

Locating domination in bipartite graphs and their complements

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Abstract

A set S of vertices of a graph G is *distinguishing* if the sets of neighbors in S for every pair of vertices not in S are distinct. A *locating-dominating set* of G is a dominating distinguishing set. The *location-domination number* of G , $\lambda(G)$, is the minimum cardinality of a locating-dominating set. In this work we study relationships between $\lambda(G)$ and $\lambda(\overline{G})$ for bipartite graphs. The main result is the characterization of all connected bipartite graphs G satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. To this aim, we define an edge-labeled graph G^S associated with a distinguishing set S that turns out to be very helpful.

Keywords: domination; location; distinguishing set; locating domination; complement graph; bipartite graph.

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1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The distance between two vertices v and w is denoted by $d_G(v, w)$. The *neighborhood* of a vertex $u \in V$ is $N_G(u) = \{v : uv \in E\}$. We write $N(u)$ or $d(v, w)$ if the graph G is clear from the context. For any $S \subseteq V$, $N(S) = \cup_{u \in S} N(u)$. A set $S \subseteq V$ is *dominating* if $V = S \cup N(S)$ (see [7]). For further notation and terminology, we refer the reader to [4].

A set $S \subseteq V$ is *distinguishing* if $N(u) \cap S \neq N(v) \cap S$ for every pair of different vertices $u, v \in V \setminus S$. In general, if $N(u) \cap S \neq N(v) \cap S$, we say that S *distinguishes* the pair u and v . A *locating-dominating set*, *LD-set* for short, is a distinguishing set that is also dominating. Observe that there is at most one vertex not dominated by a distinguishing set. The *location-domination number* of G , denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [12, 13]. Certainly, every LD-set of a non-connected graph G is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. Both, LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 3, 5, 6, 8, 9, 10]. A complete and regularly updated list of papers on locating-dominating codes is to be found in [11].

The *complement* of G , denoted by \overline{G} , has the same set of vertices of G and two vertices are adjacent in \overline{G} if and only if they are not adjacent in G . This work is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ for connected bipartite graphs.

It follows immediately from the definitions that a set $S \subseteq V$ is distinguishing in G if and only if it is distinguishing in \overline{G} . A straightforward consequence of this fact are the following results.

Proposition 1 ([9]). *Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, S is an LD-set of \overline{G} if and only if S is a dominating set of \overline{G} ;*

Proposition 2 ([8]). *Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, the following properties hold.*

- (a) *There is at most one vertex $u \in V \setminus S$ such that $N(u) \cap S = S$, and in the case it exists, $S \cup \{u\}$ is an LD-set of \overline{G} .*
- (b) *S is an LD-set of \overline{G} if and only if there is no vertex in $V \setminus S$ such that $N(u) \cap S = S$.*

Theorem 1 ([8]). *For every graph G , $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding inequality, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$ for every graph G , all cases being feasible for some connected graph G . We intend to determine graphs such that $\lambda(\overline{G}) > \lambda(G)$, that is, we want to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was completely solved in [9] for the family of block-cactus.

In this work, we carry out a similar study for bipartite graphs. For this purpose, we first introduce in Section 2 the graph associated with a distinguishing set. This graph turns out to be very helpful to derive some properties related to LD-sets and the location-domination number of G , and will be used to get the main results in Section 3.

In Table 1, the location-domination number of some families of bipartite graphs are displayed, along with the location-domination number of its complement graphs. Concretely, we consider the path P_n of order $n \geq 4$; the cycle C_n of (even) order $n \geq 4$; the star $K_{1,n-1}$ of order $n \geq 4$, obtained by joining a new vertex to $n - 1$ isolated vertices; the complete bipartite graph $K_{r,n-r}$ of order $n \geq 4$, with $2 \leq r \leq n - r$ and stable sets of order r and $n - r$, respectively; and finally, the bi-star $K_2(r, s)$ of order $n \geq 6$ with $3 \leq r \leq s = n - r$, obtained by joining the central vertices of two stars $K_{1,r-1}$ and $K_{1,s-1}$ respectively.

Proposition 3 ([9]). *Let G be a graph of order $n \geq 4$. If G is a graph belonging to one of the following classes: $P_n, C_n, K_{1,n-1}, K_{r,n-r}, K_2(r, s)$, then the values of $\lambda(G)$ and $\lambda(\overline{G})$ are known and they are displayed in Table 1.*

G	P_n	P_n	C_n	C_n
n	$4 \leq n \leq 6$	$n \geq 7$	$4 \leq n \leq 6$	$n \geq 7$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$
$\lambda(\overline{G})$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$
G	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r, s)$	
n	$n \geq 4$	$2 \leq r \leq n - r$	$3 \leq r \leq s$	
$\lambda(G)$	$n - 1$	$n - 2$	$n - 2$	
$\lambda(\overline{G})$	$n - 1$	$n - 2$	$n - 3$	

Table 1: The values of $\lambda(G)$ and $\lambda(\overline{G})$ for some families of bipartite graphs.

Notice that in all cases considered in Proposition 3, we have $\lambda(\overline{G}) \leq \lambda(G)$. Moreover, for every pair of integers (r, s) , with $3 \leq r \leq s$ we have examples of bipartite graphs with stable sets of order r and s respectively, such that $\lambda(\overline{G}) = \lambda(G)$ and such that $\lambda(\overline{G}) = \lambda(G) - 1$.

2 The graph associated with a distinguishing set

Let S be a distinguishing set of a graph G . We introduce in this section a labeled graph associated with S and study some general properties. Since LD-sets are distinguishing sets that are also dominating, this graph allows us to derive some properties related to LD-sets and the location-domination number of G .

Definition 1. Let S be a distinguishing set of cardinality k of a graph $G = (V, E)$ of order n . The so-called S -associated graph, denoted by G^S , is the edge-labeled graph defined as follows.

- i) $V(G^S) = V \setminus S$;
- ii) If $x, y \in V(G^S)$, then $xy \in E(G^S)$ if and only if the sets of neighbors of x and y in S differ in exactly one vertex $u(x, y) \in S$;
- iii) The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only vertex $u(x, y) \in S$ described in the preceding item.

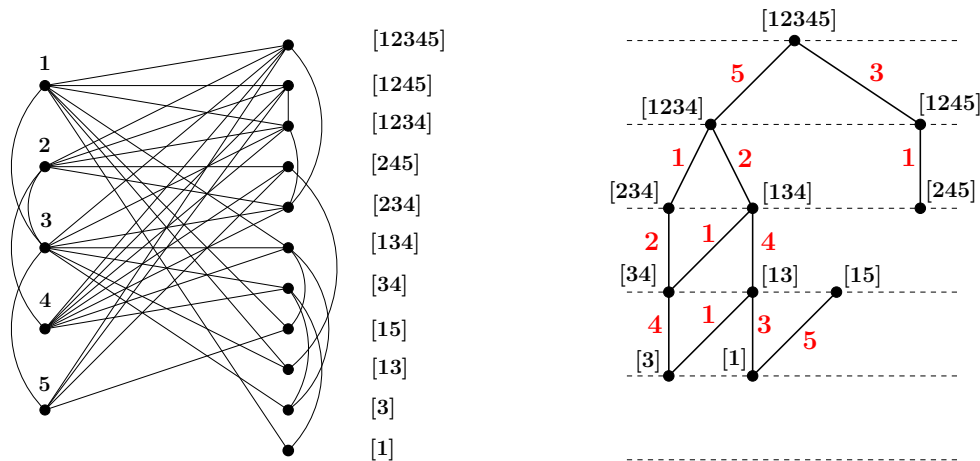


Figure 1: A graph G (left) and the graph G^S associated with the distinguishing set $S = \{1, 2, 3, 4, 5\}$ (right). The neighbors in S of each vertex are those enclosed in brackets.

Notice that if $xy \in E(G^S)$, $\ell(xy) = u \in S$ and $|N(x) \cap S| > |N(y) \cap S|$, then $N(x) \cap S = (N(u) \cap S) \cup \{u\}$. Therefore, we can represent the graph G^S with the vertices lying on $|S| + 1 = k + 1$ levels, from bottom (level 0) to top (level k), in such a way that vertices with exactly j neighbors in S are at level j . For any $j \in \{0, 1, \dots, k\}$ there are at most $\binom{k}{j}$ vertices at level j . So, there is at most one vertex at level k and, if it is so, this vertex is adjacent to all vertices of S . There is at most one vertex at level 0 and, if it is so, this vertex

has no neighbors in S . The vertices at level 1 are those with exactly one neighbor in S . See Figure 1 for an example of an LD-set-associated graph.

Next, we state some basic properties of the graph associated with a distinguishing set that will be used later.

Proposition 4. *Let S be a distinguishing set of $G = (V, E)$, $x, y \in V \setminus S$ and $u \in S$. Then,*

- (1) S is a distinguishing set of \overline{G} .
- (2) The associated graphs G^S and \overline{G}^S are equal.
- (3) The representation by levels of \overline{G}^S is obtained by reversing bottom-top the representation of G^S .
- (4) $xy \in E(G^S)$ and $\ell(xy) = u$ if and only if x and y have the same neighborhood in $S \setminus \{u\}$ and (thus) they are not distinguished by $S \setminus \{u\}$.
- (5) If $xy \in E(G^S)$ and $\ell(xy) = u$, then $S \setminus \{u\}$ is not a distinguishing set.

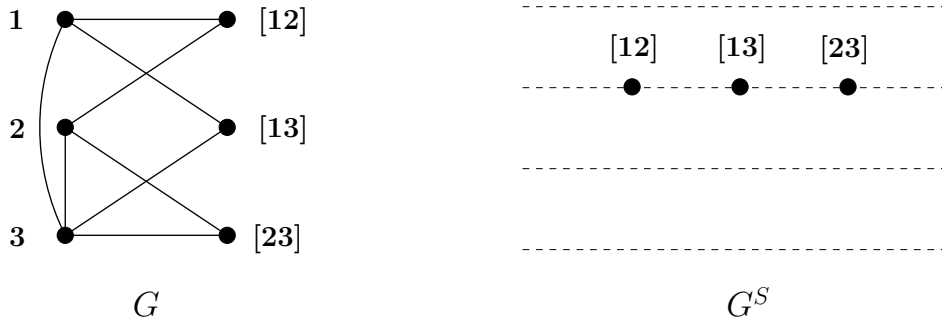


Figure 2: $S = \{1, 2, 3\}$ is distinguishing, $S' = \{1, 2\}$ is not distinguishing and G^S has no edges.

The converse of Proposition 4 (5) is not necessarily true. For example, consider the graph G of order 6 displayed in Figure 2. By construction, $S = \{1, 2, 3\}$ is a distinguishing set. However, $S' = S \setminus \{3\} = \{1, 2\}$ is not a distinguishing set, because $N(3) \cap S' = N([12]) \cap S' = \{1, 2\}$, and the S -associated graph G^S has no edge with label 3 (in fact, G^S has no edges since the neighborhood in S of all vertices not in S have the same size).

As a straight consequence of Proposition 4 (5), the following result is derived.

Corollary 1. *Let S be a distinguishing set of G and let $S' \subseteq S$. Consider the subgraph $H_{S'}$ of G^S induced by the edges with a label from S' . Then, all the vertices belonging to the same connected component in $H_{S'}$ have the same neighborhood in $S \setminus S'$, concretely, it is the neighborhood in S of a vertex from the connected component lying on the lowest level.*

For example, consider the graph shown in Figure 1. If $S' = \{1, 2\}$, then vertices of the same connected component in $H^{S'}$ have the same neighborhood in $S \setminus S'$. Concretely, the neighborhood of vertices $[1234]$, $[234]$, $[134]$ and $[34]$ in $S \setminus \{1, 2\}$ is $\{3, 4\}$; the neighborhood of vertices $[13]$ and $[3]$ in $S \setminus \{1, 2\}$ is $\{3\}$; and the neighborhood of vertices $[1245]$ and $[245]$ in $S \setminus \{1, 2\}$ is $\{4, 5\}$ (see Figure 3).

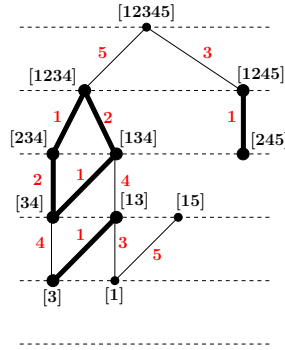


Figure 3: If $S' = \{1, 2\}$, then $H^{S'} \cong C_4 + 2K_2$ has three components. Vertices of the same component in $H^{S'}$ have the same neighborhood in $S \setminus S'$.

Proposition 5. *Let S be a distinguishing set of cardinality k of a connected graph $G = (V, E)$ of order n . Let G^S be its associated graph. Then, the following conditions hold.*

1. $|V(G^S)| = n - k$.
2. G^S is bipartite.
3. Incident edges of G^S have different labels.
4. Every cycle of G^S contains an even number of edges labeled v , for all $v \in S$.
5. Let ρ be a walk with no repeated edges in G^S . If ρ contains an even number of edges labeled v for every $v \in S$, then ρ is a closed walk.
6. If $\rho = x_i x_{i+1} \dots x_{i+h}$ is a path satisfying that vertex x_{i+h} lies at level $i + h$, for any $h \in \{0, 1, \dots, h\}$, then
 - (a) the edges of ρ have different labels;
 - (b) for all $j \in \{i + 1, i + 2, \dots, i + h\}$, $N(x_j) \cap S$ contains the vertex $\ell(x_k x_{k+1})$, for any $k \in \{i, i + 1, \dots, j - 1\}$.

Proof. 1. It is a direct consequence from the definition of G^S .

2. Take the sets $V_1 = \{x \in V(G^S) : |N(x) \cap S| \text{ is odd}\}$ and $V_2 = \{x \in V(G^S) : |N(x) \cap S| \text{ is even}\}$. Then, $V(G^S) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Since $||N(x) \cap S| - |N(y) \cap S|| = 1$ for any $xy \in E(G^S)$, it is clear that the vertices x, y are not in the same subset V_i , $i = 1, 2$.
3. Suppose that edges $e_1 = xy$ and $e_2 = yz$ have the same label $l(e_1) = l(e_2) = v$. This means that $N(x) \cap S$ and $N(y) \cap S$ differ only in vertex v , and $N(y) \cap S$ and $N(z) \cap S$ differ only in vertex v . It is only possible if $N(x) \cap S = N(z) \cap S$, implying that $x = z$.
4. Let ρ be a cycle such that $E(\rho) = \{x_0x_1, x_1x_2, \dots, x_hx_0\}$. The set of neighbors in S of two consecutive vertices differ exactly in one vertex. If we begin with $N(x_0) \cap S$, then each time we add (remove) the vertex of the label of the corresponding edge, we have to remove (add) it later in order to obtain finally the same neighborhood, $N(x_0) \cap S$. Therefore, ρ contains an even number of edges with label v .
5. Consider the vertices $x_0, x_1, x_2, x_3, \dots, x_{2k}$ of ρ . In this case, $N(x_{2k}) \cap S$ is obtained from $N(x_0) \cap S$ by either adding or removing the labels of all the edges of the walk. As every label appears an even number of times, for each element $v \in S$ we can match its appearances in pairs, and each pair means that we add and remove (or remove and add) it from the neighborhood in S . Therefore, $N(x_{2k}) \cap S = N(x_0) \cap S$, and hence $x_0 = x_{2k}$.
6. It straightly follows from the fact that $N(x_j) \cap S = N(x_{j-1} \cap S) \cup \{\ell(x_{j-1}x_j)\}$, for any $j \in \{i+1, \dots, i+h\}$. \square

In the study of distinguishing sets and LD-sets using its associated graph, a family of graphs is particularly useful, the *cactus graph* family. A *block* of a graph is a maximal connected subgraph with no cut vertices. A connected graph G is a *cactus* if all its blocks are either cycles or edges. Cactus are characterized as those connected graphs with no edge shared by two cycles.

Lemma 1. *Let S be a distinguishing set of a graph G and $S' \subseteq S$. Consider a subgraph H of G^S induced by a set of edges containing exactly two edges with label u , for each $u \in S' \subseteq S$. Then, all the connected components of H are cactus.*

Proof. We prove that there is no edge lying on two different cycles of H . Suppose, on the contrary, that there is an edge e_1 contained in two different cycles C_1 and C_2 of H . Note that C_1 and C_2 are cycles of G^S , since $S' \subseteq S$. Hence, if the label of e_1 is $u \in S' \subseteq S$, then by Proposition 5 both cycles C_1 and C_2 contain the other edge e_2 of H with label u . Suppose that $e_1 = x_1y_1$ and $e_2 = x_2y_2$ and assume w.l.o.g. that there exist $x_1 - x_2$ and $y_1 - y_2$ paths in C_1 not containing edges e_1, e_2 . Let P_1 and P'_1 denote respectively those paths (see Figure 4 a).

We have two possibilities for C_2 : (i) there are $x_1 - x_2$ and $y_1 - y_2$ paths in C_2 not containing neither e_1 nor e_2 . Let P_2 denote the $x_1 - x_2$ path in C_2 in that case (see Figure 4 b); (ii) there are $x_1 - y_2$ and $y_1 - x_2$ paths in C_2 not containing neither e_1 nor e_2 (see Figure 4 c).

In case (ii), the closed walk formed with the path P_1 , e_1 and the $y_1 - x_2$ path in C_2 would contain a cycle with exactly an edge labeled with u , which is a contradiction (see Figure 4 d).

In case (i), at least one the following cases hold: either the $x_1 - x_2$ paths in C_1 and in C_2 , P_1 and P_2 , are different, or the $y_1 - y_2$ paths in C_1 and in C_2 are different (otherwise, $C_1 = C_2$).

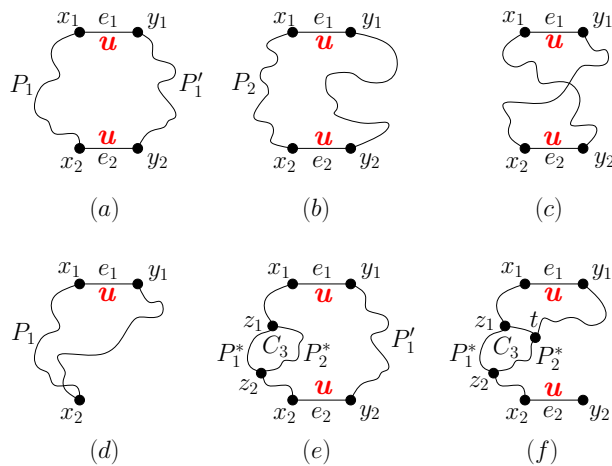


Figure 4: All connected components of the subgraph H are cactus.

Assume that P_1 and P_2 are different. Let z_1 be the last vertex shared by P_1 and P_2 advancing from x_1 and let z_2 be the first vertex shared by P_1 and P_2 advancing from z_1 in P_2 . Notice that $z_1 \neq z_2$. Take the cycle C_3 formed with the $z_1 - z_2$ paths in P_1 and P_2 . Let P_1^* and P_2^* be respectively the $z_1 - z_2$ subpaths of P_1 and P_2 (see Figure 4 e). We claim that the internal vertices of P_2^* do not lie in P_1' . Otherwise, consider the first vertex t of P_1' lying also in P_2^* . The cycle beginning in x_1 , formed by the edge e_1 , the $y_1 - t$ path contained in P_1' , the $t - z_1$ path contained in P_2^* and the $z_1 - x_1$ path contained in P_1 has exactly one appearance of an edge with label u , which is a contradiction (see Figure 4 f). By Proposition 5, the labels of edges belonging to P_1^* appear exactly two times in cycle C_3 , but they also appear exactly two times in cycle C_1 . But this is only possible if they appear exactly two times in P_1^* , since H contains exactly to edges with the same label. By Proposition 5, P_1^* must be a closed path, which is a contradiction. \square

Next, we establish some properties relating parameters of bipartite graphs having cactus as connected components.

Lemma 2. *Let H be a bipartite graph of order at least 4 such that all its connected components are cactus. Let $cc(H)$ denote the number of connected components of H . Then, $|V(H)| \geq \frac{3}{4}|E(H)| + cc(H) \geq \frac{3}{4}|E(H)| + 1$.*

Proof. Let $cc(H)$ be the number of connected components and $cy(H)$ the number of cycles of H . Since H is a planar graph with $cy(H) + 1$ faces and $cc(H)$ connected components, the equality follows from the generalization of Euler's Formula:

$$(cy(H) + 1) + |V(H)| = |E(H)| + (cc(H) + 1).$$

Let $ex(H) = |E(H)| - 4cy(H)$. Then,

$$\begin{aligned} |V(H)| &= |E(H)| - cy(H) + cc(H) = |E(H)| - \frac{1}{4}(|E(H) - ex(H)|) + cc(H) \\ &= \frac{3}{4}|E(H)| + \frac{1}{4}ex(H) + cc(H). \end{aligned}$$

But $cc(H) \geq 1$, and $ex(H) \geq 0$ because all cycles of a bipartite graph have at least 4 edges. Therefore,

$$|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}ex(H) + cc(H) \geq \frac{3}{4}|E(H)| + cc(H) \geq \frac{3}{4}|E(H)| + 1. \quad \square$$

Corollary 2. *Let S be a distinguishing set of a graph G . Consider a subgraph $H_{S'}$ of G^S induced by a set of edges containing exactly two edges with label u for each $u \in S' \subseteq S$. Let $r' = |S'|$. Then, $|V(H_{S'})| \geq \frac{3}{2}r' + 1$.*

3 The bipartite case

This section is devoted to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being U, W their stable sets and $1 \leq |U| = r \leq s = |W|$.

In the study of LD-sets, vertices with the same neighborhood play an important role, since at least one of them must be in an LD-set. We say that two vertices u and v are *twins* if either $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$.

Lemma 3. *Let S be an LD-code of G . Then, $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions hold.*

i) $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$.

ii) $r < s$ and $S = W$.

iii) $2^r \leq s$.

Proof. If S satisfies item i), then the LD-code of G is a distinguishing set of \overline{G} and it is dominating in \overline{G} because there is no vertex in G with neighbors in both stable sets. Thus, $\lambda(\overline{G}) \leq \lambda(G)$.

Next, assume that $r < s$ and $S = W$. In this case, $\lambda(G) = |W| > |U|$ and thus U is not an LD-set, but is it a dominating set since G is connected. Therefore, there exists a pair of vertices $w_1, w_2 \in W$ such that $N(w_1) = N(w_2)$. Hence, $W - \{w_1\}$ is an LD-set of $G - w_1$. Let $u \in S$ be a vertex adjacent to w_1 (it exists since G is connected), and notice that $(W \setminus \{w_1\}) \cup \{u\}$ is an LD-code of G with vertices in both stable sets, which, by the preceding item, means that $\lambda(\overline{G}) \leq \lambda(G)$.

Finally, if $2^r \leq s$ then $S \neq U$, which means that S satisfies either item i) or item ii). \square

Proposition 6. *If G has order at least 3 and $1 \leq r \leq 2$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proof. If $r = 1$, then G is the star $K_{1,n-1}$ and $\lambda(\overline{G}) = \lambda(G) = n - 1$.

Suppose that $r = 2$. We distinguish cases (see Figure 5).

- If $s \geq 2^2 = 4$ then, by Lemma 3, $\lambda(\overline{G}) \leq \lambda(G)$.
- If $s = 2$, then G is either P_4 and $\lambda(\overline{P_4}) = \lambda(P_4) = 2$, or G is C_4 and $\lambda(\overline{C_4}) = \lambda(C_4) = 2$.
- If $s = 3$, then G is either P_5 , $K_{2,3}$, $K_2(1, 2)$, or the banner P , and $\lambda(\overline{P_5}) = \lambda(P_5) = 2$, $\lambda(\overline{K_{2,3}}) = \lambda(K_{2,3}) = 3$, $2 = \lambda(\overline{K_2(1, 2)}) < \lambda(K_2(1, 3)) = 3$, and $2 = \lambda(\overline{P}) < \lambda(P) = 3$. \square

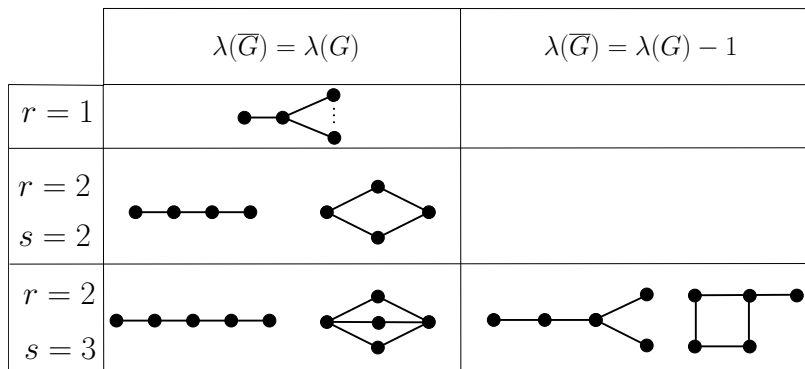


Figure 5: Some bipartite graphs with $1 \leq r \leq 2$.

From now on, we assume that $r \geq 3$.

Proposition 7. *If $r = s$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proof. If G has an LD-code with vertices at both stable sets, then $\lambda(\overline{G}) \leq \lambda(G)$ by Lemma 3. In any other case, G has at most two LD-codes, U and W .

If both U and W are LD-codes, then we distinguish the following cases.

- If there is no vertex $u \in U$ such that $N(u) = W$, then W is an LD-set of \overline{G} , and consequently, $\lambda(\overline{G}) \leq \lambda(G)$.
- Analogously, if there is no vertex $w \in W$ such that $N(w) = U$, then we derive $\lambda(\overline{G}) \leq \lambda(G)$.
- If there exist vertices $u \in U$ and $w \in W$ such that $N(u) = W$ and $N(w) = U$, then $(U - \{u\}) \cup \{w\}$ would be an LD-set of \overline{G} , and thus $\lambda(\overline{G}) \leq \lambda(G)$.

Next, assume that U is an LD-code and W is not an LD-code of G . If there is no vertex $w \in W$ such that $N(w) = U$, then U is an LD-set of \overline{G} , and so $\lambda(\overline{G}) \leq \lambda(G)$. Finally, suppose that there is a vertex $w \in W$ such that $N(w) = U$. Note that W is not a distinguishing set of G (otherwise, it would be an LD-code because W is a dominating set of size r). Therefore, there exist vertices $x, y \in U$ such that $N(x) = N(y)$. In such a case, $(U \setminus \{x\}) \cup \{w\}$ is an LD-set of \overline{G} , and thus $\lambda(\overline{G}) \leq \lambda(G)$. \square

From Lemma 3 and Proposition 7 we derive the following result.

Corollary 3. *If $\lambda(\overline{G}) = \lambda(G) + 1$, then $r < s \leq 2^r - 1$ and U is the only LD-code of G .*

Next theorem characterizes connected bipartite graphs satisfying the equation $\lambda(\overline{G}) = \lambda(G) + 1$ in terms of the graph associated with a distinguishing set.

Theorem 2. *Let $3 \leq r < s$. Then, $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if the following conditions hold.*

- i) W has no twins.*
- ii) There exists a vertex $w \in W$ such that $N(w) = U$.*
- iii) For every vertex $u \in U$, the graph G^U has at least two edges with label u .*

Proof. \Leftarrow) Condition *i)* implies that U is an LD-set of G and, hence, $\lambda(G) \leq r$. Let S be an LD-code of \overline{G} . We next prove that S has at least $r + 1$ vertices. Note that $S \neq U$, since U is not a dominating set in \overline{G} . Consider the graph G^U associated with U . Let $H_{U \setminus S}$ be the subgraph of G^U induced by the set of edges with a label in $U \setminus S \neq \emptyset$. By Corollary 1, the

vertices of the same connected component in $H_{U \setminus S}$ have the same neighborhood in $U \cap S$. Besides, W induces a complete graph in \overline{G} . Hence, $S \cap W$ must contain at least all but one vertices of the same connected component of $H_{U \setminus S}$, otherwise \overline{G} would contain vertices with the same neighborhood in S . Therefore, $S \cap W$ has at least $|V(H_{U \setminus S})| - cc(H_{U \setminus S})$ vertices, where $cc(H_{U \setminus S})$ is the number of connected components of $H_{U \setminus S}$. Lemmas 1 and 2 imply that $|V(H_{U \setminus S})| - cc(H_{U \setminus S}) \geq \frac{3}{4}|E(H_{U \setminus S})|$, and condition iii) implies that $|E(H_{U \setminus S})| \geq 2|U \setminus S|$. Therefore,

$$\begin{aligned} |S| &= |S \cap U| + |S \cap W| \geq |S \cap U| + |V(H_{U \setminus S})| - cc(H_{U \setminus S}) \\ &\geq |S \cap U| + \frac{3}{4}|E(H_{U \setminus S})| \geq |S \cap U| + \frac{3}{2}|U \setminus S| \\ &= |U| + \frac{1}{2}|U \setminus S| > |U| = r. \end{aligned}$$

\Rightarrow) By Corollary 3, U is the only LD-code of G and hence, U is not an LD-set of \overline{G} . Therefore, W has no twins and $N(w) = U$ for some $w \in W$. It only remains to prove that condition iii) holds. Suppose on the contrary that there is at most one edge in G^U with label u for some $u \in U$. We consider two cases.

If there is no edge with label u , then by Proposition 4, $U \setminus \{u\}$ distinguishes all pairs of vertices of W in \overline{G} . Let $S = (U \setminus \{u\}) \cup \{w\}$. We claim that S is an LD-set of \overline{G} . Indeed, S is a dominating set in \overline{G} , because u is adjacent to any vertex in $U \setminus \{u\}$ and vertices in $W \setminus \{w\}$ are adjacent to w . It only remains to prove that S distinguishes the pairs of vertices of the form u and v , when $v \in W \setminus \{w\}$. But $w \in N_{\overline{G}}(v) \cap S$ and $w \notin N_{\overline{G}}(u) \cap S$. Thus, S is an LD-set of \overline{G} , implying that $\lambda(\overline{G}) \leq |S| = |U| = \lambda(G)$, a contradiction.

If there is exactly one edge xy with label u , then only one of the vertices x or y is adjacent to u in G . Assume that $ux \in E(G)$. Recall that $x, y \in W$. By Proposition 4, $U \setminus \{u\}$ distinguishes all pairs of vertices of W , except the pair x and y , in \overline{G} . Let $S = (U \setminus \{u\}) \cup \{x\}$. We claim that S is an LD-set of \overline{G} . Indeed, S is a dominating set in \overline{G} , because u is adjacent to any vertex in $U \setminus \{u\}$ and vertices in $W \setminus \{x\}$ are adjacent to x . It only remains to prove that S distinguishes the pairs of vertices of the form u and v , when $v \in W \setminus \{x\}$. But $x \in N_{\overline{G}}(v) \cap S$ and $x \notin N_{\overline{G}}(u) \cap S$. Thus, S is an LD-set of \overline{G} , implying that $\lambda(\overline{G}) \leq |S| = |U| = \lambda(G)$, a contradiction. \square

Observe that condition iii) of Theorem 2 is equivalent to the existence of at least two pairs of twins in $G - u$, for every vertex $u \in U$. Therefore, it can be stated as follows.

Theorem 3. *Let $3 \leq r < s$. Then, $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if the following conditions hold:*

- i) W has no twins;
- ii) There exists a vertex $w \in W$ such that $N(w) = U$;

iii) For every vertex $u \in U$, the graph $G - u$ has at least two pairs of twins in W .

We already know that it is not possible to have $\lambda(\overline{G}) = \lambda(G) + 1$ when $s > 2^r$. However, the condition $s \leq 2^r - 1$ is not sufficient to ensure the existence of bipartite graphs satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. We next show that there are graphs satisfying this equation if and only if $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

Proposition 8. *If $r \geq 3$ and $\lambda(\overline{G}) = \lambda(G) + 1$, then $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.*

Proof. By Corollary 3, we have that $s \leq 2^r - 1$. Consider a subgraph H of G^U with exactly two edges with label u for every $u \in S$. Now inequality $\frac{3r}{2} + 1 \leq s$ immediately follows from Proposition 2 and Corollary 2. \square

Proposition 9. *For every pair (r, s) , $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(\overline{G}) = \lambda(G) + 1$.*

Proof. Let $s = \lceil \frac{3r}{2} + 1 \rceil$. Take the bipartite graph $G(r, \lceil \frac{3r}{2} + 1 \rceil)$ such that $V = U \cup W$, $U = [r] = \{1, 2, \dots, r\}$, and $W \subseteq \mathcal{P}([r]) \setminus \{\emptyset\}$ is defined as follows (see Figure 3). For $r = 2k$ even:

$$W = \{[r]\} \cup \{[r] \setminus \{i\} : i \in [r]\} \cup \{[r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k\}$$

and for $r = 2k + 1$ odd:

$$W = \{[r]\} \cup \{[r] \setminus \{i\} : i \in [r]\} \cup \{[r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k\} \cup \{[r] \setminus \{r - 1, r\}\}.$$

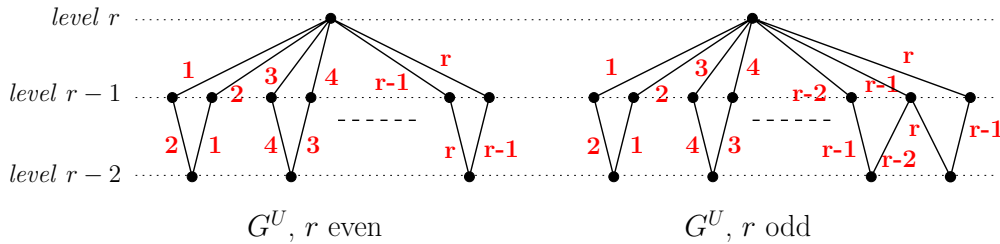


Figure 6: The labeled graph G^U , for $G = G(r, \lceil \frac{3r}{2} + 1 \rceil)$ and $U = \{1, \dots, r\}$.

By construction, W has no twins, there is a vertex w such that $N(w) = U$ and the U -associated graph, G^U has at least two edges with label u for every $u \in U$. Hence, $\lambda(\overline{G}) = \lambda(G) + 1$ by Theorem 2.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set W of the graph $G(r, \lceil \frac{3r}{2} + 1 \rceil)$ taking into account that the neighborhoods in S of vertices in W must be different and non-empty. \square

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