

## Book Review

### **Category Theory**, Second Edition

*Steve Awodey*

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Categories and functors appeared in print for the first time in 1945 in the paper *General Theory of Natural Equivalences* [4] by S. Eilenberg and S. MacLane as a tool to tackle natural isomorphisms. At the beginning categories were a convenient language in which to express certain properties encountered in Homological Algebra and Algebraic Topology and allowing one to unify various branches of mathematics. After the influential works of A. Grothendieck, [5], D. Kan, [6], and L.W. Lawvere, [7], to cite three breaking points in its development, Category Theory grew rapidly and found applications far beyond the subjects that originated it, from Algebraic Geometry and Algebraic Topology to Set Theory, and afterwards to Computer Science, Logic and Physics. Nowadays there is an intensive work in many different directions, let's cite for example one of my favorites, the relations to Homotopy Theory through Higher Topos Theory, [8], or the recent connections between Type Theory and Homotopy [2].

There are several good books on Category Theory, most of them focusing on some specialized aspect, of which undoubtedly the most influential has been MacLane's *Categories for the working mathematician*, first published in 1971, [9]. It is a superb book on Category Theory by one of his fathers, used by almost any mathematician wanting to know the basis of the theory in the last forty years. Since its appearance any new book on Category Theory is inevitably compared with it. Now, an expert on the subject, Steve Awodey, offers us a new book (in fact its 2on edition, after the 2006 edition), how does it compare with MacLane's? I think the best we can do is to cite Awodey's preface: "What is needed now, after 30 years of spreading into various other disciplines and places in the curriculum, is a book for every one else". Jokingly one could suggest a new title for Awodey's book, *Categories for every one*.

Category theory is in some sense elementary. There are few prerequisites to its study, but for its full comprehension one needs a good collection of examples to test the abstraction of its concepts and results. It is Awodey's purpose to circumvent this situation writing a book for students having only an elementary background on Calculus, Linear Algebra, Combinatorics and Logic. This objective is achieved by a very careful presentation of the main ideas and results and the systematic analysis of some easy examples, like posets and monoids, complemented with some views to applications, specially to propositional calculus and  $\lambda$ -calculus.

The book is divided in ten chapters of approximately the same length, about 25 pages each, except the chapter on Adjoint Functors (Chapter 8), which is longer. Each chapter ends with a collection of exercises, and a final section of the book provides hints for the solution of some of them, making it very useful as a textbook or for self study. A more detailed inspection of the contents yields:

1. Categories.
2. Abstract structures

3. Duality
4. Groups and categories.
5. Limits and colimits.
6. Exponentials
7. Naturality.
8. Categories of diagrams.
9. Adjoints.
10. Monads and algebras.

For those who already know MacLane's book, we could say in short that Awodey's Category Theory covers approximately the first six chapters of the former, although making easy the reading for the non mathematically trained.

Roughly speaking, the idea underlying the first 6 chapters is that of universal mapping property (UMP) and its usefulness to present some category-theoretical definitions, like products, equalizers, pullbacks, and their duals. This presentation culminates on chapters 5 and 6. The main point of chapter 5 is the introduction of limits and colimits and the identification of the above mentioned concepts as a particular limit, or colimit. Aside from this, in this chapter we find that a representable functor preserves all limits and the notion and some examples of functors creating limits. Chapter 6, is dedicated to exponentials as a construction defined by a UMP which is not a limit, and to cartesian closed categories. The relevance of cartesian closed categories is pointed out in some of the applications appearing in this chapter, with sections like Heyting algebras, propositional calculus,  $\lambda$ -calculus or variable sets. These sections are not completely self contained, but they give a hint for future reading in this field.

It is not till chapter 7 that we encounter the notion of natural transformation between two functors and of equivalence of categories, which is a more natural condition than that of isomorphism of categories. Having introduced naturality, chapter 8 is dedicated to the study of categories of diagrams. There we find one of the most useful results of Category Theory, the Yoneda lemma, and the analysis of the constructions of the previous chapters (like limits and exponentials) in categories of diagrams. In particular, we find a proof that for any small category  $\mathbf{C}$ , every object  $P$  in the functor category  $\mathbf{Sets}^{\mathbf{C}^{op}}$  is a colimit of representable functors. The chapter ends with the notion of topos and the proof that for any small category  $\mathbf{C}$ , the category of diagrams  $\mathbf{Sets}^{\mathbf{C}^{op}}$  is a topos. At this point one may wonder why natural transformations and the Yoneda lemma, so basic in Category Theory, appear so late in the book, certainly it is a question of taste and I have to admit that the chosen presentation proceeds smoothly along the text.

As we read in the introduction of chapter 9 on adjoint functors, "this chapter represents the high point of this book, the goal towards which we have been working steadily". The chapter begins with the definition of adjoint functor and the identification of many category constructions as adjoints and more specifically, identifying quantifiers as adjoints,

as recognized by Lawvere in the 60s. The two main results of the chapter are the RALP property and the Freyd adjoint theorem. The RALP property establishes that right adjoint preserve limits, a very useful property of adjoints which is applied to prove the UMP of the Yoneda embedding from a category into a cocomplete category and to base change in locally cartesian categories. The Freyd adjoint theorem answers the natural question of when does a functor have an adjoint and is applied to establish the equivalence of three properties for a functor  $U : \mathbf{C} \rightarrow \mathbf{Sets}$ , where  $\mathbf{C}$  is a small complete category: preserve limits, have a left adjoint or being representable.

The book ends with a chapter on monads and algebras. The main objective of the chapter is the Eilenberg-Moore theorem that establishes that every monad arises from an adjunction. To this end, given a monad  $T$  on a category  $\mathbf{C}$  it is introduced the Eilenberg-Moore category of  $T$ -algebras,  $\mathbf{C}^T$ , and the adjunction given by the free algebra and forgetful functors  $F : \mathbf{C} \rightleftarrows \mathbf{C}^T : U$ . The final section presents a weakened variation of  $T$ -algebras, the algebras for an endomorphism  $P : \mathbf{C} \rightarrow \mathbf{C}$  of a category  $\mathbf{C}$ , which include some very basic algebraic structures, as for example the group structures as the  $P$ -algebras for  $P(X) = 1 + X + X \times X : \mathbf{Sets} \rightarrow \mathbf{Sets}$ , thinking on them as operations without imposing the group equations, as associativity or inverses. The section concludes with a criterion for  $P$ -algebras to be equivalent to  $T$ -algebras coming from a monad  $T$ .

The writing of Awodey is very careful and thorough, providing complete details for proofs or asking the reader to do so as an exercise when the proof follows an already presented scheme or a dual result. Many concepts are first discussed for sets, introducing them in an element like presentation and looking for an element free characterization, and afterwards getting the categorical notion. Moreover, any time there is a new concept, there is a set of examples to get used to it, some of them with the warning that they presuppose some knowledge of General Topology or Type Theory. The introduction of generalized elements of an object in a category in chapter 2, permits one to recover many category concepts in a classic element style.

As an example, of the detailed discussion that the author proposes when introducing new concepts, let me mention the chapter on adjoint functors. The definition of adjoint functor occupies several pages: beginning with the example of the UMP of free monoids, the author defines a (preliminary version of an) adjunction between two functors  $F : \mathbf{C} \rightleftarrows \mathbf{D} : U$  as a natural transformation  $\eta : 1_{\mathbf{C}} \rightarrow UF$  with the corresponding UMP. From this, he derives the usual bijection  $Hom_{\mathbf{D}}(FC, D) \cong Hom_{\mathbf{C}}(C, UD)$  for any  $C \in \mathbf{C}$  and  $D \in \mathbf{D}$ , and finally he takes care of the naturality of this isomorphism in both  $C$  and  $D$ . This results in a leisurely pace of the presentation, stressing the main points of the categorical definition at the benefit of those encountering categories for the first time.

The references in the book are reduced to eight items. Due to its textbook character, it seems reasonable to give a small list of references, some of them by its historical importance, like Eilenberg-MacLane [4] or Lawvere [7], and the other by the author preferences, mainly focused on Topos Theory and Logic. I think it would have been desirable also to cite some other standard manuals, like for example Borceaux's [3].

This second edition differs from the first one in the exercises and a section on monoidal categories. I think it is a good idea to include these exercises and also the appendix sketching the solutions of some of them. Remark that exercises 4 and 5 of chapter 5,

pp. 115-116, are equal. It is almost impossible to publish a book without typographical errors, and this book is but one more example. Nevertheless we have to be thankful to the author for providing us with a list of corrections in his web page, [1], only some months after its publication.

As a resume, I can say that the book is well organized and very well written. The presentation of the material is from the concrete to the abstract, proofs are worked out in detail, and the examples and the exercises spread through the text mark a pleasant rhythm for its reading. In all, Awodey's Category Theory is a very nice and recommendable introduction to the subject.

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