Hesitant fuzzy relations and their transitive closure

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Abstract

The hesitant fuzzy relations and some of their operations are defined.
Two definition of T-transitive property and their closures are defined.

Introduction

The hesitant fuzzy sets are a generalization of fuzzy sets that allow to assign few or several membership degrees to the elements of a finite universe, introducing the concept of possible hesitation of their membership degrees.

Operations and partial order $\leq_H$ on **Lists.**

Let L <[0, 1]> be a finite sorted list of membership degrees in [0, 1], from now just noted L. Sorted list are Abstract Data Types (ADT) well know in computer programming sciences that allow at least to ask its length, ask whether it is empty, and add and remove elements keeping the order of the elements.

Definition 1. Let L be a sorted list of l membership degrees in [0, 1], let n be a natural number then L x n is a sorted list of l x n membership degrees adding very membership degree n times.

Example: {0.4, 0.8, 1} x 3 = {0.4, 0.4, 0.4, 0.8, 0.8, 0.8, 1, 1, 1}

Definition 2. Let L be a sorted list of l membership degrees in [0, 1], then L[i] is the ith element of the list L, for i = 1..l.

Definition 3. Let L_1 and L_2 be lists of length l_1 and l_2 . Then $L_1 \leq_H L_2$ if and only if $L_1x(l/l_1)[i] \leq L_2x(l/l_2)[i]$ for all i= 1.. $mcm(l_1, l_2) = l$. $\leq_H$ is a partial order on the set of Lists.

Definition 4. Let T be a triangular norm. The intersection T_H of 2 lists L_1 and L_2 of length l_1 and l_2 is a list $T_H(L_1, L_2)$ of length $l = mcm(l_1, l_2)$ such that
$T_H(L_1, L_2)[i] = T(L_1x(l/l_1)[i], L_2x(l/l_2)[i])$ for all i= 1.. l

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Let E be a finite universe of discourse.

Definition 5. A hesitant fuzzy set μ_H on a universe E is a mapping $\mu_H: E \to L$ where L is a sorted list of membership degrees in [0, 1].

Let l_i be the length of the list $L_i = \mu_H(e_i)$. So l_i is the number of membership degrees of e_i.
Let l_H be the mcm of all l_i for all i in 1.card(E).

Definition 6. A hesitant fuzzy set bag B_μ of a hesitant fuzzy set μ_H on a universe E is a mapping
$B_\mu: E \to L_B$ where L_B is a sorted list of length l_H and $B_\mu(e_i) = L_i \times (l_H/l_i)$

Example: Let E = {e_1, e_2}, Let μ_H = {{0.4, 0.6}, {0.5, 0.7, 1}}
Then B_μ = {{0.4, 0.4, 0.4, 0.6, 0.6, 0.6}, {0.5, 0.5, 0.7, 0.7, 1, 1} }
Note that{0.4, 0.6}$\leq_H${0.5, 0.7, 1}cause{0.4, 0.4, 0.4, 0.6, 0.6, 0.6}$\leq${0.5, 0.5, 0.7, 0.7, 1, 1}

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Definition 7. A hesitant fuzzy relation R_H on a universe E is a mapping R_H: ExE$\rightarrow$ L where L is a sorted list of membership degrees in [0, 1].
Let l_{ij} be the length of the list $L_{ij} = R_H(e_i, e_i)$.
Let l_R be the mcm of all l_{ij} for all i, j in 1..card(E).
Definition 8. A hesitant fuzzy relation bag B_R of a hesitant fuzzy relation R_H is a mapping B_R: E x E$\rightarrow L_B$ where L_B is a sorted list of length l_R and $B_R(e_i, e_i) = L_{ij}$ x (l_R/l_{ij})
Example: Let E ={e_1, e_2}, Let μ_H = { {0.4, 0.8}, {0.5, 0.7, 1} }

Let $R_H = \begin{pmatrix} 1 & \{0.5, 0.7, 1\} \\ \{0.4, 0.8\} & 0 \end{pmatrix}$

Then $B_R = \begin{pmatrix} \{1,1,1,1,1,1\} & \{0.5, 0.5, 0.7, 0.7, 1, 1\} \\ \{0.4, 0.4, 0.4, 0.8, 0.8, 0.8\} & \{0, 0, 0, 0, 0, 0\} \end{pmatrix}$

T-Transitive property and T-transitive closure of Hesitant fuzzy relations
Definition 9. Let T be a triangular norm. A hesitant fuzzy relation R_H on a universe E is T-transitive if and only if $T_H(R_H(a, b), R_H(b, c)) \leq_H R_H(a, c)$ for all a, b, c in E.
Definition 10. Let T be a triangular norm. Let R_H be hesitant fuzzy relation on a universe E . The T transitive closure of R_H is the $\leq_H$-lowest relation $R_H{}^T$ that is T-transitive and $\leq_H$ -contains R_H
Theorem 1. The T-transitive closure of a hesitant fuzzy relation always exists and it is unique

Example: Let E = {e_1, e_2}, Let $R_H = \begin{pmatrix} 1 & \{0.5, 0.7, 1\} \\ \{0.4, 0.8\} & 0 \end{pmatrix}$

Then $B_R = \begin{pmatrix} \{1,1,1,1,1,1\} & \{0.5, 0.5, 0.7, 0.7, 1, 1\} \\ \{0.4, 0.4, 0.4, 0.8, 0.8, 0.8\} & \{0, 0, 0, 0, 0, 0\} \end{pmatrix}$

Then $R_H{}^{Min}$, $R_H{}^{Prod}$, $R_H{}^{W}$ (closures for the t-norms min, product and Lucasiewicz) are

$$R_H{}^{Min} = \begin{pmatrix} \{1,1,1,1,1,1\} & \{0.5, 0.5, 0.7, 0.7, 1, 1\} \\ \{0.4, 0.4, 0.4, 0.8, 0.8, 0.8\} & \{0.4, 0.4, 0.4, 0.7, 0.8, 0.8\} \end{pmatrix}$$

$$R_H{}^{Prod} = \begin{pmatrix} \{1,1,1,1,1,1\} & \{0.5, 0.5, 0.7, 0.7, 1, 1\} \\ \{0.4, 0.4, 0.4, 0.8, 0.8, 0.8\} & \{0.2, 0.2, 0.28, 0.56, 0.8, 0.8\} \end{pmatrix}$$

$$R_H{}^{W} = \begin{pmatrix} \{1,1,1,1,1,1\} & \{0.5, 0.5, 0.7, 0.7, 1, 1\} \\ \{0.4, 0.4, 0.4, 0.6, 0.6, 0.6\} & \{0, 0, 0.1, 0.5, 0.8, 0.8\} \end{pmatrix}$$

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