Order types of random point sets can be realized with small integer coordinates

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Abstract

Let $S := \{p_1, \ldots, p_n\}$ be a set of $n$ points chosen independently and uniformly at random from the unit square and let $M$ be a positive integer. For every point $p_i = (x_i, y_i)$ in $S$, let $p'_i = ([Mx_i], [My_i])$. Let $S' := \{p'_i : 1 \leq i \leq n\}$. We call $S'$ the digitization of $S$ by $M$. In this paper we study the problem:

How large does $M$ have to be such that with high probability, $S$ and $S'$ have the same order type?

1 Introduction

The order type of a point set $S = \{p_1, \ldots, p_n\}$ in the plane is a mapping that assigns to each ordered triplet $(p_i, p_j, p_k)$ of points of $S$ its orientation. Many combinatorial properties of point sets only depend on the order type. For instance, point sets with the same order type have the same number of triangulations and the same rectilinear crossing number. A drawing, or realization, of $S$ is a set of $n$ points in the plane with integer coordinates and with the same order type as $S$. For numerical computations, it is very desirable to have drawings of point sets, since integer arithmetic is faster than floating point arithmetic, and is also less susceptible to rounding errors. We only refer to the works of Greene and Yao [7] and Milenkovic [9] for digitization and the use of finite precision arithmetic in computational geometry.

Aichholzer et al. [1] developed an order type data base with drawings for all sets of up to $n = 11$ points. However, for sets with a large number $n$ of points, it is infeasible to obtain such a data base, since the number of order types is large, at least $n^{4n+O(n/\log(n))}$, see [5].

Goodman et al. [6] also proved that there exist order types which require doubly exponential grid size in any drawing: Let $f(n)$ be the smallest integer $M$ such that every set of $n$ points in general position in the plane can be realized on the grid $\{(i,j) : -M \leq i, j \leq M\}$. Then, there exist constants $c_1$ and $c_2$ such that $2^{c_1 n^2} \leq f(n) \leq 2^{c_2 n^2}$. Hence, the number of bits needed to store an order type is exponential in $n$. Here, we show that order types of random point sets can be realized on an integer grid of small size.

Theorem 1 Let $\epsilon > 0$ and let $M := \lfloor n^{3+\epsilon} \rfloor$. Let $S$ be a set of $n$ points chosen independently and uniformly at random from a square with opposite corners $(0,0)$ and $(M,M)$. Let the integer grid $G := \{(i,j) : 0 \leq i, j \leq M\}$. Let $S'$ be the subset of $n$ points of $G$ which are closest to $S$. Then, the probability that $S'$ and $S$ have the same order type tends to 1 as $n$ tends to infinity.

We remark that $M$ is chosen sufficiently large in this theorem to guarantee that asymptotically almost surely no two points of $S$ are digitized to the same point of $S'$. This will be proved in Lemma 2. It also follows from the proof of Theorem 1 that asymptotically almost surely $S'$ is a set of points in general position; that is, no three points of $S'$ are on the same straight line.

Small drawings for several special point sets are known. Jarník [8] proved that a set of $n$ points in convex position has a drawing on an integer grid of size $O(n^{3/2})$. Bereg et al. [3] showed that the double circle of $n$ points can also be drawn on a grid of size $O(n^{3/2})$; Barba et al. [2] proved that the Horton set of $n$ points can be drawn on a grid of size $O(n^{(\log(n)-1)/2})$; Duque et al. [4] proved that the nested almost convex set of $n$ points can be drawn on a grid of size $O(n^{\log(3)})$.

2 The upper bound on the grid size

We prove Theorem 1 for points selected from the unit square; the result then follows by scaling. Precisely:
Let $S := \{p_1, \ldots, p_n\}$ be a set of $n$ points chosen independently and uniformly at random from the unit square. For $\epsilon > 0$ define $M := \lceil n^{3+\epsilon} \rceil$. Let

$$G' := \left\{ \left( \frac{i}{M}, \frac{j}{M} \right) \mid i, j = 0, 1, \ldots, M \right\},$$

and let $p'_i$ be the point of $G'$ closest to $p_i$. Let $S' := \{p'_1, \ldots, p'_n\}$. First we show in Lemma 2 that the minimum distance among points of $S$ is large enough, so that asymptotically almost surely no two points of $S$ have the same closest point $p'$ of $G'$. For $M_1$ an integer, assume that the unit square has been subdivided into $M_1 \times M_1$ squares; we call these squares big squares. We say that two such squares are adjacent if their boundaries intersect; note that a big square is adjacent to itself and that it has at most 8 other big squares adjacent to it. For a given point $p_i$ we say that the big squares adjacent to the square that contains $p_i$ are forbidden squares by $p_i$.

**Lemma 2** Let $M_1 \geq n$. The probability that no pair of points of $S$ lie in adjacent big squares is greater than

$$1 - \frac{9n^2}{2M_1^2}.$$

**Proof.** Let $A_k$ be the event that point $p_k$ does not lie in any of the big squares forbidden by every $p_i$ for all $i < k$. Let

$$B_k := \bigcap_{i=1}^{k} A_i.$$

Note that if $p_k$ is not in the big squares forbidden by every $p_i$, for all $i < k$, then no such $p_i$ is in the big squares forbidden by $p_k$. We are thus interested in the probability of $B_n$. Note that

$$\Pr\{A_k\} \geq 1 - \frac{9(k-1)}{M_1^2}.$$

From the inequality

$$\Pr\{B_n\} = \Pr\left\{ \bigcap_{i=1}^{n} A_i \right\} \geq \left( \sum_{i=1}^{n} \Pr\{A_i\} \right) - (n - 1),$$

we get

$$\Pr\left\{ \bigcap_{i=1}^{n} A_i \right\} \geq \left( \sum_{i=1}^{n} 1 - \frac{9(i-1)}{M_1^2} \right) - (n - 1)$$

$$\geq 1 - \frac{9n^2}{2M_1^2}.$$

\[\square\]

We now prove that $S$ and $S'$ asymptotically almost surely have the same order type. Denote with

$\Delta_{p_i,p_j,p_k}$ the triangle with vertices $p_i, p_j, p_k$; and denote with $\Delta'_{p_i',p_j',p_k'}$ the triangle with vertices $p_i', p_j', p_k'$. In particular, we will show that asymptotically almost surely, $\Delta_{p_i,p_j,p_k}$ and $\Delta'_{p_i',p_j',p_k'}$ have the same orientation, for all $1 \leq i, j, k \leq n$.

For every pair of points $(p_i, p_j)$ of $S$ we define a forbidden region such that if a third point $p_k$ avoids this region, then $\Delta_{p_i,p_j,p_k}$ and $\Delta'_{p_i',p_j',p_k'}$ have the same orientation.

Let $l_{i,j}$ be the line passing through $p_i$ and $p_j$, and let $l'_{i,j}$ be the line passing through $p_i'$ and $p_j'$. Let $w_{i,j}$ be the double wedge bounded by $l_{i,j}$ and $l'_{i,j}$, and that contains the line segment with endpoints $p_i$ and $p_j'$, see Figure 1.

The forbidden region of $p_i$ and $p_j$ is the union of all squares of area $1/M^2$, with vertices on $G'$, that intersect $w_{i,j}$. We denote it by $F_{i,j}$.

**Lemma 3** If $p_i$ and $p_j$ are at distance $d$ then the area of $F_{i,j}$ is at most $\frac{c}{M^4}$, for some constant $c$.

**Proof.** Assume $d < 1$; the case $1 \leq d \leq \sqrt{2}$ is similar.

Consider two squares $\Box p'_i$ and $\Box p'_j$, with sides parallel to the edge $p'_i p'_j$, of side length $\frac{\sqrt{2}}{M}$, and centers $p'_i$ and $p'_j$ respectively. See Figure 2. Without loss of generality, the edge $p'_i p'_j$ is drawn as a horizontal segment in Figure 2. Note that $p_i$ lies inside $\Box p'_i$, and $p_j$ lies inside $\Box p'_j$. Then the distance $\frac{p'_i p'_j}{\sqrt{2}}$ satisfies

$$d - \frac{2\sqrt{2}}{M} \leq \frac{p'_i p'_j}{\sqrt{2}} \leq d + \frac{\sqrt{2}}{M}.$$  

Observe that the double wedge $w_{i,j}$ is contained in the region bounded by the four common tangents to $\Box p'_i$ and $\Box p'_j$; that is, $w_{i,j}$ is contained in the region bounded by the double wedge defined by lines $l_{A,C}$ and $l_{B,D}$ which contains $\Box p'_i$ and $\Box p'_j$, and the rectangle with vertices $A, B, C, D$ (see Figure 2). The area of the rectangle with vertices $A, B, C, D$ is

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at most $4\sqrt{2}/M$.

Let $q$ be the intersection point of $\ell_{A,C}$ and $\ell_{B,D}$. Then the distance $\overrightarrow{p_iq} = \overrightarrow{p_iq} = \frac{\overrightarrow{p_jq}}{2}$. The area of the triangle $\Delta_{q,C,D}$ with vertices $q, C, D$ is $\sqrt{2} \frac{\overrightarrow{p_jq}}{2} = \frac{\sqrt{2}}{2M}$. The area of $w_{q,C,D}$, which is at most $\frac{\sqrt{2d}}{4M}$. Let $w_{q,C,D}$ be the convex wedge with apex $q$ and halflines $\ell_{q,C}$ and $\ell_{q,D}$, inside the unit square. Then $w_{q,C,D}$ contains a scaled triangle of $\Delta_{q,C,D}$ where the edges are scaled by a factor less than $\frac{\sqrt{2}}{2}$. Then the area of $w_{q,C,D}$ is less than $\frac{\sqrt{2d}}{4M} \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\sqrt{2d}}{8M}$. The same arguments apply to the convex wedge $w_{q,A,B}$ with apex $q$ and halflines $\ell_{q,A}$ and $\ell_{q,B}$, inside the unit square, to show that its area is less than $\frac{\sqrt{2d}}{8M}$. It follows that the area of the double wedge $w_{i,j}$ is less than $\frac{16\sqrt{2} + 4\sqrt{2}}{M} \leq \frac{17\sqrt{2}}{M}$. Finally, $F_{i,j}$ contained in $w_{i,j}$ union the squares which intersect the boundary of $w_{i,j}$. The result follows. □

Let

$$F_k := \bigcup_{i,j < k} F_{i,j}.$$ 

Let $A_k$ be the event that $p_k$ is not in $F_k$. If all the $A_k$ occur, for $1 \leq k \leq n$, then $S$ and $S'$ have the same order type.

**Lemma 4**

$$\Pr\{p_k \in F_k\} \leq \frac{Ck^2}{M},$$

for some constant $C$.

**Proof.** We show that for $1 \leq i < j < k$,

$$\Pr\{p_k \in F_{i,j}\} \leq \frac{c}{M},$$

for some constant $c$. Thereto, we use the density function $f_D(x)$ of the distance $x$ of two random points placed uniformly and independently in the unit square, see for instance [10].

$$f_D(x) = \begin{cases} 2\pi x - 8x^2 + 2x^3 & \text{for } 0 \leq x \leq 1 \\ 4x (\arcsin \left(\frac{1}{x}\right) - \arccos \left(\frac{1}{x}\right)) + 2\sqrt{x^2 - 1} - \frac{x^2}{2} - 1 & \text{for } 1 \leq x \leq \sqrt{2} \end{cases}$$

Then, by Lemma 3,

$$\Pr\{p_k \in F_{i,j}\} = \int_{x=0}^{\sqrt{2}} \Pr\{p_k \in F_{i,j} \mid \overrightarrow{p_i p_j} = x\} f_D(x) \, dx$$

$$\leq \int_{x=0}^{\sqrt{2}} \frac{c}{M} f_D(x) \, dx$$

$$= \frac{4}{3} \left(1 - \sqrt{2} + 3 \arcsinh(1)\right) \frac{c}{M} < \frac{3c}{M}.$$

Since there are less than $\binom{k}{3} < k^3$ choices for $i$ and $j$,

$$\Pr\{p_k \in F_k\} \leq \sum_{i,j < k} \Pr\{p_k \in F_{i,j}\} \leq k^3 \frac{3c}{M} \leq \frac{Ck^2}{M}$$

for some constant $C$. □

**Lemma 5**

$$\Pr\left\{ \bigcap_{i=1}^{n} A_i \right\} \geq 1 - \frac{Cn^3}{M}$$

for some constant $C$.

**Proof.** By Lemma 4, $\Pr\{A_k\} = 1 - \Pr\{p_k \in F_k\} \geq 1 - \frac{Ck^2}{M}$. From the inequality

$$\Pr\left\{ \bigcap_{i=1}^{n} A_i \right\} \geq \left(\sum_{i=1}^{n} \Pr\{A_i\}\right) - (n-1),$$

and since $\sum_{i=1}^{n} i^2 \leq n^3$, we get

$$\Pr\left\{ \bigcap_{i=1}^{n} A_i \right\} \geq \left(\sum_{i=1}^{n} 1 - \frac{Ci^2}{M}\right) - (n-1) \geq 1 - \frac{Cn^3}{M}.$$ □

Since

$$\lim_{n \to \infty} 1 - \frac{Cn^3}{M} = 1,$$

Theorem 1 follows.
References


