

# A contribution to the contractibility problem for overlapping controllers

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**Abstract:** In this work, a contribution to the contractibility of decentralized control laws, in the context of the Inclusion Principle, is offered. By means of this principle, an overlapping system can be expanded to another bigger one where the subsystems are considered as disjoint. Under a control criterion, local controllers are designed for each subsystem. After this expansion-design process, the decentralized control laws need to be contracted and implemented in the original system. However, the contractibility conditions are not satisfied by arbitrary local gain matrices. In this paper, an explicit structure for the local gain matrices ensuring contractibility is presented; moreover, the possibility of using different control criteria in accordance with these structures is discussed.

**Keywords:** Contractibility, inclusion principle, overlapping decomposition, decentralized control.

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## 1. INTRODUCTION

Numerous real systems are composed by overlapped subsystems sharing states, inputs and/or outputs. When dealing with interconnected large-scale or complex systems, the design of a centralized control law could be inappropriate due to several reasons: high dimensionality of the systems, information structure constraints, great computational effort, the presence of uncertainties, or a mixture of some of them. With the objective of controlling these systems, strategies based on the design of local controllers have been profusely developed in the literature, İftar (1993), Stanković and Šiljak (2001), Chen and Stanković (2004), Stipanović et al. (2004), Chen and Stanković (2005), Zečević and Šiljak (2005).

The *Inclusion Principle* appears as a powerful mathematical tool to work with overlapping systems. By means of this principle, an expansion-design-contraction process can be carried out in three phases: (1) expansion of the original overlapping system into a bigger one, where the subsystems are considered as disjoint; (2) local design of controllers for each decoupled subsystem; (3) contraction of the obtained local controllers to be implemented as an overlapping control law into the initial system. A rigorous treatment of the inclusion principle can be found in Ikeda et al. (1981), Ikeda et al. (1984), Šiljak (1991).

In phase (1), certain matrices involved in the expansion process, known as *complementary matrices*, play a crucial role for achieving decoupled or weakly coupled expanded systems. It has been proved that, by adding convenient complementary matrices, some desired properties are preserved, Bakule et al. (2000), Bakule et al. (2001a), Bakule et al. (2001b). In this paper, only two types of complementary matrices will be considered, which define the two most important cases of expansion: *aggregations* and *restrictions*. The result of this expansion

phase is an expanded system formed by subsystems which can be treated as decoupled.

In phase (2), using only local information, decentralized controllers for each subsystem are designed by means of standard methods. Then, a set of local controllers are obtained and joined together in a diagonal gain matrix to be later contracted to the initial system. To satisfy the inclusion principle, the expanded closed-loop system has to include the corresponding contracted one. In order to guarantee this inclusion, some conditions on the expanded gain matrices are required. Sometimes, convenient adjustments on the block diagonal gain matrix need to be done in order to assure contractibility. In any case, there exists no general methodology to properly modify the elements of the extended gain matrices, Ikeda and Šiljak (1986).

Finally, phase (3) is straightforward when the contractibility conditions are satisfied. The contracted *overlapping controller*, which has a tridiagonal structure, can be implemented into the original system.

This article attempts to be a contribution to clarify the contractibility problem in the framework of the inclusion principle. In this line, we firstly present a class of gain matrices which are contractible for implementation in the original system. Secondly, we determine if such structures can be achieved when different control criteria are used in the design of local expanded controllers. In particular, optimal control, guaranteed cost control, and  $H_\infty$  control are considered. It is worth to mention that, in the present study, the initial system consists of only two overlapped subsystems; however, the obtained results could be easily extended to a greater number of overlapped subsystems.

## 2. PRELIMINARIES

### 2.1 The inclusion principle

Consider a pair of linear systems

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$$\mathbf{S} : \dot{x}(t) = Ax(t) + Bu(t), \quad \tilde{\mathbf{S}} : \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), \quad (1)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  are the state and input of the system  $\mathbf{S}$  at time  $t \geq 0$ , while  $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ ,  $\tilde{u}(t) \in \mathbb{R}^{\tilde{m}}$  are the state and input of  $\tilde{\mathbf{S}}$ .  $A$ ,  $B$  and  $\tilde{A}$ ,  $\tilde{B}$  are constant matrices of dimensions  $n \times n$ ,  $n \times m$  and  $\tilde{n} \times \tilde{n}$ ,  $\tilde{n} \times \tilde{m}$ , respectively. We assume  $\tilde{n} \geq n$ ,  $\tilde{m} \geq m$ . By  $x(t; x_0, u)$  we denote the state behavior of  $\mathbf{S}$  for a fixed input  $u(t)$  and initial state  $x(0) = x_0$ . The notation  $\tilde{x}(t; \tilde{x}_0, \tilde{u})$  is used for the state behavior of  $\tilde{\mathbf{S}}$ .

Consider the following expansion-contraction transformations:

$$\begin{aligned} V : \mathbb{R}^n &\rightarrow \mathbb{R}^{\tilde{n}}, & U : \mathbb{R}^{\tilde{m}} &\rightarrow \mathbb{R}^m, \\ R : \mathbb{R}^m &\rightarrow \mathbb{R}^{\tilde{m}}, & Q : \mathbb{R}^{\tilde{n}} &\rightarrow \mathbb{R}^n, \end{aligned} \quad (2)$$

where  $\text{rank}(V) = n$ ,  $\text{rank}(R) = m$  so that  $UV = I_n$ ,  $QR = I_m$ , where  $I_n$ ,  $I_m$  denote the identity matrices of indicated dimensions.

**Definition 1. (Inclusion Principle)** A system  $\tilde{\mathbf{S}}$  includes the system  $\mathbf{S}$ , if there exists a quadruplet of matrices  $(V, U, R, Q)$  satisfying (2) such that, for any initial state  $x_0$  and any fixed input  $u(t)$  of  $\mathbf{S}$ , the choice  $\tilde{x}_0 = Vx_0$  and  $\tilde{u}(t) = Ru(t)$ , implies  $x(t; x_0, u) = U\tilde{x}(t; \tilde{x}_0, \tilde{u})$ , for all  $t \geq 0$ .

**Definition 2.** If a system  $\tilde{\mathbf{S}}$  includes the system  $\mathbf{S}$ , denoted by  $\tilde{\mathbf{S}} \supset \mathbf{S}$ , then  $\tilde{\mathbf{S}}$  is said to be an *expansion* of  $\mathbf{S}$ , and  $\mathbf{S}$  is a *contraction* of  $\tilde{\mathbf{S}}$ .

**Definition 3.** A the system  $\mathbf{S}$  is an *aggregation* of  $\tilde{\mathbf{S}}$ , if there exists a pair  $(U, Q)$  satisfying (2) such that, for any initial state  $\tilde{x}_0$  and any fixed input  $\tilde{u}(t)$  of  $\tilde{\mathbf{S}}$ , the choice  $x_0 = U\tilde{x}_0$  and  $u(t) = Q\tilde{u}(t)$ , implies  $x(t; x_0, u) = U\tilde{x}(t; \tilde{x}_0, \tilde{u})$  for all  $t \geq 0$ .

**Definition 4.** A the system  $\mathbf{S}$  is a *restriction* of  $\tilde{\mathbf{S}}$ , if there exists a pair  $(V, R)$  satisfying (2) such that, for any initial state  $x_0$  and any fixed input  $u(t)$  of  $\mathbf{S}$ , the choice  $\tilde{x}_0 = Vx_0$  and  $\tilde{u}(t) = Ru(t)$ , implies  $\tilde{x}(t; \tilde{x}_0, \tilde{u}) = Vx(t; x_0, u)$  for all  $t \geq 0$ .

**Definition 5. (Contractibility)** A control law  $\tilde{u}(t) = \tilde{K}\tilde{x}(t)$  for  $\tilde{\mathbf{S}}$  is contractible to the control law  $u(t) = Kx(t)$  for  $\mathbf{S}$ , if the choice  $\tilde{x}_0 = Vx_0$  and  $\tilde{u}(t) = Ru(t)$ , implies  $Kx(t; x_0, u) = Q\tilde{K}\tilde{x}(t; \tilde{x}_0, \tilde{u})$ , for all  $t \geq 0$ , any initial state  $x_0$ , and any fixed input  $u(t)$  of  $\mathbf{S}$ .

## 2.2 Complementary matrices

Suppose that the expansion transformations  $V$  and  $R$  are selected a priori. Then, the relationship between the systems  $\mathbf{S}$  and  $\tilde{\mathbf{S}}$  can be expressed as

$$\tilde{A} = VAU + M, \quad \tilde{B} = VBQ + N, \quad (3)$$

where  $M$  and  $N$  are complementary matrices to be determined. As it is well known, the complementary matrices  $M$  and  $N$  play an important role in the framework of the inclusion principle, providing freedom and flexibility to the obtention of expanded systems. Numerous papers have studied the structures of these matrices guaranteeing some properties in the expanded spaces, such as stability, controllability, observability or contractibility, Bakule et al. (2000), Bakule et al. (2001a), Bakule et al. (2001b).

By using the relationships given in (3), we can rewrite the inclusion principle in terms of complementary matrices. This approach offers the possibility to show the matrix structures and the conditions under which the inclusion principle is satisfied, Ikeda et al. (1981), Ikeda et al. (1984), Ikeda and Šiljak (1986), Šiljak (1991).

**Theorem 6.** A system  $\tilde{\mathbf{S}}$  is an *expansion* of the system  $\mathbf{S}$  if and only if  $UM^iV = 0$ ,  $UM^{i-1}NR = 0$ , for all  $i = 1, 2, \dots, \tilde{n}$ .  $\square$

**Proposition 7.** A system  $\mathbf{S}$  is an *aggregation* of the system  $\tilde{\mathbf{S}}$  if and only if  $UM = 0$  and  $UN = 0$ .  $\square$

**Proposition 8.** A system  $\mathbf{S}$  is a *restriction* of the system  $\tilde{\mathbf{S}}$  if and only if  $MV = 0$  and  $NR = 0$ .  $\square$

Analogously, Definition 5 can be rewritten in the following way.

**Proposition 9.** A control law  $\tilde{u}(t) = \tilde{K}\tilde{x}(t)$  for  $\tilde{\mathbf{S}}$  is contractible to the control law  $u(t) = Kx(t)$  for  $\mathbf{S}$ , if and only if  $K = Q\tilde{K}V$ ,  $Q\tilde{K}M^iV = 0$ ,  $Q\tilde{K}M^{i-1}NR = 0$ , for all  $i = 1, \dots, \tilde{n}$ .  $\square$

**Remark 10.** From Propositions 8 and 9, we can observe that the conditions  $Q\tilde{K}M^iV = 0$ ,  $Q\tilde{K}M^{i-1}NR = 0$ , for all  $i = 1, \dots, \tilde{n}$ , are automatically satisfied when the system is expanded by means of a restriction. Moreover, in this case the contracted controller is given by  $K = Q\tilde{K}V$ .

## 3. EXPANSION PROCESS

In order to simplify the discussion, we assume that the system  $\mathbf{S}$  given in (1) is composed by two overlapped subsystems  $\mathbf{S}_1$ ,  $\mathbf{S}_2$ . Consider the matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, \quad (4)$$

where  $A_{ii}$ ,  $B_{ij}$  for  $i = 1, 2, 3$ ,  $j = 1, 2, 3$  are  $n_i \times n_i$ ,  $n_i \times m_j$  dimensional matrices, respectively. The overlapped parts correspond to the subsystems  $A_{22}$  and  $B_{22}$ . A useful selection of the expansion transformations  $V$  and  $R$  is given by

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \quad R = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix}. \quad (5)$$

The corresponding pseudoinverse matrices  $U$  and  $Q$  are computed as

$$U = (V^T V)^{-1} V^T = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_{n_2} & \frac{1}{2}I_{n_2} & 0 \\ 0 & 0 & 0 & I_{n_3} \end{bmatrix}, \quad (6)$$

$$Q = (R^T R)^{-1} R^T = \begin{bmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2}I_{m_2} & \frac{1}{2}I_{m_2} & 0 \\ 0 & 0 & 0 & I_{m_3} \end{bmatrix}.$$

Then, the expanded matrix  $\tilde{A} = VAU + M$  adopts the form

$$\tilde{A} = \begin{bmatrix} A_{11} & \frac{1}{2}A_{12} + M_{12} & \frac{1}{2}A_{12} - M_{12} & A_{13} \\ A_{21} + M_{21} & \frac{1}{2}A_{22} + M_{22} & \frac{1}{2}A_{22} + M_{23} & A_{23} + M_{24} \\ A_{21} - M_{21} & \frac{1}{2}A_{22} - (M_{22} + M_{23} + M_{33}) & \frac{1}{2}A_{22} + M_{33} & A_{23} - M_{24} \\ A_{31} & \frac{1}{2}A_{32} + M_{42} & \frac{1}{2}A_{32} - M_{42} & A_{33} \end{bmatrix}.$$

A similar structure for the matrix  $\tilde{B} = VBQ + N$  is obtained.

**Theorem 11.** Consider a system  $\mathbf{S}$  given in (1) with the structure (4) and the transformations (5)-(6). Then,  $\tilde{\mathbf{S}} \supset \mathbf{S}$  if and only if the complementary matrices  $M$  and  $N$  have the following structures

$$M = \begin{bmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22} + M_{23} + M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{bmatrix}, \quad (7)$$

$$N = \begin{bmatrix} 0 & N_{12} & -N_{12} & 0 \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{21} & -(N_{22} + N_{23} + N_{33}) & N_{33} & -N_{24} \\ 0 & N_{42} & -N_{42} & 0 \end{bmatrix}$$

and satisfy

$$\begin{aligned} \begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} [M_{22}+M_{33}]^i \begin{bmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{bmatrix} &= 0, \\ \begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} [M_{22}+M_{33}]^i \begin{bmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{bmatrix} &= 0, \end{aligned} \quad (8)$$

for all  $i=0, 1, \dots, \tilde{n}-1$ .  $\square$

**Remark 12.** We note how the requirements (8) are, in practice, very difficult to be satisfied. To avoid these hard matrix conditions, two particular cases are considered: (i) the first column matrices in (8) are zero (aggregations) or, (ii) the last row matrices in (8) are zero (restrictions). The structure of the complementary matrices for both cases is given by the following propositions.

**Proposition 13.** Consider a system  $\mathbf{S}$  as in (1) with the structure (4) and the transformations (5)-(6). Then,  $\mathbf{S}$  is an *aggregation* of  $\tilde{\mathbf{S}}$  if and only if the matrices  $M$  and  $N$  have the following structures:

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -M_{22} & -M_{23} & -M_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{21} & -N_{22} & -N_{23} & -N_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (9)$$

**Proposition 14.** Consider a system  $\mathbf{S}$  as in (1) with the structure (4) and the transformations (5)-(6). Then,  $\mathbf{S}$  is a *restriction* of  $\tilde{\mathbf{S}}$  if and only if the matrices  $M$  and  $N$  have the following structures:

$$M = \begin{bmatrix} 0 & M_{12} & -M_{12} & 0 \\ 0 & M_{22} & -M_{22} & 0 \\ 0 & M_{32} & -M_{32} & 0 \\ 0 & M_{42} & -M_{42} & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & N_{12} & -N_{12} & 0 \\ 0 & N_{22} & -N_{22} & 0 \\ 0 & N_{32} & -N_{32} & 0 \\ 0 & N_{42} & -N_{42} & 0 \end{bmatrix}. \quad (10)$$

Since in the expansion process the basic idea is to achieve decoupled or weakly coupled expanded systems, a proper choice of the matrices  $M$  and  $N$  is required (see Fig. 1). In the expanded

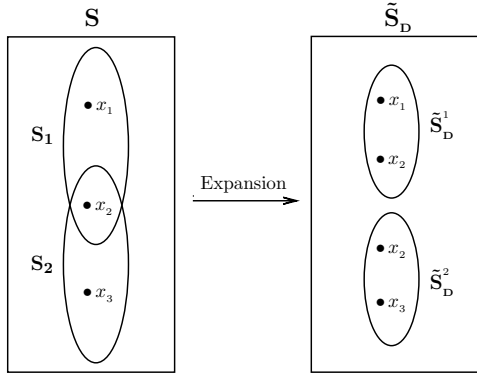


Fig. 1. Expansion-decoupling process of the system  $\mathbf{S}$ .

space  $\tilde{\mathbf{S}}$ , we denote

$$\tilde{\mathbf{S}}_1 : \dot{\tilde{x}}_1(t) = \tilde{A}_{11}\tilde{x}_1(t) + \tilde{B}_{11}\tilde{u}_1(t) + \tilde{A}_{12}\tilde{x}_2(t) + \tilde{B}_{12}\tilde{u}_2(t),$$

$$\tilde{\mathbf{S}}_2 : \dot{\tilde{x}}_2(t) = \tilde{A}_{22}\tilde{x}_2(t) + \tilde{B}_{22}\tilde{u}_2(t) + \tilde{A}_{21}\tilde{x}_1(t) + \tilde{B}_{21}\tilde{u}_1(t),$$

where  $\tilde{A}_{ij}$ ,  $\tilde{B}_{ij}$ ,  $i, j=1, 2$ ,  $i \neq j$  are the interconnection matrices. The decoupled subsystems can be expressed as

$$\begin{aligned} \tilde{\mathbf{S}}_D^1 : \dot{\tilde{x}}_1(t) &= \tilde{A}_{11}\tilde{x}_1(t) + \tilde{B}_{11}\tilde{u}_1(t), \\ \tilde{\mathbf{S}}_D^2 : \dot{\tilde{x}}_2(t) &= \tilde{A}_{22}\tilde{x}_2(t) + \tilde{B}_{22}\tilde{u}_2(t) \end{aligned} \quad (11)$$

or denoted by

$$\tilde{\mathbf{S}}_D : \dot{\tilde{x}}(t) = \tilde{A}_D\tilde{x}(t) + \tilde{B}_D\tilde{u}(t), \quad (12)$$

in a more compact notation, where  $\tilde{A}_D = \text{diag}\{\tilde{A}_{11}, \tilde{A}_{22}\}$ ,  $\tilde{B}_D = \text{diag}\{\tilde{B}_{11}, \tilde{B}_{22}\}$ .

The local control laws corresponding to the decoupled expanded subsystems  $\tilde{\mathbf{S}}_D^1$  and  $\tilde{\mathbf{S}}_D^2$ , are given by

$$\tilde{u}_1(t) = \tilde{K}_{11}\tilde{x}_1(t), \quad \tilde{u}_2(t) = \tilde{K}_{22}\tilde{x}_2(t), \quad (13)$$

where  $\tilde{x}_1(t) = [x_1^T(t), x_2^T(t)]^T$ ,  $\tilde{x}_2(t) = [x_2^T(t), x_3^T(t)]^T$ ,  $\tilde{u}_1(t) = [u_1^T(t), u_2^T(t)]^T$ ,  $\tilde{u}_2(t) = [u_2^T(t), u_3^T(t)]^T$  (see Fig. 2).

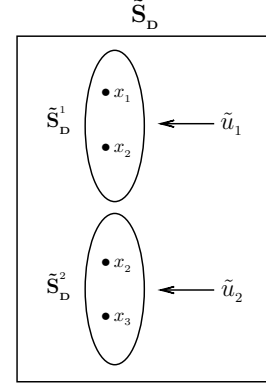


Fig. 2. Local controllers designed in  $\tilde{\mathbf{S}}_D$ .

#### 4. DESIGN OF LOCAL CONTROLLERS

In order to perform a decentralized design of the controller for the expanded system, the expanded gain matrix  $\tilde{K}_D$  is taken as a block diagonal matrix with the following structure

$$\tilde{K}_D = \begin{bmatrix} \tilde{K}_{11} & 0 \\ 0 & \tilde{K}_{22} \end{bmatrix}, \quad (14)$$

where each diagonal block matrix  $\tilde{K}_{11}$ ,  $\tilde{K}_{22}$  is independently designed by means of a certain control criterion. The goal is to implement an overlapping controller in the system  $\mathbf{S}$ , denoted by  $u(t) = Kx(t)$ , but as a contraction of a control law  $\tilde{u}_D(t) = \tilde{K}_D\tilde{x}(t)$  designed in  $\tilde{\mathbf{S}}_D$ . Then, taking into account the previous structures, the gain matrices in the expanded and initial systems have the following form:

$$\tilde{K}_D = \begin{bmatrix} \tilde{K}_{11} & 0 \\ 0 & \tilde{K}_{22} \end{bmatrix} \xrightarrow{\text{contraction}} K = Q\tilde{K}_DV = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}.$$

According to Remark 10, we can restrict our attention to *aggregations*. When an aggregation is used to expand an original system, some structural conditions on the expanded gain matrices have to be satisfied to ensure their contractibility. The purpose is to design local gain matrices  $\tilde{K}_{11}$  and  $\tilde{K}_{22}$ , as given in (14), so that the diagonal gain matrix  $\tilde{K}_D$  can be contracted to the initial system. The structure and the conditions on the local gain matrices are given by the following theorem.

**Theorem 15.** Suppose that  $\mathbf{S}$  is an aggregation of  $\tilde{\mathbf{S}}$ . Then, a block diagonal gain matrix  $\tilde{K}_D$ , designed for the decoupled expanded system, having the following structure

$$\tilde{K}_D = \begin{bmatrix} K_{11} & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22} & K_{34} \\ 0 & 0 & 0 & K_{44} \end{bmatrix}, \quad (15)$$

is contractible to the system  $\mathbf{S}$ . Moreover, the contracted gain matrix will be in the form

$$K = \begin{bmatrix} K_{11} & 0 & 0 \\ \frac{1}{2}K_{21} & K_{22} & \frac{1}{2}K_{34} \\ 0 & 0 & K_{44} \end{bmatrix}. \quad (16)$$

**Proof.** Consider  $\mathbf{S}$  an aggregation of  $\tilde{\mathbf{S}}$ , with the structures given in (4), (5), (6) and (9). Suppose that a block diagonal gain matrix

$$\tilde{K}_D = \begin{bmatrix} K_{11} & K_{12} & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix}, \quad (17)$$

has to be designed in the decoupled expanded space  $\tilde{\mathbf{S}}_D$ . From Proposition 9, and in order to obtain a contractible controller, the requirements  $Q\tilde{K}_D M^i V = 0$ ,  $Q\tilde{K}_D M^{i-1} N R = 0$ , for all  $i=1, \dots, \tilde{n}$ , have to be satisfied. By imposing the condition  $Q\tilde{K}_D M^i V = 0$ , for  $i=1$ , we obtain

$$\begin{aligned} K_{12} M_{21} &= 0, & K_{12} [M_{22} + M_{23}] &= 0, \\ K_{12} M_{24} &= 0, & [K_{22} - K_{33}] M_{21} &= 0, \\ [K_{22} - K_{33}] [M_{22} + M_{23}] &= 0, & [K_{22} - K_{33}] M_{24} &= 0, \\ K_{43} M_{21} &= 0, & K_{43} [M_{22} + M_{23}] &= 0, \\ K_{43} M_{24} &= 0. \end{aligned} \quad (18)$$

If the matrices  $M_{2j}$ ,  $j=1, \dots, 4$ , are previously selected, the conditions (18) are reduced, in practice, to the following sufficient conditions:

$$K_{12} = 0, \quad K_{22} - K_{33} = 0, \quad K_{43} = 0, \quad (19)$$

which lead to the gain matrix

$$\tilde{K}_D = \begin{bmatrix} K_{11} & 0 & 0 & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & K_{22} & K_{34} \\ 0 & 0 & 0 & K_{44} \end{bmatrix}. \quad (20)$$

It is easy to prove that the gain matrix (20) satisfies the remaining conditions  $Q\tilde{K}_D M^i V = 0$  for all  $i=2, \dots, \tilde{n}$ . Moreover, the conditions  $Q\tilde{K}_D M^{i-1} N R = 0$  are automatically satisfied too, for all  $i=1, \dots, \tilde{n}$ . Finally, from Proposition 9, the contracted gain matrix  $K = Q\tilde{K}_D V$  adopts the structure (16).  $\square$

*Remark 16.* At the end of this process, we can observe that the overlapping controller  $u(t) = [u_1^T(t), u_2^T(t), u_3^T(t)]^T$  does not need the information on the overall states of the system  $\mathbf{S}$ . Thus,  $u_1(t)$  and  $u_3(t)$  only use the information contained in  $x_1(t)$  and  $x_3(t)$ , respectively. Fig. 3 shows how the contracted overlapping controller acts on the original system  $\mathbf{S}$ .

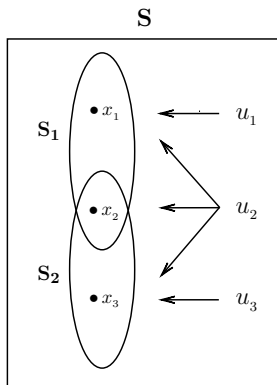


Fig. 3. Overlapping controller for the initial system  $\mathbf{S}$ .

## 5. CONTROL CRITERIA AND CONTRACTIBILITY

The first objective has been to design a gain matrix in the expanded space, possessing a contractible structure as given in (15). However, the design of a control law also depends on a control criterion. In this paper, we consider three different control criteria: (1) optimal control, (2) guaranteed cost control,

and (3)  $H_\infty$  control. For each one of them, we study the possibility of designing decentralized controllers in the expanded space which fit in with the structure provided by Theorem 15.

### 5.1 Optimal control

Consider an optimal control problem in the initial space, described by

$$J(x_0, u) = \int_0^\infty [x^T(t) Q^* x(t) + u^T(t) R^* u(t)] dt, \quad (21)$$

$$\text{s.t. } \mathbf{S}: \dot{x}(t) = Ax(t) + Bu(t).$$

Following the process described before, we are interested in designing an optimal control law for the initial system. In the expanded space, two local cost functions corresponding to the disjoint subsystems are considered

$$\tilde{J}_1(\tilde{x}_{10}, \tilde{u}_1(t)) = \int_0^\infty [\tilde{x}_1^T(t) \tilde{Q}_{11}^* \tilde{x}_1(t) + \tilde{u}_1^T(t) \tilde{R}_{11}^* \tilde{u}_1(t)] dt, \quad (22)$$

$$\tilde{J}_2(\tilde{x}_{20}, \tilde{u}_2(t)) = \int_0^\infty [\tilde{x}_2^T(t) \tilde{Q}_{22}^* \tilde{x}_2(t) + \tilde{u}_2^T(t) \tilde{R}_{22}^* \tilde{u}_2(t)] dt,$$

where  $\tilde{x}_{10}$ ,  $\tilde{x}_{20}$  are the initial states and  $\tilde{Q}_{11}^*$ ,  $\tilde{R}_{11}^*$ ,  $\tilde{Q}_{22}^*$ ,  $\tilde{R}_{22}^*$  are appropriate weighting matrices. The local control gain matrices minimizing the cost functions (22) are independently computed as

$$\tilde{K}_{11} = [\tilde{R}_{11}^*]^{-1} \tilde{B}_{11}^T \tilde{P}_{11}, \quad \tilde{K}_{22} = [\tilde{R}_{22}^*]^{-1} \tilde{B}_{22}^T \tilde{P}_{22}, \quad (23)$$

where  $\tilde{P}_{11}$ ,  $\tilde{P}_{22}$  are the solutions of the corresponding Riccati equations. However, from (15), the structures of the local gain matrices have to be in the form

$$\tilde{K}_{11} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix}, \quad \tilde{K}_{22} = \begin{bmatrix} K_{22} & K_{34} \\ 0 & K_{44} \end{bmatrix}. \quad (24)$$

Since the matrices  $\tilde{K}_{11}$  and  $\tilde{K}_{22}$  have zero blocks in their structures, they cannot be obtained from (23) due to the uniqueness of  $\tilde{P}_{11}$  and  $\tilde{P}_{22}$ . Then, by means of an optimal control criterion, it is practically impossible to obtain a contractible gain matrix  $\tilde{K}_D$ .

### 5.2 Guaranteed cost control

Consider a guaranteed cost problem described by

$$J(x_0, u) = \int_0^\infty [x^T(t) Q^* x(t) + u^T(t) R^* u(t)] dt, \quad (25)$$

$$\text{s.t. } \mathbf{S}: \dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t),$$

where  $A$ ,  $B$  are real constant matrices with appropriate dimensions and  $\Delta A(t)$  and  $\Delta B(t)$  are real-valued matrices of uncertain parameters. Norm-bounded time-varying uncertainties are considered in the form

$$\Delta A(t) = H_A F(t) E_A, \quad \Delta B(t) = H_B F(t) E_B, \quad (26)$$

where  $H_A$ ,  $E_A$ ,  $H_B$ ,  $E_B$  are known real constant matrices of appropriate dimensions and  $F(t) \in \mathbb{R}^{i \times j}$  is an unknown real time-varying matrix satisfying  $F^T(t) F(t) \leq I$ . For these types of systems, and in order to obtain a gain matrix so that  $J \leq J^*$  (guaranteed cost), an LMI approach may be appropriate. The following theorem supplies a desired gain matrix, Yu and Chu (1999), Mukaidani (2003).

**Theorem 17.** Consider a linear continuous-time uncertain system with an associated cost function as given in (25) and satisfying (26). If there exist matrices  $M > 0$ ,  $Y$ , and constants  $\alpha_1 > 0$ ,  $\alpha_2 > 0$  such that the following LMI

$$\begin{bmatrix} W_1 & M^T & M E_A^T & Y^T E_B^T & Y^T \\ M & -[Q^*]^{-1} & 0 & 0 & 0 \\ E_A M & 0 & -\alpha_1 I & 0 & 0 \\ E_B Y & 0 & 0 & -\alpha_2 I & 0 \\ Y & 0 & 0 & 0 & -[R^*]^{-1} \end{bmatrix} < 0 \quad (27)$$

is feasible, where

$$\begin{aligned} W_1 &= AM + MA^T + BY + Y^T B^T + \alpha_1 H_A H_A^T + \alpha_2 H_B H_B^T, \\ W_2 &= [M \quad ME_A^T \quad Y^T E_B^T \quad Y^T], \end{aligned}$$

then the control law  $u(t) = YM^{-1}x(t)$  is a quadratic guaranteed cost controller for the uncertain system (25). The gain matrix has the form  $K = YM^{-1}$ . Moreover, the bounded cost is given by  $J \leq \text{tr}(M^{-1})$ , where  $\text{tr}(\cdot)$  denotes the trace of the corresponding matrix.  $\square$

In this case, and using the same process as in the optimal control design, we are interested in obtaining two local gain matrices as given in (24) by means of Theorem 17, which is now applied to the disjoint subsystems. For the first local gain matrix  $\tilde{K}_{11}$ , we can impose on the matrix variables  $\tilde{Y}_{11}$  and  $\tilde{M}_{11}$  the following structural matrix conditions:

$$\tilde{K}_{11} = \begin{bmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{bmatrix} = \tilde{Y}_{11} \tilde{M}_{11}^{-1} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}. \quad (28)$$

Once the submatrix block  $K_{22}$  has been obtained, we compute the other local gain matrix  $\tilde{K}_{22}$ , taking into account that  $K_{22}$  is not a free submatrix now. Then, we can write

$$\tilde{K}_{22} = \begin{bmatrix} K_{22} & K_{34} \\ 0 & K_{44} \end{bmatrix} = \tilde{Y}_{22} \tilde{M}_{22}^{-1} = \begin{bmatrix} K_{22} & * \\ 0 & * \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & * \end{bmatrix}. \quad (29)$$

Then, if the two local LMI's are feasible, the contractibility of the gain matrix  $\tilde{K}_D$  is ensured.

### 5.3 $H_\infty$ control

Consider a class of linear continuous-time uncertain systems described by the equations

$$\begin{aligned} \mathbf{S}: \dot{x}(t) &= [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) + B_1 w(t), \\ z(t) &= Cx(t) + Du(t), \end{aligned} \quad (30)$$

where  $x(t) \in \mathbb{R}^n$  corresponds to the state,  $u(t) \in \mathbb{R}^m$  is the input control,  $w(t) \in L_2^p[0, \infty)$  the disturbance input, and  $z(t) \in \mathbb{R}^q$  is the controlled output.  $A, B, B_1, C, D$  are known, real and constant matrices of appropriate dimensions. Norm-bounded time-varying uncertainties satisfy (26).

The  $H_\infty$  control objective is to design controllers such that the closed-loop system is stable guaranteeing the disturbance attenuation of the closed-loop system from  $w(t)$  to  $z(t)$ , i.e.

$$\|z(t)\|_2 \leq \gamma \|w(t)\|_2, \quad \gamma > 0, \quad (31)$$

for all non-zero  $w(t)$ , under zero initial conditions. In this paper, an LMI approach is used. The next theorem provides a state feedback controller solving the  $H_\infty$  control problem.

**Theorem 18.** Consider a linear continuous-time uncertain system as given in (30) with norm-bounded uncertainties (26) and a scalar  $\gamma > 0$ . For given scalars  $\beta_1 > 0$ ,  $0 < \beta_2 < 1$ , suppose that there exist matrices  $X > 0$  and  $W$  such that the following linear matrix inequality

$$\begin{bmatrix} W_1 & XE_A^T & X & W^T E_B^T & W_2^T \\ E_A^T X & -I & 0 & 0 & 0 \\ X & 0 & -I & 0 & 0 \\ E_B^T W & 0 & 0 & -\beta_1 I & 0 \\ W_2 & 0 & 0 & 0 & -\beta_2 I \end{bmatrix} < 0 \quad (32)$$

holds, where

$$\begin{aligned} W_1 &= AX + XA^T + BW + [BW]^T + H_A H_A^T + (1 + \beta_1) H_B H_B^T \\ &\quad + \gamma^{-2} B_1 B_1^T, \\ W_2 &= CX + DW. \end{aligned}$$

Then, there exists a state feedback controller in the form  $u(t) = Kx(t)$  so that the resulting closed-loop system is asymptotically stable with  $H_\infty$  norm-bound  $\gamma$ . Moreover, the control gain matrix  $K$  is given by  $K = WX^{-1}$ .  $\square$

It should be noted that the expression of the gain matrix  $K$  is analogous to the gain matrix obtained by means of a quadratic guaranteed cost control. Then, the same conclusions can be extracted when an  $H_\infty$  control is used. In consequence, for this type of control criterion a contractible gain matrix can also be obtained.

## 6. CONCLUSIONS

In this paper, a contribution to the contractibility of decentralized control laws for systems composed by overlapped subsystems, has been offered. For these kinds of systems, the Inclusion Principle provides an excellent mathematical framework to design decentralized controllers, which can be finally contracted to be implemented in the original space. However, to guarantee the contractibility of the gain matrices, obtained in the expanded systems, some requirements must be satisfied. In this line, an explicit structure for the local gain matrices has been presented. This structure has proven to be compatible with the usage of guaranteed cost and  $H_\infty$  control criteria. In both cases, a convenient LMI formulation allows to impose a proper structure on the expanded local gain matrices. On the contrary, the approach is not suitable to be used together with a quadratic optimal control strategy.

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