Preserving $T$-Transitivity

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Abstract. This contribution deals with the problem of aggregating $T$-equivalence relations, in the sense that we are looking for functions that preserve reflexivity, symmetry and transitivity with respect to a given t-norm $T$. Under any extra condition on the t-norm, we obtain a complete description of those functions in terms of what we call $T$-triangular triplets.

Keywords. t-norm, additive generator, $T$-equivalence relation, preserving properties, triangular triplet.

Introduction

Fuzzy equivalence relations, together with $T$-preorders, are probably the most important kind of fuzzy relations since they measure the degree to which two points of an universe are indistinguishable, equal or equivalent, and generalize the concept of classical equivalence relations.

They were introduced in [13] under the name similarity relations (with respect to the minimum) although they are also present in [9] and in [5]. The generalization to t-norms was considered in [11]. Other names have been used for this concept in the literature (sometimes in connection with a specific t-norm), such as likeness relation, indistinguishability relation, fuzzy equality, proximity relation, etc. We shall use in the sequel the term $T$-equivalence relation which, in our opinion, reflects in the best way the mathematical motivation in the axioms we recall in Section 1. The term $T$-indistinguishability operator is also widely used in the literature [3,8,11,12].

In many situations, there can be more than one $T$-indistinguishabilities defined on a universe and, in these cases, we may need to aggregate them. The most common way to aggregate a collection of $T$-equivalence relations is calculating their minimum, which also is a $T$-equivalence relation. However, sometimes this way of aggregating fuzzy relations leads to undesirable results since the Minimum only takes the smaller value for every couple into account and disregards the information of the other values. Similar draw-
back occurs when the Minimum is replaced by the t-norm $T$, specially when it is non-
strict Archimedean. Thus, more general procedures to aggregate $T$-indistinguishability are
needed.

Several authors have dealt the problem of the aggregation of some classes of fuzzy
relations. With the same spirit as in [12,8], we revisit this topic in order to give, whatever
the t-norm $T$ we use, a characterization of those functions that combine a collection of
$T$-equivalence relations in a single one.

After a section of preliminaries containing the basic definitions related with t-norms
and $T$-equivalence relations, Section 2 introduces the concept of $T$-triangular triplet that
will be central in the study of the preservation of $T$-transitivity. Section 3 contains the
main results of the paper characterizing the functions that aggregate $T$-equivalence rela-
tions and some examples of functions aggregating $T$-equivalence relations for continu-
ous Archimedean t-norms and the minimum and drastic t-norms. The contribution ends
with a section of Concluding Remarks.

1. Preliminaries

Despite the fact that triangular norms (t-norms, for short) were first introduced in the
context of statistical metric spaces [6], they have become an important tool in many
other fields: fuzzy sets, decision making, statistics, theories of non-additive measures,
etc. Comprehensive monographs on t-norms are [1,4]. We use the set of axioms provided
by Schweizer and Sklar [10]. Thus, our requirements on a t–norm $T: [0,1] \times [0,1] \rightarrow
[0,1]$ for all $a, b, c, d$ in $[0,1]$ are:

(i) $T(a, b) = T(b, a),$
(ii) $T(T(a, b), c) = T(a, T(b, c)),$
(iii) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d,$
(iv) $T(a, 1) = a.$

The following are the four basic t–norms $T_M, T_L, T_P$ and $T_D$:

- $T_M(a, b) = \min(a, b),$ (minimum)
- $T_L(a, b) = \max(a + b - 1, 0),$ (Łukasiewicz t–norm)
- $T_P(a, b) = ab,$ (Product t–norm)
- $T_D(a, b) = \begin{cases} \min(a, b), & \text{if } a = 1 \text{ or } b = 1 \\ 0, & \text{otherwise} \end{cases}$ (drastic t–norm).

A t-norm $T$ is called Archimedean if for each $a, b \in (0,1)^2$ there is $n \geq 1$ such
that $T(a, \ldots, a) < b$. One special property of a continuous Archimedean t-norm is that
it is strictly increasing, except for the subset of $[0,1]^2$ where its value is 0. A remarkable
fact is that any continuous Archimedean t-norm $T$ can be expressed with the help of an
additive generator $^1 T(a, b) = g^{-1}(a) (g(a) + g(b)),$ where $g^{-1}$ is the pseudo-inverse$^2$
of $g$. Note that $T_L$ and $T_P$ are continuous Archimedean t-norms with additive generators
g(a) = 1 - a and $g(a) = - \log a$ respectively.

$^1$An additive generator is a continuous and strictly decreasing function $g: [0,1] \rightarrow [0,\infty]$ such that
$g(1) = 0.$

$^2$g$^{-1}(t) = \sup \{ c \in [0,1]; g(c) > t \}, \sup \emptyset = 0.$
Given a set $X$ and a t-norm $T$, we say that a fuzzy relation $E : X \times X \rightarrow [0, 1]$ is a $T$-equivalence (or a $T$-indistinguishability) if for all $x, y, z$ in $X$ the following conditions hold:

(i) $E(x, x) = 1$ (reflexivity)
(ii) $E(x, y) = E(y, x)$ (symmetry)
(iii) $E(x, y) \geq T(E(x, z), E(z, y))$ ($T$-transitivity)

As it is known, $E(x, y)$ is interpreted as the degree of indistinguishability (or similarity) between $x$ and $y$. The axioms of reflexivity, symmetry and $T$-transitivity fuzzify the ones of a crisp equivalence relation.

Given a left continuous t-norm $T$, we can define the function on $[0, 1]^2$ defined by $\overrightarrow{T}(a, b) = \sup\{c \in [0, 1]; T(a, c) \leq b\}$ that we call the residuation of $T$. It is easy to see that $\overrightarrow{T}$ is a $T$-preorder on $[0, 1]$. The biresiduation of $T$ is the function on $[0, 1]^2$ defined by $\overleftarrow{T}(a, b) = T(\overrightarrow{T}(a, b), \overrightarrow{T}(b, a)) = \min(\overrightarrow{T}(a, b), \overrightarrow{T}(b, a))$. It is an important example of $T$-equivalence on $[0, 1]$ that usually is called the natural $T$-equivalence associated to $T$, denoted by $E_T$.

If $T$ is a continuous Archimedean t-norm with additive generator $g$, then $E_T(x, y) = g^{(-1)}(\{ g(x) - g(y) \})$ for all $x, y \in [0, 1]$. In particular, for the Łukasiewicz t-norm, $E_T(x, y) = 1 - |x - y|$ and for the Product t-norm, $E_T(x, y) = \min(\frac{x}{y}, \frac{y}{x})$.

Complete information on indistinguishability operators can be found in the recent monograph [8].

2. $T$-triangular triplets

**Definition 1.** We say that a triplet $(a, b, c) \in [0, \infty]^3$ is triangular if and only if

$$a \leq b + c, \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$  

Being $a, b, c \in [0, \infty]^m, \ m \geq 1$, we say that $(a, b, c)$ is a (m-dimensional) triangular triplet if $(a_i, b_i, c_i)$ is triangular for all $i = 1, \ldots, m$, where $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m), c = (c_1, \ldots, c_m)$.

Note that if $(a, b, c)$ is triangular then so is any reordering of $a, b, c$.

**Proposition 1.** A triplet $(a, b, c) \in [0, \infty]^3$ is triangular if and only if it is of one of the following forms:

(i) $(\infty, \infty, c), \ c \in [0, \infty]$
(ii) $c = \sqrt{a^2 + b^2 + \lambda ab}, \ 0 \leq a, b \leq \infty, -2 \leq \lambda \leq 2$

**Definition 2.** Let $T$ be a t-norm. We say that $(a, b, c) \in [0, 1]^3$ is $T$-triangular if and only if

$$a \geq T(b, c), \ b \geq T(a, c) \ \text{and} \ c \geq T(a, b).$$

\footnote{Reflexive and $T$-Transitive}
Being $a, b, c \in [0, 1]^m$, $m \geq 1$, we say that $(a, b, c)$ is a $(m$-dimensional) $T$-triangular triplet if $(a_i, b_i, c_i)$ is $T$-triangular for all $i = 1, \ldots, m$, where $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m), c = (c_1, \ldots, c_m)$.

**Proposition 2.** Let $T$ be a left continuous t-norm. A triplet $(a, b, c) \in [0, 1]^3$ is $T$-triangular if and only if $T(a, b) \leq c \leq E_T(a, b)

**Proof:**

$\Rightarrow$) Suppose that $(a, b, c)$ is $T$-triangular. From $T(a, c) \leq b$ and $T(b, c) \leq a$ we deduce $c \leq \overline{T}(a, b)$ and $c \leq \overline{T}(b, a)$, hence $c \leq \min(\overline{T}(a, b), \overline{T}(b, a)) = E_T(a, b)$. Then $T(a, b) \leq c \leq E_T(a, b)$.

$\Leftarrow$) Reciprocally, assuming $T(a, b) \leq c \leq E_T(a, b)$ we have to prove that $(a, b, c)$ is $T$-triangular. From $c \leq \overline{T}(a, b)$ and, applying left continuity and monotonicity of $T$, we obtain $T(a, c) \leq b$. Similarly, from $c \leq \overline{T}(b, a)$ we obtain $T(b, c) \leq a$. Thus, the triplet $(a, b, c)$ is $T$-triangular.

**Example 1.**

- A triplet is $T_M$-triangular if and only if there exists a reordering $(a, b, c)$ such that $a = b$ and $c \geq a$.
- A triplet is $T_L$-triangular if and only if there exists a reordering $(a, b, c)$ such that $\max(a + b - 1, 0) \leq c \leq 1 - |a - b|$.
- A triplet is $T_P$-triangular if and only if it is $(0, 0, 0)$ or there exists a reordering $(a, b, c)$ with $a, b, c > 0$, such that $ab \leq c \leq \min(\frac{a}{b}, \frac{b}{a})$.

For the drastic t-norm, which is not left continuous, we have the following result.

**Example 2.** A triplet $(a, b, c)$ is $T_D$-triangular if and only if $a, b$ and $c$ are different from 1 or one of them is 1 and the other two coincide.

**Remark 1.** Denoting by $\Delta(T)$ the set of $T$-triangular triplets, observe that $T_1 \leq T_2$ implies $\Delta(T_1) \supset \Delta(T_2)$. Thus we have $[0, 1]^3 \supset \Delta(T_D) \supset \Delta(T_L) \supset \Delta(T_P) \supset \Delta(T_M) \supset \{(a, a, a); a \in [0, 1]\}$.

3. Aggregating $T$-equivalence relations

**Definition 3.** We say that a function $F : [0, 1]^m \rightarrow [0, 1], m \geq 1$, aggregates $T$-equivalence relations if for any set $X$ and any collection of $T$-equivalence relations on $X$, $(E_1, \ldots, E_m)$, then $F(E_1, \ldots, E_m)$ is also a $T$-equivalence relation on $X$, where $F(E_1, \ldots, E_m)$ is the fuzzy binary relation $F(E_1, \ldots, E_m)(x, y) = F(E_1(x, y), \ldots, E_m(x, y))$.

**Example 3.** Any t-norm $T$ aggregates $T$-equivalence relations (for any $m \geq 1$).

In [3] an aggregation method with respect to $E_T$ is introduced. Being $g$ an additive generator of $T$ (continuous Archimedean), then the corresponding aggregation function coincides with the quasi-arithmetic mean generated by $g$. Next proposition states that this function aggregates $T$-equivalence relations.
Proposition 3. Let $T$ be a continuous Archimedean $t$-norm with $g$ as additive generator. The quasi-arithmetic mean generated by $g$, $M_g(a_1, \ldots, a_m) = \frac{g^{-1}(g(a_1) + \ldots + g(a_m))}{m}$, aggregates $T$-equivalence relations.

The main result in this contribution is collected in the following proposition.

Proposition 4. A function $F : [0, 1]^m \rightarrow [0, 1], m \geq 1$, aggregates $T$-equivalence relations if and only if the following conditions hold:

(i) $F(1, \ldots, 1) = 1$.

(ii) $F$ transforms $m$-dimensional $T$-triangular triplets into 1-dimensional $T$-triangular triplets.

Proof:

$\Rightarrow$) Assuming that $F$ satisfies (i) and (ii), we must prove that $F(E_1, \ldots, E_m)$ is a $T$-equivalence relation for all $T$-equivalence relations $E_1, \ldots, E_m$.

We know that, for each $i = 1, \ldots, m$, it is $E_i(x, y) \geq T(E_i(x, z), E_i(z, y))$. Thus, the triplet $(a, b, c)$, where $a_i = E_i(x, y), b_i = E_i(x, z), c_i = E_i(z, y), i = 1, \ldots, m$, is $T$-triangular, and from (ii) we have that $(F(a), F(b), F(c))$ so is, and, consequently, we can write:

$$F(E_1, \ldots, E_m(x, y)) = F(E_1(x, y), \ldots, E_m(x, y)) \geq$$

$$T(F(E_1(x, z), \ldots, E_m(x, z)), F(E_1(z, y), \ldots, E_m(z, y))) =$$

$$T(F(E_1, \ldots, E_m)(x, z)), F(E_1, \ldots, E_m)(z, y)).$$

Hence, $F(E_1, \ldots, E_m)$ is $T$-transitive. reflexivity and symmetry follow immediately from (i) and symmetry of $T$.

$\Rightarrow$) Reciprocally, let us suppose that $F$ aggregates $T$-equivalence relations. We have to prove that it satisfies conditions (i) and (ii).

First, it is $F(1, \ldots, 1) = 1$ because $F(1, \ldots, 1) = F(E(x, x), \ldots, E(x, x)) = F(E, \ldots, E)(x, x) = 1$, where $E$ is a $T$-equivalence relation on a set $X$ and $x \in X$.

Now, let us prove that $(F(a), F(b), F(c))$ is $T$-triangular whenever $(a, b, c)$ also is.

There exist a set $X$, $T$-equivalence relations $E_1, \ldots, E_m$ on $X$ and $x, y, z \in X$ such that $E_i(x, y) = a_i, E_i(x, z) = b_i$ and $E_i(z, y) = c_i$ for all $i = 1, \ldots, m$. Then we can write

4If $(a, b, c)$ is a $T$-triangular triplet in $[0, 1]^m$ then $(F(a), F(b), F(c))$ is a $T$-triangular triplet in $[0, 1]$.

5It is sufficient we consider a 3-element set $X = \{x, y, z\}$ and define $E_i(x, y) = E_i(y, x) = E_i(1, z) = 1, E_i(x, z) = E_i(z, y) = 0, E_i(z, y) = E_i(y, z) = 1$. Note that each $E_i$ is a $T$-equivalence relation on $X$. 

\[ F(a) = F(E_1(x, y), \ldots, E_m(x, y)) = F(E_1, \ldots, E_m)(x, y) \geq T(F(E_1, \ldots, E_m)(x, z), F(E_1, \ldots, E_m)(y, z)) = T(F(b), F(c)). \]

Similarly, we obtain \( F(b) \geq T(F(a, F(c))) \) and \( F(c) \geq T(F(a, F(b))) \) and we have proved that \((F(a), F(b), F(c))\) is \(T\)-triangular.

Next, an immediate consequence is shown.

**Proposition 5.** A function \( F : [0, 1]^m \rightarrow [0, 1], m \geq 1 \), aggregates \(T_M\)-equivalence relations (similarity relations) if and only if it is increasing in each variable and \( F(1, \ldots, 1) = 1 \).

**Proof:** Obvious, because \( F \) transforms \(m\)-dimensional \(T_M\)-triangular triplets into 1-dimensional \(T_M\)-triangular triplets if and only if it is increasing in each variable.

When \( T \) is a continuous Archimedean t-norm, a characterization of those functions that aggregate \(T\)-equivalence relations can be formulated in terms of an additive generator of \( T \) as follows.

**Proposition 6.** If \( T \) is a continuous Archimedean t-norm with additive generator \( g \), then \( F : [0, 1]^m \rightarrow [0, 1] \) aggregates \(T\)-equivalence relations if and only if the function \( G = gFg^{-1}(a) \) transforms (ordinary) triangular triplets of \([0, \infty]^m\) (with elements in \([0, g(0)]^m\)) into (ordinary) triangle triplets of \([0, \infty]\) (with elements in \([0, g(0)]\)) and \( G(0, \ldots, 0) = 0 \).

**Proof:** Straightforward.

**Example 4.** A function \( F : [0, 1]^m \rightarrow [0, 1], m \geq 1 \), aggregates \(T_L\)-equivalence relations if and only if \( G(a_1, \ldots, a_m) = 1 - F(\max(1 - a_1, 0), \ldots, \max(1 - a_m, 0)) \) transforms triangular triplets of \([0, \infty]^m\) (with elements in \([0, 1]^m\)) into triangle triplets of \([0, \infty]\) (with elements in \([0, 1]\)) and \( G(0, \ldots, 0) = 0 \).

Under increasingness, subadditivity\(^7\) is equivalent to the property of transforming triangular triplets into triangle triplets.

**Proposition 7.** Consider \( G : [0, \infty]^m \rightarrow [0, \infty] \). Then:

(i) If \( G \) transforms triangular triplets of \([0, \infty]^m\) into triangular triplets of \([0, \infty]\) then it is subadditive.

(ii) If \( G \) is increasing and subadditive then it transforms triangular triplets of \([0, \infty]^m\) into triangular triplets of \([0, \infty]\).

\(6\) \(g^{-1}(a_1, \ldots, a_m) = (g^{-1}(a_1), \ldots, g^{-1}(a_m)).\)

\(7\) \(G(a + b) \leq G(a) + G(b).\)
Thus, from the two previous propositions, we can enunciate the following result.

**Proposition 8.** An increasing function $F : [0, 1]^m \to [0, 1]$, with $F(1, \ldots , 1) = 1$, aggregates $T$-equivalence relations ($T$ is a continuous Archimedean t-norm with additive generator $g$) if and only if the function $G = gFg^{-1}$ is subadditive.

Consequences of the previous propositions are two known results concerning the role of weighted arithmetic means and ordered weighted arithmetic means (OWA operators) in this approach. More details on these classes of aggregation functions can be found in the recent monograph [2].

**Proposition 9.** Any weighted quasi-arithmetic mean $F(a_1, \ldots , a_m) = g^{-1}(\Sigma w_i g(a_{i(k)}))$ where the components of the weighting list $(w_1, \ldots , w_m)$ are non-negative real numbers satisfying $\Sigma w_i = 1$ and $g$ is an additive generator of a given continuous Archimedean t-norm $T$, aggregates $T$-equivalence relations.

**Proposition 10.** An ordered weighted quasi-arithmetic mean $F(a_1, \ldots , a_m) = g^{-1}(\Sigma w_i g(a_{(m-i)}))$ where $a_{(k)}$ denotes the $k$-largest input in the list $(a_1, \ldots , a_m)$ and $g$ is an additive generator of a given continuous Archimedean t-norm $T$, aggregates $T$-equivalence relations.

**Example 5.** Related to Proposition 2 and recalling that the fuzzy relation $E_T$ on $[0, 1]$ defined by $E_T(a, b) = \begin{cases} T(a, b), & \text{if } a \neq b \\ 1, & \text{otherwise} \end{cases}$ is a (decomposable) $T$-equivalence relation [7], and calculating the weighted mean of $E_T$ and $E_T$,

(i) For the Łukasiewicz t-norm $T_L$, the fuzzy relations

$$E(a, b) = \begin{cases} (2\lambda - 1)(\max(a, b) - 1) + \min(a, b), & \text{if } a + b \geq 1 \text{ and } a \neq b \\ (1 - \lambda)(1 - \max(a, b) + \min(a, b)), & \text{if } a + b < 1 \text{ and } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

are $T_L$-equivalence relations between $E_T$ and $E_T$ for any $\lambda \in [0, 1]$.

(ii) For the Product t-norm $T_P$, the fuzzy relations

$$E(a, b) = \begin{cases} \min(a, b) \max(a, b)^{2\lambda - 1}, & \text{if } a \neq b \\ 1, & \text{if } a = b \end{cases}$$

are $T_P$-equivalence relations between $E_T$ and $E_T$ for any $\lambda \in [0, 1]$.

In this way we have a way to go smoothly from $E_T$ to $E_T$.

Let us end this section by characterizing the functions that aggregate $T_D$-equivalence relations.

**Proposition 11.** A function $F : [0, 1]^m \to [0, 1]$, $m \geq 1$, aggregates $T_D$-equivalence relations if and only if
\( F(1, 1, \ldots, 1) = 1 \)

(ii) If there exists \( a_i \in [0, 1] \), \( a_i \neq 1 \), with \( F(a_1, a_2, \ldots, a_i, \ldots, a_m) = 1 \), then
\( F(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m), x \neq a_i, x \neq 1 \) is a constant function.

Proof:

\( \Leftarrow \) Straightforward.

\( \Rightarrow \) If there exists \( a_i \in [0, 1] \) with \( F(a_1, a_2, \ldots, a_i, \ldots, a_m) = 1 \) and \( (a_i, b_i, c_i) \)
is a \( T_D \)-triangular triplet with \( b_i \neq 1 \) and \( c_i \neq 1 \), then according to Example 2.
\( F(a_1, a_2, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_m) = F(a_1, a_2, \ldots, a_{i-1}, c_i, a_{i+1}, \ldots, a_m). \)

For example, every \( F : [0,1]^m \longrightarrow [0,1], m \geq 1 \) satisfying \( F(a_1, a_2, \ldots, a_m) = 1 \)
if and only if \( (a_1, a_2, \ldots, a_m) = (1,1,\ldots,1) \) aggregates \( T_D \)-equivalence relations.

4. Conclusions

In this contribution we revisit the problem of the aggregation of \( T \)-equivalence relations. After introducing the concept of \( T \)-triangular triplet, we characterize those functions that transform any collection of \( T \)-equivalence relations into a single one. The interest of this characterization is that we do not assume any extra condition on the t-norm \( T \).

Considering Functions \( F : [0,1]^m \longrightarrow [0,1], m \geq 1 \) transforming \( m \)-dimensional \( T \)-triangular triplets into 1-dimensional \( T \)-triangular triplets but not satisfying necessarily the property \( F(1,1,\ldots,1) = 1 \), we obtain the functions that preserve \( T \)-transitivity and hence more general \( T \)-transitive relations than \( T \)-equivalence relations, in particular \( T \)-preorders.

Acknowledgments

This work was supported by the projects TIN2013-42795-P and TIN2014-56381-REDT (LODISCO) (Spanish Government).

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