The connectivity of the flip graph of Hamiltonian paths of the grid graph

Frank Duque*1, Ruy Fabila-Monroy12, David Flores-Peñaloza13, Carlos Hidalgo-Toscano52, and Clemens Huemer44

1Universidad de Antioquia
2Departamento de Matemáticas, Cinvestav
3Facultad de Ciencias, Universidad Nacional Autónoma de México
4Departament de Matemàtiques, Universitat Politècnica de Catalunya

Abstract

Let $G_{n,m}$ be the grid graph with $n$ columns and $m$ rows. Let $H_{n,m}$ be the graph whose vertices are the Hamiltonian paths in $G_{n,m}$, where two vertices $P_1$ and $P_2$ are adjacent if we can obtain $P_2$ from $P_1$ by deleting an edge in $P_1$ and adding an edge not in $P_1$. In this paper we show that $H_{n,2}$, $H_{n,3}$ and $H_{n,4}$ are connected.

1 Introduction

Let $m$ and $n$ be two positive integers, the grid graph $G_{n,m}$ is the graph whose vertex set is the set of points in the plane given by

$$\{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$ 

Two vertices $(i, j)$ and $(k, l)$ of $G_{n,m}$ are adjacent if $i = k$ and $|j - l| = 1$, or $j = l$ and $|i - k| = 1$. Let $H_{n,m}$ be the graph whose vertices are all the Hamiltonian paths of $G_{n,m}$, where two such paths $P$ and $Q$ are adjacent if there exist edges $e \in P$ and $f \notin P$ such that $Q = P - e + f$. We call this graph the flip graph of Hamiltonian paths of $G_{n,m}$; the operation of replacing $e$ with $f$ in $P$ is called a flip. In this paper we study the following problem, proposed by Hurtado [2]:

Problem 1 For what values of $n$ and $m$ is $G_{n,m}$ connected?

Hamiltonian paths in grid graphs have been studied from different points of view. Itai, Papadimitriou and Szwarcfiter [3] gave necessary and sufficient conditions for a Hamiltonian path to exist between two vertices of $G_{n,m}$. They also showed that the problem of finding a Hamiltonian path (or cycle) between two vertices is NP-complete. Collins and Kromp [1] gave generating functions for counting the number of Hamiltonian paths in $G_{n,m}$ that start in the lower left corner and end in the upper right corner, for $1 \leq m \leq 5$. Jacobsen [4] presented an algorithm for enumerating all the Hamiltonian paths in $G_{n,m}$, and calculated the number of distinct Hamiltonian paths in $G_{n,n}$ for $1 \leq n \leq 17$.

One motivation for studying the connectedness of $H_{n,m}$ is generating random Hamiltonian cycles in $G_{n,m}$: if $H_{n,m}$ is connected, we choose a random Hamiltonian path and then can perform a random walk until we reach a Hamiltonian path whose endpoints are adjacent in $G_{n,m}$. Any such path can be completed into a Hamiltonian cycle.

In this paper we show that $H_{n,2}$, $H_{n,3}$ and $H_{n,4}$ are connected.

1.1 Preliminaries

We start by introducing some notation. Given a vertex $v$ in $G_{n,m}$, we denote by $v_x$ and $v_y$ its $x$ and $y$ coordinates, respectively. Let $R$ be the boundary rectangle of $G_{n,m}$. That is, $R$ is the subgraph of $G_{n,m}$ induced by the vertices of degree at most three. Consider the subgraphs of $R$ induced by the following sets of vertices:

$$\{(i, 1) : 1 \leq i \leq n\};$$

$$\{(i, m) : 1 \leq i \leq n\};$$

$$\{(1, j) : 1 \leq j \leq m\}$$

$$\{(m, j) : 1 \leq j \leq m\}.$$ 

We refer to them as the bottom, top, left and right sides of $R$, respectively.

Given a subgraph $H$ of $G_{n,m}$ with the same vertex set as $G_{n,m}$, denote by $H_{i}[j]$ the subgraph of $H$ induced by the vertices with $x$-coordinate at least $i$ for
1 \leq i \leq n$. Analogously, denote by $H_{\rightarrow}[i]$ the subgraph of $H$ induced by the vertices with $x$-coordinate at most $i$ for $1 \leq i \leq n$. Let $F$ be another subgraph of $G_{n,m}$ with the same vertex set as $G_{n,m}$. If $H_{\rightarrow}[n] = F_{\rightarrow}[n]$, denote by $\text{same}_{\rightarrow}(H, F)$ the minimum integer $i \in [1, n]$ such that $H_{\rightarrow}[i] = F_{\rightarrow}[i]$. If $H_{\rightarrow}[1] = F_{\rightarrow}[1]$, denote by $\text{same}_{\rightarrow}(H, F)$ the maximum integer $i \in [1, n]$ such that $H_{\rightarrow}[i] = F_{\rightarrow}[i]$.

A cyclic Hamiltonian path of $G_{n,m}$ is a Hamiltonian path of $G_{n,m}$ such that its endpoints are adjacent. Note that adding the edge joining the endpoints of a cyclic Hamiltonian path produces a Hamiltonian cycle of $G_{n,m}$. Given a cyclic Hamiltonian path $P$, we denote by $P_e$ the associated Hamiltonian cycle.

In the following lemmas it is convenient to assume that the Hamiltonian paths are oriented from one of its endpoints to the other. We refer to them as the first and last vertices, respectively. In what follows let $P$ be such a Hamiltonian path and let $s$ and $t$ be its first and last vertices, respectively.

**Lemma 1** Suppose that both endpoints of $P$ lie on $R$. Let $\ell$ be a directed vertical line in the plane and let $e_1, \ldots, e_k$ be the horizontal edges of $P$ that intersect $\ell$, in their order of intersection with $\ell$. Then for all $1 \leq i \leq k - 1$, the edges $e_i$ and $e_{i+1}$ have opposite directions.

**Lemma 2** Let $e := (t, x)$ be an edge not in $P$ such that $x \neq s$. Let $Q$ be the Hamiltonian path obtained from $P$ by a flip that adds $e$. Then the edge removed in the flip is the directed edge $(x, y)$. Assuming that $s$ is the first vertex of $Q$, then the last vertex of $Q$ is $y$.

Given two oriented Hamiltonian paths $P$ and $Q$ of $G_{n,m}$ we say that the endpoints of $Q$ are **monotonically to the right of $P$** if the three following properties hold.

- The first vertex of $Q$ is not to the left of the first vertex of $P$.
- The last vertex of $Q$ is not to the left of the last vertex of $P$.
- Either the first vertex of $Q$ is to the right of the first vertex of $P$ or the last vertex of $Q$ is to the right of the last vertex of $P$.

**Lemma 3** Suppose that $t$ is not on the right side of $R$ and that the right edge of $t$ is not in $P$. Then $P$ is adjacent in $H_{n,m}$ to a Hamiltonian path $Q$ whose endpoints are monotonically to the right of $P$.

2 $H_{n,2}$ and $H_{n,3}$ are Connected

**Theorem 4** $H_{n,2}$ is connected.

**Proof.** Let $P$ be a Hamiltonian path in $G_{n,2}$. Note that all cyclic Hamiltonian paths in $G_{n,2}$ are adjacent. We prove that $H_{n,2}$ is connected by showing that there exists a sequence $Q_1, Q_2, \ldots, Q_k$ of Hamiltonian paths of $G_{n,2}$ such that $Q_1 = P$, $Q_k$ is cyclic and $Q_i$ is adjacent to $Q_{i+1}$ in $H_{n,2}$ for all $1 \leq i \leq k - 1$.

Let $Q_1 = P$ and suppose that $Q_i$ has been defined. If the endpoints $s$ and $t$ of $Q_i$ have the same $x$-coordinate then either both are on the left side of $R$ or both are on the right side of $R$. In both cases $Q_i$ is cyclic. In this case set $Q_{k} = Q_i$ and we are done.

Suppose without loss of generality that $s$ is to the right of $t$. Note that the right edge of $t$ is not contained in $Q_i$; as otherwise, $s$ cannot be to the right of $t$. Apply Lemma 3 to obtain a path $Q_{i+1}$ adjacent to $Q_i$ such that its last vertex is to the right of $t$ and its first vertex is not to the left of $s$. Note that since the endpoints of the $Q_i$ move monotonically to the right, then eventually both have the same $x$-coordinate; the result follows.

**Lemma 5** Every Hamiltonian path $P$ of $G_{n,3}$ is connected in $H_{n,3}$ to a Hamiltonian path whose endpoints are on the right side of $R$.

**Lemma 6** Let $P$ be a Hamiltonian path of $G_{n,3}$ whose endpoints lie on the right side of $R$. Then $P$ contains the following sets of edges.

- $L := \{(1,1), (1,2), (2,1), (2,3)\}$
- $B := \{(2i - 1, 1), (2i, 1) : 1 \leq i \leq \lceil \frac{n}{2} \rceil \}$
- $M := \{(2i, 2), (2i + 1, 2) : 1 \leq i \leq \lceil \frac{n}{2} \rceil \}$ and
- $T := \{(2i, 1), (2i, 3) : 1 \leq i \leq \lceil \frac{n}{2} \rceil \}$

**Theorem 7** $H_{n,3}$ is connected.

**Proof.** Let $P$ be a Hamiltonian path in $G_{n,3}$. By Lemma 5, we can assume that the endpoints of $P$ lie on the right side of $R$. Let $Q$ be the Hamiltonian path in $H_{n,3}$ with set of edges $B \cup T \cup M \cup V \cup \{(n, 1), (n, 2)\}$, where

- $B = \{(i, 1), (i + 1, 1) : 1 \leq i \leq n - 1\}$
- $T = \{(n - 2i + 1, 1), (n - 2i + 2, 3) : 1 \leq i \leq \lfloor n/2 \rfloor\}$
- $M = \{(2i, 2), (n - 2i + 1, 2) : 1 \leq i \leq \lfloor n/2 \rfloor\}$
- $V = \{(i, 2), (i, 3) : 1 \leq i \leq n\}$

We prove that $H_{n,3}$ is connected by showing that there exists a sequence of paths $P = P_1, \ldots, P_k = Q$ in $G_{n,3}$ such that $P_i$ and $P_{i+1}$ are adjacent in $H_{n,3}$ and $\text{same}_{\rightarrow}(P_i, Q) \geq \text{same}_{\rightarrow}(P_{i+1}, Q)$ for every $1 \leq i \leq k - 1$. We have two cases:

- $n$ is even. If $n$ is even, $P$ is a cyclic Hamiltonian path in $G_{n,3}$. By Lemma 6, the set of edges $D = \{P, \emptyset \}$ consists of only edges from $T$ and $V$. Furthermore, $D$ can be partitioned in sets of the form $\{(2i, 3), (2i +$
3 $H_{n,4}$ is Connected

**Lemma 8** Suppose that each vertex of $G_{n,4}$ is colored with one of two colors in such a way that adjacent vertices receive different colors. Then the endpoints of every Hamiltonian path of $G_{n,4}$ have different colors.

**Lemma 9** Let $P$ be a Hamiltonian path of $G_{n,4}$. Then there exists a sequence $P = P_1, \ldots, P_l$ of Hamiltonian paths of $G_{n,4}$ such that for every $1 \leq i < t$ the following three properties hold.

1) $P_i$ and $P_{i+1}$ are adjacent in $H_{n,4}$.
2) Either the first vertices of $P_i$ and $P_{i+1}$ are equal and the last vertex of $P_{i+1}$ is to the right of the last vertex of $P_i$, or the last vertices of $P_i$ and $P_{i+1}$ are equal and the first vertex of $P_{i+1}$ is to the right of the first vertex of $P_i$.
3) Let $j$ be the smallest $x$-coordinate of the endpoints of $P_i$ and $P_{i+1}$. Then samecc$(P_i, P_{i+1}) \geq j$.

Furthermore, $P_1$ is cyclic and its endpoints lie on the right side of $R$.

**Proof.** We proceed by induction on $i$. Let $P_i := P$ and assume that $i \geq 1$ and that $P_1, \ldots, P_i$ have been defined. We construct $P_{i+1}$. Let $s$ and $t$ be the first and last vertices of $P_i$, respectively. Assume that the right edges of both $s$ and $t$ are in $P_i$, otherwise Lemma 3 gives a path $P_{i+1}$ that satisfies all three conditions. It can be shown that if $s_x = t_x$ we can obtain $P_{i+1}$.

Assume that $t_x < s_x$.

We consider the following cases.

- **$t \notin R$**

Assume without loss of generality that $t_y = 3$ and denote the right edge of $t$ by $e$. The right edge of $(t_x, 4)$ is in $P_i$, if it has different orientation than $e$, we perform a flip where we add the edge $(t, (t_x, 4))$ and we are done. Suppose that it has the same orientation. If the right edge of $(t_x, 2)$ is in $P_i$, it has different orientation than $e$, thus we can perform a flip where we add the edge $(t, (t_x, 2))$ and we are done. Suppose the right edge of $(t_x, 2)$ does not belong to $P_i$. Then $P_i$ contains the edges $((t_x - 1, 4), (t_x, 4))$ and $((t_x + 1, 2), (t_x, 2))$. The subgraph of $P_i$ induced by the vertices with $x$-coordinate less than $t_x$ is a Hamiltonian path on $G_{t_x-1,4}$ with endpoints $(t_x - 1, 4)$ and $(t_x - 1, 2)$. These vertices have the same color, a contradiction to Lemma 8.

- **$s \notin R$**

Assume without loss of generality that $s_y = 3$ and denote the right edge of $s$ by $e$. The right edge of $(s_x, 4)$ is in $P_i$, if it has different orientation than $e$, we perform a flip where we add the edge $(s, (s_x, 4))$ and we are done. Suppose that it has the same orientation as $e$ and let $\ell$ be a vertical line that passes through the middle point of $e$. Since $t_x < s_x$, $P_i$ must cross $\ell$ two times from right to left. This implies that the edge $((s_x, 2), (s_x + 1, 2))$ has different
orientation than $e$. Thus we can make a flip where we add the edge $(s, (s_x, 2))$ to obtain $P_{i+1}$.

- \{s, t\} \subset R

Assume without loss of generality that $t_y = 4$ and denote the right edge of $t$ by $e$. If the edge $((t_x, 3), (t_x + 1, 3))$ is in $P_i$, it has different orientation than $e$ by Lemma 1, thus we can get $P_{i+1}$ by performing a flip where we add the edge $(t, (t_x, 3)).$ Assume $((t_x, 3), (t_x + 1, 3))$ is not in $P_i$ and let $\ell$ be a vertical line that passes through the middle point of $e$. Since $t_x < s_x$, $P_i$ must intersect $\ell$ two more times. Note that $t_x > 1$, otherwise the vertex $(t_x, 3)$ has no incident edge. The subgraph of $P_i$ induced by the vertices with $x$-coordinate less than $t_x$ is a Hamiltonian path on $G_{t_x-1,4}$ with endpoints $(t_x, 3)$ and $(t_x, 1)$. These vertices have the same color, a contradiction to Lemma 8.

It follows that $P_i$ is a Hamiltonian path whose endpoints lie on the right side of $R$. There is only left to prove that $P_i$ is cyclic. There are three possible configurations for the endpoints of $P_i$ that do not correspond to a cyclic path. It can be shown that any such configuration can be taken to a configuration of a cyclic path using at most 3 flips.

\[
\begin{align*}
H_1 & = \{((i, j), (i + 1, j)) : 1 \leq i \leq n - 2, 1 \leq j \leq 4\} \\
H_2 & = \{((n - 1, 1), (n, 1)), ((n - 1, 4), (n, 4))\} \\
V_1 & = \{((n, j), (n, j + 1)) : 1 \leq j \leq 3\} \\
V_2 & = \{((1, 1), (1, 2)), ((1, 3), (1, 4)), ((n - 1, 2), (n - 1, 3))\}
\end{align*}
\]

**Lemma 10** Let $S$ be a cyclic Hamiltonian path in $G_{n,4}$ whose endpoints lie on the right side of $R$. Then $S$ is connected in $H_{n,4}$ to a cyclic Hamiltonian path $S'$ whose endpoints lie on the right side of $R$ such that:

- If $S_{c,1} \neq Q[1]$, then $\text{same}_{\rightarrow}(S', Q) \geq 1$
- If $S_{c,1} = Q[1]$, then $\text{same}_{\rightarrow}(S', Q) \geq \text{same}_{\rightarrow}(S, C)$

**Proof.** If $S_{c,1} \neq Q_{c,1}$, $S_c$ must contain the left side of $R$. Let $S_1 = S_c \setminus \{(1, 2), (1, 3)\}$. Figure 4 shows two flips that transform $S_1$ into a Hamiltonian path $S_2$ such that $\text{same}_{\rightarrow}(S_2, Q) \geq 1$. By Lemma 9, we can transform $S_3$ into a cyclic Hamiltonian path $S'$ such that $\text{same}_{\rightarrow}(S', Q) \geq 1$.

Suppose that $S_{c,1} = Q[1]$ and let $m := \text{same}_{\rightarrow}(S_c, Q)$. Let $H$ be the set of horizontal edges of $S$ whose left endpoint has $x$-coordinate $m$. We claim that $H$ consists exactly of two edges of $R$. There cannot be 4 edges in $H$, otherwise $m$ would be bigger. Furthermore, since the endpoints of $S$ lie on the right side of $R$, there cannot be 3 edges in $H$. Thus, there are exactly two edges in $H$. The $y$-coordinates of the edges in $H$ cannot be $\{2, 3\}$, as such a configuration cannot be completed to a Hamiltonian path which is cyclic (see Figure 5a). Similarly, the $y$-coordinates cannot be $\{1, 2\}$ or $\{3, 4\}$, as this configuration cannot be completed to a path (see Figure 5b).

![Figure 5: Forbidden configurations of horizontal edges.](image)

The positions of the edges in $H$ imply that $((m, 2), (m, 3)) \notin S$. Let $S_1 = S_c \setminus \{(m, 2), (m, 3)\}$. Figure 6 shows two flips that transform $S_1$ into a Hamiltonian path $S_2$ such that $\text{same}_{\rightarrow}(S_2, Q) > m$. By Lemma 9, we can transform $S_3$ into a cyclic Hamiltonian path $S'$ such that $\text{same}_{\rightarrow}(S', Q) > m$.

![Figure 6: The flips that take $S$ to $S_3$ when $S_{c,1} = Q[1]$.](image)

**Theorem 11** $H_{n,4}$ is connected.

**Proof.** Let $P$ be a Hamiltonian path in $G_{n,4}$. By Lemma 9, we can assume that the endpoints of $P$ lie on the right side of $R$. The result follows from a repeated application of Lemma 10.

**References**


