On the Accuracy of the Adaptive Cross Approximation Algorithm

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***Abstract*—This contribution identifies an often ignored source of uncertainty in the accuracy of the adaptive cross approximation algorithm, and proposes a combination of adaptations that reduce this uncertainty with negligible additional computational cost.**

I. Introduction

The adaptive cross approximation (ACA) [1] is an algorithm that computes a low rank approximation to matrices or matrix sub-blocks. Due to its efficiency and its black-box nature (no prior knowledge of the matrix content or origin is necessary), it is gaining ground as a tool in accelerated electromagnetic simulations. As with all approximations, the ACA introduces an error that can only be known up to an order of magnitude, to be chosen by the user. With the growing popularity of the ACA, so grows the importance of assessing the reliability of the algorithm under all circumstances. In this contribution, we intend to draw attention to one aspect of the ACA algorithm that introduces a variation in the actual error around the user-chosen error threshold of about one order of magnitude. This variation is triggered by an initial choice inside the algorithm which is not left to the user but instead hard-wired in the algorithm formulation. There is presently no known recipe for optimizing this choice. This means that the ACA algorithm is, to some extent, a randomized algorithm, with a probability distribution for the true residual error. The reason that these observations have not been made before, even though the ACA algorithm has been receiving ample attention for some time now, is twofold: Firstly, since the initial setting is hard-wired, the user will always obtain identical results on identical problems and the algorithm will seem perfectly deterministic to her. Secondly, the ACA is typically used to compress off-diagonal blocks of a larger linear system. How exactly the relative error in the blocks propagates into the solution of the linear system cannot be known but it will often be considerably smaller, as the matrix is often dominated by the non-approximated on-diagonal blocks. So it is unlikely, though not impossible, that an error that is one order of magnitude larger than the user intended, will result in an error of the same magnitude in the final solution. In Section II we illustrate the phenomenon with a simple numerical example and in Section III we propose an adaptation to the ACA algorithm and the ACA convergence criterion which, although it does not solve the problem, seems to dampen its effect, at negligible extra computational cost. Remains to be mentioned that the possible unreliability of the estimated error in the ACA algorithm has been noted before, although without linking it to the initial choice and consequently without addressing its statistical nature, notably in [2] and [3].

II. Numerical Example

As an illustration, the ACA algorithm is invoked to compress the mutual interaction matrix of the two square PEC plates shown in Fig. 1, in the EFIE formulation and using RWG basis functions. The plates are discretized into 1160 basis functions each, and the working frequency is such that the plate edges span two wave lengths. Fig. 2 shows the distribution of the true relative error after convergence of the ACA algorithm with a threshold of 10^{-4}, when all 1160 possible random choices of the initial column are tried. Note the variation in the true error of more than one order of magnitude. The number of iterations needed for convergence similarly shows a wide distribution, in this case ranging from $k = 41$ to 58 steps, with an average over all 1160 initial choices of $\bar{k} = 37.8$ steps.

III. Proposed Adaptation of the ACA

The first intervention that we attempted was based on the observation that the conventional ACA estimation of the residual error sometimes shows sharp negative peaks leading to a highly premature declaration of convergence, as was noted in [2], [3]. An example of this phenomenon is observed at step 32 in Fig. 3. Considering that the global convergence rate is theoretically expected and observed in practice to

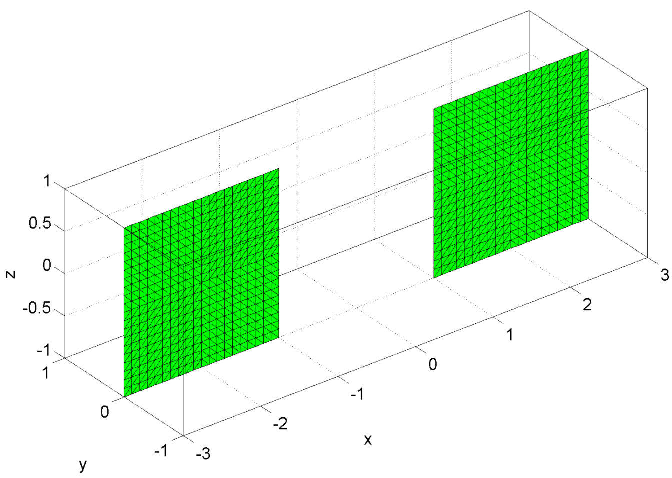

Fig. 1. Two coplanar $2\lambda \times 2\lambda$ PEC plates at a mutual distance of 2λ.

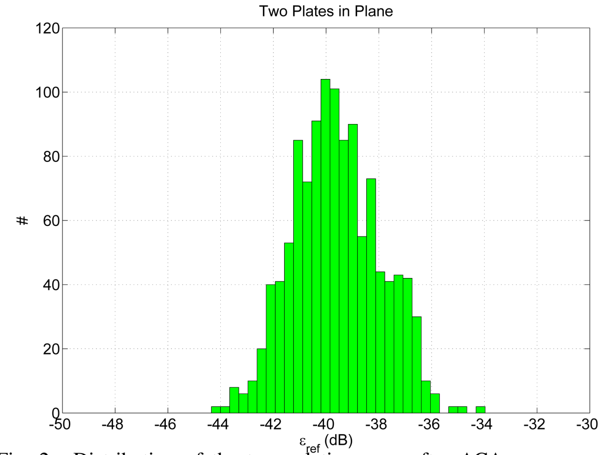

Fig. 2. Distribution of the true relative error after ACA convergence with threshold $\tau = 10^{-4}$, for the two plates in Fig. 1.

be exponential, a cumulative linear fit is computed of the logarithm of the estimated residual error versus the step index, including all estimates up to the current step. The resulting curve is also shown in Fig. 3. This fit is then used to estimate the current error. This slightly narrows the distribution of true relative errors, but it also leads to an increase of the average number of steps before convergence, to $\bar{k} = 40.4$ in the example. Considering that the computational cost of the ACA scales with k^2, this approach unacceptably increases the computational effort.

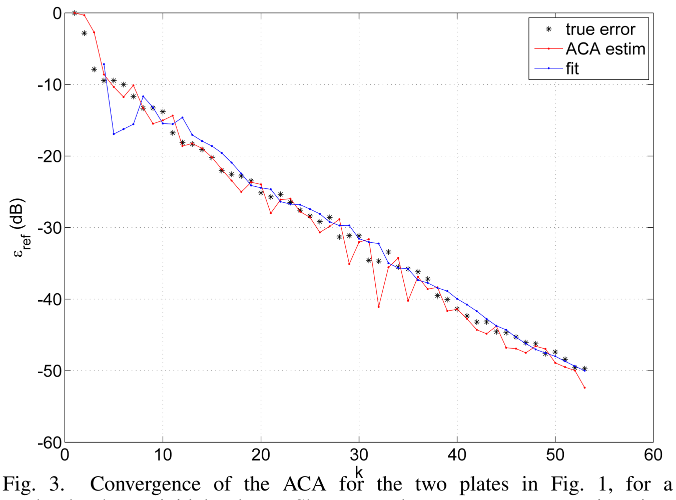

Fig. 3. Convergence of the ACA for the two plates in Fig. 1, for a randomly chosen initial column. Shown are the true error at every iteration, the conventional ACA estimator and the cumulative linear fit proposed in this paper.

Subsequently we explored an entirely different idea; observing that changing the initial column, the sequence of selected columns and rows often changes drastically, regularly resulting in a zero overlap between the columns and rows used in two equivalent ACA decompositions, we conjectured that the choice of the next column or row to be included is not unique. The ACA algorithm prescribes to choose that column/row whose common element in the current row/column shows the largest absolute error. We ran a series of experiments picking the elements corresponding to the nth largest error instead, and we found the counter-intuitive outcome that up to $n \approx 50$ the distribution of the true error narrowed and at the same time the average number of steps decreased (For example, with $n = 10$, the extremes of the distribution decreased to $\{-44$ dB, -36 dB$\}$ and $\bar{k}$ decreased to 36.5). The effect was too small to provide a substantial improvement to the ACA efficiency, but it was persistent.

The most surprising result was obtained when we combined the two approaches hereabove: Picking the 10th best element to select the rows and columns and using the linear fit to estimate the residual error. The distribution of the true error after convergence is shown in Fig. 4. There is a considerable narrowing and a left-shift of the distribution, which now spans approximately $\{-45$ dB, -38 dB$\}$, but this has not been at the expense of the convergence rate which, with $\bar{k} = 38.3$, is only half a step above the conventional ACA.

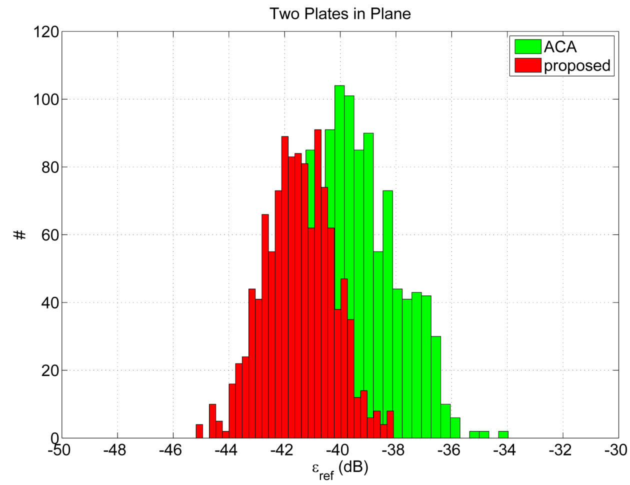

Fig. 4. Distribution of the true relative error as in Fig. 2, with and without the adaptations proposed in this paper.

IV. Conclusions

The aim of this contribution was to draw attention to the often overlooked statistical uncertainty in the ACA algorithm. Although no definitive solution has been found to this problem, a combination of two adaptations to the ACA algorithm has been proposed, that reduces its impact without compromising the efficiency of the ACA. Further investigation is necessary to understand the mechanisms behind these adaptations and perhaps to reduce the remaining uncertainty.

Acknowledgment

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