

since the polynomial xp_m^2 is odd and has degree $2m+1$, so the leading term is the only one contributing to the trace.

Conversely, if the preintersection numbers satisfy $\alpha_i = 0$ for $i = 0, \dots, m-1$, then again, by (3), the parity of the predistance polynomial p_i coincides with the parity of its index i for $i = 0, \dots, m$. Then, for any $i < m$ we have that $\text{tr } \mathbf{A}^{2i+1} = n \langle \mathbf{A}^i, \mathbf{A}^{i+1} \rangle = n \langle x^i, x^{i+1} \rangle_G = 0$, as the expressions of x^i and x^{i+1} in terms of the basis p_0, \dots, p_m have polynomials with distinct parity. Thus, Γ has no odd cycles of length smaller than $2m + 1$, and since $\alpha_m \neq 0$, it follows (from the necessity part of the proof) that the odd-girth is indeed $2m + 1$. Moreover, in the case when $m = d + 1$, this implies that Γ has no odd cycles and, hence, it is bipartite. \square

From this result 4.4 and the relations between the preintersection numbers and the polynomial coefficients described in Subsections 3.3 and 3.4, we can write the following Proposition:

Proposition 4.5. *Let Γ be a graph with $d + 1$ distinct eigenvalues. Then,*

- (a)(i) *If Γ is bipartite, then $\omega_i^j = 0$ for every $i + j$ odd in the matrix Ω .*
- (a)(ii) *If Γ satisfies $\omega_i^j = 0$ for every $i + j$ odd and it is not bipartite, then it has odd girth $2d + 1$.*
- (b) *If Γ is not bipartite, then it has odd girth $2m + 1$ if and only if $\omega_i^j = 0$ for every $i + j$ odd and $i \leq m$.*

Proof. Remember the equation (12) derived from the three-terms-recurrence rule and lets proceed by induction:

$$p_i = \frac{1}{\gamma_i} [(x - \alpha_{i-1})p_{i-1} - \beta_{i-2}p_{i-2}].$$

Given p_{i-1} and p_{i-2} such that satisfy $\omega_i^j = 0$, the polynomial p_i satisfies that if and only if $\alpha_{i-1} = 0$. Proposition 4.4 completes the proof. \square

Example 4.6. • *Let consider the Petersen graph. Its predistance polynomials are:*

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_2(x) = x^2 - 3.$$

This graph satisfies the property $\omega_i^j = 0$ for $i + j$ odd, but it is not bipartite. Thus, by 4.5(a)(ii), it has maximal odd girth $2d + 1 = 5$.

- Let consider the cubic graph Γ represented in Figure 6 in the page 39. The initial predistance polynomials are:

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_2(x) = \frac{1}{1.111}[x^2 - 3],$$

$$p_3(x) = \frac{1}{2.823}[(x - 0.75)p_2 - 2x] = \frac{1}{1.111 \cdot 2.823}[x^3 - 0.75x^2 - 5x - 2.25].$$

We can see the predistance polynomial p_3 does not satisfy the property $\omega_i^j = 0$ for $i + j$ odd; it is satisfied up to p_2 . Then it is not bipartite by 4.5(a)(i) and it has odd girth 5 by 4.5(b).

Extending the study of the girth of the graph, there is the following result:

Proposition 4.7. (a) A regular graph Γ has girth $2m + 1$ if and only if $\alpha_0 = \dots = \alpha_{m-1} = 0$, $\alpha_m \neq 0$ and $\gamma_1 = \dots = \gamma_m = 1$.

(b) A regular graph Γ has girth $2m$ if and only if $\alpha_0 = \dots = \alpha_{m-1} = 0$, $\gamma_1 = \dots = \gamma_{m-1} = 1$ and $\gamma_m > 1$.

This is a natural extension of the distance-regular graphs properties. By the translation of this kind of properties from the distance-regular graphs we can see combinatorial properties from an algebraic point of view.

In the same way we used the equivalences between information given by the preintersection numbers and the predistance polynomials, we can rewrite this in the following terms:

Proposition 4.8. Let Γ be a regular graph with $d + 1$ distinct eigenvalues,

(a) If $m < d$, then Γ has girth $2m + 1$ if and only if the polynomial coefficients satisfies $\omega_i^j = 0$ for every $i + j$ odd and $i \leq m$, $\omega_{m+1}^j \neq 0$ for some $m + j$ odd, and $\omega_1^1 = \dots = \omega_m^m = 1$.

(b) The graph Γ has girth $2m$ if and only if the polynomial coefficients satisfies $\omega_i^j = 0$ for every $i + j$ odd and $i \leq m$, $\omega_1^1 = \dots = \omega_{m-1}^{m-1} = 1$, and $\omega_m^m < 1$.

In Proposition 4.8(a), the condition $m < d$ is needed for just one of the implications, because if $m = d$, we can not consider the polynomial p_{d+1} , they are only defined up to p_d . The reversal implication holds even in the case in which $m = d$:

A regular graph Γ of girth $2m + 1$ satisfies that the polynomial coefficients hold $\omega_i^j = 0$ for every $i + j$ odd and $i \leq m$, $\omega_{m+1}^j \neq 0$ (in the case in that p_{m+1} exists) for some $m + j$ odd, and $\omega_1^1 = \dots = \omega_m^m = 1$.

4.3 Internal properties of the pieces of information

In this last subsection of the chapter we write about the existent internal properties and equalities that the preintersection numbers have between them and also with the spectrum or the predistance polynomials.

One of the most difficult relationship between the different pieces of information is the one between the spectrum and the preintersection numbers. In this following property is written a simple description of how the preintersection numbers grow and how they are related with the distinct eigenvalues.

Proposition 4.9. *Let Γ be a graph with distinct eigenvalues $\lambda_0 > \dots > \lambda_d$, and preintersection numbers α_i , β_i , and γ_i . Then,*

(a) $\gamma_i > 0$ for $i = 1, \dots, d$, and $\beta_i > 0$ for $i = 0, \dots, d - 1$.

(b) $\sum_{i=0}^d \alpha_i = \sum_{i=0}^d \lambda_i$.

Proof. (a) First note that, since $p_0 = q_0 = 1$ and $p_i = q_i - q_{i-1}$ for $i = 1, \dots, d$, by Lemma 2.1(c) we have that $p_i(\lambda_0) > 0$ for every $i = 0, \dots, d$. Thus, by Lemma 2.1(a2), we only need to prove the condition on the γ_i 's. Moreover, by the theory of orthogonal polynomials, we know that all the zeros of p_i are between λ_d and λ_0 . Consequently, the leading coefficient ω_i of p_i must be positive, as $\lim_{x \rightarrow \infty} p_i(x) = \infty$. Thus, the conclusion is obtained since by Lemma 3.5 we have $\omega_i = (\gamma_1 \dots \gamma_i)^{-1}$ for $i = 1, \dots, d$. To prove (b) just use Lemma 2.1(d) and consider the trace of the recurrence matrix \mathbf{R} . \square

In contrast with the above, we know that there are graphs such that $\lambda_0 + \dots + \lambda_d < 0$ and, hence, by Proposition 4.9(b), some of their preintersection numbers α_i must be negative. An example is the cubic graph Γ with 12 vertices and $d = 10$ of Figure 6 (no. 3.83 in [7]), which has spectrum $\text{sp } \Gamma = \{3^1, 1.7321^1, 1.4812^1, 1.2143^1, 1^2, -0.3111^1, -1^1, -1.5392^1, -1.7321^1, -2.1701^1, -2.6751^1\}$ ($\lambda_0 + \dots + \lambda_d = -1$) and preintersection numbers as shown in the next table.

i	0	1	2	3	4	5	6	7	8	9	10
β_i	3	2	1.138	0.434	0.587	0.316	0.253	0.559	0.0514	0.643	
α_i	0	0	0.750	-0.257	-0.382	-0.051	-0.849	-0.097	0.082	-0.570	0.287
γ_i		1	1.111	2.823	2.794	2.632	3.595	2.537	2.865	2.925	2.722

Note that the α_i 's sum up to -1 , in concordance with Lemma 4.9(b). Notice also that, contrarily to the case of the intersection numbers b_i 's and c_i 's, the β_i 's and the γ_i 's do not show a monotone behaviour and, even more, $\gamma_6 > \lambda_0$.

The Hoffman polynomial was introduced by Hoffman [21] in order to improve a characterization of the regular and connected graph. It says, a graph is regular and connected if and only if exists a polynomial $H(x)$ such that $H(\mathbf{A}) = \mathbf{J}$.

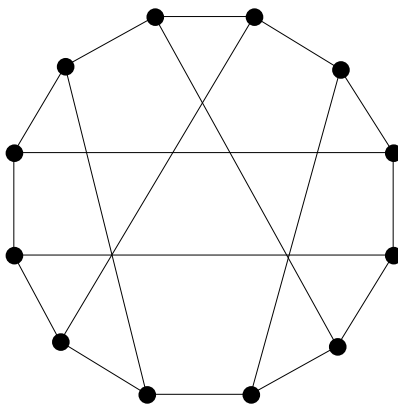


Figure 6: A cubic graph with negative preintersection numbers.

In Lemma 2.1 we give the relation between the Hoffman polynomial and the predistance polynomials of the graph. However, we can write an expression of this polynomial in terms of the eigenvalues and number of vertices:

$$H(x) = \sum_{i=0}^d p(x) = \frac{n}{\pi_0} \prod_{i=1}^d (x - \lambda_i) = \frac{n}{\pi_0} (x^d - (\lambda_1 + \dots + \lambda_d)x^{d-1} + \dots + (-1)^d \lambda_1 \dots \lambda_d), \quad (24)$$

where $\pi_0 = \prod_{i=1}^d (\lambda_0 - \lambda_i)$. This property is going to be used to write relations between the eigenvalues and the predistance polynomials in Chapter 3. This polynomial, together with the formula (3), is useful also to give relation between the coefficients of the predistance polynomials and the preintersection parameters; we are able to express each one of these coefficients in terms of the eigenvalues.

We introduce now the concept of *m-partially* distance regularity in graphs. We are not going to study this concept in depth, but we will use properties of this kind of graphs in order to prove the Proposition 4.12.

Definition 4.10. *A graph Γ with diameter D is called m -partially distance-regular, for some $0 \leq m \leq D$, if its predistance polynomials satisfy $p_i(\mathbf{A}) = \mathbf{A}_i$ for every $i \leq m$ (see Dalfo, Van Dam, Fiol, Garriga, and Gorissen [8]).*

A *m-partially distance-regular* graph works like a distance-regular graph on any subgraph formed by a vertex and the set of vertices at distance at most m from it. Thus, we can give an alternative characterization: we have that Γ is *m-partially distance-regular* when the intersection numbers a_i, b_i, c_i up to c_m are well-defined, that is, the distance matrices satisfy the recurrence

$$\mathbf{A}\mathbf{A}_i = b_{i-1}\mathbf{A}_{i-1} + a_i\mathbf{A}_i + c_{i+1}\mathbf{A}_{i+1}, \quad i = 0, \dots, m-1.$$

In this case, these intersection numbers are of course equal to the corresponding preintersection numbers $\alpha_i, \beta_i, \gamma_i$ up to γ_m .

As first example of this graphs we see that every m -partially distance-regular graph with $m \geq 2$ is regular. Also, if a graph of diameter D is D -partially distance-regular, obviously, is distance-regular. In [1], Abiad, van Dam and Fiol obtain the following lemma, which some properties in aim to increase the value of m for that we can write a graph is m -partially distance-regular.

Lemma 4.11. *Let Γ be a regular graph, and let $m \leq D$ be a positive integer. Suppose that Γ is $(m-1)$ -partially distance-regular, and any of the following conditions holds:*

- (i) $\bar{c}_m \geq \gamma_m$,
- (ii) $c_{m-1} \geq \gamma_m$,
- (iii) $k_{m-1}(\overline{a_{m-1}^2} - \alpha_{m-1}^2) + \bar{k}_m(\bar{c}_m^2 - \gamma_m^2) \geq 0$,
- (iv) $\bar{c}_m^2 \geq \gamma_m^2$,
- (v) a_{m-1} is well-defined, and $c_m(u, v) \leq \gamma_m$ for every pair of vertices $u, v \in V$ at distance m .

Then, Γ is m -partially distance-regular with intersection numbers $a_{m-1} = \alpha_{m-1}$ and $c_m = \gamma_m$.

Recall that the overline in a parameter denote the average value amount all the values calculated taking each vertex of the graph as origin vertex.

By using this lemma 4.11 (i) and, together with the mean number of walks of distance ℓ (see section 2.5), we can also obtain some related results. With this aim, remember $\bar{c}(\ell)_k$ represents the mean number of walks of length ℓ between vertices at distance k . We can extend the notation being $c(\ell)_{uv}$ the number of walks of length ℓ between the vertices u and v , and $c(\ell)_k$ the number of walks of length ℓ joining every pair of vertices at distance k if it is well defined.

Proposition 4.12. *Let Γ be a regular graph with $d+1$ distinct eigenvalues, $\lambda_0 > \dots > \lambda_d$, and girth $g \geq 2d-2$.*

- (i) *If $\alpha_{d-1} < \gamma_d$, then $\bar{c}(d)_{d-1} \geq \alpha_{d-1}\gamma_{d-1}$, with equality if and only if Γ is distance-regular.*
- (ii) *If $\alpha_{d-1} > \gamma_d$, then $\bar{c}(d)_{d-1} \leq \alpha_{d-1}\gamma_{d-1}$, with equality if and only if Γ is distance-regular.*
- (iii) *If $\alpha_{d-1} = \gamma_d$, then $c(d)_{d-1} = \alpha_{d-1}\gamma_{d-1}$ is well-defined.*

Proof. Necessity in (i) and (ii) is clear since, when Γ is distance-regular, $\alpha_{d-1} = a_{d-1}$, $\gamma_{d-1} = c_{d-1}$, and, from the hypothesis on the girth, $\gamma_i = c_i = 1$ for $i = 1, \dots, d-2$. So, the number of d -walks between every pair of vertices u, v at distance $d-1$ is $a_{d-1}c_{d-1}$.

On the other hand, as Γ has girth at least $2d-2$, we have $\alpha_{d-1} + \alpha_d = \lambda_0 + \dots + \lambda_d$. This implies

$$\gamma_d - \alpha_{d-1} = -(\lambda_1 + \dots + \lambda_d). \quad (25)$$

Let us use also the Hoffman polynomial of the graph Γ . Remember the Hoffman polynomial wrote in (24):

$$H(x) = \sum_{i=0}^d p(x) = \frac{n}{\pi_0} (x^d - (\lambda_1 + \dots + \lambda_d)x^{d-1} + \dots).$$

The factor $\frac{n}{\pi_0}$ can be calculated as the leading coefficient of p_d , which is $\frac{1}{\gamma_d \gamma_{d-1}}$ because $\gamma_i = c_i = 1$ for $i = 1, \dots, d-2$. Let us consider two vertices u and v at distance $d-1$. As $H(\mathbf{A}) = \mathbf{J}$, we have:

$$1 = \frac{1}{\gamma_d \gamma_{d-1}} ((\mathbf{A}^d)_{uv} - (\lambda_1 + \dots + \lambda_d)(\mathbf{A}^{d-1})_{uv}).$$

Thus,

$$(\lambda_1 + \dots + \lambda_d)(\mathbf{A}^{d-1})_{uv} + \gamma_d \gamma_{d-1} = (\mathbf{A}^d)_{uv} \geq 0.$$

This and (25) imply that if u, v are two vertices at distance $d-1$, then

$$(\alpha_{d-1} - \gamma_d)c_{d-1}(u, v) + \gamma_{d-1}\gamma_d = c(d)_{uv}. \quad (26)$$

Thus, by taking averages over all vertices u, v at distance $d-1$, we have

$$(\alpha_{d-1} - \gamma_d)\bar{c}_{d-1} + \gamma_{d-1}\gamma_d = \bar{c}(d)_{d-1}.$$

Now, for proving sufficiency in case (i), let us assume that $\bar{c}(d)_{d-1} \leq \alpha_{d-1}\gamma_{d-1}$, and aim to prove equality. Then by the hypothesis that $\alpha_{d-1} < \gamma_d$, we obtain that

$$\bar{c}_{d-1} = \frac{\gamma_{d-1}\gamma_d - \bar{c}(d)_{d-1}}{\gamma_d - \alpha_{d-1}} \geq \frac{\gamma_{d-1}\gamma_d - \alpha_{d-1}\gamma_{d-1}}{\gamma_d - \alpha_{d-1}} = \gamma_{d-1}.$$

Then, by Lemma 4.11 (i), Γ is $(d-1)$ -partially distance-regular, it means that $p_i(\mathbf{A}) = \mathbf{A}_i$ for every $i = 0, \dots, d-1$. Also, we know that the Hoffman polynomial is expressed as $H = p_0 + \dots + p_d$ and $H(\mathbf{A}) = \mathbf{J}$ because it is regular. Thus we conclude that $p_d(\mathbf{A}) = \mathbf{A}_d$ and Γ is distance-regular. The proof of sufficiency for case (ii) is similar.

Finally, if the hypothesis in (iii) holds, then (26) gives

$$c(d)_{uv} = (\mathbf{A}^d)_{uv} = \gamma_{d-1}\gamma_d = \gamma_{d-1}\alpha_{d-1}$$

for every pair of vertices u, v at distance $d-1$ and, hence, $c(d)_{d-1} = \alpha_{d-1}\gamma_{d-1}$, as claimed. \square

5 Distance mean-regular graphs

In this Chapter we introduce the concept of distance mean-regular graph. These graphs can be seen as a generalization of distance-regular graphs. Let Γ be a graph with vertex set V , diameter D , adjacency matrix \mathbf{A} , and adjacency algebra \mathcal{A} . Then, Γ is *distance mean-regular* when, for a given $u \in V$, the averages of the intersection numbers $p_{ij}^h(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|$ (number of vertices at distance i from u and distance j from v) computed over all vertices v at a given distance $h \in \{0, 1, \dots, D\}$ from u , do not depend on u . In this section, we study some properties and characterizations of these graphs. For instance, it is shown that a distance mean-regular graph is always distance degree-regular, and we give a condition for the converse to be also true.

Some algebraic and spectral properties of distance mean-regular graphs are also investigated. We show that, for distance mean regular-graphs, the role of the distance matrices of distance-regular graphs is played by the so-called distance mean-regular matrices. These matrices are computed from a sequence of orthogonal polynomials evaluated at the adjacency matrix of Γ and, hence, they generate a subalgebra of \mathcal{A} . Some other algebras associated to distance mean-regular graphs are also characterized. Also, we present some other results which show properties of the graph in terms of the structure of these algebras.

Since their introduction by Biggs [28] in the early 70s, most of generalizations proposed for distance-regular graphs are basically intended for regular graphs. For instance, Weichsel [40] introduced the so-called *distance-polynomial graphs*, as those having their distance matrices in the adjacency algebra of the graph. Another example is the *distance-degree regular* or *super-regular* graphs, proposed by Hilano and Nomura [39], and characterized by the independence of the number of vertices at a given distance from every vertex.

The motivation for studying and characterizing distance mean-regular graphs is that they generalize both the vertex-transitive and the distance-regular graphs (see Fig. 7). This allows us to unify some properties which are common to both families, and to apply techniques used in one family to the other one. For instance, we introduce a family of orthogonal polynomials, which is a generalization of the distance polynomials for distance-regular graphs, and is now also related to vertex-transitive graphs.

5.1 Definition and examples

In this section, we introduce the concept of distance mean-regular graph and give some of their basic properties. The reader will soon realize that most of such properties are similar to those of distance-regular graphs. To have a first idea of how distance mean-regular graphs compare with other well-known classes of graphs, see the Venn diagram of Fig.7.

First, recall that a graph $\Gamma = (V, E)$ with diameter D is *distance-regular* if, for any two vertices $u, v \in V$ at distance $h = \text{dist}(u, v)$, the numbers (*intersection parameters*)

$$p_{ij}^h(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|, \quad h, i, j = 0, \dots, D,$$

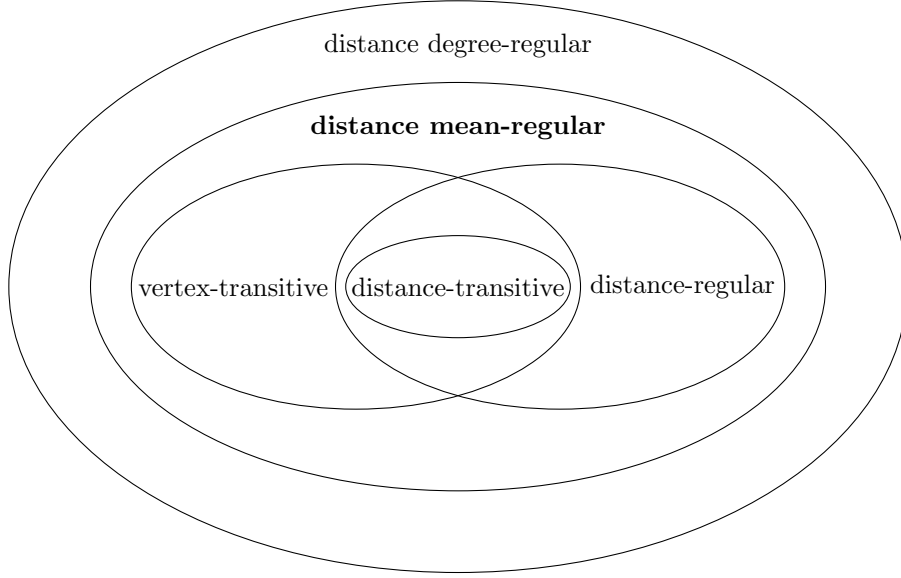


Figure 7: Comparison of different classes of graphs with high symmetry and/or regularity.

only depend on h, i, j . Inspired by this definition, we consider the following generalization:

Definition 5.1. Given a graph Γ with a vertex $u \in V$ we consider the averages

$$\bar{p}_{ij}^h(u) = \frac{1}{|\Gamma_h(u)|} \sum_{v \in \Gamma_h(u)} |\Gamma_i(u) \cap \Gamma_j(v)|, \quad h, i, j = 0, \dots, D.$$

If these numbers do not depend on the vertex u , but only on the integers h, i, j , we say that Γ is distance mean-regular with parameters \bar{p}_{ij}^h .

Notice that all the vertices of a distance mean-regular graph must have the same eccentricity since $\bar{p}_{hh}^0 = k_h(u)$ for every $u \in V$, where $k_h(u)$ represents the number of vertices at distance h from the vertex u . Also, as in the case of distance-regular graphs, we will show that, $\bar{p}_{ij}^h = \bar{p}_{ji}^h$ for every i, j, h (see Proposition 5.12). Thus, when $i = 1$ we use the abbreviated notations $\bar{a}_h = \bar{p}_{1h}^h$, $\bar{b}_h = \bar{p}_{1,h+1}^h$, and $\bar{c}_h = \bar{p}_{1,h-1}^h$. In due course, we shall show that, under a general condition, the invariance of these *intersection numbers* also suffices for having distance mean-regularity.

The *intersection mean-matrix* $\bar{\mathbf{B}}$ of Γ , with entries $(\bar{\mathbf{B}})_{hj} = \bar{p}_{1j}^h$, has the tridiagonal form

$$\bar{\mathbf{B}} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & & & & \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 & & & \\ & \bar{c}_2 & \bar{a}_2 & \bar{b}_2 & & \\ & & & \ddots & & \\ & & & & \bar{c}_{d-1} & \bar{a}_{d-1} & \bar{b}_{d-1} \\ & & & & & \bar{c}_d & \bar{a}_d \end{pmatrix},$$

(notice that $\bar{a}_0 = 0$, and $\bar{p}_{1j}^h = 0$ for $j \neq h-1, h, h+1$), with corresponding intersection mean-diagram of Fig.8 (where ω_{ij} denotes the number of edges between vertex sets $\Gamma_i(u)$ and $\Gamma_j(u)$, which will be used later). As the row sums of $\bar{\mathbf{B}}$ are equal to the degree of Γ (see Lemma 5.3(i)), the same information can be represented by the *intersection mean-array*, which is

$$\iota(\Gamma) = \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{d-1}; \bar{c}_1, \bar{c}_2, \dots, \bar{c}_d\}.$$

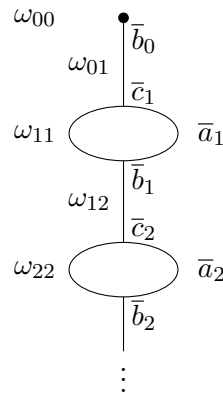


Figure 8: Mean-intersection diagram.

Some instances of distance mean-regular graphs are the distance-regular graphs, and the vertex-transitive graphs. Thus, a first example of distance mean-regular (vertex-transitive) graph is the circulant (or Cayley graph) $\Gamma = \text{Cay}(\mathbb{Z}_8; \pm 4)$ shown in Fig. 9. This graph

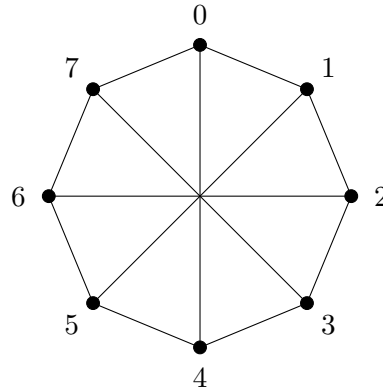


Figure 9: A vertex-transitive distance mean-regular graph.

has diameter $D = 2$, and intersection mean-matrix

$$\bar{\mathbf{B}} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & 0 \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 \\ 0 & \bar{c}_2 & \bar{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} \end{pmatrix}.$$

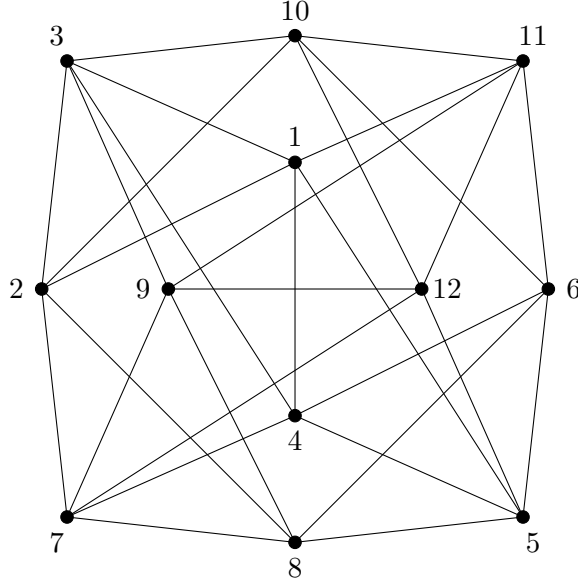


Figure 10: A distance mean-regular graph which is neither vertex-transitive, nor distance-regular.

A second example of distance mean-regular, but not vertex-transitive, graph Γ is shown in Fig. 10.

Now, Γ has again diameter $D = 2$, and intersection mean-matrix

$$\overline{\mathbf{B}} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & 0 \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 \\ 0 & \bar{c}_2 & \bar{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 \\ 1 & \frac{6}{5} & \frac{14}{5} \\ 0 & \frac{14}{6} & \frac{16}{6} \end{pmatrix}$$

In fact, we can easily check that Γ is neither vertex-transitive, nor distance-regular by comparing the subgraphs induced by $\Gamma_1(1)$ (a 4-path and a singleton), and $\Gamma_1(2)$ (a 2-path and a 3-path).

5.2 A first algebraic characterization

We present first the following basic results about interlacing and equitable partitions. It can be found in Haemers [37], Fiol [33], or Brouwer and Haemers [27]. Given a graph Γ on n vertices, and with eigenvalues $\theta_1 \geq \theta_2 \geq \dots \geq \theta_n$, let $\mathcal{P} = \{U_1, \dots, U_m\}$ be a partition of its vertex set V . Let \mathbf{T} be the characteristic matrix of \mathcal{P} , whose columns are the characteristic vectors of U_1, \dots, U_m , and consider the matrix $\mathbf{S} = \mathbf{T}\mathbf{D}^{-1}$ where $\mathbf{D} = \text{diag}(|U_1|, \dots, |U_m|) = \mathbf{T}^\top \mathbf{T}$, satisfying $\mathbf{S}^\top \mathbf{T} = \mathbf{I}$. Then, the so-called *quotient matrix* of \mathbf{A} with respect to \mathcal{P} , is

$$\mathbf{B} = \mathbf{S}^\top \mathbf{A} \mathbf{T} = \mathbf{D}^{-1} \mathbf{T}^\top \mathbf{A} \mathbf{T}, \quad (27)$$

and its element b_{ij} equals the average row sum of the block $\mathbf{A}_{i,j}$ of \mathbf{A} with rows and columns indexed by U_i and U_j , respectively. Moreover, the eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ of \mathbf{B} interlace those of \mathbf{A} , that is,

$$\theta_i \geq \mu_i \geq \theta_{n-m+i}, \quad i = 1, \dots, m. \quad (28)$$

If the interlacing is *tight*, that is, there exists some k , $0 \leq k \leq m$, such that $\theta_i = \mu_i$, $i = 1, \dots, k$, and $\mu_i = \theta_{n-m+i}$, $i = k+1, \dots, m$, then \mathcal{P} is an *equitable* (or *regular*) partition of \mathbf{A} , that is, each block of the partition has constant row and column sums. In the graph Γ , this means that the bipartite induced subgraph $\Gamma[U_i, U_j]$ is biregular for each $i \neq j$, and that the induced subgraph $\Gamma[U_i]$ is regular for each $i \in \{1, \dots, m\}$.

From this algebraic results, it is clear that the intersection mean-matrix $\overline{\mathbf{B}}$ of a distance mean-regular graph Γ is, in fact, the quotient matrix of the distance partition with respect to any vertex, computed as in (27). This leads us to the following result.

Proposition 5.2. *A graph Γ is distance mean-regular if and only if the quotient matrix \mathbf{B} , evaluated in (27), with respect to the distance partition of every vertex, is independent of the origin vertex chosen in the partition and, in this case, \mathbf{B} turns out to be the intersection mean-matrix $\overline{\mathbf{B}}$ of Γ . Moreover, Γ is distance-regular if and only if the interlacing is tight.*

Proof. We only need to prove the sufficiency of the second statement. If the interlacing is tight, the distance partition of every vertex is equitable. Moreover, the graph is clearly regular. Then, distance-regularity follows from the result of Godsil and Shawe-Taylor [36] (see also [34]). \square

5.3 Some properties

As it could be expected, some combinatorial properties of distance mean-regular graphs are similar to those of distance-regular graphs. The following result, which is not exhaustive, shows some simple examples.

Lemma 5.3. *Let Γ be a distance mean-regular graph with diameter D and parameters as above. Then, the following holds:*

- (i) *The graph Γ is regular with degree $k = \bar{b}_0$, with $k = \bar{a}_i + \bar{b}_i + \bar{c}_i$ for every $i = 0, \dots, D$.*
- (ii) *For every vertex u and $i = 0, \dots, D-1$, we have $k_i(u)\bar{b}_i = \bar{c}_{i+1}k_{i+1}(u)$.*
- (iii) *Γ is distance-degree regular, that is, $k_i(u) = k_i$ for every $u \in V$ and $i = 0, \dots, D$.*

Proof. (i) The first statement is clear since, for $v \in \Gamma_i(u)$, we have

$$|\Gamma_{i-1}(u) \cap \Gamma_1(v)| + |\Gamma_i(u) \cap \Gamma_1(v)| + |\Gamma_{i+1}(u) \cap \Gamma_1(v)| = |\Gamma_1(v)| = k,$$

and, computing the average on $\Gamma_i(u)$, we obtain the result.

(ii) This follows by counting in two ways the edges between $\Gamma_i(u)$ and $\Gamma_{i+1}(u)$:

$$|\Gamma_i(u)|\bar{b}_i = \sum_{v \in \Gamma_i(u)} |\Gamma_1(v) \cap \Gamma_{i+1}(u)| = \sum_{w \in \Gamma_{i+1}(u)} |\Gamma_1(w) \cap \Gamma_i(u)| = \bar{c}_{i+1}|\Gamma_{i+1}(u)|.$$

(iii) By induction using (ii) and starting from $k_0(u) = 1$ (or, by (i), $k_1(u) = k$) for every $u \in V$. \square

Also, the intersection mean-numbers often gives similar information to that provided for the corresponding parameters of distance-regular graphs. For instance, recalling that the *odd-girth* of a graph is the minimum length of an odd cycle, we have the following result.

Lemma 5.4. *Let Γ be a distance mean-regular graph with intersection mean-numbers as above. Then, Γ has odd-girth $2m + 1$ if and only if $\bar{a}_i = 0$ for all $i < m$ and $\bar{a}_m \neq 0$.*

Proof. First, $\bar{a}_i = 0$ for $i < m$ if and only if $a_i(u, v) = 0$ for every pair of vertices u, v at distance $i < m$ or, equivalently, Γ contains no odd cycle of length $\ell < 2i + 1$. Moreover, $\bar{a}_m \neq 0$ if and only if there are some vertices u, v such that $a_m(u, v) \geq 1$ and, hence, there is an odd cycle of length $2m + 1$ constructed by the two paths joining each vertex u and v with the origin vertex and the edge joining u and v . This two paths cannot share any vertex but the origin vertex, because if they have a common vertex w in Γ_j for some $1 \leq j \leq m$, then taking w as origin vertex we have $a_{m-i}(u, v) \neq 0$. \square

The same consequence can be found in [23, Lemma 3.7], where the conditions are given on the so-called preintersection numbers (another generalization of the intersection numbers of distance-regular graphs, see [32]) instead of the intersection mean-numbers.

In contrast with the above, some other well-known properties of distance-regular graphs are not shared by the distance mean-regular graphs. For instance, the intersection numbers b_i and c_i of a distance-regular graph satisfy the monotonic properties $b_0 \geq b_1 \geq b_2 \geq \dots$ and $c_1 \leq c_2 \leq c_3 \leq \dots$, whereas this is not the case for the intersection mean-parameters \bar{b}_i and \bar{c}_i (see, e.g. the truncated tetrahedron studied in Subsection 13).

Since every distance-regular graph is also distance mean-regular, one natural question would be the following:

Question 5.5. *If the intersection mean-parameters of a distance mean-regular graph are integers, is there always a distance-regular graph with these parameters?*

In fact, Brouwer [25] gave a negative answer to this question by noting that the Cayley graph $\Gamma = \text{Cay}(\mathbb{Z}_{21}; \{\pm i, i = 1, \dots, 5\})$, with diameter $D = 2$, has the parameters of a nonexistent strongly regular graph. Indeed, as a distance mean-regular, Γ has the integer intersection mean-matrix

$$\bar{\mathbf{B}} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & 0 \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 \\ 0 & \bar{c}_2 & \bar{a}_2 \end{pmatrix} = \begin{pmatrix} 0 & 10 & 0 \\ 1 & 6 & 3 \\ 0 & 3 & 7 \end{pmatrix}.$$

However, Γ has $d+1 = 11$ distinct eigenvalues, namely $\text{sp}(\Gamma) = \{10, 5.690^2, 0.368^2, -0.198^2, -0.394^2, -0.476^2, -1.505^2, -1.554^2, -1.682^2, -2^2, -3.246^2\}$ and, hence, it cannot be distance-regular ($D \neq d$).

5.4 The intersection mean-parameters and their properties

Let Γ be a distance mean-regular graph with diameter D . For $i = 0, \dots, D$, let $\overline{\mathbf{B}}_i$ be the *proper intersection- i mean-matrix* with entries $(\overline{\mathbf{B}}_i)_{hj} = \overline{p}_{ij}^h$. In particular, notice that $\overline{\mathbf{B}}_0 = \mathbf{I}$, and $\overline{\mathbf{B}}_1 = \overline{\mathbf{B}}$. In general, $\overline{\mathbf{B}}_i$ can be easily computed in the following way.

Lemma 5.6. *Let \mathbf{S} and \mathbf{T} be the matrices corresponding to a distance partition with respect to a given vertex of a distance mean-regular graph Γ . Then, its proper intersection- i mean-matrix can be computed from its i -distance matrix \mathbf{A}_i by the formula*

$$\overline{\mathbf{B}}_i = \mathbf{S}^\top \mathbf{A}_i \mathbf{T}, \quad i = 0, \dots, D. \quad (29)$$

Proof. Let $U_i = \Gamma_i(u)$, with $k_i = |\Gamma_i(u)|$, $i = 0, \dots, D$, be the distance partition with respect to vertex u . Then, the matrices \mathbf{T} and \mathbf{S} have entries

$$(\mathbf{T})_{wi} = \begin{cases} 1 & \text{if } w \in U_i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (\mathbf{S})_{vi} = \begin{cases} 1/k_i & \text{if } v \in U_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $h, j = 0, \dots, D$,

$$\begin{aligned} (\mathbf{S}^\top \mathbf{A}_i \mathbf{T})_{hj} &= \sum_{v, w \in V} (\mathbf{S}^\top)_{hv} (\mathbf{A}_i)_{vw} (\mathbf{T})_{wj} = \sum_{v \in \Gamma_h(u), w \in \Gamma_i(v) \cap \Gamma_j(u)} \frac{1}{k_h} \\ &= \frac{1}{k_h} \sum_{v \in \Gamma_h(u)} |\Gamma_i(v) \cap \Gamma_j(u)| = \overline{p}_{ji}^h = \overline{p}_{ij}^h = (\overline{\mathbf{B}}_i)_{hj}, \end{aligned}$$

where we have used the symmetric property $\overline{p}_{ji}^h = \overline{p}_{ij}^h$, which is proved later in Proposition 5.12. \square

We will see an example of this matrix construction in Subsection 5.6.

5.5 A family of orthogonal polynomials

From the proper intersection mean-matrix $\overline{\mathbf{B}}$ of a distance mean-regular graph Γ with n vertices and diameter D , we can construct an orthogonal sequence of polynomials by using the three-term recurrence

$$x\overline{p}_i = \overline{b}_{i-1}\overline{p}_{i-1} + \overline{a}_i\overline{p}_i + \overline{c}_{i+1}\overline{p}_{i+1}, \quad i = 0, 1, \dots, D, \quad (30)$$

initiated with $\overline{p}_0 = 1$ and $\overline{p}_1 = x$, and where, by convention, $\overline{b}_{-1} = \overline{c}_{i+1} = 0$.

According to the results in Cámara, Fàbrega, Fiol, Garriga [30], these polynomials, which here will be called the *distance mean-polynomials*, are orthogonal with respect to the scalar product

$$\langle f, g \rangle_\star = \frac{1}{n} \sum_{i=0}^D w_i f(\mu_i) g(\mu_i) \quad (31)$$

where $\mu_0 > \mu_1 > \dots > \mu_D$ are the distinct eigenvalues of $\overline{\mathbf{B}}$, and their *pseudo-multiplicities* w_i are computed with the formulas

$$w_i = \frac{\pi_0 \overline{p}_D(\mu_0)}{\pi_i \overline{p}_D(\mu_i)} \quad (32)$$

where $\pi_i = \prod_{j \neq i} |\mu_i - \mu_j|$, $i = 0, \dots, D$. Moreover, these polynomials are normalized in such a way that $\|\overline{p}_i\|_\star^2 = \overline{p}_i(\lambda_0)$ for $i = 0, \dots, D$.

Because of the orthogonality, the constants in (30) are the Fourier coefficients of $x\overline{p}_i$ in terms of the basis $\overline{p}_0, \dots, \overline{p}_D$. Namely,

$$\overline{a}_i = \frac{\langle x\overline{p}_i, \overline{p}_i \rangle_\star}{\|\overline{p}_i\|_\star^2}, \quad \overline{b}_i = \frac{\langle x\overline{p}_{i+1}, \overline{p}_i \rangle_\star}{\|\overline{p}_i\|_\star^2}, \quad \overline{c}_i = \frac{\langle x\overline{p}_{i-1}, \overline{p}_i \rangle_\star}{\|\overline{p}_i\|_\star^2}. \quad (33)$$

Lemma 5.7. *The distance mean-polynomials of a distance mean-regular graph satisfy the following:*

$$(i) \quad \overline{p}_i(k) \overline{b}_i = \overline{c}_{i+1} \overline{p}_{i+1}(k).$$

$$(ii) \quad \overline{p}_i(k) = k_i.$$

Proof. (i) Using (33) and the normalization condition of the polynomials, we have

$$\overline{b}_i = \frac{\langle x\overline{p}_{i+1}, \overline{p}_i \rangle_\star}{\|\overline{p}_i\|_\star^2} = \frac{\langle \overline{p}_{i+1}, x\overline{p}_i \rangle_\star}{k_i} = \frac{k_{i+1}}{k_i} \frac{\langle \overline{p}_{i+1}, x\overline{p}_i \rangle_\star}{\|\overline{p}_{i+1}\|_\star^2} = \frac{k_{i+1}}{k_i} \overline{c}_{i+1},$$

and the result follows.

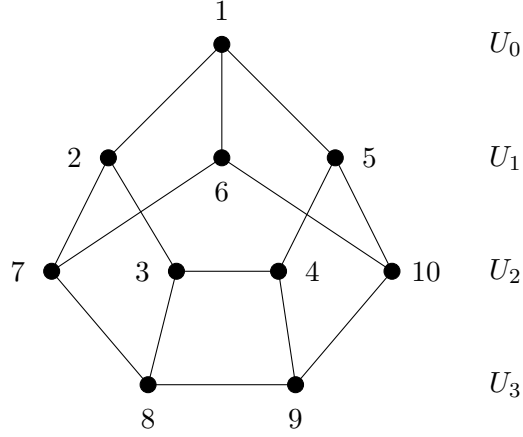
(ii) From Lemma 5.3 and (i), we get

$$\frac{\overline{b}_i}{\overline{c}_{i+1}} = \frac{k_{i+1}}{k_i} = \frac{\overline{p}_{i+1}(k)}{\overline{p}_i(k)}.$$

Hence, $\overline{p}_{i+1}(k) = \frac{k_{i+1}}{k_i} \overline{p}_i(k)$, and the result follows by applying induction from $\overline{p}_0(k) = 1 = k_0$. \square

By evaluating the distance-mean polynomials at \mathbf{A} or $\overline{\mathbf{B}}$, we obtain, respectively, the so-called *distance mean-matrices*

$$\overline{\mathbf{A}}_i = \overline{p}_i(\mathbf{A}), \quad i = 0, \dots, D, \quad (34)$$

Figure 11: The prism $\Gamma = C_5 \times K_2$

and the *intersection mean-matrices*

$$\overline{\mathbf{B}}_i = \overline{p}_i(\overline{\mathbf{B}}), \quad i = 0, \dots, D. \quad (35)$$

Of course, both families of matrices satisfy a three-term recurrence like (30). For instance, the distance mean-matrices satisfy

$$\mathbf{A}\overline{\mathbf{A}}_i = \overline{b}_{i-1}\overline{\mathbf{A}}_{i-1} + \overline{a}_i\overline{\mathbf{A}}_i + \overline{c}_{i+1}\overline{\mathbf{A}}_{i+1}, \quad i = 0, 1, \dots, D, \quad (36)$$

starting with $\overline{\mathbf{A}}_0 = \mathbf{I}$ and $\overline{\mathbf{A}}_1 = \mathbf{A}$ (by convention, $\overline{\mathbf{A}}_{-1} = \overline{\mathbf{A}}_{i+1} = \mathbf{0}$). Besides, because of (34) and (35), both $\overline{\mathbf{A}}_i$ and $\overline{\mathbf{B}}_i$ have constant row sum $\overline{p}_i(\lambda_0)$. For instance, $\overline{\mathbf{A}}_i \mathbf{j} = \overline{p}_i(\lambda_0) \mathbf{j} = k_i \mathbf{j}$, where \mathbf{j} is the all-1 vector.

Concerning the intersection mean-matrices, notice that $\overline{\mathbf{B}}_0 = \overline{p}_0(\overline{\mathbf{B}}) = \mathbf{I}$, with $\mathbf{I}_{hj} = \overline{p}_{0j}^h$, and $\overline{\mathbf{B}}_1 = \overline{p}_1(\overline{\mathbf{B}}) = \overline{\mathbf{B}}$, with $(\overline{\mathbf{B}})_{hj} = \overline{p}_{1j}^h$. In general, in subsection 5.8 we will show that, under some conditions, the intersection- i mean matrix is proper for every $i = 0, \dots, D$. That is, it satisfies (29) and, hence,

$$(\overline{\mathbf{B}}_i)_{hj} = \overline{p}_{ij}^h. \quad (37)$$

5.6 An Example

As an example, consider the prism $\Gamma = C_5 \times K_2$ shown in Fig.11, which is a distance mean-regular graph with spectrum (here and henceforth, numbers are rounded at three decimals)

$$\text{sp } \Gamma = \{3, 1.618^2, 1^1, -0.382^2, -0.618^2, -2.618^2\}.$$

Its quotient matrix with respect to the distance partition $\mathcal{P} = \{1\} \cup \{2, 6, 5\} \cup \{7, 3, 4, 10\} \cup$

$\{8, 9\}$ can be obtained from the matrices

$$\mathbf{T}^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{matrix},$$

$\mathbf{D} = \text{diag}(1, 3, 4, 2)$, and

$$\mathbf{S}^\top = \mathbf{D}^{-1}\mathbf{T}^\top = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{matrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{matrix},$$

satisfying $\mathbf{S}^\top \mathbf{T} = \mathbf{I}$. Then, the (proper) intersection mean-matrix is

$$\bar{\mathbf{B}} = \mathbf{S}^\top \mathbf{A} \mathbf{T} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} \bar{a}_0 & \bar{b}_0 & 0 & 0 \\ \bar{c}_1 & \bar{a}_1 & \bar{b}_1 & 0 \\ 0 & \bar{c}_2 & \bar{a}_2 & \bar{b}_2 \\ 0 & 0 & \bar{c}_3 & \bar{a}_3 \end{pmatrix},$$

with eigenvalues

$$\mu_1 = 3, \quad \mu_2 = 1.402, \quad \mu_3 = -0.433, \quad \mu_4 = -2.469,$$

which interlace the eigenvalues of \mathbf{A} .

The distance mean-polynomials, and their values at $\lambda_0 = 3$, are

$$\begin{aligned} \bar{p}_0(x) &= 1, & \bar{p}_0(3) &= 1, \\ \bar{p}_1(x) &= x, & \bar{p}_1(3) &= 3, \\ \bar{p}_2(x) &= \frac{1}{3}(2x^2 - 6), & \bar{p}_2(3) &= 4, \\ \bar{p}_3(x) &= \frac{1}{6}(2x^3 - x^2 - 12x + 3), & \bar{p}_4(3) &= 2. \end{aligned}$$

From p_3 we then compute the pseudo-multiplicities by using (32), which are $w_0 = 1$, $w_1 = 3.085$, $w_2 = 3.575$, and $w_3 = 2.340$ (notice that, as required, they sum up to $n = 10$). Moreover, besides $\bar{p}_0(\bar{\mathbf{B}}) = \mathbf{I}$ and $\bar{p}_1(\bar{\mathbf{B}}) = \bar{\mathbf{B}}$, we have the other proper intersection mean-matrices:

$$\bar{\mathbf{B}}_2 = \mathbf{S}^\top \mathbf{A}_2 \mathbf{T} = \bar{p}_2(\bar{\mathbf{B}}) = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 2 & \frac{2}{3} & \frac{4}{3} \\ 1 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix},$$

$$\bar{\mathbf{B}}_3 = \mathbf{S}^\top \mathbf{A}_3 \mathbf{T} = \bar{p}_3(\bar{\mathbf{B}}) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

5.7 Characterizations

In Proposition 5.2 we write the first characterization of distance mean-regularity in terms of the matrix \mathbf{B} , that is the intersection mean-matrix $\overline{\mathbf{B}}$. And this matrix \mathbf{B} takes special relevance in Subsections 5.4 and 5.5, and later in 5.8. In this subsection, we give several distinct (combinatorial and algebraic) characterizations of distance mean-regular graphs.

5.7.1 Characterization on the number of edges

Distance mean-regular graphs deal with the invariance of the number of edges whose end vertices are at a given distance from each vertex u . More precisely, for a graph $\Gamma = (V, E)$ with diameter D , a vertex u , and integers $i, j = 0, \dots, D$, let us consider the following parameters:

$$\omega_{ij}(u) = |\{vw \in E : \text{dist}(u, v) = i, \text{dist}(u, w) = j\}|.$$

Notice that $\omega_{ij}(u) = 0$ if $|i - j| > 1$. Besides, some trivial values are $\omega_{00}(u) = 0$ (since there are no loops), and $\omega_{01} = \delta(u)$, the degree of u . If these numbers do not depend on u , we say that they are *well defined*, and represent them as ω_{ij} .

Lemma 5.8. *A graph Γ with diameter D is distance mean-regular if and only if the numbers ω_{ij} are well defined for every $i, j = 0, \dots, D$.*

Proof. If Γ is distance mean-regular, then ω_{ij} is well defined since, by Lemma 5.3, $\omega_{ii} = \bar{a}_i k_i / 2$ and $\omega_{i, i+1} = k_i \bar{b}_i = k_{i+1} \bar{c}_{i+1}$. Conversely, if the ω_{ij} are well defined, Γ is regular with degree $k = \omega_{01}$. Moreover, for any vertex u ,

$$k_i(u) = \frac{1}{k} \sum_{v \in \Gamma_i(u)} \delta(v) = \frac{1}{k} (\omega_{i-1, i} + 2\omega_{i, i} + \omega_{i, i+1}),$$

so that k_i is well defined. Then, $\bar{a}_i = \frac{2}{k_i} \omega_{i, i}$, $\bar{b}_i = \frac{1}{k_i} \omega_{i, i+1}$, and $\bar{c}_i = \frac{1}{k_i} \omega_{i-1, i}$ are also well defined, and Γ is distance mean-regular. \square

Although every distance mean-regular is super-regular (Lemma 5.3(iii)), the converse is not true. A counterexample is, for instance, the graph of Fig. 12 on $n = 12$ vertices and diameter $D = 2$, which is super-regular with $k_1 = 5$ and $k_2 = 6$, but not distance mean-regular.

However, we have the following characterization of those super-regular graphs which are distance mean-regular:

Proposition 5.9. *Let Γ be a super-regular graph. Then, Γ is distance mean-regular if and only if ω_{ii} is well defined for every $i = 0, \dots, D$.*

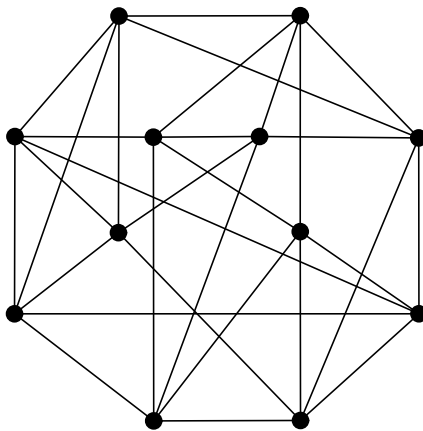


Figure 12: A super-regular, but not distance mean-regular, graph.

Proof. Necessity follows from Lemma 5.8. To prove sufficiency, let Γ be a super-regular graph with well-defined ω_{ii} for every $i = 0, \dots, D$. Then, $\bar{a}_i = 2\omega_{ii}/k_i$ is well defined. Let us now proceed by induction on i . Since Γ is regular, $\bar{b}_0 = k$ is well defined. Now suppose that $\bar{b}_0, \dots, \bar{b}_i$ are well defined. Then, from Lemma 5.3(ii), $\bar{c}_{i+1} = \bar{b}_i \frac{k_i}{k_{i+1}}$ is well defined, and, from Lemma 5.3(i), so is $\bar{b}_{i+1} = k - \bar{a}_{i+1} - \bar{c}_{i+1}$. This completes the induction step. \square

As a consequence of this result, we have the following family of distance mean-regular graphs:

Corollary 5.10. *Every δ -regular graph Γ with diameter $D = 2$ and equal number τ of triangles through any vertex is distance mean-regular.*

Proof. Let n and m be, respectively, the numbers of vertices and edges of Γ . Then, Γ is super-regular with $k_1 = \delta$ and $k_2 = n - \delta$. Moreover, a simple counting gives $\omega_{00}(u) = 0$, $\omega_{11}(u) = \delta$, and $\omega_{22}(u) = m - \delta^2 + \tau$ for every vertex u . Thus, Proposition 5.9 applies. \square

5.7.2 Characterization on the number of triples

In the final result of this previous subsection (Corollary 5.10) we give a characterization of distance mean-regularity in terms of the number τ of triangles through each vertex for graphs with diameter $D = 2$. This is not enough for graphs of diameter larger than 2, but we can generalize the parameter τ in some directions. One of these possible ways to extend the number of triangles through a vertex, could be to consider the number of closed cycles of different lengths through each vertex in the graph.

However, in this subsection we consider other parameters that characterize distance mean-regularity, the numbers of triples of vertices at some given distances between them. More precisely, given a graph Γ with diameter D , three integers $h, i, j = 0, \dots, D$, and a fixed vertex u , let $t_{hij}(u)$ be the number of triples $u, v, w \in V$ such that $\text{dist}(u, v) = h$, $\text{dist}(u, w) = i$, and $\text{dist}(v, w) = j$. Note that these numbers can be computed in two ways:

$$t_{hij}(u) = \sum_{v \in \Gamma_h(u)} |\Gamma_j(v) \cap \Gamma_i(u)| = \sum_{w \in \Gamma_i(u)} |\Gamma_j(w) \cap \Gamma_h(u)|.$$

Thus, if Γ is distance mean-regular t_{hij} is *well defined* (that is, it does not depend on u) since

$$t_{hij} = k_h \bar{p}_{ji}^h = k_i \bar{p}_{jh}^i. \quad (38)$$

In particular, with $j = 1$ and $i = h + 1$, the equation above gives $k_h \bar{p}_{1, h+1}^h = k_{h+1} \bar{p}_{1h}^{h+1}$, which corresponds to Lemma 5.3(ii)-(iii).

Conversely, if the numbers t_{hij} are well defined for $h, i, j = 0, \dots, D$, then Γ is super regular with $k_h = t_{hh0}$, and (38) yields that the intersection numbers $\bar{p}_{ji}^h = t_{hij}/k_h$ are also well defined. Hence, we have proved the following characterization.

Proposition 5.11. *A graph Γ with diameter D is distance mean-regular if and only if the numbers of triples t_{hij} are well defined for $h, i, j = 0, \dots, D$.*

5.7.3 Characterization on the distance matrices

Within the vector space of real $n \times n$ matrices, here we use the standard scalar product

$$\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{n} \text{tr}(\mathbf{M}\mathbf{N}^\top) = \frac{1}{n} \text{sum}(\mathbf{M} \circ \mathbf{N}),$$

where ‘ \circ ’ stands for the entrywise or Hadamard product, and $\text{sum}(\cdot)$ denotes the sum of the entries of the corresponding matrix.

In terms of the distance matrices of a graph, we have the following characterization of distance mean-regular graphs.

Proposition 5.12. *A graph Γ with diameter D and distance matrices $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_D$ is distance mean-regular if and only if, for any $h, i, j = 0, \dots, D$, the matrix $\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h$ has the eigenvector \mathbf{j} . Then, the corresponding eigenvalue is $\lambda = k_h \theta$, where $k_h = \|\mathbf{A}_h\|^2$ is the (common) number of vertices at distance h from any vertex, and*

$$\theta = \bar{p}_{ij}^h = \bar{p}_{ji}^h = \frac{\langle \mathbf{A}_i \mathbf{A}_j, \mathbf{A}_h \rangle}{\|\mathbf{A}_h\|^2}. \quad (39)$$

Proof. Let u, v be two vertices at distance h . Then,

$$(\mathbf{A}_i \mathbf{A}_j)_{uv} = \sum_{w \in V} (\mathbf{A}_i)_{uw} (\mathbf{A}_j)_{wv} = |\Gamma_i(u) \cap \Gamma_j(v)| = p_{ij}^h(u, v).$$

Thus, $\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h$ has right eigenvalue \mathbf{j} , if and only if all its row sums have constant value (eigenvalue), say $\lambda_{ij}^h = \sum_{v \in \Gamma_h(u)} p_{ij}^h(u, v)$. In particular, when $h = i$ and $j = 0$, the above means that the number of vertices at distance h from every vertex u , $|\Gamma_h(u)|$, has constant value k_h . Consequently, $\lambda_{ij}^h = k_h \bar{p}_{ij}^h$, where

$$\bar{p}_{ij}^h = \frac{1}{k_h} \sum_{v \in \Gamma_h(u)} (\mathbf{A}_i \mathbf{A}_j)_{uv} = \frac{1}{k_h} \frac{1}{n} \text{sum}(\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h) = \frac{\langle \mathbf{A}_i \mathbf{A}_j, \mathbf{A}_h \rangle}{\|\mathbf{A}_h\|^2}.$$

Finally, for any i, j, h , we have

$$(\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h) \mathbf{j} = \lambda_{ij}^h \mathbf{j} \iff \mathbf{j}^\top (\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h) = ((\mathbf{A}_j \mathbf{A}_i \circ \mathbf{A}_h) \mathbf{j})^\top = \mathbf{j}^\top \lambda_{ji}^h,$$

so that, computing the (standard) inner products by \mathbf{j}^\top and by \mathbf{j} in the corresponding equalities, we get $\text{sum}(\mathbf{A}_i \mathbf{A}_j \circ \mathbf{A}_h) = n \lambda_{ij}^h = n \lambda_{ji}^h$, and (39) follows. \square

5.8 Algebraic structures defined

The matrix algebras obtained by different constructions from a distance-regular graph, and the properties retrieved, have been well studied by Godsil in [35] or Brouwer, Cohen and Neumann at [26] for example. In this section, we show that, as in the theory of distance regular-graphs, distance mean-regular graphs have associated some matrix (and polynomial) algebras, from which we can retrieve some of their main parameters. We study the properties of each individual algebra, as well as the existing relation between the different algebras.

Let us first recall some basic concepts about associative algebras (see e.g. [38]). Let \mathcal{A} be a finite-dimensional (associative) algebra over a field K . Then, its bilinear multiplication from $\mathcal{A} \times \mathcal{A}$ to \mathcal{A} , denoted by ' \star ', is completely determined by the multiplication of basis elements of \mathcal{A} . Conversely, once a basis for \mathcal{A} has been chosen, the products of basis elements can be set arbitrarily, and then extended in a unique way to a bilinear operator on \mathcal{A} , so giving rise to a (not necessarily associative) algebra. Thus, \mathcal{A} can be specified, up to isomorphism, by giving its dimension (say d), and specifying d^3 *structure coefficients* $c_{h,i,j}$, which are scalars and determine the multiplication in \mathcal{A} via the following rule:

$$\mathbf{e}_i \star \mathbf{e}_j = \sum_{h=1}^d c_{h,i,j} \mathbf{e}_h, \quad h, i, j = 0, 1, \dots, d,$$

where $\mathbf{e}_1, \dots, \mathbf{e}_d$ form a basis of \mathcal{A} .

A *representation* of an associative algebra \mathcal{A} is a vector space V equipped with a linear mapping $\varrho : \mathcal{A} \rightarrow \text{End } V$ preserving the product and the unit. A representation is *faithful* when ϱ is injective. Then, distinct elements of $\mathbf{x} \in \mathcal{A}$ are represented by distinct elements $\varrho(\mathbf{x}) \in \text{End } V$.

5.8.1 Algebras from distance mean-regular graphs

Let Γ be a distance mean-regular graph with diameter D , adjacency matrix \mathbf{A} , and $d + 1$ distinct eigenvalues. Then, we can consider the following vector spaces over \mathbb{R} :

- $\mathcal{A} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d)$;
- $\mathcal{D} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_D)$;
- $\overline{\mathcal{D}} = \text{span}(\mathbf{I}, \mathbf{A}, \overline{\mathbf{A}}_2, \dots, \overline{\mathbf{A}}_D)$;
- $\overline{\mathcal{B}} = \text{span}(\mathbf{I}, \overline{\mathbf{B}}, \overline{\mathbf{B}}_2, \dots, \overline{\mathbf{B}}_D)$.

As is well known, \mathcal{A} is an algebra with the ordinary product of matrices, and it is called the *adjacency algebra* of Γ . Moreover Γ is distance-regular if and only if $\mathcal{D} = \mathcal{A}$, which implies $D = d$ (see e.g. [26, 29]). In this case, \mathcal{A} is the so-called *Bose-Mesner algebra* of Γ , where multiplication on the basis $\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}_d$ is again the usual product of matrices since

$$\mathbf{A}_i \mathbf{A}_j = \sum_{h=0}^d p_{ij}^h \mathbf{A}_h, \quad h, i, j = 0, 1, \dots, d. \quad (40)$$

To obtain an algebra from $\overline{\mathcal{D}}$, we define the *star product* ‘ \star ’ in the following way (recall that $\langle \mathbf{M}, \mathbf{N} \rangle = \frac{1}{n} \text{tr}(\mathbf{M}\mathbf{N})$).

$$\mathbf{A}_i \star \mathbf{A}_j = \sum_{h=0}^D \overline{p}_{ij}^h \mathbf{A}_h, \quad h, i, j = 0, 1, \dots, D. \quad (41)$$

Note that, since the intersection mean-numbers \overline{p}_{ij}^h are Fourier coefficients (see (39)), the product $\mathbf{A}_i \star \mathbf{A}_j$ is just the orthogonal projection of $\mathbf{A}_i \mathbf{A}_j$ on $\overline{\mathcal{D}}$. Then, we can enunciate the main result of this section.

Theorem 5.13. *Let Γ be a distance mean-regular graph with diameter D , adjacency matrix \mathbf{A} , and $d + 1$ distinct eigenvalues. Then the following holds:*

- (i) *The vector space $\overline{\mathcal{D}}$ is a subalgebra of \mathcal{A} with the ordinary product of matrices, and*

$$\dim \overline{\mathcal{D}} = D \leq d = \dim \mathcal{A}.$$

- (ii) *The vector space \mathcal{D} is a commutative (but not necessarily associative) algebra with the star product ‘ \star ’.*

- (iii) *The algebra $\overline{\mathcal{D}}$ is isomorphic to $\overline{\mathcal{B}}$ via $\phi: \overline{\mathcal{D}} \xrightarrow{\sim} \overline{\mathcal{B}}$, where*

$$\phi(\overline{\mathbf{A}}_i) = \overline{\mathbf{B}}_i, \quad i = 0, 1, \dots, D.$$

(iv) If the algebra (\mathcal{D}, \star) is associative, then it is faithfully represented by $\overline{\mathcal{B}}$.

Proof. (i) is a direct consequence of (34), (40), and the well-known fact that $D \leq d$.

(ii) It can be easily checked that, the ‘star product’ of two matrices $\mathbf{X} = \sum_i x_i \mathbf{A}_i$ and $\mathbf{Y} = \sum_j y_j \mathbf{A}_j$ is a linear combination in terms of the basis as:

$$\mathbf{X} \star \mathbf{Y} = \sum_h \left(\sum_{i,j} x_i y_j \overline{p}_{ij}^h \right) \mathbf{A}_h.$$

Thus, such an operation is bilinear, and the vector space generated by \mathcal{D} is closed under it. Moreover, the dimension of this algebra cannot be smaller than D , because the distance matrices \mathbf{A}_i are clearly independent. Finally, the commutative property follows from the equalities $\overline{p}_{ij}^h = \overline{p}_{ji}^h$ for $h, i, j = 0, \dots, D$.

(iii) This is a consequence of the fact that, both $\overline{\mathcal{D}}$ and $\overline{\mathcal{B}}$, are isomorphic to the algebra $\mathbb{R}_D[x]$ with basis $\{\overline{p}_0, \overline{p}_1, \dots, \overline{p}_D\}$.

(iv) The equation (41) indicates that left-multiplication ‘ \star ’ by \mathbf{A}_i can be seen as a linear mapping ϕ_i of \mathcal{D} with respect to the basis $\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}_D$. Moreover, ϕ_i is fully represented by the matrix $\overline{\mathbf{B}}_i^\top$. Then, the result follows since (\mathcal{D}, \star) is commutative. \square

Notice that (iv) is similar to the result for distance-regular graphs given by Biggs in [29, Prop. 21.1]. Some interesting consequences of this result are shown in the following proposition where the associativity condition on $\overline{\mathcal{D}}$ has been translated into a commutativity requirement in $\overline{\mathcal{B}}$. This is because, from (41), (37), and (39), it can be checked that, for any $i, j, k = 0, \dots, D$,

$$(\mathbf{A}_i \star \mathbf{A}_j) \star \mathbf{A}_k = \mathbf{A}_i \star (\mathbf{A}_j \star \mathbf{A}_k) \iff \sum_{\ell=0}^D (\overline{\mathbf{B}}_k \overline{\mathbf{B}}_i)_{\ell j} \mathbf{A}_\ell = \sum_{\ell=0}^D (\overline{\mathbf{B}}_i \overline{\mathbf{B}}_k)_{\ell j} \mathbf{A}_\ell.$$

In fact, it can be shown that the commutativity of the $\overline{\mathbf{B}}_i$'s is equivalent to that of the \mathbf{A}_i 's (both with respect to the usual product of matrices).

Proposition 5.14. *Let Γ be a distance mean-regular graph with diameter D , and proper intersection mean-matrices $\overline{\mathbf{B}}_i$, $i = 0, \dots, D$, that commute with each other. Then, the following holds:*

(i) The matrices $\overline{\mathbf{B}}_i$ satisfy

$$\overline{\mathbf{B}}_i \overline{\mathbf{B}}_j = \sum_{h=0}^D \overline{p}_{ij}^h \overline{\mathbf{B}}_h, \quad h, i, j = 0, \dots, D \quad (42)$$

and, in particular,

$$\overline{\mathbf{B}} \overline{\mathbf{B}}_i = \overline{b}_{i-1} \overline{\mathbf{B}}_{i-1} + \overline{a}_i \overline{\mathbf{B}}_i + \overline{c}_{i+1} \overline{\mathbf{B}}_{i+1}, \quad i = 0, \dots, D. \quad (43)$$

(ii) The matrix $\overline{\mathbf{B}}_i$ is the distance mean-polynomial of degree i at $\overline{\mathbf{B}}$:

$$\overline{\mathbf{B}}_i = \overline{p}_i(\overline{\mathbf{B}}), \quad i = 0, \dots, D.$$

(iii) Every parameter \overline{p}_{ij}^h is well determined by the parameters \overline{a}_i , \overline{b}_i and \overline{c}_i .

Proof. All the results are sustained by the fact, under the hypotheses, $\overline{\mathbf{B}}$ is a faithful representation of (\mathcal{D}, \star) . Indeed, let us check that the (linear) mapping $\Psi : \mathcal{D} \rightarrow \overline{\mathbf{B}}$ defined by $\Psi(\mathbf{A}_i) = \overline{\mathbf{B}}_i$, for $i = 0, \dots, D$ is an algebra isomorphism. First, using (41), (37), always and (39),

$$\Psi(\mathbf{A}_i \star \mathbf{A}_j) = \Psi\left(\sum_{h=0}^D \overline{p}_{ij}^h \mathbf{A}_h\right) = \sum_{h=0}^D (\overline{\mathbf{B}}_i)_{hj} \overline{\mathbf{B}}_h$$

so that, for any $r, s = 0, \dots, D$,

$$(\Psi(\mathbf{A}_i \star \mathbf{A}_j))_{rs} = \sum_{h=0}^D (\overline{\mathbf{B}}_i)_{hj} (\overline{\mathbf{B}}_h)_{rs} = \sum_{h=0}^D (\overline{\mathbf{B}}_i)_{hj} (\overline{\mathbf{B}}_s)_{rh} = (\overline{\mathbf{B}}_s \overline{\mathbf{B}}_i)_{rj}.$$

Moreover,

$$(\Psi(\mathbf{A}_i) \Psi(\mathbf{A}_j))_{rs} = (\overline{\mathbf{B}}_i \overline{\mathbf{B}}_j)_{rs} = \sum_{h=0}^D (\overline{\mathbf{B}}_i)_{rh} (\overline{\mathbf{B}}_j)_{hs} = \sum_{h=0}^D (\overline{\mathbf{B}}_i)_{rh} (\overline{\mathbf{B}}_s)_{hj} = (\overline{\mathbf{B}}_i \overline{\mathbf{B}}_s)_{rj}.$$

Thus, $\Psi(\mathbf{A}_i \star \mathbf{A}_j) = \Psi(\mathbf{A}_i) \Psi(\mathbf{A}_j)$, as claimed.

Taking the above into mind, (i) and (ii) are direct consequences of (41) and the results of Subsection 2.1.

(iii) By the same results, the parameters \overline{a}_i , \overline{b}_i and \overline{c}_i determine the distance mean-polynomials $\overline{p}_0, \overline{p}_1, \dots, \overline{p}_D$ which, according to (ii), give the intersection mean-matrices $\overline{\mathbf{B}}_i$ with entries \overline{p}_{ij}^h . Alternatively, by (42) and (ii), we have

$$\overline{p}_i \overline{p}_j = \sum_{h=0}^D \overline{p}_{ij}^h \overline{p}_h, \quad h, i, j = 0, \dots, D.$$

Thus, \overline{p}_{ij}^h is just the Fourier coefficient of $\overline{p}_i \overline{p}_j$, with respect to the scalar product in (31), in terms of the basis $\{\overline{p}_h : h = 0, \dots, D\}$:

$$\overline{p}_{ij}^h = \frac{\langle \overline{p}_i \overline{p}_j, \overline{p}_h \rangle_\star}{\|\overline{p}_h\|_\star^2}.$$

This completes the proof. □

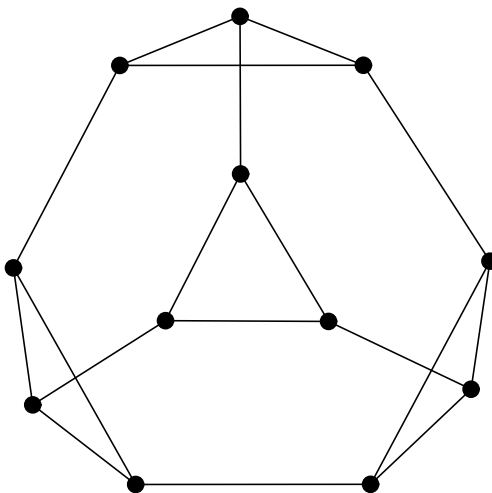


Figure 13: The truncated tetrahedron

Notice that the equalities $(\overline{\mathbf{B}}_i \overline{\mathbf{B}}_j)_{rs} = (\overline{\mathbf{B}}_j \overline{\mathbf{B}}_i)_{rs}$ for every i, j, r, s , which can be written as

$$\sum_{h=0}^D \overline{p}_{sh}^r \overline{p}_{ij}^h = \sum_{h=0}^D \overline{p}_{ih}^r \overline{p}_{sj}^h,$$

are like the ones satisfied by the parameters p_{ij}^h of an association scheme with $D = d$ classes (or a distance-regular graph with diameter D); see e.g. Brouwer, Cohen and Neumaier [26, Lemma 2.1.1(vi)].

5.8.2 An example

We end this section with an example showing that the conditions of Theorem 5.13 (or Proposition 5.14) do not always hold. The *truncated tetrahedron* $\Gamma = K_4[\Delta]$ shown in Fig. 13, is a vertex-transitive (and Cayley) graph with diameter $D = 3$, and spectrum

$$\text{sp } \Gamma = \{3, 2^3, 0^2, -1^3, -2^3\}.$$

Thus, it is a distance mean-regular graph with proper intersection mean-matrices

$$\overline{\mathbf{B}} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} & \frac{3}{2} \end{pmatrix}, \quad \overline{\mathbf{B}}_2 = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & \frac{4}{3} & \frac{2}{3} & 2 \\ 1 & \frac{1}{2} & 1 & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}, \quad \text{and} \quad \overline{\mathbf{B}}_3 = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \\ 1 & \frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Note that, since $\overline{b}_1 = \frac{4}{3} < \frac{3}{2} = \overline{b}_2$, the monotonic property $\overline{b}_0 \geq \overline{b}_1 \geq \overline{b}_2$ does not hold.

However, as these matrices do not commute, they are not all equal to the intersection mean-matrices obtained from the distance mean-polynomials (35), which turn out to be:

$$\bar{p}_1(\bar{\mathbf{B}}) = \bar{\mathbf{B}}, \quad \bar{p}_2(\bar{\mathbf{B}}) = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & \frac{4}{3} & \frac{2}{3} & 2 \\ 1 & \frac{1}{3} & \frac{1}{2} & 2 \\ 0 & \frac{3}{2} & 2 & \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad \bar{p}_3(\bar{\mathbf{B}}) = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & \frac{3}{2} & 2 & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

6 On the isoperimetric problem in Johnson graphs

The Johnson graph $J(n, m)$ has the m -subsets of $[n]$ as vertices and two subsets are adjacent in the graph if they share $m - 1$ elements, where $[n]$ denotes the set $\{1, \dots, n\}$. Shapozenko asked about the isoperimetric function $\mu_{n,m}(k)$ of Johnson graphs, that is, the cardinality of the smallest boundary of sets with k vertices in $J(n, m)$ for each $1 \leq k \leq \binom{n}{m}$. We give an upper bound for $\mu_{n,m}(k)$ and show that, for each given k such that the solution to the Shadow Minimization Problem in the Boolean lattice is unique, and each sufficiently large n , the given upper bound is tight. We also show that the bound is tight for the small values of $k \leq m - 1$ and for all values of k for some cases: it is tight for $m = 2$ and for $n = 2m - 2$.

Let $G = (V, E)$ be a graph. Given a set $X \subset V$ of vertices, we denote by

$$\partial X = \{y \in V \setminus X : d(X, y) = 1\} \quad \text{and} \quad B(X) = \{y \in V : d(X, y) \leq 1\} = X \cup \partial X,$$

the *boundary* and the *ball* of X respectively, where $d(X, y)$ denotes $\min\{d(x, y) : x \in X\}$.

We write ∂_G and B_G when the reference to G has to be made explicit. The *vertex-isoperimetric function* (we will call it simply *isoperimetric function*) of G is defined as

$$\mu_G(k) = \min\{|\partial X| : X \subset V, |X| = k\},$$

that is, $\mu_G(k)$ is the size of the smallest boundary among sets of vertices with cardinality k .

The isoperimetric function is known only for a few classes of graphs. One of the seminal results is the exact determination of the isoperimetric function for the n -cube obtained by Harper [59] in 1966 (and by Hart with the edge-isoperimetric function at [61] in 1976.) Analogous results were obtained for cartesian products of chains by Bollobás and Leader [48] and Bezrukov [43], cartesian products of even cycles by Karachanjan [62] and Riordan [67] (see also Bezrukov and Leck at [45]) and some other cartesian products by Bezrukov and Serra [46].

The Johnson graph $J(n, m)$ has the m -subsets of $[n] = \{1, 2, \dots, n\}$ as vertices and two m -subsets are adjacent in the graph whenever their symmetric difference has cardinality 2. It follows from the definition that, for $m = 1$, the Johnson graph $J(n, 1)$ is the complete graph K_n . For $m = 2$ the Johnson graph $J(n, 2)$ is the line graph of the complete graph on n vertices, also known as the triangular graph $T(n)$. Thus, for instance, $J(5, 2)$ is the complement of the Petersen graph, displayed in Figure 6. Also, $J(n, 2)$ is the complement of the Kneser graph $K(n, 2)$, the graph which has the 2-subsets of $[n]$ as vertices and two pairs are adjacent whenever they are disjoint.

Also, it is important to see that each Johnson graph $J(n, m)$ is isomorphic to the graph $J(n, n - m)$. Thus, when we prove any result for some case of $J(n, m)$, the same result is automatically proved for the isomorphic graph $J(n, n - m)$. For example, Theorem 6.3 (where the case $m = 2$ is solved) apply also in the case $J(n, n - 2)$. Or Property 6.13 (which solves the case $n = 2m - 2$) also solves the case $J(2m + 2, m)$.

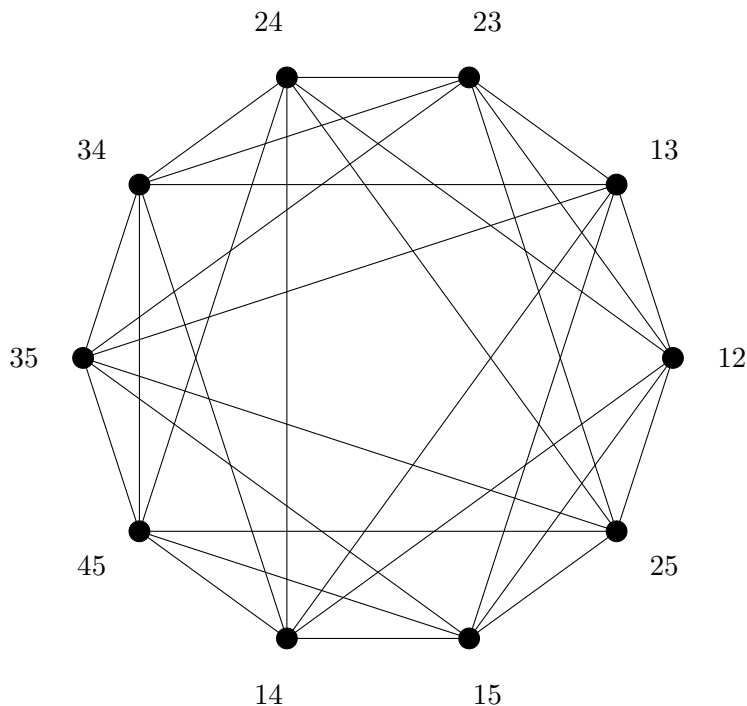


Figure 14: The Johnson graph $J(5, 2)$.

Johnson graphs arise from the association schemes named after Johnson, who introduced them, see e.g. [51]. The Johnson graphs are one of the important classes of distance-transitive graphs; see e.g. Brouwer, Cohen, Neumaier [50, Chapter 9] or Godsil [58, Chapter 11].

Given a family S of m -sets of an n -set, its *lower shadow* $\Delta(S)$ is the family of $(m-1)$ -sets which are contained in some m -set in S . The *upper shadow* $\nabla(S)$ of S is the family of $(m+1)$ -sets which contain some m -set in S . The *ball* of S in the Johnson graph $J(n, m)$ can be written as

$$B(S) = \nabla(\Delta(S)) = \Delta(\nabla(S)). \quad (44)$$

These equalities establish a connection between the isoperimetric problem in the Johnson graph with the Shadow Minimization Problem (SMP) in the Boolean lattice, which consists in finding, for a given k , the smallest cardinality of $\Delta(S)$ among all families S of m -sets with cardinality k . The latter problem is solved by the well-known Kruskal–Katona theorem [64, 63], which establishes that the initial segments in the colex order provide a family of extremal sets for the SMP.

Recall that the *colex order* in the set of m -subsets of $[n]$ is defined as $X \leq Y$ if and only if $\max((X \setminus Y) \cup (Y \setminus X)) \in Y$ (we follow here the terminology from Bollobás [47, Section 5];

we also use $\binom{[n]}{m}$ to denote the family of m -subsets of an n -set, and $[k, l] = \{k, k+1, \dots, l\}$ for integers $k < l$.) The computation of the boundary of initial segments in the colex order (the family of the first m -subsets in this order) provides the following upper bound for the isoperimetric function of Johnson graphs:

Proposition 6.1. *Let $\mu_{n,m} : [N] \rightarrow \mathbb{N}$ denote the isoperimetric function of the Johnson graph $J(n, m)$, where $N = \binom{[n]}{m}$. Let*

$$k = \binom{k_0}{m} + \binom{k_1}{m-1} + \dots + \binom{k_r}{m-r}, \quad k_0 > \dots > k_r \geq m-r > 0,$$

be the m -binomial representation of k . Then

$$\mu_{n,m}(k) \leq f(k, n, m), \quad (45)$$

where

$$f(k, n, m) = \binom{k_0}{m-1}(n-k_0) + \sum_{i=1}^r \left(\binom{k_i}{m-i-1}(n-k_0-1) - \binom{k_i}{m-i} \right). \quad (46)$$

Proof. The initial segment I of length k in the colex order is the disjoint union

$$I = I_0 \cup \dots \cup I_r,$$

where I_0 consists of all m -sets in $\binom{[k_0]}{m}$ and, for $j > 0$, I_j consists of all sets containing $\{k_{j-1} + 1, \dots, k_0 + 1\}$ and $m-j$ elements in $[k_j]$. The right hand side of (45) is the cardinality of ∂I as can be shown by induction on r . If $r = 0$ then ∂I consists of the $\binom{k_0}{m-1}(n-k_0)$ sets obtained by replacing one element in $[k_0]$ by one element in $[k_0 + 1, n]$ from a set in I . Suppose that $r > 0$ and write $I = I' \cup I_r$ as the disjoint union of $I' = I_0 \cup \dots \cup I_{r-1}$ and I_r . We have $\partial I = (\partial I' \setminus I_r) \cup (\partial I_r \setminus B(I'))$, the union being disjoint. Since $I_r \subset \partial I'$ we have,

$$|\partial I' \setminus I_r| = |\partial I'| - |I_r| = |\partial I'| - \binom{k_r}{m-r},$$

while

$$|\partial I_r \setminus B(I')| = \binom{k_r}{m-r-1}(n-k_0-1),$$

since the only sets in $\partial I_r \setminus B(I')$ are those obtained from a set in I_r by replacing one element in $[k_r]$ by one element in $[k_0 + 2, n]$. \square

In general, the family of initial segments in the colex order does not provide a solution to the isoperimetric problem in $J(n, m)$. A simple example is as follows.

Example 6.2. Take $n = 3(m + 1)/2$. The ball $B(\{\mathbf{x}\})$ of radius one in $J(3(m + 1)/2, m)$ has cardinality

$$|B_1| = 1 + m(n - m) = \frac{(m + 2)(m + 1)}{2} = \binom{m + 2}{m},$$

and its boundary has cardinality

$$|\partial B_1| = \binom{m}{2} \binom{n - m}{2} = \frac{m(m - 1)(m + 3)(m + 1)}{16}.$$

On the other hand, according to (45) and the m -binomial decomposition of $|B_1|$, the initial segment I of length $|B_1|$ has cardinality

$$\begin{aligned} |\partial I| &= \binom{m + 2}{m - 1} \frac{m - 1}{2} = \frac{(m + 2)(m + 1)m(m - 1)}{12} \\ &= |\partial B_1| + \frac{(m + 1)m(m - 1)^2}{48}, \end{aligned}$$

which shows that the unit ball can have, as a function of m , an arbitrarily smaller boundary than the initial segment in the colex order.

In his monograph on discrete isoperimetric problems, Leader [65] mentions the isoperimetric problem for Johnson graphs as one of the intriguing open problems in the area. Later on, in his extensive monograph on isoperimetric problems, Harper [60] attributes the problem to Shapozenko, and recalls that it is still open. Recently, Christofides, Ellis and Keevash [54] have obtained a lower bound for the isoperimetric function of Johnson graphs which is asymptotically tight for sets with cardinality $\frac{1}{2} \binom{n}{m}$. The Johnson graphs $J(n, 2)$ provide a counterexample to a conjecture of Brouwer on the 2-restricted connectivity of strongly regular graphs, see Cioabâ, Kim and Koolen [52] and Cioabâ, Koolen and Li [53], where the connectivity of the more general class of strongly regular graphs and distance-regular graphs is studied. It is also worth mentioning that the edge version of the isoperimetric problem, where the minimization is for the number of edges leaving a set of given cardinality, has also been studied, see e.g. Ahlswede and Katona [41] or Bey [42]. We will only deal with the vertex isoperimetric problem in this paper and refer to it simply as the isoperimetric problem.

We call a set S of vertices of $J(n, m)$ *optimal* if $|\partial(S)| = \mu_{n,m}(|S|)$. Our first result shows that initial segments in the colex order are optimal sets in $J(n, 2)$.

Theorem 6.3. For each $n \geq 3$ and each $1 \leq k \leq \binom{n}{2}$ we have

$$\mu_{n,2}(k) = f(k, n, 2).$$

In particular, the initial segments in the colex order are optimal sets of $J(n, 2)$ for each $n \geq 3$.

The following theorem allows one to show that the inequality (45) is also tight in $J(n, m)$ for very small sets.

Theorem 6.4. *For $k < m - 1$ and $n \geq 2(m - 1)$ the initial segment of length k of the colex order in $J(n, m)$ is an optimal set.*

Our last result, Theorem 6.5, extends Theorem 6.4 in an asymptotic way, by showing that the inequality (45) is tight for a large number of small cardinalities and gives a lower bound for all small cardinalities.

Theorem 6.5. *Let k, m be positive integers and let*

$$k = \binom{k_0}{m} + \binom{k_1}{m-1} + \cdots + \binom{k_r}{m-r}, \quad k_0 > \cdots > k_r \geq m-r > 0,$$

be the m -binomial representation of k .

There is $n(k, m)$ such that, for all $n \geq n(k, m)$, the following holds.

(i) *If $r < m - 1$ then*

$$\mu_{n,m}(k) = f(k, n, m),$$

and the initial segment in the colex order with length k is the only (up to automorphisms) optimal set with cardinality k of the Johnson graph $J(n, m)$.

(ii) *If $r = m - 1$ then*

$$\mu_{n,m}(k) \leq f(k - k_r + 1, n, m) + k_r + 1,$$

where ℓ is the length of the longest sequence $k, k - 1, \dots, k - \ell + 1$ of integers whose m -binomial representation has length $r = m - 1$.

The proof of Theorem 6.5 provides the estimation

$$n(k, m) \leq m + k + 1 + \mu_{m+k+1,m}(k) - f(k, m + k + 1, m)$$

for the value of $n(k, m)$ above, for which the statement of Theorem 6.5 holds. This upper bound for $n(k, m)$ is not tight but we make no attempt to optimize its value in this chapter.

Example 6.2 shows that the initial segment in the colex order of length $\binom{m+2}{m}$ can fail to be an optimal set in $J(n, m)$ if $n = 3(m + 1)/2$. In the last section, we describe another infinite family of examples for which the initial segment in the colex order fails again to be an optimal set in $J(n, m)$ for every fixed m and all n large enough.

Proposition 6.6. *Let m be a positive integer. For each integer k of the form*

$$k = \binom{t}{m} + 3 \binom{t}{m-1},$$

with t sufficiently large with respect to m , there is a set S with cardinality k such that

$$|B(S)| < f(k, n, m)$$

for all $n \geq t + 3$.

When $n = t + 3$ the set S in Proposition 6.6 can be described as the ball in $J(n, m)$ of $\binom{[t]}{m}$, the family of m -subsets of the first t symbols. Such sets are clear candidates to be optimal sets, but other choices for t result in sets with larger boundary than the corresponding initial segment in colex order. The examples described in Proposition 6.6 are closely related to the non-unicity of solutions to the Shadow Minimization Problem in the Boolean lattice (see Theorem 6.7 below.)

Standard compression techniques are used to prove the above results. These tools cannot fully solve the isoperimetric problem of Johnson graphs. It is because, as pointed out in [54], for instance, optimal sets in Johnson graphs do not have the nested property (the ball of an optimal set is not optimal.) However, these techniques are still useful to show that the colex order provides a sequence of extremal sets for small cardinalities.

This section of the work is organized as follows. Subsection 6.1 recalls the shifting techniques and compression of sets. The proofs of Theorems 6.3, 6.4 and 6.5 are given in Subsections 6.2, 6.3 and 6.4 respectively. In the proof of Theorem 6.5, we use a result by Füredi and Griggs [57] which characterizes the cardinalities for which the Shadow Minimization Problem for the Boolean lattice has a unique solution. The statement below is a rewriting of a combination of Proposition 2.3 and Theorem 2.6 in [57].

Theorem 6.7 ([57]). *Let*

$$k = \binom{k_0}{m} + \binom{k_1}{m-1} + \cdots + \binom{k_r}{m-r}, \quad k_0 > \cdots > k_r \geq m-r > 0,$$

be the m -binomial representation of k . The initial segment in the colex order is the unique (up to automorphisms) solution to the Shadow Minimization Problem in the Boolean lattice if and only if $r < m - 1$.

Finally in Section 6.5 we prove Proposition 6.6. The result describes an infinite family of examples which show that the initial segments in colex order may fail to be optimal sets. The nature of this example shows that the isoperimetric problem in Johnson graphs still has many intriguing open questions.

6.1 Shifting techniques

Shifting techniques are one of the key tools in the study of set systems. They were initially introduced in the original proof of the Erdős-Ko-Rado theorem [55] and have been particularly used by Frankl and Füredi [56] in the solution of the isoperimetric problem for hypercubes.

In what follows, we identify subsets of $[n]$ with their characteristic vectors $\mathbf{x} = (x_1, \dots, x_n) \in \{0, 1\}^n$, where $x_i = 1$ if i is in the corresponding set and $x_i = 0$ otherwise. We denote the *support* of \mathbf{x} by

$$\bar{\mathbf{x}} = \{i : x_i = 1\}.$$

and the ℓ_1 -norm of \mathbf{x} by

$$|\mathbf{x}| = \sum_i x_i.$$

The support \bar{S} of $S \subset \{0, 1\}^n$ is the union of the supports of its vectors. We often identify a set $S \subset \{0, 1\}^n$ with the subset in $\{0, 1\}^{n'}$, $n' > n$, obtained by adding zeros to the right in the coordinates of its vectors. Thus, the initial segment of length k is considered to be a subset of $\{0, 1\}^n$ for each sufficiently large n .

The sum $\mathbf{x} + \mathbf{y} = (x_1 + y_1 \pmod{2}, \dots, x_n + y_n \pmod{2})$, of characteristic vectors is meant to be performed in the field \mathbb{F}_2^n and corresponds to the symmetric difference of the corresponding sets. We also denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the unit vectors with 1 in the i -th coordinate and zero everywhere else.

With the above notation, the set of vertices of the Johnson graph $J(n, m)$ are all vectors of $\{0, 1\}^n$ with norm m , and the neighbors of \mathbf{x} in $J(n, m)$ are the vectors

$$\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j,$$

for each pair i, j such that $x_i + x_j = 1$.

We next recall the definition of the shifting transformation.

Definition 6.8. *Let $i, j \in [n]$. For a set $S \subset \{0, 1\}^n$ define*

$$S_{ij} = \{\mathbf{x} \in S : x_i = 1 \text{ and } x_j = 0\},$$

and

$$T_{ij}(\mathbf{x}, S) = \begin{cases} \mathbf{x} + \mathbf{e}_i + \mathbf{e}_j, & \text{if } \mathbf{x} \in S_{ij} \text{ and } \mathbf{x} + \mathbf{e}_i + \mathbf{e}_j \notin S \\ \mathbf{x} & \text{otherwise} \end{cases}$$

The ij -shift of S is defined as

$$T_{ij}(S) = \{T_{ij}(\mathbf{x}, S) : \mathbf{x} \in S\}.$$

It follows from the definition that the shifting T_{ij} of a set preserves its cardinality and the norm of its elements. Moreover, it sends every vertex to a vertex at distance at most 1. The main property of the shifting transformation is that it does not increase the cardinality of the ball of a set. This property follows from the analogous ones for upper and lower shadows. We include a direct proof here for completeness.

Lemma 6.9. *Let $i, j \in [n]$ and write $T = T_{ij}$. For each set S of vertices in the Johnson graph $J(n, m)$ we have*

$$B(T(S)) \subseteq T(B(S)). \tag{47}$$

In particular,

$$|B(T(S))| \leq |T(B(S))| = |B(S)|. \tag{48}$$

Proof. We will show that,

$$\text{for each } \mathbf{y} \in T(S), \text{ we have } B(\mathbf{y}) \subseteq T(B(S)), \quad (49)$$

which is equivalent to (47). Then, we observe that (48) follows since

$$|\partial(T(S))| = |B(T(S))| - |T(S)| \leq |T(B(S))| - |T(S)| = |B(S)| - |S| = |\partial S|.$$

Let \mathbf{x} be the element in S such that $\mathbf{y} = T(S, \mathbf{x})$. We consider two cases.

Case 1. $(x_i, x_j) \neq (1, 0)$. In this case we certainly have $\mathbf{y} = \mathbf{x}$. Moreover, for each $\mathbf{z} \in \partial \mathbf{x}$ such that $z_i = 1$ and $z_j = 0$, we have $\mathbf{z} + \mathbf{e}_i + \mathbf{e}_j \in B(\mathbf{x})$. Therefore, since $B(\mathbf{x}) \subseteq B(S)$, the transformation $T(B(S), \cdot)$ leaves all vectors in $B(\mathbf{x})$ invariant. Hence, $B(\mathbf{x}) \subseteq T(B(S))$.

Case 2. Suppose now that $x_i = 0$ and $x_j = 1$. Then $\mathbf{z} = \mathbf{x} + \mathbf{u} + \mathbf{v}$ is the only neighbor of \mathbf{x} with $z_i = 1$ and $z_j = 0$.

Case 2.1 If $\mathbf{y} = \mathbf{x}$ then, by the definition of $T_{uv}(S, \cdot)$, we have $\mathbf{z} \in S$ and $T_{uv}(S, \mathbf{z}) = \mathbf{z}$. Observe that every neighbor \mathbf{z}' of \mathbf{x} is left invariant by $T_{uv}(B(S), \cdot)$. This is clearly the case if $(z'_i, z'_j) \neq (0, 1)$ and, if $(z_i, z_j) = (0, 1)$, because we then have $\mathbf{z}'' = \mathbf{z}' + \mathbf{u} + \mathbf{v} \in B(\mathbf{z}) \subset B(S)$. Hence

$$B(\mathbf{y}) = B(\mathbf{x}) \subseteq T_{uv}(B(S), B(\mathbf{x})) \subseteq T_{uv}(B(S)).$$

Case 2.2 Suppose that $\mathbf{y} \neq \mathbf{x}$. Then $\mathbf{y} \notin S$ but $\mathbf{y} \in B(\mathbf{x}) \subseteq B(S)$. Each neighbor \mathbf{z} of \mathbf{y} distinct from \mathbf{x} is of the form $\mathbf{z} = \mathbf{z}' + \mathbf{u} + \mathbf{v}$ for some neighbor \mathbf{z}' of \mathbf{x} and therefore it belongs to $T_{uv}(B(S))$. For \mathbf{x} itself we have $T_{uv}(B(S), \mathbf{x}) = \mathbf{x}$ because $\mathbf{y} = \mathbf{x} + \mathbf{u} + \mathbf{v} \in B(S)$. Thus we again have $B(\mathbf{y}) \subset T_{uv}(B(S))$. This completes the proof of (49). \square

The weight of a vector $\mathbf{x} \in \{0, 1\}^n$ is

$$w(\mathbf{x}) = \sum_{i=1}^n ix_i,$$

and the weight of a set S is

$$w(S) = \sum_{\mathbf{x} \in S} w(\mathbf{x}).$$

We note that, if $i > j$ then $w(T_{ij}(S)) \leq w(S)$. Moreover, equality holds if and only if $T_{ij}(S) = S$. Thus, successive application of transformations T_{ij} using pairs i, j with $i > j$ eventually produces a set which is stable by any of such transformations. This fact leads to the following definition.

Definition 6.10. We say that a set S is compressed if $T_{ij}(S) = S$ for each pair $i, j \in [n]$ with $i > j$.

Every set can be compressed by keeping its cardinality and without increasing its boundary (see Lemma 6.9). Therefore, in what follows we can restrict our attention to compressed sets in our study of optimal sets.

6.2 On the case $m = 2$

Theorem 6.3 follows from the following proposition which characterizes compressed optimal sets in $J(n, 2)$.

Proposition 6.11. *A set S of vertices of the graph $J(n, 2)$ with cardinality $\binom{t-1}{2} < |S| \leq \binom{t}{2}$ is optimal if and only if, up to isomorphism, $S \subseteq \binom{[t]}{2}$.*

Proof. Write $V(J(n, 2))$ as the disjoint union

$$V(J(n, 2)) = S \cup \partial S \cup \tilde{S},$$

where \tilde{S} is the set of vertices at distance two from S . Let $t = |\tilde{S}|$ be the amount of elements of $[n]$ in the support \bar{S} of S . The only vectors in \bar{S} are the ones which have both nonzero coordinates in $[n] \setminus |\tilde{S}|$. Therefore

$$|\tilde{S}| = \binom{n-t}{2},$$

and

$$|\partial S| = \binom{n}{2} - |S| - \binom{n-t}{2}.$$

Hence, for a given cardinality $|S|$, $|\partial S|$ is an increasing function of t alone. The optimal value is therefore obtained when t is the smallest possible, which is the smallest t such that $|S| \leq \binom{t}{2}$. This is achieved for any subset $S \subset \binom{[t]}{2}$ if $|S| > \binom{t-1}{2}$. \square

As a consequence of the above proposition, we can see that the solution to the isoperimetric problem in $J(n, 2)$ is unique, up to isomorphisms, for sets of cardinality $\binom{t}{2}$ (and it is also unique for sets of cardinality $\binom{t}{2} - 1$).

6.3 Small sets and the case $n = 2m - 2$

In this section, we prove Theorem 6.4, which talks about optimal sets with small cardinality. To do this, we prove and use the Proposition 6.13, which also constitute an important result by itself. This Proposition 6.13 solves the whole isoperimetric problem for graph of the form $J(2m - 2, m)$.

Consider the partition

$$V(J(n, m)) = \{\mathbf{x} \in V(J(n, m)) : x_n = 0\} \cup \{\mathbf{x} \in V(J(n, m)) : x_n = 1\} = V_0 \cup V_1.$$

The subgraph of $J(n, m)$ induced by V_0 is isomorphic to $J(n - 1, m)$ and the subgraph induced by V_1 is isomorphic to $J(n - 1, m - 1)$. There is an edge in $J(n, m)$ joining $\mathbf{x} \in V_0$ with $\mathbf{y} \in V_1$ if and only if $\bar{\mathbf{y}} \setminus \{n\} \subset \bar{\mathbf{x}}$.

Lemma 6.12. *Let S be a set of vertices in $J(n, m)$. Let $S_0 = S \cap \{x_n = 0\}$ and $S_1 = S \cap \{x_n = 1\}$. If S is compressed then*

$$B(S) = B(S_0) \quad \text{and} \quad |B(S)| = |B'(S'_0)| + |\Delta(S'_0)|,$$

where $S'_0 = \{\mathbf{x} \in \{0, 1\}^{n-1} : (\mathbf{x}, 0) \in S_0\}$ and B' denotes the ball in $J(n-1, m)$.

Proof. Let $\mathbf{x} \in S_1$ and $i \notin \bar{\mathbf{x}}$. Since S is compressed, we have

$$\mathbf{y} = \mathbf{x} + \mathbf{e}_n + \mathbf{e}_i \in S_0,$$

which implies $\mathbf{x} = \mathbf{y} + \mathbf{e}_n + \mathbf{e}_i \in B(S_0)$. Hence, $S_1 \subset B(S_0)$. Moreover, if $j \in \bar{\mathbf{x}}$ then

$$\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j = (\mathbf{x} + \mathbf{e}_n + \mathbf{e}_i) + \mathbf{e}_n + \mathbf{e}_j \in B(S_0),$$

so that $B(\mathbf{x}) \subset B(S_0)$. Hence $B(S) = B(S_0) \cup B(S_1) = B(S_0)$. This proves the first part of the sentence.

For the second part, we just note that $B(S_0)$ is the disjoint union $(B(S_0) \cap V_0) \cup (B(S_0) \cap V_1)$. Since the subgraph induced by V_0 is isomorphic to $J(n-1, m)$, we have $|B(S_0) \cap V_0| = |B'(S'_0)|$. On the other hand, there is an edge in $J(n, m)$ joining $x \in V_1$ with $y \in V_0$ if and only if $\bar{x} \setminus \{n\} \subset \bar{y}$. It follows that $B(S_0) \cap V_1 = \Delta(S'_0)$. \square

The next proposition considers the case of Johnson graphs $J(n, m)$ when $n = 2m - 2$.

Proposition 6.13. *For each $k = 1, \dots, \binom{2m-2}{m}$ the initial segment of length k in the colex order is an optimal set of the graph $J(2m-2, m)$.*

Proof. Let us recall (44) and express the ball of a set S as

$$B(S) = \nabla(\Delta(S)).$$

By the Kruskal–Katona theorem, ΔS is minimized by the initial segment I_k , $k = |S|$, in the colex order of $\binom{[2m-2]}{m}$. Moreover, ΔI_k is also an initial segment of the colex order in $\binom{[2m-2]}{m-1}$.

The automorphism of the Boolean lattice given by taking complements sends the middle level $\binom{[2m-2]}{m-1}$ to itself, and the initial segments in the colex order are exchanged by the initial segments in the lexicographic order. It follows that the initial segments of the colex order are also a solution to the minimization of the *upper* shadow (as well as the initial segments in the lexicographic order). Hence, I_k minimizes $|\nabla(\Delta(S))|$. \square

We use the Proposition 6.13 as the base case of the induction for the proof of Theorem 6.4.

Proof of Theorem 6.4. The proof is by induction on n . For $n = 2m - 2$ the result follows from Proposition 6.13. Let S be a compressed set of cardinality $k \leq m - 1$ in $J(n, m)$, $n \geq 2m - 1$ and consider its decomposition $S = S_0 \cup S_1$. Since S is compressed, every element in S_1 gives rise to at least $m - 1$ elements in S . Since $k < m - 1$ we have $S_1 = \emptyset$.

By Lemma 6.12, the cardinality of the ball of S is

$$|B(S)| = |B'(S'_0)| + |\Delta(S'_0)|.$$

By the induction hypothesis, the initial segment in the colex order minimizes the ball $|B'(S'_0)|$ in $J(n - 1, m)$ as well as, by the Kruskal–Katona theorem, the lower shadow $|\Delta(S'_0)|$. \square

6.4 Optimal sets for large n

In this subsection, we give the proof of Theorem 6.5. In what follows we call a positive integer k *critical* if its m -binomial representation has length m (namely, it has $r = m - 1$).

Proof of Theorem 6.5. Let S be an optimal set with cardinality k in $J(n, m)$. We may assume that S is compressed. Let n_0 be such that the support of every element in S is contained in $[n_0]$. Since S is compressed, if the support of $\mathbf{x} \in S$ contains n_0 , then we have $T_{n_0 i}(\mathbf{x}) \in S$ for each $i \in [n_0 - 1] \setminus \bar{\mathbf{x}}$. It follows that $n_0 \leq m + k + 1$. For each $n \geq n_0$ every element in $\Delta(S)$ gives rise to $n - n_0$ distinct vectors in ∂S which have a coordinate in $[n_0 + 1, n]$ and therefore are disjoint from the ball $B_0(S)$ in $J(n_0, m)$. Moreover, every two such vectors, which only differ in their coordinate from $[n_0 + 1, n]$, come from a unique element in $\Delta(S)$. Therefore, we have

$$|B(S)| = |B_0(S)| + (n - n_0)|\Delta S|,$$

where B_0 denotes the ball of S in $J(n_0, m)$ and B denotes the ball of S in $J(n, m)$. Similarly, when I is the initial segment of length k in the colex order. We have

$$|B(I)| = |B_0(I)| + (n - n_0)|\Delta I|$$

Hence,

$$|B(S)| = |B(I)| + (|B_0(S)| - |B_0(I)|) + (n - n_0)(|\Delta(S)| - |\Delta(I)|). \quad (50)$$

If $|\Delta(S)| > |\Delta(I)|$ then we have $|B(S)| > |B(I)|$ for each sufficiently large n . By Theorem 6.7, if the m -binomial representation of k has less than m terms, then the initial segment in the colex order is the unique solution to the Shadow Minimization Problem. It follows that, if $S \neq I$ then $|B(S)| > |B(I)|$ for all $n > n_0 + |B_0(S)| - |B_0(I)|$. This proves the first part of Theorem 6.5 and gives the estimate:

$$n(k, m) \leq m + k + 1 - \mu_{m+k+1, m}(k) + f(k, m + k + 1, m).$$

On the other hand, we have $\mu_{n, m}(k) \leq \mu_{n, m}(k + 1) - 1$, since otherwise an optimal set X with cardinality $k + 1$ satisfies $|\partial(X \setminus \{x\})| < \mu_{n, m}(k)$ for every $x \in X$, a contradiction with

the definition of $\mu_{n,m}$. Suppose that k is a critical integer and let $k, k-1, \dots, k-\ell+1$ be the longest decreasing sequence of critical integers. By the above remark, we have $\mu_{n,m}(k) \leq \mu_{n,m}(k-\ell) + \ell$ and $\mu_{n,m}(k-\ell) = f(k-\ell, n, m)$, the cardinality of the boundary of an initial segment in colex order with length $k-\ell$. The value of ℓ is clearly ¹² k_r . This proves the second part of the statement. \square

6.5 Initial segments which are not optimal

We conclude the chapter by proving Proposition 6.6, which shows that there are values of k for which the initial segment of length k in the colex order fails to be an optimal set of $J(n, m)$ for all sufficiently large n .

We prove first that, for each m and each integer t sufficiently large with respect to m ,

$$g(t, m) = \binom{t}{m} + 3 \binom{t}{m-1}$$

is a critical cardinality, namely, the m -binomial expansion of $g(t, m)$ has m terms. This means that the solution of the Minimal Shadow Problem is not unique for $k = g(t, m)$. This fact is also used in the proof of Proposition 6.6.

Lemma 6.14. *There is an infinite strictly increasing integer sequence $\{\lambda_i\}_{i \geq 0}$, $\lambda_i + 1 < \lambda_{i+1}$ such that, for each t and each $m \geq 1$,*

$$g(t, m) = \sum_{i=0}^{m-1} \binom{t - \lambda_i}{m - i} + 1. \quad (51)$$

Proof. By induction on m . For $m = 1$ we have

$$g(t, 1) = t + 3 = \binom{t+2}{1} + 1, \quad (52)$$

and for $m = 2$,

$$g(t, 2) = \binom{t+2}{2} + \binom{t-2}{1} + 1, \quad (53)$$

giving $\lambda_0 = -2$ and $\lambda_1 = 2$. By using $\binom{n}{m} = \sum_{j=0}^{n-1} \binom{j}{m-1}$ and induction, for $m \geq 3$ we

¹L: previously: $k_r + 1$

²L: if the last element is $\binom{2}{1}$, so $k_r = 2$, then there are 2 consecutive elements that are critical, those mentioned that have $\binom{2}{1}$, and $k-1$ which will have $\binom{1}{1}$. The value of $k-2$ will not be critical. Hence, the value of l is, precisely, 2, as we have $k, k-1$ as the longest decreasing sequence of critical numbers, hence $l = 2$ which is the same as k_r .

have

$$\begin{aligned}
g(t, m) &= \binom{t}{m} + 3 \binom{t}{m-1} \\
&= \sum_{j=0}^{t-1} \binom{j}{m-1} + 3 \sum_{j=0}^{t-1} \binom{j}{m-2} \\
&= \sum_{j=0}^{t-1} g(j, m-1) \\
&= \sum_{j=0}^{t-1} \left(\sum_{i=0}^{m-2} \binom{j-\lambda_i}{m-1-i} + 1 \right) \\
&= \sum_{i=0}^{m-2} \sum_{j=0}^{t-1} \binom{j-\lambda_i}{m-1-i} + t.
\end{aligned}$$

By using $\lambda_0 = -2$ and $\lambda_i > 0$ for $i \in [1, m-2]$, we can write

$$\begin{aligned}
g(t, m) &= \sum_{j=0}^{t-1} \binom{j-\lambda_0}{m-1} + \sum_{i=1}^{m-2} \left(\sum_{j=\lambda_i}^{t-1} \binom{j-\lambda_i}{m-1-i} + \sum_{j=0}^{\lambda_i-1} \binom{j-\lambda_i}{m-1-i} \right) + t \\
&= \binom{t-\lambda_0}{m} + \sum_{i=1}^{m-2} \binom{t-\lambda_i}{m-i} + t + \sum_{i=1}^{m-2} \sum_{j=0}^{\lambda_i-1} \binom{j-\lambda_i}{m-1-i},
\end{aligned}$$

which shows that (51) holds with

$$\begin{aligned}
\lambda_{m-1} &= - \sum_{i=1}^{m-2} \sum_{j=0}^{\lambda_i-1} \binom{j-\lambda_i}{m-1-i} + 1 \\
&= - \sum_{i=1}^{m-2} \sum_{\ell=1}^{\lambda_i} \binom{-\ell}{m-i-1} + 1 \\
&= - \sum_{i=1}^{m-2} \sum_{\ell=1}^{\lambda_i} (-1)^{m-i-1} \binom{m-i-2+\ell}{m-i-1} + 1 \\
&= \sum_{i=1}^{m-2} (-1)^{m-i} \binom{m-i-1+\lambda_i}{m-i} + 1 \\
&= \sum_{j=2}^{m-1} (-1)^j \binom{\lambda_{m-j} + j - 1}{j} + 1 \tag{54}
\end{aligned}$$

We observe that the sequence is uniquely determined once λ_1 is fixed. The first values of the sequence are

$$-2, 2, 4, 7, 14, 51, 928, 409625, \dots$$

It remains to show that the sequence is increasing. We will in fact show that $\lambda_m \geq \max\{\lambda_{m-1} + 2, \lambda_{m-1}^2/4\}$ for all $m \geq 2$. The above inequality holds for $m \leq 7$ as shown by the first values of the sequence. By using (54) (with λ_m instead of λ_{m-1}), we have,

$$\lambda_m \geq \sum_{j=2, j \text{ even}}^{2\lfloor m/2 \rfloor - 1} \left(\binom{\lambda_{m-j+1} + j - 1}{j} - \binom{\lambda_{m-j} + j}{j+1} \right).$$

For $j = 2$ we have

$$\begin{aligned} \binom{\lambda_{m-1} + 1}{2} - \binom{\lambda_{m-2} + 2}{3} &= \frac{\lambda_{m-1}^2}{4} + \frac{\lambda_{m-1}(\lambda_{m-1} + 2)}{4} - \binom{\lambda_{m-2} + 2}{3} \\ &\geq \frac{\lambda_{m-1}^2}{4} + \frac{\lambda_{m-2}^4}{64} + \frac{2\lambda_{m-2}^2}{16} - \binom{\lambda_{m-2} + 2}{3} \\ &> \frac{\lambda_{m-1}^2}{4} \end{aligned}$$

where the last inequality holds (for $m \geq 4$) because the largest root of the polynomial $x^4/64 + x^2/8 - \binom{x+2}{3}$ is smaller than $\lambda_4 = 14$.

On the other hand, for $j \geq 4$,

$$\begin{aligned} \binom{\lambda_{m-j+1} + j - 1}{j} - \binom{\lambda_{m-j} + j}{j+1} &= \frac{1}{j!} \left(\prod_{t=0}^{j-1} (\lambda_{m-j+1} + t) - \frac{\prod_{t=0}^j (\lambda_{m-j} + t)}{j+1} \right) \\ &\stackrel{\lambda_{m-j+1} > \lambda_{m-j} + 1}{>} \frac{1}{j!} \prod_{t=2}^j (\lambda_{m-j} + t) \left(\lambda_{m-j+1} - \frac{\lambda_{m-j}(\lambda_{m-j} + 1)}{j+1} \right) \\ &\stackrel{\lambda_{m-j+1} \geq \lambda_{m-j}^2/4}{\geq} \frac{1}{j!} \prod_{t=2}^j (\lambda_{m-j} + t) \left(\frac{\lambda_{m-j}^2}{4} - \frac{\lambda_{m-j}(\lambda_{m-j} + 1)}{j+1} \right). \end{aligned}$$

The right-hand side is nonnegative if $m - j \geq 2$, as then $\lambda_{m-j} \geq 4$. If $m - j = 1$ then it follows by induction on $j \geq 1$ that

$$\binom{\lambda_2 + j - 1}{j} - \binom{\lambda_1 + j}{j+1} = \binom{3 + j}{j} - \binom{2 + j}{j+1} > 0.$$

This completes the proof. \square

For t larger than λ_{m-1} the equality (51) in Lemma 6.14 provides the m -binomial expansion of $g(t, m)$. Hence this binomial expansion has length m and, by Theorem 6.7, $g(t, m)$ is a critical cardinality. The proof of Proposition 6.6 uses this fact by choosing two distinct optimal sets for the SMP problem which have different boundaries in the Johnson graph.

Proof of Proposition 6.6. Let $S = B_0(\binom{[t]}{m})$, the ball of $\binom{[t]}{m}$ in the Johnson graph $J(n_0, m)$, $n_0 = t + 3$. The cardinality of S is

$$k = \binom{t}{m} + 3 \binom{t}{m-1} = g(t, m).$$

Let $I(k)$ denote the initial segment of length k in the colex order.

By Lemma 6.14 the m -binomial expansion of k has m terms and can be written as

$$\binom{t}{m} + 3\binom{t}{m-1} = \binom{t-\lambda_0}{m} + \binom{t-\lambda_1}{m-1} + \dots + \binom{t-\lambda_{m-1}}{1}. \quad (55)$$

We note that the shadows of S and of $I(k)$ have the same cardinality:

$$\begin{aligned} |\Delta S| &= \binom{t}{m-1} + 3\binom{t}{m-2} \\ &= \binom{t-\lambda_0}{m-1} + \binom{t-\lambda_1}{m-2} + \dots + \binom{t-\lambda_{m-2}}{1} + \binom{t-\lambda_{m-1}}{0} \\ &= |\Delta(I(k))|. \end{aligned}$$

It can be readily checked that the boundary of S in $J(n_0, m)$ has cardinality

$$|\partial S| = |\partial^2 \binom{[t]}{m}| = 3\binom{t}{m-2},$$

On the other hand, the boundary in $J(n_0, m)$ of the initial interval $I(k)$ as given by the function $f(k, n_0, m)$ in (46) is

$$\begin{aligned} |\partial I(k)| &= \binom{t-\lambda_0}{m-1} + \sum_{i=1}^{m-1} \left(\binom{t-\lambda_i}{m-i-1} (n-t+\lambda_0-1) - \binom{t-\lambda_i}{m-i} \right) \\ &= \binom{t+2}{m-1} - \sum_{i=1}^{m-1} \binom{a_i}{m-i} = \binom{t+2}{m-1} + \binom{t+2}{m} - \binom{t}{m} - 3\binom{t}{m-1} \\ &= \binom{t+1}{m-2} + 2\binom{t}{m-2}, \end{aligned}$$

which is strictly larger than $3\binom{t}{m-2}$. Thus $|B_0(I(k))| > |B_0(S)|$. Moreover, by (50), for all $n \geq n_0$, we have

$$|B(I(k))| = |B(S)| + (|B_0(I(k))| - |B_0(S)|) > |B(I(k))|.$$

Therefore the initial segment in colex order I fails to be an optimal set for all $n \geq t+3$. \square

7 Conclusions

In this chapter we summarize the contributions of this work in the subjects we have dealt with.

7.1 On pieces of information of a graph

In Chapters 2 and 3, we first study the properties of each piece of information: the spectrum, the preintersection numbers, the predistance polynomials, the Bose-Mesner algebras and the closed walks. Then, we give the specific formulas and procedures with which we can obtain each piece of information in terms of the others. This is the importance of this part of the work.

The preintersection numbers and predistance polynomials are extensions of the well-known parameter intersection numbers and distance polynomials in distance-regular graphs, respectively. We extend them applying the same techniques to the spectrum as if the graph were distance-regular. It is as if we asked which intersection numbers and distance polynomials this graph would have if it were distance-regular.

These results can be seen as tools that can be used in the area of Graph Theory to develop future works. The results given in this part of the work extend the properties of distance-regular graphs to any other graph. This allows us to relate the combinatorial and algebraic properties of any graph. Or, simply, we can study the combinatorial properties by deriving it from an algebraic point of view, which is much more powerful.

7.2 Characterizations of distance-regularity in terms of the pieces of information

In this chapter, we apply the results given in Chapter 3. In Subsection 4.1, we show how different results, such as Theorem 4.2 and Theorem 4.3, that deal with characterizations of distance-regularity, became trivially equivalent to each other.

In Subsection 4.2, we show and prove some results that deal with the combinatorial properties of the graph in terms of the different pieces of information. Here, we use the results given in Chapter 3, not only for proving these results, but also to establish relationships between them. These relationships help us to better understand the properties of the graph and how the pieces of information are intrinsically related with these combinatorial properties.

For example, in Proposition 4.5, we relate the girth of the graph (which is a well studied combinatorial parameter) with the structure of the proper coefficients matrix of the predistance polynomials (see also Example 4.6, where we show the simplicity with which the girth can be determined and check if the graph is bipartite).

A similar case can be seen in Propositions 4.7 and 4.8, where the same combinatorial

property can be determined from two different pieces of information.

In Subsection 4.3, we write properties and relationships that the proper pieces of information have between them. We show how the tools described in Chapter 3 are also useful to show the internal properties of the different parameters.

For example, Proposition 4.9 deals with the behavior of the preintersection numbers and a relation between them and the spectrum of the graph. This proposition gives the inequalities $\gamma_i > 0$ for $i = 1, \dots, d$, and $\beta_i > 0$ for $i = 0, \dots, d - 1$. The example that follows this property, given by the Figure 6, shows how these inequalities cannot be written of case of α_i and they do not even have a monotone behavior.

7.3 Distance mean-regular graphs

In Chapter 5, we construct a natural extension of the distance regular graphs. In the general case, the preintesection numbers or the predistance polynomials are extended from the intersection numbers by taking the spectrum of the graph as a base. That is, these parameters are the parameters that a distance-regular graph would have with the given spectrum. This extension, unlike in the case of the preintersection numbers or the predistance polynomials, are the intersection mean-numbers which are taken as a base for this extension.

According to these parameters, we construct the distance mean-polynomials and the pseudo-spectrum. They can be seen as the polynomials and spectrum a distance-regular graph would have if it were distance-regular. Thus, the most important properties of this graph are the algebraic properties.

Obviously, the intersection mean-numbers have a lot of combinatorial properties, and the distance mean-regularity of a graph can be easily characterized by them. For example, some of the first lemmas with some basic properties are Lemma 5.3 and Lemma 5.4.

Moreover, the pseudo-spectrum built from this combinatorial parameters have many algebraic properties. For example, we can read the first properties in Subsection 5.2. The pseudo-spectrum interlaces the real spectrum of the graph, and this interlacing is tight if and only if the graph is distance-regular; that is, if the pseudo-spectrum matches the real spectrum of the graph.

In such a subsection, we find the first important relationship between the intersection mean-numbers and the pseudo-spectrum of the graph. The matrix $\overline{\mathbf{B}}$ of parameters can be calculated as a quotient matrix of the adjacency matrix \mathbf{A} . As shown in Equality (27):

$$\mathbf{B} = \mathbf{S}^\top \mathbf{A} \mathbf{T} = \mathbf{D}^{-1} \mathbf{T}^\top \mathbf{A} \mathbf{T},$$

where the matrices \mathbf{S} and \mathbf{T} depend only on the distance between the vertices of the graph. In Proposition 5.2, we prove that the graph is distance mean-regular if and only if \mathbf{B} is well defined and, in this case, we have that $\mathbf{B} = \overline{\mathbf{B}}$. From this matrix $\overline{\mathbf{B}}$ we calculate the pseudo-spectrum of the distance mean-regular graph. Subsection 5.5 contains details

with which we calculate the multiplicities of the eigenvalues of the pseudo-spectrum and the distance mean-polynomials.

An important part of the study of these graphs is their characterizations. In Subsection 5.2 we give the first characterization in terms of the intersection mean-numbers matrix and the adjacency matrix, but many others are possible. Indeed, we also provide a characterization in terms of the number of edges, and we extend this result to the number of triangles (see Subsection 5.7.1). This characterization can then be extended in many directions. One of them could be the study of the number of closed walks of different lengths (note that a triangle is a closed walk of length 3); we had done in Subsection 2.5 in the study of general information of graphs. Thus, we chose to give the characterization of the distance mean-regular graphs in terms of the number of the triples t_{hij} in a vertex u as, defined as:

$$t_{hij}(u) = \sum_{v \in \Gamma_h(u)} |\Gamma_j(v) \cap \Gamma_i(u)| = \sum_{w \in \Gamma_i(u)} |\Gamma_j(w) \cap \Gamma_h(u)|.$$

That is, the number of triangles on u of edge size h , i and j respectively. Thus, in Subsection 5.7.2, we have:

Proposition. 5.11. *A graph Γ with diameter D is distance mean-regular if and only if the numbers of triples t_{hij} are well defined for $h, i, j = 0, \dots, D$.*

Furthermore, going back to the algebraic tools, we write another characterization in terms of the distance matrices. These results are in Proposition 5.12 of Subsection 5.7.3. Some relevant equalities have been given there. In Equality (39), we express the relationship between the general intersection mean-numbers and the distance matrices:

$$\bar{p}_{ij}^h = \bar{p}_{ji}^h = \frac{\langle \mathbf{A}_i \mathbf{A}_j, \mathbf{A}_h \rangle}{\|\mathbf{A}_h\|^2}.$$

Subsection 5.7.3, with results like Proposition 5.12, is a perfect introduction to the final part of Chapter 5. One of the main results is the study of the matrix algebras defined for each distance mean-regular graph (see Subsection 5.8):

- $\mathcal{A} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d)$, where \mathbf{A} is the adjacency matrix and $d + 1$ is the number of distinct eigenvalues;
- $\mathcal{D} = \text{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_D)$, where \mathbf{A}_i is the i -distance matrix;
- $\bar{\mathcal{D}} = \text{span}(\mathbf{I}, \mathbf{A}, \bar{\mathbf{A}}_2, \dots, \bar{\mathbf{A}}_D)$, where $\bar{\mathbf{A}}_i = \bar{p}_i(\mathbf{A})$;
- $\bar{\mathcal{B}} = \text{span}(\mathbf{I}, \bar{\mathbf{B}}, \bar{\mathbf{B}}_2, \dots, \bar{\mathbf{B}}_D)$, where $\bar{\mathbf{B}}_i = \bar{p}_i(\mathbf{B})$.

We study the existing relationship between them. See Theorem 5.13 and Proposition 5.14.

7.4 Isoperimetric problem in Johnson graphs

In this section we present the results obtained in the study of the vertex-isoperimetric problem for the Johnson graph $J(n, m)$. The used techniques are mainly of a combinatorial nature. For example, we use the shifting (see Subsection 6.1), which was used before by some authors in the study of similar problems.

One of the first results is the case $m = 2$. In Proposition 6.11 of Subsection 6.2, we explicitly describe the optimal sets in the graph $J(n, 2)$:

Proposition. 6.11. *A set S of vertices of the graph $J(n, 2)$ with cardinality $\binom{t-1}{2} < |S| \leq \binom{t}{2}$ is optimal if and only if, up to isomorphism, $S \subseteq \binom{[t]}{2}$.*

From this proposition, Theorem 6.3 follows. This theorem relates the $J(n, 2)$ and its optimal sets with the colexicographic order, which have been an important sequence of vertices in this problem (and also in other similar problems like the Minimal Shadow Problem, or the vertex-isoperimetric problem in n -cubes).

The initial segments of the colexicographic order is also the optimal solution in some other graphs. In Proposition 6.13, we show that this order is the optimal solution for Johnson graphs of the form $J(2m - 2, m)$. This result is explained and proved in Subsection 6.3. In this subsection, we also show that the colex order also gives the optimal solution for small cardinals. That is, the initial segment of length k of the colex order is optimal for $k \leq m - 1$ for every $J(n, m)$. This result is in Theorem 6.4, and its proof and the used techniques are explained in Subsection 6.3. These two results are related because the case $J(2m - 2, m)$ constitutes the basis of the proof of Theorem 6.4.

In Subsection 6.4, we study the optimal sets for large enough values of n . In this case, the colex order is optimal again. We can see how the corresponding Theorem 6.5 is divided into two cases: $r < m - 1$, and $r = m - 1$, where r is the length of the m -binomial expansion:

$$k = \binom{k_0}{m} + \binom{k_1}{m-1} + \cdots + \binom{k_r}{m-r}, \quad k_0 > \cdots > k_r \geq m-r > 0.$$

This kind of cardinals with maximal length have a unique solution in the shadow minimization problem, and the shadow of the sets are used in the proof of Theorem 6.5. We express the ball of a set as:

$$|B(S)| = |B_0(S)| + (n - n_0)|\Delta S|,$$

(see more details in Subsection 6.4).

The boundary of the initial segment of the colex order, which is the optimal solution for all of these subfamilies of graphs, can be determined with the function described in Proposition 6.1:

$$f(k, n, m) = \binom{k_0}{m-1}(n - k_0) + \sum_{i=1}^r \left(\binom{k_i}{m-i-1}(n - k_0 - 1) - \binom{k_i}{m-i} \right).$$

However, an important conclusion is that the colex order is not optimal in all graphs $J(n, m)$. In Example 6.2 we give a first example of a set that can have a smaller boundary than the initial segment of the colex order. This set is a ball of a single vertex. We study this kind of counterexamples in Subsection 6.5. We give an infinity family of cardinals for which the colex order is not optimal. We prove that the ball of a single vertex in a subgraph $J(t, m)$ for $t \leq n + 3$ and t sufficiently large with respect to m , is a set of cardinality

$$g(t, m) = \binom{t}{m} + 3 \binom{t}{m-1},$$

with a smaller boundary than the initial segment of the colex order of the same size. This result is contained in Proposition 6.6, but the proof and details can be seen in Subsection 6.5. For this proof, we use Lemma 6.14 (of a combinatorial nature). It shows a very curious property: no matter how large is the value of t is in the expression

$$g(t, m) = \binom{t}{m} + 3 \binom{t}{m-1},$$

its m -binomial expansion always has maximal length, and also, the differences between each two consecutive parameters k_i and k_{i+1} is always the same. This infinity sequence of differences is

$$-2, 2, 4, 7, 14, 51, 928, 409625, \dots$$

In conclusion, we prove that, in many cases, the initial segment of the colexicographic order is optimal. But also, we show that this is not the optimal solution for every cardinality k and every graph $J(n, m)$, giving an explicit infinity family of counterexamples.

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References

- [1] A. Abiad, E.R. van Dam, and M.A. Fiol, Some spectral and quasi-spectral characterizations of distance-regular graphs, *J. Combin. Theory Ser. A* **143** (2016) 1–18.
- [2] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974, second edition, 1993.
- [3] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin-New York, 1989.
- [4] A.E. Brouwer and W.H. Haemers, The Gewirtz graph: An exercise in the theory of graph spectra, *European J. Combin.* **14** (1993) 397–407.
- [5] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, 2012; available online at <http://homepages.cwi.nl/~aeb/math/ipm/>.
- [6] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga, Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes, *Electron. J. Combin.* **16(1)** (2009) #R83.
- [7] D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs. Theory and Application*, VEB Deutscher Verlag der Wissenschaften, Berlin, second edition, 1982.
- [8] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga, and B.L. Gorissen, On almost distance-regular graphs, *J. Combin. Theory Ser. A* **118** (2011), no. 3, 1094–1113.
- [9] C. Dalfó, M.A. Fiol, and E. Garriga, Characterizing (ℓ, m) -walk-regular graphs, *Linear Algebra Appl.* **433** (2010), no. 9, 1821–1826.
- [10] E.R. van Dam, The spectral excess theorem for distance-regular graphs: A global (over)view, *Electron. J. Combin.* **15(1)** (2008) #R129.
- [11] E.R. van Dam and M.A. Fiol, A short proof of the odd-girth theorem, *Electron. J. Combin.* **19(3)** (2012) #P12.
- [12] E.R. van Dam and W.H. Haemers, Spectral characterizations of some distance-regular graphs, *J. Algebraic Combin.* **15** (2002) 189–202.
- [13] E.R. van Dam and W.H. Haemers, An odd characterization of the generalized Odd graphs, *J. Combin. Theory Ser. B* **101** (2011) 486–489.
- [14] E.R. van Dam, J.H. Koolen, and H. Tanaka, Distance-regular graphs, preprint, 2014, arXiv:1410.6294.
- [15] M.A. Fiol, An eigenvalue characterization of antipodal distance-regular graphs, *Electron. J. Combin.* **4** (1997), no. 1, #R30.

- [16] M.A. Fiol, Algebraic characterizations of distance-regular graphs, *Discrete Math.* **246** (2002) 111–129.
- [17] M.A. Fiol, S. Gago, and E. Garriga, A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.* **432** (2010), no. 9, 2418–2422.
- [18] M.A. Fiol and E. Garriga, From local adjacency polynomials to locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **71** (1997) 162–183.
- [19] C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.
- [20] W.H. Haemers and E. Spence, Graphs cospectral with distance-regular graphs, *Linear Multilinear Algebra* **39** (1995), no. 1-2, 91–107.
- [21] A.J. Hoffman, On the polynomial of a graph, *Amer. Math. Monthly* **70** (1963) 30–36.
- [22] G.-S. Lee and C.-W. Weng, The spectral excess theorem for general graphs, *J. Combin. Theory, Ser. A* **119** (2012) 1427–1431.
- [23] A. Abiad, E.R. van Dam, M.A. Fiol, Some spectral and quasi-spectral characterizations of distance-regular graphs, preprint, arXiv:1404.3973v3 [math.CO], 2014.
- [24] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin Cummings Lecture Notes Series, Vol. **58**, Benjammin-Cummings, London, 1993.
- [25] A.E. Brouwer, Personal communication, 2015.
- [26] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin-New York, 1989.
- [27] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, 2012; available online at <http://homepages.cwi.nl/~aeb/math/ipm/>.
- [28] N. Biggs, Intersection matrices for linear graphs, in: D.J.A. Welsh (Ed.), *Combinatorial Mathematics and its Applications*, Academic Press, London, 1971, pp. 15–23.
- [29] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974, second edition, 1993.
- [30] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga, Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes, *Electron. J. Combin.* **16** (2009), #R83.
- [31] E.R. van Dam, J.H. Koolen, and H. Tanaka, Distance-regular graphs, preprint (2014); arXiv:1410.6294 [math.CO].
- [32] M.A. Fiol and E. Garriga, From local adjacency polynomials to locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **71** (1997) 162–183.

- [33] M.A. Fiol, Eigenvalue interlacing and weight parameters of graphs, *Linear Algebra Appl.* **290** (1999), 275–301.
- [34] M.A. Fiol, Pseudo-distance-regularized graphs are distance-regular or distance-biregular, *Linear Algebra Appl.* **437** (2012) 2973–2977.
- [35] C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.
- [36] C.D. Godsil and J. Shawe-Taylor, Distance-regularised graphs are distance-regular or distance-biregular, *J. Combin. Theory Ser. B* **43** (1987), no. 1, 14–24.
- [37] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995) 593–616.
- [38] M. Hazewinkel, N. Gubareni, N.M. Gubareni, V.V. Kirichenko, *Algebras, Rings and Modules*, Volume 1, Springer, Heidelberg 2004.
- [39] T. Hilano and K. Nomura, Distance degree regular graphs, *J. Combin. Theory Ser. B* **37** (1984), 96–100.
- [40] P. Weichsel, On distance-regularity in graphs, *J. Combin. Theory, Ser. B* **32** (1982) 156–161.
- [41] R. Ahlswede and G.O. Katona, Graphs with maximal number of adjacent pairs of edges. *Acta Math. Acad. Sci. Hungar.* **32** (1978) 97–120.
- [42] C. Bey, Remarks on an edge-isoperimetric problem, General theory of information transfer and combinatorics, Lecture Notes in Comput. Sci., 4123, Springer, Berlin (2006) 971–978.
- [43] S.L. Bezrukov, An isoperimetric problem for Manhattan lattices. Proc. Int. Conf. Finite and Infinite Sets, Budapest (1990).
- [44] S. L. Bezrukov, Isoperimetric problems in discrete spaces. Extremal problems for finite sets (Visegrád, 1991), Bolyai Soc. Math. Stud., 3, János Bolyai Math. Soc., Budapest, (1994) 59–91.
- [45] S.L. Bezrukov, U. Leck, A simple proof of the Karakhanyan-Riordan theorem on the even discrete torus, *SIAM J. Discrete Math.* **23** (2009) 1416–1421.
- [46] S. L. Bezrukov, O. Serra, A local-global principle for vertex-isoperimetric problems, *Discrete Math.* **257** (2002) 285–309.
- [47] B. Bollobás, *Combinatorics, Set systems, Hypergraphs, Families of vectors and Combinatorial Probability*. Cambridge Univ. Press (1986). An isoperimetric inequality on the discrete torus, *SIAM J. Appl. Math.* **3** (1990) 32–37.
- [48] B. Bollobás, I. Leader, Compressions and isoperimetric inequalities. *J. Combin. Theory Ser. A* **56** (1991), no. 1, 47–62.

- [49] B. Bollobás, I. Leader, Isoperimetric inequalities for r -sets, *Combinatorics, Probability and Computing* 13 (2004), pp. 277–279.
- [50] A.E. Brouwer, A. Cohen and A. Neumaier, Distance-Regular Graphs, Springer-Verlag 1989.
- [51] P. J. Cameron, *Permutation groups*. London Mathematical Society Student Texts, 45. Cambridge University Press, Cambridge, 1999.
- [52] S.M. Cioabva, K. Kim and J.H. Koolen, On a conjecture of Brouwer involving the connectivity of strongly regular graphs, *Journal of Combinatorial Theory, Series A* 119 (2012), 904–922.
- [53] S.M. Cioabva, J.H. Koolen and W. Li, Disconnecting strongly regular graphs, *European J. Combinatorics* 38 (2014), 1–11.
- [54] D. Christofides, D. Ellis, P. Keevash, An approximate isoperimetric inequality for r -sets. *Electron. J. Combin.* 20, no. 4, Paper 15, 12 pp, (2013).
- [55] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)* 12 (1961) 313–320.
- [56] P. Frankl, Z. Füredi, A short proof for a theorem of Harper about Hamming-spheres, *Disc. Math.* 34, no. 3, (1981) 311–313.
- [57] Z. Füredi; J. R. Griggs, Families of finite sets with minimum shadows. *Combinatorica* 6, (1986) 355–363.
- [58] C. D. Godsil, Algebraic combinatorics. Chapman and Hall Mathematics Series I Chapman & Hall. New York, (1993).
- [59] L.H. Harper, Optimal numberings and isoperimetric problems on graphs. *J. Comb. Theory* 1 (1966) 385–393.
- [60] L. H. Harper, Global methods for combinatorial isoperimetric problems. Cambridge Studies in Advanced Mathematics, 90. Cambridge University Press, Cambridge, (2004).
- [61] S. Hart, A note on the edges of the n -cube. *Disc. Math.* 14 (1976) 157–163.
- [62] V.M. Karachanjan, A discrete isoperimetric problem on multidimensional torus (in Russian), *Doklady AN Arm. SSR*, vol. LXXIV, No. 2, (1982) 61–65.
- [63] G. O. H. Katona, A theorem of finite sets. Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, (1968) 187–207.
- [64] J. B. Kruskal, The number of simplices in a complex. Mathematical optimization techniques. Univ. of California Press, Berkeley, Calif., (1963) 251–278.

- [65] I. Leader, Discrete isoperimetric inequalities. Probabilistic combinatorics and its applications (San Francisco, CA, 1991), Proc. Sympos. Appl. Math., 44, Amer. Math. Soc., Providence, RI, (1991) 57–80.
- [66] Michael Mörs, A Generalization of a Theorem of Kruskal. *Graphs and Combinatorics* 1 (1985) 167–183.
- [67] O. Riordan, An ordering on the discrete even torus, 110–127 *SIAM J. Discr. Math.*, 11, No. 1, (1998).