Matching points with diametral disks

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Abstract

We consider matchings between a set $R$ of red points and a set $B$ of blue points with diametral disks. In other words, for each pair of matched points $p \in R$ and $q \in B$, we consider the diametral disk defined by $p$ and $q$. We prove that for any $R$ and $B$ such that $|R| = |B|$, there exists a perfect matching such that the diametral disks of the matched point pairs have a common intersection. More precisely, we show that a maximum weight perfect matching has this property.

1 Introduction

Let $n \geq 2$, and $R$ and $B$ be disjoint point sets in the plane such that $|R| = |B| = n$. The elements of $R$ are called red points and the elements of $B$ blue points.

Geometric matching of red and blue points in the plane with pairwise disjoint geometric objects asks for pairing the points in such a way that each pair is associated with a geometric object that covers both points of the pair, and all associated objects are pairwise disjoint. In all pairs the two points are restricted to be of different colors, or in all pairs the two points are restricted to be of the same color. This is well studied in discrete and computational geometry, starting from the classic result that $n$ red points and $n$ blue points can always be perfectly matched with $n$ pairwise non-crossing segments, where each segment connects a red point with a blue point [9]. The study has been continued by using pairwise disjoint segments [3, 7], rectangles and squares [1, 2, 5, 6], and more general geometric objects [4].

More formally, for $R = \{p_1, \ldots, p_n\}$ and $B = \{q_1, \ldots, q_n\}$, a matching of $R \cup B$ is a partition of $R \cup B$ into $n$ pairs such that each pair consists of a red and a blue point. A point $p \in R$ and a point $q \in B$ are matched if and only if the pair $(p, q)$ is in the matching. We use $pq$ to denote the segment connecting $p$ and $q$, and $|pq|$ to denote its length. The diametral disk of $pq$, denoted $D_{pq}$, is the disk with diameter equal to $|pq|$ that is centered on the midpoint of $pq$. For a matching $\mathcal{M}$, we use $D_{\mathcal{M}}$ to denote the set of disks associated with the matching, that is: $D_{\mathcal{M}} = \{D_{pq} \mid (p, q) \in \mathcal{M}\}$.

In this paper, we prove that for any $R$ and $B$ as above, there always exists a matching $\mathcal{M}$ such that all disks in $D_{\mathcal{M}}$ have a common intersection (see Figure 1). More precisely, we show that any maximum matching satisfies this property. A matching $\mathcal{M}$ of $R \cup B$ is maximum if it maximizes the sum of the squared distances between the matched points, that is, it maximizes $\sum_{(p, q) \in \mathcal{M}} |pq|^2$.

Observe that our result goes in the direction opposite to that of known results on matching red and blue points: Our goal is that all matching objects have a common intersection, whereas in prior work it is required that all matching objects are pairwise disjoint. When matching points of different colors with segments, it is not always possible to guarantee that all matching segments are pairwise intersecting (e.g., two red and two blue points not in convex position). This also happens when matching with axis-aligned rectangles, where for any two matched points the associated rectangle is the one with minimum-area that contains both points. Our result is also motivated from Tverberg’s theorem [10], which states that any $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into $r$ subsets such that the convex hull of all of the subsets

Figure 1: Example for a set of $n = 4$ red and blue points, showing a matching and the associated disks, with a common intersection.
have a point in common.

We begin by introducing some additional notation. After that, we consider a maximum matching $\mathcal{M}$ of $R \cup B$, and prove in Section 2 that any pair of disks in $D_\mathcal{M}$ intersect. Finally, in Section 3, we prove that all disks in $D_\mathcal{M}$ must intersect. Due to space constraints, some proofs have been omitted.

**Notation:** For a point $p$, let $x(p)$ and $y(p)$ denote the $x$- and $y$-coordinates of $p$, respectively. Given three different points $p$, $q$, and $r$, let $\ell(p,q)$ denote the line containing both $p$ and $q$. $\Delta pqr$ the triangle with vertex set $\{p, q, r\}$, $C(pqr)$ the angle at $q$ in the triangle $\Delta pqr$, and $C_{pq}$ the circle bounding $D_{pq}$.

2 Any two disks intersect

**Lemma 1** Let $p_1, p_2 \in R$ and $q_1, q_2 \in B$ such that $\{(p_1, q_1), (p_2, q_2)\}$ is a maximum matching for $\{(p_1, p_2, q_1, q_2)\}$. Suppose further that $y(p_1) = y(p_2)$, and $x(p_1) < x(p_2)$. Then, $x(q_2) \leq x(q_1)$.

**Proof.** Assume w.l.o.g. $p_1 = (-1, 0)$ and $p_2 = (1, 0)$. Given a constant $c$, the points $x = (x, y)$ that satisfy $|x^2| - |y^2| = c$ are those such that $(x + 1)^2 + y^2 = (x - 1)^2 + y^2 = 4x = c$. Then, the locus of such points is the vertical line $x = c/4$. Since $\{(p_1, q_1), (p_2, q_2)\}$ is a maximum matching, we have that

$$|p_1 q_1|^2 + |p_2 q_2|^2 \geq |p_1 q_2|^2 + |p_2 q_1|^2,$$

$$|p_1 q_1|^2 - |p_2 q_1|^2 \geq |p_1 q_2|^2 - |p_2 q_2|^2.$$

Let $d_1 = |p_1 q_1|^2 - |p_2 q_1|^2$ and $d_2 = |p_1 q_2|^2 - |p_2 q_2|^2$. Note that the vertical line through $q_1$ is the line $x = d_1/4$, thus $x(q_1) = d_1/4$, and analogously, $x(q_2) = d_2/4$. Since $d_2 \leq d_1$, we have $x(q_2) \leq x(q_1)$. □

**Lemma 2** Let $p_1, p_2 \in R$ and $q_1, q_2 \in B$ such that $\{(p_1, q_1), (p_2, q_2)\}$ is a maximum matching for $\{(p_1, p_2, q_1, q_2)\}$. Then, $D_{p_1 q_1} \cap D_{p_2 q_2} \neq \emptyset$.

**Proof.** Assume w.l.o.g. that $y(p_1) = y(p_2)$ and $x(p_1) < x(p_2)$. Let $\bar{q}_1$ and $\bar{q}_2$ be the orthogonal projections of $q_1$ and $q_2$ on $\ell(p_1, p_2)$, respectively. By Thales’ theorem, $C_{p_1 q_1}$ contains $\bar{q}_1$, which implies $p_1 \bar{q}_1 = D_{p_1 q_1} \cap \ell(p_1, p_2)$. Similarly, $C_{p_2 q_2}$ contains $\bar{q}_2$, and $p_2 \bar{q}_2 = D_{p_2 q_2} \cap \ell(p_1, p_2)$. By Lemma 1, $x(q_2) \leq x(q_1)$, which implies that segments $p_1 \bar{q}_1$ and $p_2 \bar{q}_2$ have a point in common. Hence, $D_{p_1 q_1} \cap D_{p_2 q_2} \neq \emptyset$. □

Next, observe that in any maximum matching $\mathcal{M}$ of $R \cup B$, where $|R| = |B| \geq 2$, $\{(p_1, q_1), (p_2, q_2)\}$ is a maximum matching of $\{(p_1, p_2, q_1, q_2)\}$ for every pair $(p_1, q_1), (p_2, q_2) \in \mathcal{M}$. That is, any two pairs of a maximum matching form a maximum matching for the four points involved. Therefore, applying Lemma 2 we can conclude that in any maximum matching $\mathcal{M}$ the disks $D_\mathcal{M}$ are pairwise intersecting.

3 All disks intersect

In this section, we strengthen the previous result by showing that in any maximum matching $\mathcal{M}$, all disks in $D_\mathcal{M}$ have a point in common. This requires considerably more effort and the help of several lemmas.

**Lemma 3** Let $A$, $B$, and $C$ be three points in the plane in general position. Let $h_{AB}$, $h_{BC}$, and $h_{CA}$ be three lines that are perpendicular to $\ell(A, B)$, $\ell(B, C)$, and $\ell(C, A)$, respectively. Let the points $A' = h_{AB} \cap h_{CA}$, $B' = h_{AB} \cap h_{BC}$, and $C' = h_{BC} \cap h_{CA}$. Then, the three circles $C_{AA'}$, $C_{BB'}$, and $C_{CC'}$ have a point in common (see Figure 2).

**Proof.** Without loss of generality assume that $A = (a, 0)$, $B = (b, 0)$, and $C = (c, 0)$, for some $a < 0$, and $b, c > 0$. Observe that triangles $\Delta ABC$ and $\Delta A'B'C'$ are similar, so that $\Delta A'B'C'$ is obtained from $\Delta ABC$ by first a rotation of $\pi/2$ radians, after that a scaling of factor $\lambda$, and finally a translation with vector $(\alpha, \beta)$, for some $\lambda \geq 0$, $\alpha, \beta \in \mathbb{R}$. Consider that the rotation is counter-clockwise. The case where it is clockwise is similar. Then, $A' = \lambda \cdot (0, a) + (\alpha, \beta) = (\lambda a + \beta, \alpha)$, $B' = \lambda \cdot (b, 0) + (\alpha, \beta) = (\lambda b + \beta, \alpha)$, and $C' = \lambda \cdot (c, 0) + (\alpha, \beta) = (\lambda c + \alpha, \beta)$. The points $(x, y)$ of $C_{AA'}$ are those such that the scalar product between vectors $(x, y) - A = (x - a, y)$ and $(x, y) - A' = (x - \alpha, y - \lambda a - \beta)$ equals zero. That is,

$$(x - a)(x - a) + (y - y - \lambda a - \beta) = 0.$$  (1)

Similarly, the points $(x, y)$ of $C_{BB'}$ satisfy that the scalar product between $(x, y) - B = (x - b, y)$ and $(x, y) - B' = (x - \alpha, y - \lambda b - \beta)$ equals zero. That is,

$$(x - b)(x - a) + (y - y - \lambda b - \beta) = 0.$$  (2)

One solution to the system formed by equations (1) and (2) is the point $(\alpha, 0) = h_{AB} \cap \ell(A, B)$, which is
one of the intersection points between $C_{AA'}$ and $C_{BB'}$. The other intersection point (considering multiplicity) can be found as follows. Subtracting (1) and (2):

$$(b-a)(x-a) + y(\lambda(b-a)) = 0$$
$$x = -\lambda y + \alpha. \quad (3)$$

Using equation (3) in equation (1), we obtain

$$(\lambda^2 y - \lambda^2 \alpha + y - \lambda \alpha - \beta) = 0$$
$$y(\lambda^2 y - \lambda^2 \alpha + y - \lambda \alpha - \beta) = 0$$
$$y = \lambda \alpha + \beta$$
$$1 + \lambda^2.$$ 

Then,

$$x = -\lambda \left( \frac{\lambda \alpha + \beta}{1 + \lambda^2} \right) + \alpha$$
$$= \frac{-\lambda \alpha - \lambda \beta + \alpha + \lambda^2 \alpha}{1 + \lambda^2} = \frac{-\lambda \beta + \alpha}{1 + \lambda^2}. \quad (4)$$

The points $(x, y)$ of $C_{CC'}$ satisfy that the scalar product between vectors $(x, y) - C = (x, y - c)$ and $(x, y) - C' = (x + \lambda c - \alpha, y - \beta)$ equals zero. That is,

$$x(x + \lambda c - \alpha) + (y - c)(y - \beta) = 0. \quad (6)$$

Hence, to show the lemma it suffices to prove that $(x, y) = \left( \frac{-\lambda \beta + \alpha}{1 + \lambda^2}, \frac{\lambda \alpha + \beta}{1 + \lambda^2} \right)$ satisfies equation (6).

$$x(x + \lambda c - \alpha) = \frac{-\lambda \beta + \alpha}{1 + \lambda^2} \left( \frac{-\lambda \beta + \alpha}{1 + \lambda^2} + \lambda c - \alpha \right) = -\lambda \left( \frac{-\lambda \beta + \alpha}{1 + \lambda^2} \right) \left( \frac{\lambda \alpha + \beta - c - \lambda^2 c}{1 + \lambda^2} \right)$$

$$(y - c)(y - \beta) = \frac{\lambda \alpha + \beta}{1 + \lambda^2} - c \left( \frac{\lambda \alpha + \beta}{1 + \lambda^2} - \beta \right)$$

Therefore,

$$(x, y) = \left( \frac{-\lambda \beta + \alpha}{1 + \lambda^2}, \frac{\lambda \alpha + \beta}{1 + \lambda^2} \right) \text{ satisfies equation (6) and is common to } C_{AA'}, C_{BB'}, \text{ and } C_{CC'}. \quad \square$$

**Lemma 4** Let $A$, $B$, $P$, and $R$ be four points in the plane such that $\ell(A, B)$ is horizontal, $B$ is to the right of $A$, $P$ belongs to $\ell(A, B)$, and $R$ is above $\ell(A, B)$. Let $C_1$ be the circle through the points $A$, $P$, and $R$, and $C_2$ be a circle through $B$ and $P$. If $C_1$ and $C_2$ are tangent, let $O = P$, otherwise let $O$ be the other intersection point than $P$ between $C_1$ and $C_2$. Then, if $C_2$ does not enclose $R$, the points $O$ and $B$ are in the same half-plane bounded by $\ell(A, R)$.

**Lemma 5** Let $\ell$ be a line, and $R, C \in \ell$ two points. Let $h$ be a half-line with apex point $H$ such that the line containing $h$ intersects $\ell$ at $R$ and is perpendicular to $\ell$. Let $\delta$ be the half-plane bounded by $\ell$ such that $\delta \cap h$ is a half-line. Then, for any two points $X, Y \in h$ with $|XH| \leq |YH|$, we have $D_{XC} \cap \delta \subseteq D_{YC} \cap \delta$.

**Lemma 6** Let $p_1, p_2, p_3 \in R$ and $q_1, q_2, q_3 \in B$ such that $(p_1, q_1), (p_2, q_2), (p_3, q_3)$ is a maximum matching for $\{p_1, q_1, p_2, q_2, p_3, q_3\}$. Then, the disks $D_{p_1q_1}, D_{p_2q_2}, D_{p_3q_3}$ have a point in common.

**Proof.** The idea is to reduce the disks $D_{p_1q_1}, D_{p_2q_2},$ and $D_{p_3q_3}$ as much as possible so that each of the new three disks is contained in its corresponding original disk, and the new disks still have a point in common. For every $\varepsilon_1 \in [0, |p_1q_1|], \varepsilon_2 \in [0, |p_2q_2|], \varepsilon_3 \in [0, |p_3q_3|], \mu_1, \mu_2, \mu_3 \in [0, |p_1q_1|]$, $q_1 \in [0, \varepsilon_1], q_2 \in [0, \varepsilon_2], q_3 \in [0, \varepsilon_3]$. Let $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be a maximal point of the set $[0, |p_1q_1|] \times [0, |p_2q_2|] \times [0, |p_3q_3|]$ such that the conditions of Lemma 1 are satisfied pairwise, that is:

- in the direction from $p_1$ to $p_2$, $q_2(\varepsilon_2)$ is not to the right of $q_1(\varepsilon_1)$;
- in the direction from $p_2$ to $p_3$, $q_3(\varepsilon_3)$ is not to the right of $q_2(\varepsilon_2)$; and
- in the direction from $p_3$ to $p_1$, $q_1(\varepsilon_1)$ is not to the right of $q_3(\varepsilon_3)$.

The point $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is maximal if there does not exist any other point $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) \in [0, |p_1q_1|] \times [0, |p_2q_2|] \times [0, |p_3q_3|]$ such that $\varepsilon_1 \leq \varepsilon'_1, \varepsilon_2 \leq \varepsilon'_2, \varepsilon_3 \leq \varepsilon'_3$, and the above three conditions are also satisfied by using $(\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$ instead of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Let $\tilde{p}_1 = q_1(\varepsilon_1), \tilde{p}_2 = q_2(\varepsilon_2), \tilde{p}_3 = q_3(\varepsilon_3)$. Note that $D_{p_1\tilde{p}_1} \subseteq D_{p_1q_1}, D_{p_2\tilde{p}_2} \subseteq D_{p_2q_2}, D_{p_3\tilde{p}_3} \subseteq D_{p_3q_3}$. We prove now that $D_{p_1\tilde{p}_1}, D_{p_2\tilde{p}_2}, D_{p_3\tilde{p}_3}$ have a point in common, which implies the lemma.

If $p_1, p_2,$ and $p_3$ belong to the same line $\ell$ (assuming w.l.o.g. that they appear in this order in $\ell$), then the points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ belong to the same line $\ell'$ perpendicular to $\ell$. By Thales’ theorem, the point $\ell \cap \ell'$ is common to $D_{p_1\tilde{p}_1}, D_{p_2\tilde{p}_2}, D_{p_3\tilde{p}_3}$, hence, assume from now on that $p_1, p_2,$ and $p_3$ are in general position. Then, there are two cases to consider:

**Case 1:** $\tilde{p}_i = p_i$ for some $i \in \{1, 2, 3\}$. Assume w.l.o.g. $\tilde{p}_3 = p_3$ (see Figure 3a). Let $s_1$ and $s_2$ be the orthogonal projections of $\tilde{p}_1$ on $\ell(p_1,p_3)$, and $\tilde{p}_2$ on $\ell(p_2,p_3)$, respectively. Since in the direction from $p_3$ to $p_1$, point $\tilde{p}_1$ is not to the right of $p_2$ we have $p_3 \in p_1s_1$. Similarly, since in the direction from $p_2$ to $p_3$, point $\tilde{p}_2$ is not to the right of $p_2$ we have $p_3 \in p_2s_2$. By Thales’ theorem $p_1s_1 \subset D_{p_1\tilde{p}_1}$ and $p_2s_2 \subset D_{p_2\tilde{p}_2}$, hence $p_3 = \tilde{p}_3$ is common to $D_{p_1\tilde{p}_1}, D_{p_2p_2}, D_{p_3\tilde{p}_3}$.

**Case 2:** $\tilde{p}_1 \neq p_1, \tilde{p}_2 \neq p_2,$ and $\tilde{p}_3 \neq p_3$. By construction of $\tilde{p}_1, \tilde{p}_2,$ and $\tilde{p}_3$, at least two pairs of lines
among \((\ell(p_1, p_2), \ell(\tilde{p}_1, \tilde{p}_2)), (\ell(p_2, p_3), \ell(\tilde{p}_2, \tilde{p}_3)), \) and 
\((\ell(p_3, p_1), \ell(\tilde{p}_3, \tilde{p}_1))\) form perpendicular lines. Assume w.l.o.g. that \(\ell(\tilde{p}_1, \tilde{p}_2)\) is perpendicular to \(\ell(p_1, p_2)\), and that \(\ell(\tilde{p}_3, \tilde{p}_1)\) is perpendicular to \(\ell(p_3, p_1)\) (see Figure 3b). Let \(s_{12} = \ell(p_1, p_2) \cap \ell(\tilde{p}_1, \tilde{p}_2), \) \(s_{13} = \ell(p_1, p_3) \cap \ell(\tilde{p}_1, \tilde{p}_3), \) and let \(s^*\) be the point of \(\ell(p_1, \tilde{p}_3)\) such that \(\ell(s^*, \tilde{p}_2)\) is perpendicular to \(\ell(p_2, p_3).\) By Thales’ theorem, \(s_{12}, s_{13} \in C_{p_1 \tilde{p}_1}, s_{12} \in C_{p_2 \tilde{p}_2}, \) and \(s_{13} \in C_{p_3 \tilde{p}_3}.\) If \(C_{p_1 \tilde{p}_1}\) and \(C_{p_2 \tilde{p}_2}\) are tangent at \(s_{12},\) let \(O = s_{12},\) otherwise let \(O\) be the intersection point other than \(s_{12}\) between \(C_{p_1 \tilde{p}_1}\) and \(C_{p_2 \tilde{p}_2}.\) Note that \(O = C_{p_1 \tilde{p}_1} \cap C_{p_2 \tilde{p}_2} \cap C_{p_3 \tilde{p}_3}^{s^*},\) by Lemma 3. If \(C_{p_2 \tilde{p}_2}\) contains \(s_{13},\) then we are done since \(s_{13}\) is common to \(D_{p_1 \tilde{p}_1}, D_{p_2 \tilde{p}_2},\) and \(D_{p_3 \tilde{p}_3}.\) Hence, assume \(C_{p_2 \tilde{p}_2}\) does not contain \(s_{13}.\) Under this assumption, by Lemma 4, \(O\) and \(p_2\) are in the same half-plane \(\mathcal{H}\) bounded by \(\ell(p_1, p_3).\) Since \(p_3\) is not to the right of \(\ell(s^*, \tilde{p}_2)\) in the direction from \(p_2\) to \(p_3,\) we have that \(p_3\) is on the half-line \(h \subset \ell(s^*, \tilde{p}_1)\) with apex \(s^*\) and such that \(h \cap \mathcal{H}\) is a half-line. By Lemma 5, \(D_{p_3 \tilde{p}_3} \cap \mathcal{H} \subseteq D_{p_3 \tilde{p}_3} \cap \mathcal{H},\) which implies that \(O\) is common to \(D_{p_1 \tilde{p}_1}, D_{p_2 \tilde{p}_2},\) and \(D_{p_3 \tilde{p}_3}.\) \(\Box\)

**Theorem 7** Given a set \(R\) of \(n \geq 2\) red points and a set \(B\) of \(n\) blue points, in any maximum matching of \(R\) and \(B,\) the disks have a common intersection.

**Proof.** Let \(\mathcal{M} = \{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}\) be a maximum matching of \(R \cup B.\) If \(n = 2,\) then \(D_{p_1 q_1} \cap D_{p_2 q_2} \neq \emptyset\) by Lemma 2, implying the theorem. Otherwise, if \(n \geq 3,\) for every different \(i, j, k \in \{1, 2, \ldots, n\}\) the matching \(\{(p_i, q_i), (p_j, q_j), (p_k, q_k)\}\) must be maximum for \(\{(p_i, q_i), (p_j, q_j), (p_k, q_k)\}.\) Then, by Lemma 6, \(D_{p_i q_i} \cap D_{p_j q_j} \cap D_{p_k q_k} \neq \emptyset.\) The result follows by Helly’s theorem. \(\Box\)

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**References**


