

Constrained observability method in static structural system identification

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Complete List of Authors:	Lei, Jun; Tongji University, Department of Bridge Engineering Nogal, María; University of Dublin Trinity College, Department of Civil, Structural & Environmental Engineering Lozano Galant, Jose Antonio; University of Castilla-La Mancha, Department of Civil Engineering Xu, Dong; Tongji University, Department of Bridge Engineering Turmo, JÓse; Universitat Politècnica de Catalunya, Department of Civil and Environmental Engineering
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11 5 **Authors:**

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13 6 Jun Lei, María Nogal , Jose Antonio Lozano-Galant, **Dong Xu**(Corresponding Author),
14 Jose Turmo
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16

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18
19 9 **Corresponding author:**

20
21 10 Dong Xu,
22 11 Prof., Ph.D.
23 12 Department of Bridge Engineering at Tongji University
24 13 1239, Siping Road, Shanghai, China
25 14 Post: 200092
26 15 Email: xu_dong@tongji.edu.cn
27 16 Tel: 021-65980953
28 17 Fax: 021-65980953
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Constrained observability method in static structural system identification

Summary

Identifiability of parameters in structural system identification (SSI) is of primary importance in any SSI method. It depends on the number and the location of the measurements, which is directly linked with sensor configuration. In this paper, under the framework of SSI by observability method (OM), the number of necessary measurements to identify all parameters of structural system was clarified first. Then an example was solved step by step to show the lacking constraints among unknowns in SSI by OM. In a frame example, it was found that no measurement set having as many measurements as the number of unknowns was able to identify all parameters. To further understand this phenomenon, the observability of a simply supported beam was analyzed in an exhaustive way with respect to 252 possible measurement sets. It turned out that three quarters of these sets were not able to identify all the parameters. In order to solve this issue, for the very first time, SSI by constrained observability method (COM), which appends the nonlinear constraints to SSI by OM, was proposed. With SSI by COM applied, the observability of structural parameters with respect to the 252 sets was greatly improved. Finally, the efficacy of this method was verified by a 13-storey frame building.

Keyword: structural system identification; stiffness method; observability method; nonlinear constraint; essential set; static

Section 1: Introduction

Structural System Identification (SSI) has long been an intriguing topic in the field of civil engineering. SSI can be conceptualized as the process of simulating the structural behavior by mathematical models. Based on the type of the excitation, methods for SSI can be categorized as static^[1-8] or dynamic^[9-12]. The dynamic SSI requires the use of mass, stiffness and damping properties while the static method only requires the use of stiffness properties^[13]. It was pointed out by Sanayei^[14] that static SSI might be more interesting than dynamic SSI in cases when only the estimation of stiffnesses is targeted. In many cases, the estimation of stiffness is sufficient for condition assessment of the structure. In the non-destructive test of bridges, slow moving load can be applied as a quasi-static load^[15-18]. In curvature-based methods^[15,18], the curvatures of the displacement influence lines of damaged and undamaged beams under moving load are obtained first and the damages are located by the irregular variation in curvatures. Boumechra^[16] approximated the inverse of the global stiffness matrix by a Neumann series. SSI is posed as the problem of finding the correction coefficients of the stiffness of each element that minimize the difference between the measured response and the predicted response under moving load. This method can accurately localize and quantify the damage with the displacements of selected nodes. Providing that the structure behaves linearly, Maxwell law of reciprocal deflection can be applied to limit the number of sensors installed in the structure subjected to this type of load, which might make the method more attractive^[17]. This strategy was also adopted by other researchers to achieve dense measurements without using many sensors^[18,19]. The elastic damage load theorem is derived by Choi^[19] for statically determinate beam. The location of

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3 81 damage in statically determinate beams is reflected by the variation in the shape of deflection.
4 82 However, this method is not able to quantify the extent of damage and its application is limited
5 83 to statically determinate structures. From the point of the interpretability of the model, SSI
6 84 methods can be classified as physics-based models (e.g. finite element models (FEM)) or non-
7 85 physics based models (the neural networks models^[20–22], autoregressive models^[23–25] or rational
8 86 polynomial models^[26–28]). Physics-based models and non-physics based models are respectively,
9 87 called parametric models and non-parametric models. However, in effect, both types of models
10 88 have parameters. In the physics-based models, the parameters represent the structural
11 89 characteristics, which might be elastic modulus, inertias, areas, mass or damping ratios.
12 90 Conversely, the parameters of the non-parametric models play the role of weight factors of the
13 91 adopted basis functions. These parameters have no physical meaning and are determined by
14 92 minimizing the discrepancy between the predicted structural response and the real-life structural
15 93 response. From a statistical perspective, SSI methods can be categorized as probabilistic
16 94 methods^[8,12,29–32] or deterministic methods^[13,33,34]. In probabilistic SSI, each structural parameter
17 95 is treated as random variable and is assumed to follow a prescribed distribution using prior
18 96 information. A set of parameters will be formed by sampling each distribution once. A set of
19 97 parameters represents a possible model for the structure. All these models are designated as trial
20 98 models^[8,12,29,30]. Many trial models will be generated and be evaluated by the discrepancy
21 99 between the response (e.g. strains, deflections or frequencies) calculated from them and the
22 100 measured one in real-life. A specified proportion of the generated models which have the lowest
23 101 discrepancy will be retained as the probable models. The mean and standard deviation of the
24 102 assumed distribution of each structural parameter, will be updated by the statistical inference
25 103 from the values of that parameter in all these probable models. These updated distributions will
26 104 be used to generate new trial models and restart the process until the discrepancy between the
27 105 response calculated from the estimated parameters and the measured response reaches a
28 106 specified threshold. The strengths of probabilistic methods are that they not only provide the
29 107 estimation of the parameters but also a measure to assess the confidence in the estimates.
30 108 However, the computation cost of the probabilistic methods increases exponentially with the
31 109 number of parameters due to the combinatorial consideration linked with the generation of
32 110 parameter set. In contrast to the probabilistic method, the deterministic methods try to pinpoint a
33 111 best model yielding the closest response to the one measured. The main drawback of these
34 112 methods is that evaluation of the confidence in the estimates is not available in the theoretical
35 113 formulation. However, this might be overcome by the post analysis of the estimates obtained by
36 114 using different sets of data generated by Monte-Carlo method.

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43 115 A vital issue for all the methods mentioned above is whether the parameters within the
44 116 structural system can be identified or not. In practice, large amount of data is required to train
45 117 the non-parametric models, i.e. to obtain the parameters. In contrast, fewer measurements are
46 118 required by the parametric models since the mathematical relationships, i.e. the system of
47 119 equations, are well defined in these models. However, in the context of the parametric methods,
48 120 one might ask: Are the parameters within the structure identifiable given a particular
49 121 measurement set? A generalized interpretation of this question is: Is the available information
50 122 sufficient to specify a unique solution of a system of equations? In fact, this issue is addressed
51 123 by the Observability Method (OM)^[35,36]. This mathematical tool provides the information
52 124 whether all the unknowns or a subset of the unknowns can be uniquely determined or not. Even
53 125 though OM was originally conceived to deal with linear system of equations, it have been also
54 126 applied to inequalities^[35] and to nonlinear systems^[36]. It has been applied extensively to

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3 127 parameter estimation in different engineering fields, including water transport network^[37-39],
4 128 traffic network^[40-43] or power systems^[44-46].

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6 129 OM was first introduced into the static SSI by Lozano-Galant^[33]. This is a deterministic static
7 130 SSI method which has the advantage of less computation cost than those statistical methods. In
8 131 the tailor-made implementation of OM in static SSI, the required system of equations is derived
9 132 by algebraic operations on the nodal equilibrium equations obtained from direct stiffness
10 133 method, which makes SSI by OM a physics-based method, and any subset of the solution is
11 134 regarded as observable if the corresponding rows in the null space of the acquired coefficient
12 135 matrix are composed of only zeros. The efficacy of this method was verified by its application
13 136 in the identification of trusses, beams, frame structures^[33]. Also, this innovative method can be
14 137 applied in cable-stayed bridges^[47], wherein the structural audacity and lightness makes these
15 138 structures sensitive to dynamic and static load cases in both service and construction stage^[48,49].
16 139 It can also help the decision making during the maintenance of structures^[50]. A peculiarity of
17 140 this method is that the identification of parameters is carried out in a recursive manner. That is,
18 141 the consecutive identifications rely on the existing information in the measurements and/or the
19 142 identification results from the preceding steps. However, in this method, the parameters to be
20 143 identified should be activated by the external load. Or in other words, the displacements should
21 144 be sensitive to the parameters under the given load. For instance, in the case of simply
22 145 supported beams subjected to vertical loads, the vertical deflections have nothing to do with the
23 146 axial stiffnesses. Hence, it is impossible to identify the axial stiffnesses of the simply supported
24 147 beams by just measuring these vertical deflections in this case.

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27 148 As mentioned above, the identifiability of the structure by both, parametric method and non-
28 149 parametric method relies on the available information. This is due to the fact that the
29 150 information carried by the measurements bridges the mathematical models with the real-life
30 151 structure. From the point of identifiability, more sensors, i.e. more information, are generally
31 152 desirable. On the converse, using less sensors means reducing the chance of measuring and
32 153 processing redundant measurements. In any SSI method, different measurement sets might be
33 154 used for parameter estimation. However, with regard to the selection of measurements, one
34 155 might be faced with the limit induced by the extended dimension of structures, the operational
35 156 costs, the accessibility. A more practical question is posed as how to find the optimal number of
36 157 measurements and the location of the sensors for successful identifications. A similar issue was
37 158 addressed in the parameter estimation of power system under OM^[46]. In this problem, the
38 159 system can be specified by n parameters and m potential measurements ($m > n$). It is preferable
39 160 to choose n out of m measurements to identify all the n parameters of the system. If a
40 161 measurement set of n measurements is able to identify all the n parameters, and the drop of any
41 162 measurement in this set fails to do so, then this set is defined as the essential set. Generally, such
42 163 a set is not unique. OM is introduced into the aforementioned disciplines to serve two purposes:
43 164 (1) In the preliminary stage of designing the measurement configuration, provide a solution of
44 165 essential set to ensure that all the parameters of the system can be identified; (2) In the operation
45 166 stage of the monitoring system, if any of the measuring devices is out of service, determine the
46 167 observability of the system parameters. In the context of SSI, it was pointed out that if the
47 168 number of measurements was less than the number of unknown parameters, the system would
48 169 become indeterminate^[13]. This is to say, in this case, only a subset of the parameters might be
49 170 identified, which is referred as partial observability in this paper. Sanayei^[51] also concluded that
50 171 the number of measurement must be greater than or equal to the number of parameters, as a
51 172 necessary condition for the solution to exist. Moreover, even if the number of measurements
52 173 equals to the number of unknowns, if those are not properly chosen, the system might still be
53 174 indeterminate. An increase of the number of measurements or a better placement of the sensors

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3 175 can lead to an increase in the number of the identified parameters, which will be referred as
4 176 increase of observability in this paper. Eventually, this process will lead to the identification of
5 177 all parameters and to make the system determinate, which is referred as full observability.

7 178 A wide range of proposals and methods dealing with the placement of sensors in SSI can be
8 179 found. Sanayei^[52] used a heuristic method to seek near-optimal placement of sensors for
9 180 structures under non-destructive test. It was pointed out that the structure was identifiable on the
10 181 premise that the number of measurements was more than the number of unknown parameters.
11 182 Cha^[53] proposed a multi-objective genetic algorithm for optimal placement of control devices
12 183 and sensors in seismically excited civil structures. A series of optimal sensor placement
13 184 techniques for monitoring vibration response, including effective influence method, the driving
14 185 point residue kinetic energy and modified variance method, are discussed by Chang^[54]. Jin^[55]
15 186 applied an improved harmony search optimization algorithm and the modal assurance criterion
16 187 to find the optimal sensor placement for the identification of the mode parameters. Malings^[56]
17 188 proposed the sensor placement as an optimization problem with respect to the conditional
18 189 entropy and the value of information.

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22 190 It is pointed out that the majority of the existing literature on measurement selection focuses on
23 191 dynamic SSI. In static SSI by OM, the identifiability of the structural parameters also relies on a
24 192 proper measurement selection. To deal with this issue, the observability trees method was
25 193 proposed by Lozano-Galant^[2]. The order of identification in the sequence of identified
26 194 parameters with any given measurement set can be analyzed graphically by the observability
27 195 tree technique. A forward and backward strategy was adopted then to find the essential sets, i.e.
28 196 to select those measurements able to maintain the observability flow and thereby to identify the
29 197 desired structural parameters. Note that the notion of essential set in power systems and the
30 198 notion of essential set in SSI by OM are used interchangeably in this paper. It was found that
31 199 improper selection of the measurement set cuts off the observability flow and thus leads to fail
32 200 to identify all the unknowns even with some measurement sets which have more measurements
33 201 than the essential sets. However, the underlying reason for this deficiency is not clear and the
34 202 pertinent solution remained to be found at that time. For the aforementioned reasons, the aim of
35 203 this paper is twofold: (1) to clarify the underlying reason why some measurement sets expected
36 204 to be able to identify all structural parameters turn out to be unqualified to do so in SSI by OM
37 205 and (2) to provide a method to alleviate or solve this problem.

41 206 The rest of this paper is organized as follows. In section 2, the essential set is conceptualized
42 207 first and two simple examples are used to illustrate the deficiency of SSI by OM. And then the
43 208 observability of the structural parameters in a simply supported beam is analyzed in an
44 209 exhaustive way for 252 enumerated measurement sets. Based on the analysis of the result, two
45 210 reasons of the unqualification of some measurement sets to be essential sets are revealed. In the
46 211 following section, the constrained observability method (COM) is proposed as the solution to
47 212 this deficiency. The effectiveness and robustness of COM is justified by the improvement of
48 213 observability on the two examples proposed in Section 2. Next, in section 4, a 13-storey frame
49 214 with a large number of Degrees Of Freedom (DOFs) is studied to further illustrate the strength
50 215 of this COM. Finally, some conclusions are drawn in section 5.

55 216 Section 2: Inadequacy of OM

56 217 Concerning beam element in 2D structural models, the nodal equilibrium equations from direct
57 218 stiffness method can be expressed as:

$$K \cdot \delta = \left[\sum_{j=1}^{N_e} K_e(E_j A_j, E_j I_j) \right] \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_{N_n} \\ v_{N_n} \\ w_{N_n} \end{Bmatrix} = \begin{Bmatrix} H_1 \\ V_1 \\ M_1 \\ \vdots \\ H_{N_n} \\ V_{N_n} \\ M_{N_n} \end{Bmatrix} = f, \quad (1)$$

219 where N_e and N_n , respectively indicate the number of elements and the number of nodes in the
 220 FEM; u_i , v_i and w_i , respectively, indicate the horizontal, vertical and rotational displacements
 221 of the node i ; H_i , V_i and M_i , respectively, indicate the external horizontal forces, vertical forces
 222 and moments applied on node i . The mechanical parameters for element j include elastic
 223 modulus, E_j , areas, A_j , and inertias, I_j .

224 In either static or dynamic SSI, the target parameters to be identified are axial stiffnesses and
 225 flexural stiffnesses. In SSI by OM, when the axial stiffness, $E_j A_j$, appears in the equation, it is
 226 regarded as one unknown instead of being regarded as a product of the unknown modulus, E_j ,
 227 and the unknown area, A_j . This principle goes the same in the case of flexural stiffness. For
 228 simplicity, the axial stiffness and the flexural stiffness will be, respectively, denoted as EA_j and
 229 EI_j in SSI by OM. In addition, truss elements are treated as a degenerate case of beam elements
 230 and their flexural stiffnesses, EI_j , is set as null.

231 In Eq. (1), the symbol Σ represents the operation of assembling all element stiffness matrices
 232 into the global stiffness matrix K . In fact, the information of the geometry, or the element
 233 connectivity, is introduced into the equations by this operation. Once the geometry of the
 234 structure, the boundary conditions and the load case are specified, then the nodal displacements,
 235 δ , will be uniquely determined by any particular set of axial stiffnesses, EA_j , and flexural
 236 stiffnesses, EI_j . In the case of there being N_A unknown axial stiffnesses and N_F unknown
 237 flexural stiffnesses, this is saying that these $N_A + N_F$ mechanical parameters uniquely specify a
 238 set of displacements of the structure. Conversely, in the inverse analysis of this problem, due to
 239 the linearity of this system, it is expected that $N_A + N_F$ measurements of the displacements will
 240 suffice to identify all the unknown parameters. For simplicity, the ability to identify all the
 241 parameters of the structure is referred as full observability in this paper. Meanwhile, if merely a
 242 subset of these parameters are identifiable, then it is referred as partial observability. Hence, in
 243 the essential sets (the sets containing the necessary and sufficient number of measurements to
 244 achieve full observability), the number of required measurements is $N_A + N_F$. However, this
 245 statement has been verified elsewhere^[2] and it was found that $N_A + N_F$ measurements led to full
 246 observability in SSI by OM under certain conditions. This is to say, even with $N_A + N_F$
 247 measurements, full observability might not be achieved due to improper selection of the
 248 measurement set, which is related with the placement and type of measuring devices. An
 249 explanation provided formerly was that the recursive steps stopped too early without identifying
 250 all parameters. However, the underlying cause for this premature end of the recursive steps was
 251 not uncovered at that time. This is one of the major interests of this paper, which can also be
 252 employed for the sensor placement strategy.

253 Eq. (1) can be rearranged for practical purposes and the system of equations to be solved by OM
 254 can be transformed into:

$$B \cdot z = D, \quad (2)$$

255 where B and D are a matrix and a vector whose entries are all known, respectively, and z is the
 256 vector of unknowns. This is the result of applying static condensation technique together with
 257 the separation of each column, with regard to different unknowns, into several columns and the
 258 merger (addition) of those resultant columns related with the same unknowns in the equilibrium
 259 equations (Eq. (1)), derived from the direct stiffness method. After these algebraic operations,
 260 the information of measurements, is absorbed in the coefficient matrix B and in the vector D .
 261 Apart from the information of measurements, the vector D contains the information of both the
 262 external loads and the boundary conditions, which are all assumed to be known. The unknowns
 263 z can be always of two types: (1) monomials of degree one, e.g. $\{EA_j, EI_j, u_i, v_i, w_i, H_i, V_i, M_i\}$,
 264 or (2) monomials of degree two, e.g. $\{EI_j u_i, EI_j v_i, EI_j w_i, EA_j u_i, EA_j v_i\}$. Note that both, EA_j
 265 and EI_j are regarded as monomials of degree one. (For more technical details, readers are
 266 strongly recommended to review^[33,57]). The occurrence of these components of the unknown z
 267 is due to the fact that Eq. (2) is essentially established by nodal force equilibrium.

268 If the vertical deflection, v_i , or the rotation, w_i , appearing in these products, $EI_j v_i$ or $EI_j w_i$, are
 269 measured, then the flexural stiffness, EI_j , can be uncoupled and separated from those and the
 270 vertical deflection, v_i , or the rotation, w_i , will be absorbed in B . Otherwise, these products
 271 themselves will appear in z in the form of monomials with degree of two and, as per
 272 requirements of OM, be regarded as linear in z . This is to say, even though the physical
 273 unknowns for a given problem might be EI_j and w_i , z may contain three different
 274 unknowns, EI_j, w_i and $EI_j w_i$. Due to the limit of sensor investment, it is not likely to measure all
 275 displacements in the structure. As a consequence, these products, $EI_j v_i$ or $EI_j w_i$, and the
 276 flexural stiffness, EI_j , which is obtained by the uncoupling of these products, both appear in the
 277 unknowns z . Likewise, the simultaneous occurrence of $EA_j u_i$ and EA_j is ascribed to the nodal
 278 equilibrium of axial forces. Furthermore, it is worth mentioning that in frame structures, due to
 279 the coupling of the axial displacements and the vertical deflections from different members, the
 280 product of axial stiffness and vertical deflection, $EA_j v_i$, and the product of flexural stiffness and
 281 axial displacement, $EI_j u_i$, also appear in the unknowns z .

282 From a mathematical point, it is straightforward that if the unknowns of a system are coupled,
 283 then these coupled unknowns should satisfy certain constraints. Nevertheless, this requirement
 284 is not satisfied in SSI by OM and therefore sometimes leads to the failure to identify all
 285 parameters, or in other words, to achieve full observability.

286 In order to clarify this deficiency induced by the lack of constraints, SSI by OM on a 4-node
 287 simply supported beam depicted in Figure 1, is solved step by step. Meanwhile, this analysis
 288 also sheds light on the peculiarity of this method. For simplicity, it is assumed that the flexural
 289 stiffnesses of elements 1 and 3, EI_1 and EI_3 , the length of element, L , and the external vertical
 290 load at node 2, V_2 , are known. Since the axial stiffness is not activated by this load case, the
 291 areas of the elements are not considered, i.e. $N_A=0$. Correspondingly, the terms associated with
 292 axial behavior are removed from the general equation by OM. Thus, the target parameter is the
 293 flexural stiffness of the element 2 (in red in Figure 1), I_2 , i.e. $N_F=1$. Then it is anticipated that
 294 one measurement suffices to achieve full observability since $N_A + N_F = 1$. Assume that the
 295 vertical deflection of the node 2, v_2 , is measured.

296 In the first step, the system of equations given by OM is as follows:

$$B_1 \cdot z_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4EI_1}{L} & \frac{2EI_1}{L} & 0 & 0 \\ \frac{12v_2}{L^3} & -\frac{12}{L^3} & \frac{6}{L^2} & \frac{6}{L^2} & 0 & 0 & 0 & -\frac{6EI_1}{L^2} & -\frac{6EI_1}{L^2} & 0 & 0 \\ \frac{6v_2}{L^2} & -\frac{6}{L^2} & \frac{4}{L} & \frac{2}{L} & 0 & 0 & 0 & \frac{2EI_1}{L} & \frac{4EI_1}{L} & 0 & 0 \\ \frac{12v_2}{L^3} & \frac{12}{L^3} & -\frac{6}{L^2} & -\frac{6}{L^2} & 0 & 0 & \frac{12EI_3}{L^3} & 0 & 0 & \frac{6EI_3}{L^2} & \frac{6EI_3}{L^2} \\ \frac{6v_2}{L^2} & -\frac{6}{L^2} & \frac{2}{L} & \frac{4}{L} & 0 & 0 & \frac{6EI_3}{L^2} & 0 & 0 & \frac{4EI_3}{L} & \frac{2EI_3}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6EI_3}{L^2} & 0 & 0 & \frac{2EI_3}{L} & \frac{4EI_3}{L} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & \frac{6EI_1}{L^2} & \frac{6EI_1}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -\frac{12EI_3}{L^3} & 0 & 0 & -\frac{6EI_3}{L^2} & -\frac{6EI_3}{L^2} \end{pmatrix} \cdot \begin{pmatrix} EI_2 \\ EI_2v_3 \\ EI_2w_2 \\ EI_2w_3 \\ V_1 \\ V_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} M_1 + \frac{6EI_1v_2}{L^2} \\ V_2 - \frac{12EI_1v_2}{L^3} \\ M_2 + \frac{6EI_1v_2}{L^2} \\ V_3 \\ M_3 \\ M_4 \\ 0 \\ \frac{12EI_1v_2}{L^3} \\ 0 \end{pmatrix} = D_1 \quad (3)$$

297 In Eq. (3), it is seen that the vertical deflection v_2 , flexural stiffnesses EI_1 and EI_3 are absorbed
 298 in both the matrix B_1 and the vector D_1 . The solution of this system can be expressed as the sum
 299 of a particular solution z_{p1} and any linear combination of the bases in the null space of B_1 , as
 300 presented in Eq. (4). This particular solution as well as the bases of the vectorial space of any
 301 possible mathematical solution can be obtained by many commercial packages, e.g. Matlab^[58]
 302 and Maple^[59]. For this structure, it can be seen that, in the null space of the matrix B_1 , i.e. V_{n1} ,
 303 the element of the rows associated with $\{V_1, V_4, w_1, w_4\}$, given in bold, are all null. Hence the
 304 values of them are not affected by the coefficients $\rho_{1,1}, \rho_{1,2}$ and $\rho_{1,3}$. This is saying
 305 $\{V_1, V_4, w_1, w_4\}$ are constant, unique, known and, hence, observable. Another thing is that z_1
 306 contains $\{EI_2, v_3, w_2, w_3, EI_2v_3, EI_2w_2, EI_2w_3\}$ and these unknowns should satisfy certain
 307 constraints, e.g. $\{EI_2 \cdot v_3 = EI_2v_3\}$. This gives rise to the nonlinearity of this problem.
 308 However, as linearity is assumed, these constraints are not considered in SSI by OM yet, which
 309 leads to the failure of the identification of EI_2 .

$$z_1 = z_{p1} + V_{n1} \cdot \rho_1 = \begin{pmatrix} EI_2 \\ EI_2v_3 \\ EI_2w_2 \\ EI_2w_3 \\ V_1 \\ V_4 \\ v_3 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} \frac{1}{v_2} & -\frac{L}{v_2} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & -L \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho_{1,1} \\ \rho_{1,2} \\ \rho_{1,3} \end{pmatrix} \quad (4)$$

310 In the next recursive step, the unknowns observed previously are incorporated into the input of
 311 SSI by OM. This is, the up-to-date measurement set for the current recursive step is $\{w_1, w_2$ and
 312 $v_2\}$ and the reactions, V_1 and V_4 , are also regarded as known. Note that a renewed input will
 313 update Eq. (2) and thus new parameters might be observed. The updated system of equations
 314 obtained in the next recursive step is as follows:

$$B_2 \cdot z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{12v_2}{L^3} + \frac{6w_2}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} & 0 & 0 & 0 \\ \frac{6v_2}{L^2} + \frac{4w_2}{L} & -\frac{6}{L^2} & \frac{2}{L} & 0 & 0 & 0 \\ \frac{12v_2}{L^3} - \frac{6w_2}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} & \frac{12EI_3}{L^3} & \frac{6EI_3}{L^2} & \frac{6EI_3}{L^2} \\ \frac{6v_2}{L^2} + \frac{2w_2}{L} & -\frac{6}{L^2} & \frac{4}{L} & \frac{6EI_3}{L^2} & \frac{4EI_3}{L} & \frac{2EI_3}{L} \\ 0 & 0 & 0 & \frac{6EI_3}{L^2} & \frac{2EI_3}{L} & \frac{6EI_3}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{12EI_3}{L^3} & -\frac{6EI_3}{L^2} & -\frac{6EI_3}{L^2} \end{pmatrix} \cdot \begin{pmatrix} EI_2 \\ EI_2v_3 \\ EI_2w_3 \\ v_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} M_1 + \frac{6EI_1v_2}{L^2} - \frac{4EI_1w_1}{L} - \frac{2EI_1w_2}{L} \\ V_2 - \frac{12EI_1v_2}{L^3} + \frac{6EI_1w_1}{L^2} + \frac{6EI_1w_2}{L^2} \\ M_2 + \frac{6EI_1v_2}{L^2} - \frac{2EI_1w_1}{L} - \frac{4EI_1w_2}{L} \\ V_3 \\ M_3 \\ M_4 \\ V_1 + \frac{12EI_1v_2}{L^3} - \frac{6EI_1w_1}{L^2} - \frac{6EI_1w_2}{L^2} \end{pmatrix} = D_2 \quad (5)$$

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2
3 315 In Eq. (5), it can be seen that the unknowns observed from the previous recursive step are
4 316 absorbed in both B_2 and D_2 . The null space, V_{n2} , of the updated matrix B_2 and the general
5 317 solution of Eq. (5), z_2 , are given as:

$$z_2 = z_{p2} + V_{n2} \cdot \rho_2 = \begin{pmatrix} EI_2 \\ EI_2 v_3 \\ EI_2 w_3 \\ v_3 \\ w_3 \\ w_4 \end{pmatrix}_{\text{particular}} + \begin{pmatrix} \frac{1}{w_2} & 0 \\ \frac{v_2 + w_2 L}{w_2} & 0 \\ 1 & 0 \\ 0 & -L \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \rho_{2,1} \\ \rho_{2,2} \end{pmatrix} \quad (6)$$

13 318 Again, z_2 can be expressed as the sum of a particular solution, z_{p2} , and any linear combination
14 319 of the two bases in the null space, V_{n2} , in which the coefficients are given as $\rho_{2,1}$ and $\rho_{2,2}$. In
15 320 the null space V_{n2} , as given in Eq. (6) no variable is observable since no null row exists. In other
16 321 words, the recursive steps end as per the criterion of null row in the null space. However, the
17 322 main reason of the premature end of the recursive steps is that the constraints, i.e. $EI_2 v_3 = EI_2 \cdot$
18 323 v_3 and $EI_2 w_3 = EI_2 \cdot w_3$, are not incorporated into the process. This is illustrated in a geometric
19 324 way in Figure 2.a. In this figure, the abscissa in ρ_1 -axis and the ordinate in ρ_2 -axis, indicate the
20 325 values of the two coefficients $\rho_{2,1}$ and $\rho_{2,2}$ in Eq. (6).

21 326 With a given set of $\rho_{2,1}$ and $\rho_{2,2}$, the value of each unknown are specified by Eq. (6) and can be
22 327 represented by dots in the space in Figure 2.a. To present the value of all unknowns in the same
23 328 range, these values are normalized by the accurate values obtained from SAP 2000^[60]. The
24 329 solution of this problem is represented as the point in Figure 2.a, where the values of all
25 330 normalized unknowns are one. If infinite sets of ρ_1 and ρ_2 are provided, all the possible
26 331 normalized values obtained by these sets will yield six planes. These planes are shown in Figure
27 332 2.a. For a given set, the equation $\{\rho_1 = \rho_{2,1}, \rho_2 = \rho_{2,2}\}$ specifies a vertical line where z can take
28 333 any value and the intersections of this vertical line with the six planes indicate a specific
29 334 solution of Eq. (5). For illustration, when $\rho_{2,1} = -22.00, \rho_{2,2} = 4.20 \times 10^{-4}$, as indicated by
30 335 the vertical line in Figure 2.a, it can be seen that the 6 solutions are deviated from the solution to
31 336 the problem. When the parameters are chosen as $\rho_{2,1} = -22.22, \rho_{2,2} = 4.12 \times 10^{-4}$, the six
32 337 intersections of the vertical line and the six planes will occur at the solution of the problem. To
33 338 clarify this, the variations of the normalized solution against $\rho_{2,1}$ with fixed $\rho_{2,2}$ of $4.12 \times$
34 339 10^{-4} and against $\rho_{2,2}$ with fixed $\rho_{2,1}$ of -22.22 , respectively, are plotted in Figure 2.b and 2.c.
35 340 Evidently, the solution of the problem comes from a particular solution in the general solution.
36 341 However, SSI by OM is incapable of detecting this solution in the general solution. The reason
37 342 is that observability treats the coupled unknowns, $\{EI_2 v_3, EI_2 w_3\}$, as independent of the
38 343 corresponding component variables, $\{EI_2, v_3, w_3\}$ though they should satisfy the constraints of
39 344 $\{EI_2 v_3 = EI_2 \cdot v_3, EI_2 w_3 = EI_2 \cdot w_3\}$. In Figure 2.b and 2.c, these constraints are gradually
40 345 satisfied by adjusting $\rho_{2,1}$ and $\rho_{2,2}$.

41 346 Consider the FEM of a one-story, one-bay frame depicted in Figure 3. In this frame, each
42 347 column and each beam are divided into two elements. The end nodes of the columns are
43 348 clamped. The possible measurements within this structure include 5 horizontal deflections, 5
44 349 vertical deflections and 5 rotations. For simplicity, it is assumed that the areas of the columns
45 350 and the beam are known, and the flexural inertias of the columns are identical. Hence, the two
46 351 unknowns here are the flexural stiffnesses of the columns and the beam. Due to the requirement
47 352 of essential set, two measurements should be used and sufficient to identify the two unknown
48 353 flexural stiffnesses. However, it is found that measuring any two displacements out of the 15

354 possible measurements, which is $C_{15}^2 = 105$ possible combinations, cannot ensure the
355 identifiability of the two parameters. Half of the possible measurement sets can only identify
356 one flexural stiffnesses and the other half cannot identify any parameter. This means that no
357 essential set exists in this structure. A closer inspection of the general solution for this structure
358 shows that the constraints between the variables are also missing.

359 From the simply-supported beam example, an explanation for the observability method being
360 incapable of ensuring identifiability is provided algebraically. The algebraic analysis shows that
361 the nonlinear constraints between the unknowns are neglected in the traditional OM. In the
362 graphical illustration of the general solution of Eq. (6), it is shown that the exact solution is
363 obtained when the constraints are satisfied. Moreover, an example of a frame is given to clarify
364 that essential set may not exist in some structures due to the same reason.

365 In order to get more knowledge of this phenomenon, an exhaustive examination of observability
366 of the structural parameters is carried out in a slightly more complicated structure depicted in
367 [Figure 4.a](#). This is a 15 m simply supported beam with 5 evenly divided elements. A vertical
368 concentrated force is applied downwards at node 3 with a magnitude of 100 kN. The flexural
369 stiffnesses of the five elements are deliberately chosen as distinct values of $1.5 \times 10^6 \text{ kN} \cdot \text{m}^2$,
370 $1.2 \times 10^6 \text{ kN} \cdot \text{m}^2$, $1.1 \times 10^6 \text{ kN} \cdot \text{m}^2$, $1.4 \times 10^6 \text{ kN} \cdot \text{m}^2$ and $1 \times 10^6 \text{ kN} \cdot \text{m}^2$. In this
371 structure, potential measurements include six rotations, $w_1 \sim w_6$, and four vertical deflections,
372 $v_2 \sim v_5$, which are ten in total. All the exact displacements are calculated in SAP 2000 and used
373 as the input in SSI by OM. In the identification problem, the five flexural stiffnesses, from EI_1
374 to EI_5 , are assumed as unknown, i.e. $N_F=5$, whereas the axial stiffnesses are disregarded due to
375 the load case, i.e. $N_A=0$, as assumed before. Therefore, 5 measurements will suffice the
376 requirement of the essential set on the number of measurements.

377 In this structure, the full observability is defined as the ability to identify the 5 unknown flexural
378 stiffnesses. Prior to further discussion, the peculiarity of SSI by OM is qualitatively illustrated
379 by an observability analysis performed on an empirically chosen essential measurement set. In
380 the authors' experience, one straightforward essential set for this structure is measuring all the
381 vertical deflections plus one rotation. For instance, measuring the four vertical deflections
382 $v_2 \sim v_5$ and the rotation w_1 . The observability flow (the identification sequence of the unknowns)
383 for this set is provided in [Figure 4.b](#). In this figure, the measurements are enclosed by solid
384 boxes whereas the estimates are enclosed by dashed boxes. It can be seen that the whole
385 structure is identified through 4 recursive steps. In recursive step 1, the flexural stiffness of
386 element 1, EI_1 , and rotation of node 2, w_2 are identified first. Meanwhile, the vertical
387 reactions, V_1, V_5 , are also obtained directly since this is a statically determinate structure.
388 Furthermore, the estimation of the remaining rotation of element 1, w_2 , is used to update the
389 original measurement set and to initiate the subsequent recursive identification process. In
390 recursive step 2, EI_2 and w_3 can be obtained by observability with two measurements, v_2 and
391 v_3 , and the estimated rotation of node 2, w_2 . Likewise, $\{EI_3, w_4\}$, $\{EI_4, w_5\}$ and $\{EI_5, w_6\}$ are
392 observed in pairs consecutively in recursive steps 3, 4 and 5. Thus, the 5 inertias are all
393 identified progressively through recursive step 1 to recursive step 5.

394 In [Figure 4.b](#), the estimation obtained in each step is given in dashed box. It should be noted that
395 the identification of each flexural stiffness is accompanied by the identification of a
396 displacement. The identification of the flexural stiffness and associated displacement is based
397 on the three DOFs of that element, which might be three measurements like step 1, or two
398 measurements plus one estimation of displacement like steps 2~5. That is to say, the

399 information of the remaining DOF, which is referred as the estimated displacement here, is
 400 somehow contained in the other three DOFs. Thus, redundancies might arise in the
 401 measurement set if the information of the four displacements of the same element is provided by
 402 real-life measurements and/or those estimated displacements. This will be exemplified and
 403 clarified in the following paragraph.

404 In the essential set proposed in [Figure 4.b](#), the redundancy of the measurements is avoided by
 405 empirical (artificial) selection of the measurement set. To check the possibility of detecting
 406 essential set with no special attention paid to avoid redundancy, an exhaustive investigation of
 407 all potential essential sets, which should have 5 measurements, is carried out by SSI by OM. As
 408 mentioned before, potential measurements include the four vertical deflections, $v_2 \sim v_5$, and six
 409 rotations, $w_1 \sim w_6$. All the essential sets should be composed of 5 out of these 10 potential
 410 measurements, resulting in $C_{10}^5 = 252$ possibilities. As clarified by the first example, the same
 411 number of measurements as the number of unknowns does not necessarily mean that the
 412 structure is fully observable. Bearing this in mind, it is interesting to know how many sets out of
 413 these 252 sets can achieve full observability. The same procedure presented in the first example
 414 is carried out on all of these 252 enumerated measurement sets. The number of identified
 415 flexural stiffnesses by OM, $N_{F,OM}$, for each set are collected. As there are five unknown flexural
 416 stiffnesses, $N_{F,OM}$ might take the value of 1,2,3,4 and 5. The frequency of the occurrence of
 417 each case is given as the histogram in [Figure 5](#).

418 It can be seen that the occurrence of full observability, $N_{F,OM} = 5$, occupies 25.4% of the 252
 419 sets whereas the occurrence of partial observability, $N_{F,OM} \leq 4$, occupies 74.6% of the 252 sets.
 420 To distinguish among the sets achieving different levels of observability, they are classified into
 421 different patterns with regard to the physical location of the measurements. This can be carried
 422 out by taking the subscript of the measurement, i.e. the node number, which indicates the
 423 location of the measurement. For instance, assume three measurement sets: (1)
 424 $\{w_1, v_2, w_2, v_3, w_3\}$; (2) $\{w_1, v_2, v_3, v_4, w_4\}$; (3) $\{w_1, w_2, w_3, v_4, w_4\}$. These three sets are,
 425 respectively, classified as $\{1,2,2,3,3\}$, $\{1,2,3,4,4\}$ and $\{1,2,3,4,4\}$. From the last 2 sets, it can be
 426 seen that no distinction is made between measuring the rotation and measuring the deflection of
 427 a node. If the same number shows twice in a pattern, then it is saying that both the deflection and
 428 the rotation of the node associated with that number are measured. Since the vertical
 429 deflection of node 1 and node 6 are null (boundary conditions), v_1 and v_6 will be added to each
 430 pattern automatically. To understand the relationship between the location of the measurement
 431 and the number of identified parameters, the most representative patterns are listed in [Table 1](#).

432 All the patterns related with $N_{F,OM} = 5$, i.e. full observability, are listed in the first column of
 433 [Table 1](#). Note that the empirically chosen measurement set adopted to illustrate the observability
 434 flow belongs to the first pattern, $\{1,1,2,3,4,5,6\}$. As indicated by the indices of these patterns,
 435 the measurements yielding full observability are taken at physically dispersed locations. These
 436 patterns, or the geometrically distributed placement of sensors, maintain the observability flow
 437 and thus the full observability is achieved.

438 In comparison, if the measurements are taken in an intensive way at a local area, redundancy in
 439 the measurement set will emerge. For instance, assume the measurement set of
 440 $\{v_3, w_3, v_4, w_4, w_6\}$, which is the first pattern in the column of $N_{F,OM} = 1$ in [Table 1](#). This set
 441 measures all the displacements of element 3 and thus the identification of EI_3 is as expected.
 442 Nevertheless, due to the improper selection of measurements, the observability flow is

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3 443 terminated. In SSI by OM, the recursive process is maintained by continuously providing
4 444 information of the displacements of the nodes which are adjacent to the previously identified
5 445 elements. To identify the flexural stiffness of the elements next to the element 3, e.g. EI_2 , apart
6 446 from the known v_3 and w_3 , at least one measurement or the estimation of v_2 or w_2 is required.
7 447 Due to the violation of this requirement, the identification of EI_2 is not executed. Similarly, the
8 448 identification of EI_4 is not executed due to the lack of any information of v_5 or w_5 . The
9 449 identification of EI_5 , which should follow the identification of EI_4 in this set, is not executed
10 450 either. This renders the available information of w_6 and v_6 unutilized and ineffective. By
11 451 analogy, the information of v_1 is also ineffective due to the failure in the identification of EI_1 .
12 452 From the previous analysis, it can be seen that the effectiveness of a measurement is defined in
13 453 the context of a particular measurement set and might be different for another set. These
14 454 unutilized measurements are referred as ineffective measurements and all of them are given in
15 455 bold in Table 1. From this measurement set, it can be deduced that one reason for partial
16 456 observability is that the number of effective measurements is less than the number of unknowns.

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20 457 However, in the last 2 patterns in the column of $N_{F,OM} = 3$, all the measurements are involved
21 458 in the identification but they still lead to partial observability. In fact, the defect of these patterns
22 459 is not as intuitive as that revealed by ineffective measurements. For the measurement set of
23 460 $\{v_1, w_1, v_2, v_3, v_5, v_6, w_6\}$, which corresponds with the 5th pattern of $N_{F,OM} = 3$, the
24 461 identifications of the pairs of $\{EI_1, w_2\}$ and $\{EI_5, w_5\}$ are, respectively, enabled by the
25 462 information of $\{v_1, w_1, v_2\}$ and $\{v_5, v_6, w_6\}$. Likewise, the identification of I_2 and w_5 is enabled
26 463 by the information of $\{v_2, v_3, w_2\}$. Hence, the up-to-date information of the measurements is
27 464 composed of $\{v_1, w_1, v_2, w_2, v_3, w_3, v_5, w_5, v_6$ and $w_6\}$. However, the lack of any information of
28 465 v_4 or w_4 terminates the observability flow, and thereby fails the identification of both EI_3 and
29 466 EI_4 , despite of the available information of v_3, w_3, v_5, w_5 . Truthfully, this cannot be resolved
30 467 without introducing any new information in SSI by OM.

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34 468 With respect to $N_{F,OM} = 1$ and 2, all the patterns yielding partial observability can be categorized
35 469 as cases of the ineffective measurements or of the premature end of the recursive steps. It should
36 470 be noted that the case of $N_{F,OM} = 4$ does not exist. The reason is that if $N_{F,OM} = 4$, the direct
37 471 measurements and those estimated ones which come together with the identifications of these
38 472 four inertias will certainly yield the full observability.

42 473 Section 3: SSI by COM

43 474 From the two examples in section 2, it is found that the nonlinear constraints among the
44 475 unknowns is lacking, and the two reasons for the partial observability are: (1) the premature end
45 476 of the recursive steps and (2) the ineffective measurements due to redundancy in the
46 477 measurement sets. It is expected that the supplement of the nonlinear constraints to SSI by OM
47 478 might improve the performance of the original method in dealing with partial observability.
48 479 Therefore, SSI by COM, which incorporates the nonlinear constraints through an optimization
49 480 routine with SSI by OM, is proposed with the aim of exploiting the information contained in the
50 481 measurements to the maximum extent. The following points have to be taken into account when
51 482 implementing the optimization:

52 483 3.1 The condition when the optimization should be applied

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55 484 Whenever is possible, appropriate measurement sets have to be chosen in order to avoid
56 485 optimization. It is less desirable to employ COM than just OM due to the fact that the

486 computation cost is normally higher for the first option. When optimization is required, SSI by
 487 OM will be carried out first and then the general equation from the last step of SSI by OM will
 488 be used for COM.

489 3.2 Decoupling of the unknowns and generating the new unknown

490 As mentioned before, the unknowns in Eq. (2) can always be of two types: (1) monomials of
 491 degree one, or single unknowns, V_s , and (2) monomials of degree two, or coupled unknowns, V_c .
 492 The single unknowns, V_s , contains flexural stiffnesses, EI_j , axial stiffnesses, EA_j , or the
 493 displacements, u_i , v_i or w_i . The unknowns V_c are coupled with some of V_s and have to be
 494 uncoupled in two steps. In first step, the existent single variables, V_{s1} , in z are picked out first.
 495 Subsequently, V_c are decomposed into two single unknowns, and then a comparison between
 496 these single unknowns and the existing V_{s1} is carried out. If there exists any new single
 497 unknown, which is always the case, these new variables will be designated as V_{s2} . These new
 498 single unknowns are added to the unknowns z to form the new unknowns z^* , i.e. $z^* = z \cup V_{s2}$.
 499 For instance, in the case of $z = \{EI_2w_2, EI_2w_3, EI_2v_3, EI_2\}$, coupled unknowns V_c include
 500 EI_2w_2, EI_2w_3 and EI_2v_3 . The components of EI_2w_2, EI_2w_3 and EI_2v_3 are EI_2, w_2, w_3 and v_3 .
 501 Since EI_2 exists in z , it is categorized as V_{s1} whereas w_2, w_3 and v_3 are categorized as V_{s2} . The
 502 new unknown z^* is $\{EI_2w_2, EI_2w_3, EI_2v_3, EI_2, w_2, w_3$ and $v_3\}$.

503 3.3 Rearrangement of the equation $B \cdot z = D$

504 To consider the additional unknowns V_{s2} in z^* , an adaption on the coefficient matrix B of Eq. (2)
 505 is made. A null matrix, Ω , is added to the matrix B and thereby V_{s2} is included in z^* without
 506 violating the equation. In addition, during the optimization, it is inevitable to have residuals in
 507 these equations. Thus, a more realistic form of the system of equations is given as:

$$\epsilon = B^* \cdot z^* - D \quad (7)$$

508 where $B^* = [B^{N_{eq} \times N_z} \mid \Omega^{N_{eq} \times N_{s2}}]$, $z^* = \begin{Bmatrix} z^{N_z \times 1} \\ V_{s2}^{N_{s2} \times 1} \end{Bmatrix}$, ϵ is a residual vector with a dimension of
 509 N_{eq} times 1. Ω is a null matrix with the dimension of N_{eq} times N_{s2} . The matrix B, and vectors z
 510 and D are directly obtained from the system of equations in the last step of SSI by OM. Here,
 511 N_{eq} and N_z , respectively, denotes the number of equations and the number of unknowns in this
 512 system. N_{s2} denotes the number of new single unknowns in V_{s2} .

513 3.4 Constraints

514 When the decoupling of V_c is accomplished, the constraints between V_c and relevant V_s are
 515 applied by imposing the equality between the value of V_c and the product of the values of
 516 corresponding V_s in the optimization. For the same example presented in 3.2, the matrix form of
 517 the corresponding constraints, i.e. $EI_2w_2 = EI_2 \cdot w_2$, $EI_2w_3 = EI_2 \cdot w_3$ and $EI_2v_3 = EI_2 \cdot v_3$, is
 518 given in Eq. (8).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} EI_2w_2 \\ EI_2w_3 \\ EI_2v_3 \\ EI_2 \\ w_2 \\ w_3 \\ v_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{pmatrix} EI_2w_2 \\ EI_2w_3 \\ EI_2v_3 \\ EI_2 \\ w_2 \\ w_3 \\ v_3 \end{pmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} EI_2w_2 \\ EI_2w_3 \\ EI_2v_3 \\ EI_2 \\ w_2 \\ w_3 \\ v_3 \end{pmatrix} \quad (8)$$

519 3.5 Objective function

520 The objective function is defined as minimizing the square sum of the residuals in Eq. (7),
521 which is expressed as:

$$\min \left(\sum_{k=1}^{N_{eq}} \epsilon_k^2 \right) \quad (9)$$

522 ϵ_k is the k^{th} component of the residual vector ϵ in Eq. (7).

523 3.6 Other remarks

524 In the optimization, the active-set algorithm^[61] is adopted to find the optimal solution to satisfy
525 Eq. (9) and the optimization toolbox of the commercial package Matlab^[58] is used to implement
526 this program. For the sake of better convergence and computational efficiency, as indicated by
527 Eq. (10), the element in row i and column k of the coefficient matrix B^* in Eq. (7), $B_{i,k}^*$, will be
528 multiplied by the numerical value of the k^{th} unknown obtained from direct stiffness method, \bar{z}_k^* .

$$\bar{\bar{B}}_{i,k}^* = B_{i,k}^* \cdot \bar{z}_k^*, \quad \forall i, k = 1, 2, \dots, (N_z + N_{s2}) \quad (10)$$

529 The vector D in Eq. (7) will remain the same. Consequently, in Eq. (11), the corresponding
530 value of the solution minimizing the objective function, will be normalized by \bar{z}_k^* , and thus the
531 numerical value of the normalized unknowns, $\bar{\bar{z}}_k^*$, are supposed to be around 1. However, in case
532 of some very small or zero values of \bar{z}_k^* , which might lead to instability of the solution, a
533 threshold z_{th} of 10^{-6} is chosen for $\bar{\bar{z}}_k^*$. If the value of $\bar{\bar{z}}_k^*$ is larger than the selected threshold,
534 the original value will be used directly in Eq. (10); otherwise, the threshold will be used in Eq.
535 (10).

$$\bar{\bar{z}}_k^* = z_k^* / \bar{z}_k^*, \quad k = 1, 2, \dots, (N_z + N_{s2}) \quad (11)$$

$$\bar{\bar{z}}_k^* = \begin{cases} z_k^* & \text{if } \bar{z}_k^* \geq z_{th} \\ z_{th} & \text{if } \bar{z}_k^* < z_{th} \end{cases} \quad (12)$$

536 For this reason, an intuitive guess for $\bar{\bar{z}}_k^*$ of all ones will be used to initialize the optimization. A
537 concurrent advantage of this guess is that the nonlinear constraints are satisfied automatically
538 since the product of ones is always one. It is worth well doing so due to the fact that initial
539 values satisfying or closely satisfying many of the constraints reduce the work involved in
540 finding a first feasible solution. Nevertheless, regardless of the initial values recommended here,
541 the same solution might be attained at a higher cost of time and computation capacity if this
542 normalization is not carried out, which is verified later. In addition, the lower bound and the
543 upper bound for each element in $\bar{\bar{z}}_k^*$, respectively, are set as 0.1 and 10.

544 3.5 Proposed algorithm

545 This proposed method combines SSI by OM with optimization and thereby includes the
546 nonlinear constraints in the identification process. The algorithm for SSI by COM is
547 summarized as follows:

548 **Step1: Apply SSI by OM and check whether the full observability is achieved or not.**
549 Initiate SSI by OM with the given measurement set. Form the general equation (Eq. (2)), and

550 check the null space of coefficient matrix B to see if any parameters are observed. If so, update
 551 the input by incorporating observed variables and reinitiate the previous procedure until the end
 552 of the recursive process, i.e. no new variable is observable. If not, the OM is ended without
 553 estimating any unknown. If full observability is achieved at the end of the recursive process, i.e.
 554 the structure is fully observable, it is not necessary to perform optimization; otherwise, go to
 555 step 2. It is highlighted that several recursive steps can be done in step 1 until no further
 556 unknowns are observable by OM;

557 **Step2: Obtain the equation $B \cdot z = D$** ^[33,57]. Extract the updated general equation $B \cdot z = D$
 558 from the last step of SSI by OM.

559 **Step3: Analyze the unknowns z and generate the new unknowns z^*** . Divide z into V_c and V_{s1} .
 560 Then compare every component of the coupled unknowns V_c with the existent single unknowns
 561 V_{s1} and collect the single unknowns V_{s2} which were not present in V_{s1} . Generate the new
 562 unknowns z^* by adding the former unknowns z and the additional single unknowns V_{s2} .

563 **Step4: Form the new matrix B^*** . Analyze the dimension of matrix B and append the null
 564 matrix Ω to contain V_{s2} in the z^* without violating the equations.

565 **Step5: Obtain the normalized unknown \bar{z}^*** . Multiply the column of the matrix B^* with the
 566 expected value \bar{z}^* determined by Eq. (12) so as to obtain the \bar{B}^* and the normalized unknown \bar{z}^* .

567 **Step6: Store the constraint information**. Check every element in \bar{z}^* and build the matrix that
 568 relates the nonlinear relationship between V_c and V_s .

569 **Step7: Optimization**. Choose the initial values of \bar{z}^* and set the bounds for the solution. The
 570 objective is to minimize the square sum of the residual vector, ϵ . In the optimization process, the
 571 nonlinear constraints are imposed by ensuring the equality between the coupled unknowns V_c
 572 and the product of corresponding single unknowns V_s .

573 A summary of the procedure is shown in the flow chart in Figure 6.

574 3.6. Application of the algorithm

575 First, both structure depicted in Figure 1 and Figure 3 are re-analyzed by COM. As expected,
 576 after the application of COM, the first example becomes identifiable with the given
 577 measurement set. Also, 45 out of the 105 measurement sets becomes capable of identifying all
 578 parameters for the frame. However, these structures are far from proving the strength and
 579 robustness of COM. In order to do this, the observability of the structure depicted in Figure 4.a
 580 is re-analyzed by the constrained observability method.

581 When COM is applied in this structure, the influence of the normalization and the choice of
 582 initial values on the accuracy of the result of the optimization are checked here. Note that the
 583 flexural stiffnesses of the five elements are deliberately chosen as distinct values of $\{1.5 \times$
 584 $10^6 \text{ kN} \cdot \text{m}^2, 1.2 \times 10^6 \text{ kN} \cdot \text{m}^2, 1.1 \times 10^6 \text{ kN} \cdot \text{m}^2, 1.4 \times 10^6 \text{ kN} \cdot \text{m}^2 \text{ and } 1 \times 10^6 \text{ kN} \cdot \text{m}^2\}$.
 585 Instead of normalizing the columns of the matrix B with the real inertias, these columns are
 586 normalized by $1 \times 10^6 \text{ KN} \cdot \text{m}^2$. Therefore, the normalized estimate of these inertias should be
 587 $[1.5, 1.2, 1.1, 1.4, 1]$. Meanwhile, instead of using all ones as the initial values, random numbers
 588 generated by uniform distribution on $[0.8, 1.2]$ are used. It should be noted that some of the
 589 normalized estimates do not lie in the sampling interval of this distribution. According to the

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3 590 result, the optimization still converges to the same solution as that obtained by using the
4 591 recommended normalization and initial values but with more iterations in the computation. In
5 592 the following comparison, the normalization factor and the initial values are taken as
6 593 recommended. The frequency of occurrence of the number of observed flexural stiffnesses by
7 594 COM, $N_{F,COM}$, equal to 1 to 5 is presented in Figure 7.a.

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10 595 Comparing the results presented in Figure 5 and Figure 7.a, a drastic increase, in the number of
11 596 the measurement sets yielding full observability, from 64 for SSI by OM to 162 for SSI by
12 597 COM is seen. In fact, all of the previous 64 sets by OM are contained in the new 162 sets by
13 598 COM due to the equivalence of the step 1 (recursive process) in SSI by COM and SSI by OM.
14 599 In other words, the improvement of the observability level occurs in the remaining 188 (=252-
15 600 64) sets. The numbers of observed flexural stiffness by OM, $N_{F,OM}$, and by COM, $N_{F,COM}$, are
16 601 plotted in Figure 7.b. For better visualization, the measurement sets are sorted so as to cluster
17 602 those sets with the same value of $N_{F,OM}$ in an ascending order. In this figure, the number of
18 603 identified flexural stiffnesses, N_F , for different measurement sets by COM and OM are,
19 604 respectively, represented by circles and points. The abscissa of the markers is the numbering of
20 605 the sets whereas the ordinate is the number of identified flexural stiffnesses. If an increase in the
21 606 number of identified parameters is attained by COM, then a position of the circle higher than the
22 607 one of the dot for that measurement set should be expected. It can be seen that the majority of
23 608 the sets yielding $N_{F,OM}=3$ by OM, alter to yield $N_{F,COM}=5$ by COM. With regard to these
24 609 measurement sets, it is found that the discontinuity of the observability flow can be overcome
25 610 by applying the nonlinear constraint. For instance, in the final step of the observability analysis
26 611 on the measurement set $\{v_1, w_1, v_2, v_3, v_5, v_6, w_6\}$, which was used previously, the up-to-date
27 612 information, apart from the original measurements, contains the estimated displacements
28 613 $\{w_2, w_3, w_5\}$ and flexural stiffnesses $\{EI_1, EI_2, EI_5\}$. Due to the lack of the information of w_4 or
29 614 v_4, EI_3 and EI_4 could not be identified despite the information of $\{v_3, w_3, v_5, w_5\}$. Nevertheless,
30 615 with the help of the nonlinear constraint, the structure can always be solved in this case. An
31 616 explanation for this is that the compatibility requirement of the displacements at the mutual
32 617 node of these two adjacent elements, i.e. the displacements of the node 4, and the imposition of
33 618 the nonlinear constraints force the solution. This deduction can be justified by another case.
34 619 From the result of the set $\{v_1, w_1, v_3, w_3, v_5, w_6, v_6\}$ by OM, a poor performance, $N_{F,OM}=1$, of
35 620 this set occurs since merely the identification of $\{w_5, EI_5\}$ is attainable. It should be noticed that
36 621 the condition of the previous deduction is satisfied here for both pairs of elements 1,2 and
37 622 elements 3,4 since all the displacements of nodes 1,3,5 are known. As expected, this set yields
38 623 the full observability by COM, which further verifies the deduction. In effect, each
39 624 measurement set shifting from partial observability towards full observability, due to the
40 625 adoption of COM, has been checked in an exhaustive way. It is found that the observability
41 626 flow of each of these sets are maintained by the same mechanism. Moreover, for the same
42 627 reason, three measurement sets, though not yielding the full observability, identify more inertias,
43 628 switching from the case of $N_{F,OM}=1$ to the case of $N_{F,COM}=3$.

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51 629 Generally, measurements can be regarded as an approach to get information of the static
52 630 response of the structure, i.e. displacements. In the context of essential measurements, if some
53 631 of the sensors are placed too intensively at a local area, then the other sensors will be placed
54 632 sparsely in the remaining areas. This leads to the loss of information of those areas, or leads to
55 633 the partial observability. Bearing this in mind, a reexamination has been performed on those
56 634 patterns where no improvement is obtained with COM. For instance, if the information obtained
57 635 by measurement is considered as a geometric constraint on the deflection shape of the structure,

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3 636 the estimation of the parameters EI_1 , EI_2 and EI_3 is not fixed in the pattern $\{1,1,4,4,5,6,6\}$. That
4 637 is, a relation can be established among them but it is impossible to get a unique estimate of these
5 638 parameters.

7 639 From the comparison of the results of observability analyses obtained by OM and by COM, it
8 640 can be seen that COM, as an extended version of OM, enhances the performance of the original
9 641 method. The number of essential sets soars from 64 to 162. Nevertheless, if the sensors are
10 642 placed too intensively, there still exists possibility of partial observability. Hence, distributed
11 643 placement of sensors is strongly recommended.

14 644 In the preliminary stage of condition assessment of any structure, the location of the possible
15 645 measurements can be determined according to the accessibility of the location. The final
16 646 location of the measurements will be a specific combination of some possible measurements.
17 647 Prior to instrumentation, the capability of a given measurement set to identify all parameters in
18 648 the structural system will be examined using the COM. Among all possible measurements sets,
19 649 the ones more likely to produce full observability are the ones that include at least one
20 650 measurement for every mechanical property at the location of the element with the unknown
21 651 parameter to be identified.

24 652 Section 4: Application in a building structure

26 653 To test the performance of the proposed method, a large structure^[33] previously analyzed by OM
27 654 is re-analyzed by COM. This structure is a 13-storey frame with a height of 39 m and a width of
28 655 32 m. This structure is modelled by beam elements with three DOFs per node in SAP2000.
29 656 Eight different sections (from I to VIII) of different inertias and different areas (see Table 2) are
30 657 used to model the structure and Young's modulus is assumed as 35000 MPa. The geometry of
31 658 the structure and the load case is described in Figure 8. Also, element properties are indicated by
32 659 Roman numerals in this figure. However, in the following analysis, each of these 16 mechanical
33 660 parameters is perturbed by random numbers. Apart from the choice of the values of the
34 661 parameters, another major difference between the previous analysis and the current one lies in
35 662 the number of measurements. At that time, 156 vertical deflections were measured. In the
36 663 current analysis, the number of measurements are chosen the same as the number of unknowns.
37 664 Also, the suggestion of distributed placement of sensors is followed here and the locations of
38 665 these sensors are indicated by crosses in Figure 8. To illustrate the robustness of COM, 4 sets of
39 666 the 16 mechanical parameters are synthesized by the product of the intact values and random
40 667 numbers evenly distributed on $[0.8,1.2]$, referred as perturbation factors later. The nodal
41 668 displacements calculated by SAP2000 using these 4 generated parameter sets are used as the
42 669 input of SSI by COM. Initial values of all ones are used here. Regarding the results from COM,
43 670 the estimates of axial stiffnesses (flexural stiffnesses) are normalized by the product of Young's
44 671 modulus and the intact areas (inertias). In other words, an accurate estimation is characterized
45 672 by the closeness between these normalized estimates and the perturbation factors. The
46 673 normalized estimates and the perturbation factors for the axial stiffnesses and flexural
47 674 stiffnesses are given in Figure 9.a and 9.b. In these two figures, it can be seen that the estimates
48 675 are close to the perturbation factors. Their ratios are also provided in Figure 9.c. In this figure,
49 676 for each of these parameters, the deviation of the ratio between the estimate and the true value is
50 677 within 2 percent, which is acceptable. It should be also noted that by OM, no inertia or area can
51 678 be identified whereas all these parameters are obtainable by COM.

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3 679 This example shows the applicability of SSI by COM presented in this paper and its ability to
4 680 identify the mechanical parameters of a large structure using essential sets.

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6 681 In this example, the structural system is defined by 16 structural parameters. One concern is that
7 682 this assumption will lead to an average estimate of the structural parameters, which might not
8 683 identify some localized damage appropriately. However, the modelling assumption that the
9 684 flexural stiffness and the axial stiffness of each element being different in such a structure could
10 685 be a formidable task for the proposed method to solve. In order to deal with the trade-off
11 686 between the issue of the number of parameters and the computation cost, it is advisable to use
12 687 some engineering techniques, e.g. visual inspections, acoustic emission or impact method
13 688 (hammering), before making modelling assumption with the aim of reducing the unknowns in
14 689 the model.

17 18 19 690 Conclusion:

20 691 One crucial step in SSI is using the adequate number of measurements to achieve full
21 692 observability of the whole structure. From the formulation of the identification problem by
22 693 observability, the number of the measurements should be equal to the number of unknowns in
23 694 the structure. However, measurement sets with this number of measurements do not inevitably
24 695 lead to the successful identification of the whole structure. This is verified by a simply
25 696 supported beam with only one unknown. By the analysis of this structure, the lack of nonlinear
26 697 constraints, which is essentially induced by the linearity of the standard OM, is presented for the
27 698 first time.

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30 699 To further understand the mechanism, a thorough examination of the observability of structural
31 700 parameters in a simply supported beam with more unknowns is carried out with respect to 252
32 701 enumerated measurement sets. Before this exhaustive check, SSI by OM is applied on the same
33 702 structure with an empirically chosen essential set so as to manifest the peculiarity of the method
34 703 and conceptualize the consecutive observability flow. Subsequently, all the 252 sets are
35 704 categorized into 70 patterns by the physical location of the measurements and corresponding
36 705 identification results are discussed with respect to these patterns. It is found that the reasons of
37 706 partial observability are: (1) the premature end of the recursive steps and (2) the redundant
38 707 information due to intensive placement of sensors.

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41 708 To address the issue of partial observability, SSI by COM, where the lacking nonlinear
42 709 constraints are appended to the original SSI by OM through an optimization routine, is proposed
43 710 with the aim of fully exploiting the information in measurements. In the first step of the
44 711 proposed method, SSI by OM is performed until no more parameters are observed. If full
45 712 observability is achieved, then the algorithm returns. Otherwise, the general equation (Eq. (2))
46 713 from the last recursive step will be extracted. The structure of the unknowns in this equation is
47 714 analyzed first and the relation between variables is identified as the nonlinear constraints used
48 715 next in the optimization. In the last step of the SSI by COM, an optimization is performed on
49 716 the observability equation with the acquired nonlinear constraints and all the parameters
50 717 observed in the first step. The efficacy of the proposed method to improve identifiability is
51 718 justified by reanalyzing the same structure with the 252 enumerated sets. The number of sets
52 719 achieving full observability by COM roughly doubles that number by OM. Nevertheless, it is
53 720 strongly recommended to place the sensor in a dispersed way since the structure is still not fully
54 721 observable in case of extremely intensive placement of sensors. The strength and robustness of
55 722 this method is further testified in a 13-storey building where the real mechanical parameters are

723 perturbed by random numbers. It is seen that the axial stiffnesses and the flexural stiffnesses of
724 all structural members can be estimated accurately.

725 This method is able to identify mechanical properties for linear systems. For non-linear systems,
726 the load test will induce a deformation increment that will provide the information of secant
727 mechanical properties of the different elements. It is highlighted that COM is not suitable to
728 identify structures with geometrical non-linear behaviour. As the use of an essential set provides
729 the only possible solution that satisfies the equations, measurement or modelling errors will
730 affect the estimates. These effects are not studied in this paper but it will be addressed in the
731 near future. The importance of this research is that the supplement of the nonlinear constraints
732 to the OM fully exploits the information provided by measurements. In contrast with SSI by
733 OM, the range of the measurement sets qualified to be essential sets are greatly enlarged by
734 applying the supplementary constraints via optimization. Besides, in some other sets, even
735 though full observability is not achieved, number of identified parameters are also increased.
736 Furthermore, the sharp decrease in the number of measurements used in the identification of the
737 mechanical parameters in a 13-storey building also justifies the efficacy of the COM.

738

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833 Table 1. Examples of patterns identifying different number of parameters by OM, and
834 corresponding identified flexural stiffnesses.

Full observability		Partial Observability					
$N_{F,OM}=5$		$N_{F,OM}=3$		$N_{F,OM}=2$		$N_{F,OM}=1$	
Pattern	Identified	Pattern	Identified	Pattern	Identified	Pattern	Identified
1,1,2,3,4,5,6	$EI_1 \sim EI_5$	1,1,2,3,4, 6,6	$EI_1 \sim EI_3$	1,1,2,2,3,3, 6	EI_1, EI_2	1,3,3,4,4,6,6	EI_3
1,2,2,3,4,5,6	$EI_1 \sim EI_5$	1,1,3,4,5,6,6	$EI_3 \sim EI_5$	1,1,2,2,3, 5,6	EI_1, EI_2	1,1,2, 4,4,6,6	EI_1
1,2,3,3,4,5,6	$EI_1 \sim EI_5$	1,1,2,2,3,4, 6	$EI_1 \sim EI_3$	1,1,2,3,3, 5,6	EI_1, EI_2	1,1,3,3,4,4,6	EI_3
1,2,3,4,4,5,6	$EI_1 \sim EI_5$	1,1,2,3,3,4, 6	$EI_1 \sim EI_3$	1,1,4,5,5,6,6	EI_4, EI_5	1,1,3,5,5,6,6	EI_5
1,2,3,4,5,5,6	$EI_1 \sim EI_5$	1,1,2,3,5,6, 6	EI_1, EI_2, EI_5	1,2,4,4,5,5,6	EI_4, EI_5	1,1,2,2, 4,6,6	EI_1
1,2,3,4,5,6,6	$EI_1 \sim EI_5$	1,1,2,4,5,6, 6	EI_1, EI_4, EI_5	1,1,2,2,5,6, 6	EI_1, EI_5	1,3,3,5,5,6,6	EI_5

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836 Table 2. Element types of different structural members and associated geometric property

Structural members	Element Type	Areas (m ²)		Inertia (m ⁴)	
outer bottom column	I	A ₁	0.563	I ₁	0.026
outer intermediate column	II	A ₂	0.360	I ₂	0.011
outer top column	III	A ₃	0.250	I ₃	0.005
Interior bottom column	IV	A ₄	0.360	I ₄	0.011
Interior intermediate column	V	A ₅	0.250	I ₅	0.011
Interior top column	VI	A ₆	0.160	I ₆	0.002
Core	VII	A ₇	1.800	I ₇	5.400
beam	VIII	A ₈	0.180	I ₈	0.005

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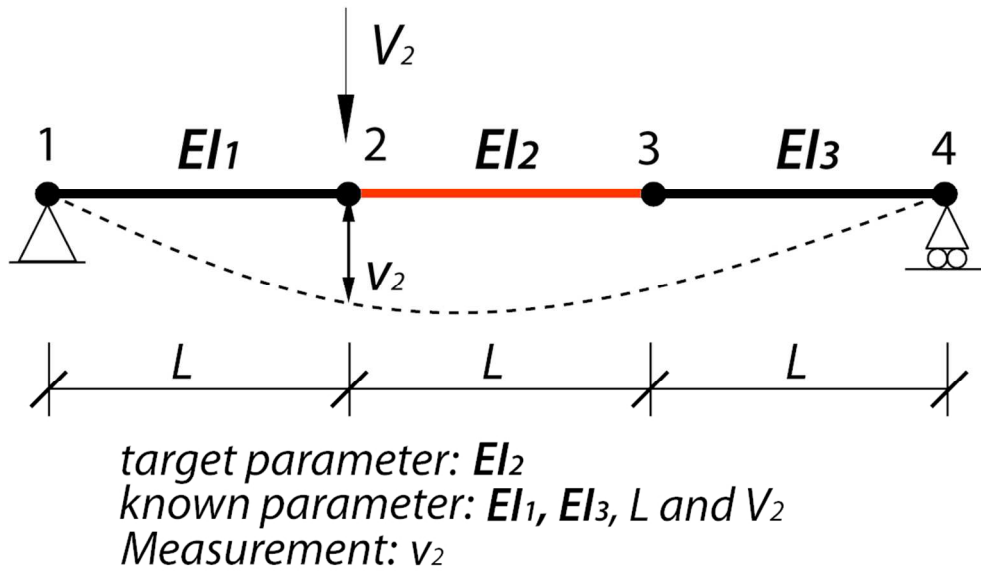


Figure 1. 4-node simply supported beam

100x67mm (300 x 300 DPI)

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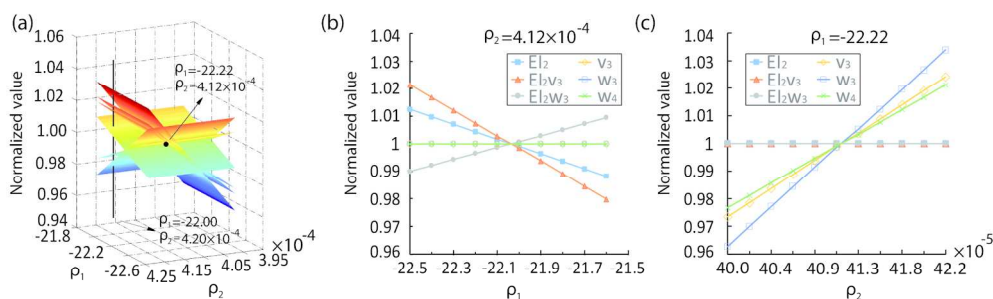


Figure 2. (a) geometric explanation of the null space, (b) variation of the normalized estimates with the variation of ρ_1 when $\rho_2=4.12 \times 10^{-5}$, (c) variation of the normalized estimates with the variation of ρ_2 when $\rho_1=-22.22$

169x62mm (300 x 300 DPI)

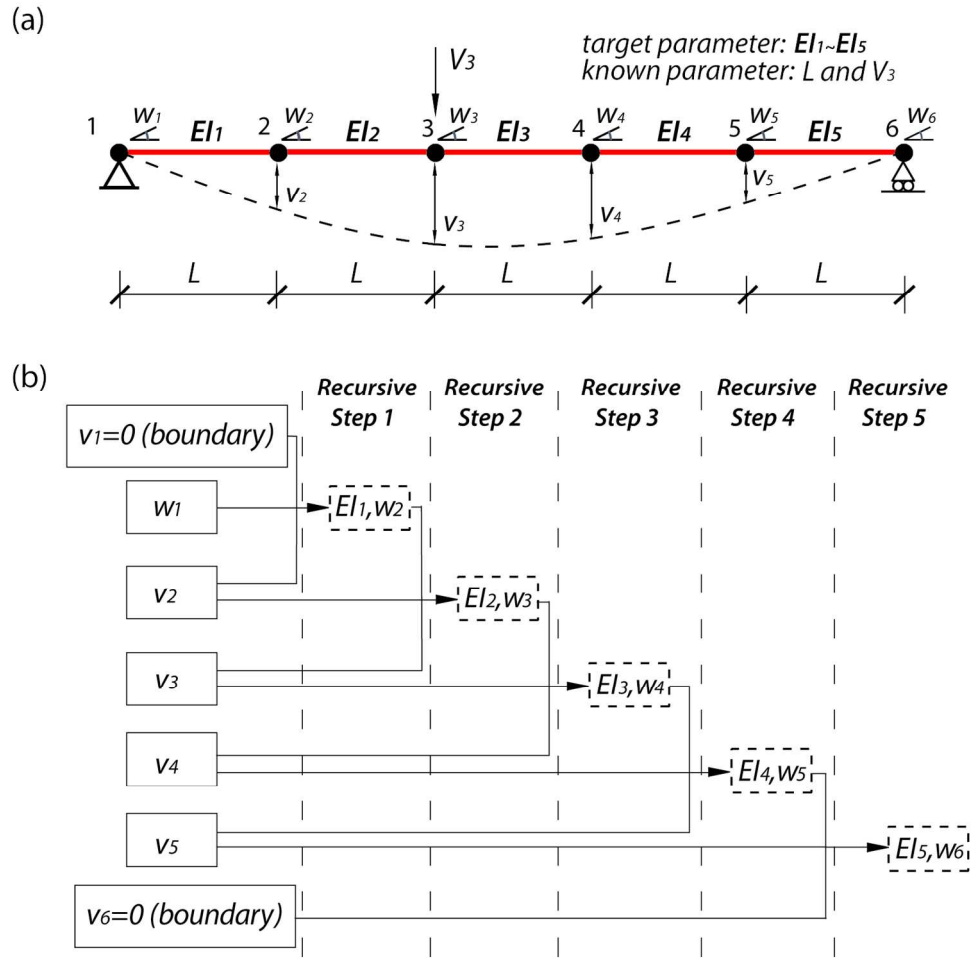


Figure 4. (a) 6-node simply supported beam with 5 target parameters, $EI_1 \sim EI_5$, and ten potential measurements, $w_1 \sim w_6$ and $v_2 \sim v_5$; (b) Observability flow for measurement set of $w_1, v_2 \sim v_5$

141x139mm (300 x 300 DPI)

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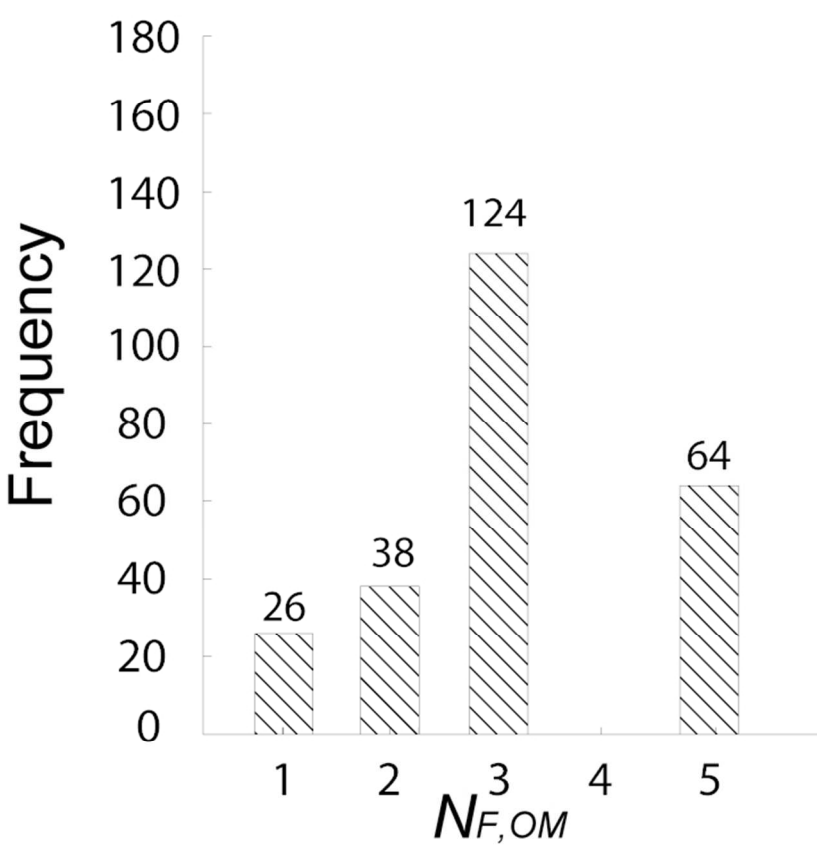


Figure 5. Frequency of the occurrence of the number of identified flexural stiffnesses by OM, $N_{F,OM}$, equal to 1 to 5

74x71mm (300 x 300 DPI)



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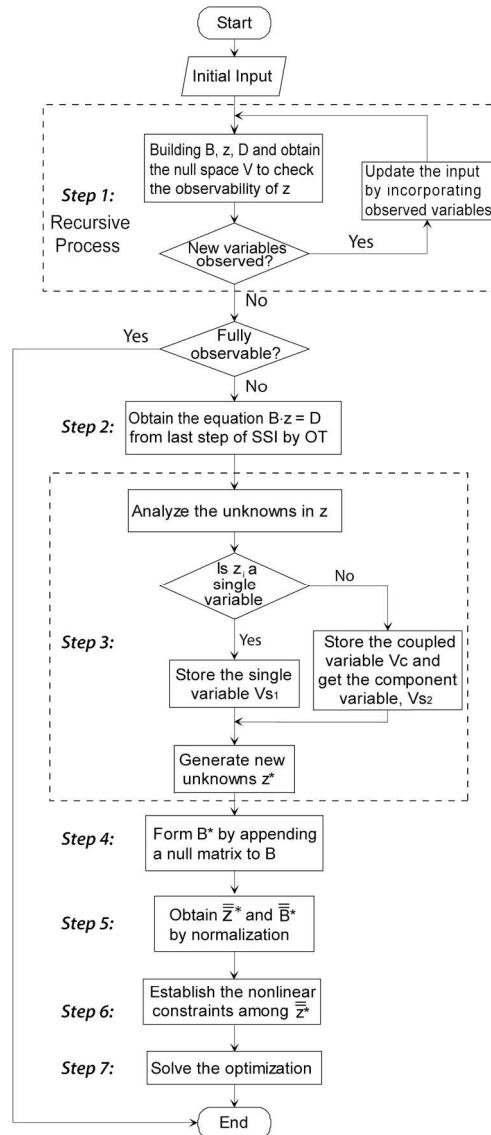


Figure 6. Flow chart of SSI by COM

85x205mm (300 x 300 DPI)

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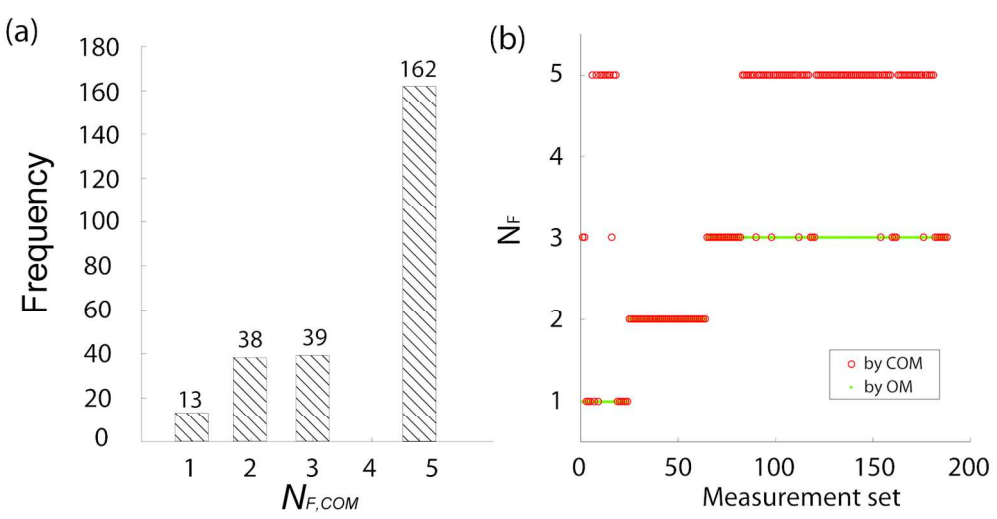


Figure 7. (a) Frequency of the occurrence of the number of identified flexural stiffnesses by COM, $N_{F,OM}$, equal to 1 to 5 ; (b) Number of observed flexural stiffnesses, N_F , by OM and COM for the 188 sets with partial observability by OM

148x77mm (300 x 300 DPI)

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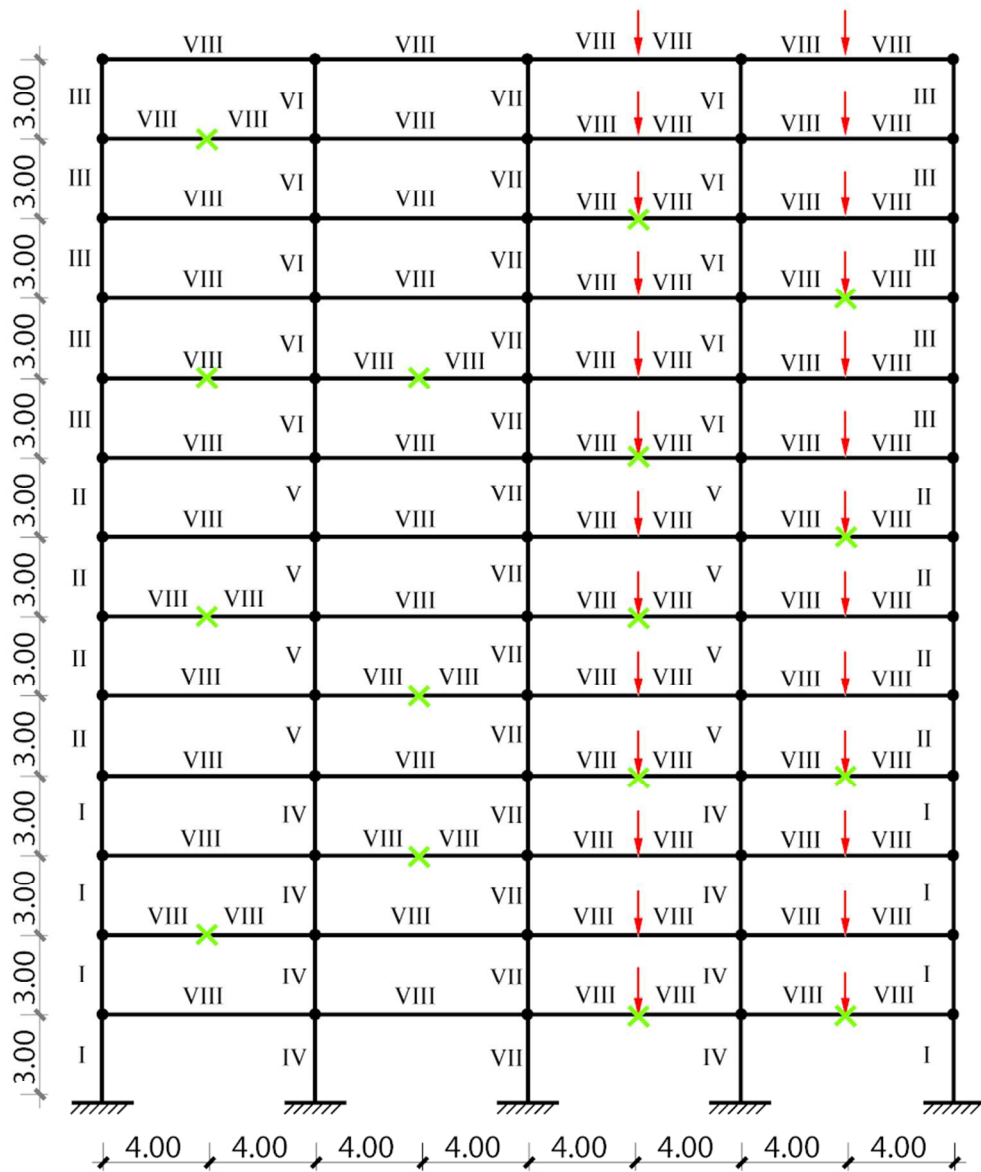


Figure 8. Element type (Roman numerals and see Table 2) of the frame building, the placement of sensors (indicated by cross) and the external load (indicated by arrows)

82x97mm (300 x 300 DPI)

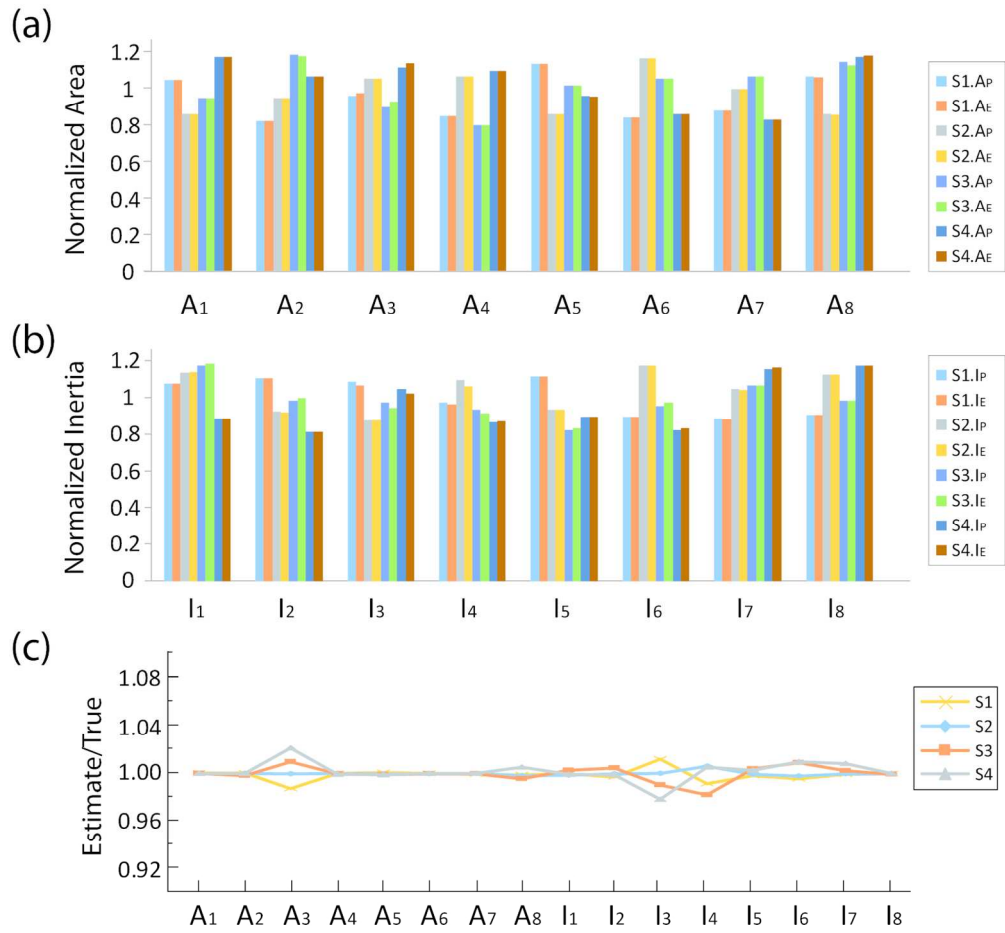


Figure 9. (a) Perturbation factor for set 1~set 4, $S1.A_p \sim S4.A_p$, and the normalized estimate for set 1~set 4, $S1.A_e \sim S4.A_e$, of the eight areas ($A_1 \sim A_8$); (b) perturbation factor for set 1~set 4, ($S1.I_p \sim S4.I_p$) and the normalized estimate ($S1.I_e \sim S4.I_e$) of the eight inertias ($I_1 \sim I_8$); (c) the ratio between the normalized estimates and the perturbation factors for all parameters ($A_1 \sim A_8$ and $I_1 \sim I_8$).

136x129mm (300 x 300 DPI)