Metric-locating-dominating partitions in graphs

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Abstract

A partition $\Pi = \{S_1, \ldots, S_k\}$ of the vertex set of a connected graph G is a *metric-locating partition* of G if for every pair of vertices u, v belonging to the same part S_i, $d(u, S_j) \neq d(v, S_j)$, for some other part S_j. The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G. A metric-locating partition Π is called *metric-locating-dominanting* if for every vertex v of G, $d(v, S_j) = 1$, for some part S_j of Π. The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of a metric-locating-dominating partition of G.

In this paper we show, among other results, that $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G)+1$. We also characterize all connected graphs of order $n \geq 7$ satisfying any of the following conditions: $\eta_p(G) \geq n-1$, $\eta_p(G) = n-2$ and $\beta_p(G) = n-2$. Finally, we present some tight Nordhaus-Gaddum bounds for both the partition dimension $\beta(G)$ and the partition metric-location-domination number $\eta(G)$.

Keywords: dominating partition, locating partition, location, domination, metric location.

AMS subject classification: 05C12, 05C35, 05C69.

1 Introduction

Domination and location in graphs are two important subjects that have deserved a lot of attention, usually separately, but sometimes also both together. There are mainly two types of location, the metric location and the neighbor location. In this work, we are just interested in the metric location, and study both concepts in the particular context of vertex partitions, i.e., we consider those partitions of the vertex set of a certain graph that are both dominating and metric-locating,

Metric location in sets was simultaneously introduced by P. Slater [14] and F. Harary and R. A. Melter [10]. In [9], M. A. Henning y O. Oellermann introduced the so-called *metric-locating-dominating* sets, by merging the concepts of a metric-locating set and a dominating set. In [3], G. Chartrand, E. Salehi and P. Zhang, brought the concept of metric-location to the ambit of vertex partitions. In this work, starting from both works [3, 9], we introduce the so-called *metric-locating-dominating partitions*.

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1.1 Basic terminology

All the graphs considered are undirected, simple, finite and connected. Let $v \in V$ be a vertex of a graph $G = (V, E)$. The *open neighborhood* of v is $N_G(v) = \{w \in V : vw \in E\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$ (we will denote $N(v)$ and $N[v]$ if the graph G is clear from the context). The *degree* of v is $\deg(v) = |N(v)|$. The minimum degree (resp. maximum degree) of G is $\delta(G) = \min\{\deg(u) : u \in V\}$ (resp. $\Delta(G) = \max\{\deg(u) : u \in V\}$). If $N_G[v] = V(G)$ (resp. $\deg(v) = 1$), then v is called *universal* (resp. a *leaf*).

Let $W \subseteq V$ be a subset of vertices of a graph G. The open neighborhood of W is $N(W) = \cup_{v \in W} N(v)$, and the closed neighborhood of W is $N[W] = \cup_{v \in W} N(v) = N(W) \cup W$. The subgraph of G induced by W, denoted by $G[W]$, has W as vertex set and $E(G[W]) = \{vw \in E(G) : v \in W, w \in W\}$.

The *complement* of G, denoted by $\overline{G}$, is the graph on the same vertices as G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let G_1, G_2 be two graphs having disjoint vertex sets. The (disjoint) *union* $G = G_1 + G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* $G = G_1 \vee G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

The distance between vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$, or $d(v, w)$ if the graph G is clear from the context. The diameter of G is $\operatorname{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$. The distance between a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, denoted by $d(v, S)$ is the minimum of the distances between v and the vertices of S, that is to say, $d(v, S) = \min\{d(v, w) : w \in S\}$.

Let $u, v \in V(G)$ a pair of vertices such that $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, i.e., such that either $N(u) = N(v)$ or $N[u] = N[v]$. In both cases, u and v are said to be *twins*. Let W be a set of vertices of a graph G. If the vertices of W are pairwise twins, then W is called a *twin set* of G.

1.2 Metric dimension and partition dimension

A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $d(v, x) \neq d(w, x)$. A set of vertices $S \subseteq V(G)$ is a *metric-locating set* of G, if every pair of distinct vertices of G are resolved by some vertex in S. The *metric dimension* $\beta(G)$ of G is the minimum cardinality of a metric-locating set. Metric-locating sets were first defined by [10] and [14], and they have since been widely investigated (see [2, 6, 11, 12] and their references).

Let $G = (V, E)$ be a graph of order n. If $\Pi = \{S_1, \ldots, S_k\}$ is a partition of V, we denote by $r(u|\Pi)$ the vector of distances between a vertex $u \in V$ and the elements of Π, that is, $r(u, \Pi) = (d(u, S_1), \ldots, d(u, S_k))$. The partition Π is called a *metric-locating partition* of G, an *ML-partition* for short, if for any pair of distinct vertices $u, v \in V$, $r(u, \Pi) \neq r(v, \Pi)$. Observe that it is enough to check that the vectors of distances of every pair of vertices belonging to the same part are different, to prove that a given partition is metric-locating. The *partition dimension* $\beta_p(G)$ of G is the minimum cardinality of an ML-partition of G. Metric-locating partitions were introduced in [3], and further studied in [4, 5, 7, 8, 13, 15]. Next, some known results concerning this parameter are shown.

Theorem 1 ([3]). *Let G be a graph of order $n \geq 2$ and diameter* $\mathrm{diam}(G) = d$*. Then,*

(1) $\beta_p(G) \leq \beta(G) + 1$.

(2) $\beta_p(G) \leq n - d + 1$. *Moreover, this bound is sharp.*

(3) $\beta_p(G) = 2$ *if and only if G is isomorphic to the path P_n.*

(4) $\beta_p(G) = n$ *if and only if G is isomorphic to the complete graph K_n.*

(5) If $n \geq 6$, then $\beta_p(G) = n - 1$ if and only if G is isomorphic to either the star $K_{1,n-1}$, or the complete split graph $K_{n-2} \vee \overline{K_2}$, or the graph $K_1 \vee (K_1 + K_{n-2})$.

Remark 2 ([4]). *Notice that the restriction $n \geq 6$ of Theorem 1(5) is tight, since $\beta_p(C_4) = 3$ and $\beta_p(C_4 \vee K_1) = 4$. Thus, in [3], the condition $n \geq 3$ of Theorem 3.3 is wrong.*

Proposition 3 ([4]). *Given a pair of integers a, b such that $3 \leq a \leq b + 1$, there exists a graph G with $\beta_p(G) = a$ y $\beta(G) = b$.*

The remaining part of this paper, consisting of two more sections, is organized as follows. In Section 2, we introduce the partition metric-location-domination number, and show some basic properties for this new parameter. Section 3 is devoted to the characterization of all graphs G satisfying any of the following conditions: $\eta_p(G) \geq n - 1$, $\eta_p(G) = n - 2$ and $\beta_p(G) = n - 2$. Finally, in Section 4 some tight Nordhaus-Gaddum bounds for both the partition dimension $\beta_p(G)$ and the partition metric-location-domination number are shown.

2 Partition metric-location-domination number

A set $D \subseteq V$ of a graph $G = (V, E)$ is a *dominating set* if $d(v, D) = 1$, for every vertex $v \in V \setminus D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

A set $S \subseteq V$ is a *metric-locating-dominating set*, *MLD-set* for short, if it is both dominating and metric-locating. The *metric-location-domination number* $\eta(G)$ of G, *MLD-number* for short, is the minimum cardinality of an MLD-set of G (see [9]).

As a straightforward consequence of these definitions it holds that (see [1]):

$$max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \gamma(G) + \beta(G).$$

A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is called *dominating* if for every $v \in V$, $d(v, S_j) = 1$ for some $j \in \{1, \ldots, k\}$. The *partition domination number* $\gamma_p(G)$ equals the minimum cardinality of a dominating partition in G.

Proposition 4. *For any non-trivial graph G, $\gamma_p(G) = 2$.*

Proof. Let S be a dominating set of cardinality $\gamma(G)$. Consider the partition $\Pi = \{S, V \setminus S\}$ and observe that Π is a dominating partition of G. □

Let $\Pi = \{S_1, \ldots, S_k\}$ be a partition of the vertex set V of a graph $G = (V, E)$. The partition Π is called a *metric-locating-dominating partition* of G, *MLD-partition* for short, if it is both metric-locating and dominating. The *partition metric-location-domination number* $\eta_p(G)$ of G, *partition MLD-number* for short, is the minimum cardinality of an MLD-partition of G. An MLD-partition of cardinality $\eta_p(G)$ is called an *η_p-partition.*

Proposition 5. *Let G be a graph. Then, $\eta_p(G) = 2$ if and only if G is isomorphic to K_2.*

Proof. Certainly, $\eta_p(K_2) = 2$. Conversely, let G be a graph with $\eta_p(G) = 2$. Take an η_p-partition $\Pi = \{S_1, S_2\}$. Suppose that for some $i \in \{1, 2\}$, $|S_i| \geq 2$. Assume w.l.o.g. that $i = 1$ and take $u, v \in S_1$. As Π is a dominating partition, $r(u, \Pi) = (0, 1) = r(v, \Pi)$, contradicting that Π is an ML-partition. So, $|S_1| = |S_2| = 1$ and thus $\cong K_2$. □

Proposition 6. *Let P_n and C_n denote the path and the cycle of order $n \geq 3$, respectively. Then, $\eta_p(P_n) = \eta_p(C_n) = 3$.*

Proof. According to Proposition 5, it is sufficient to show, in both cases, the existence of an MLD-partition of cardinality 3. Assume that $V(P_n) = V(C_n) = \{1, \ldots, n\}$. Consider the following sets of vertices:

$$S_1 = \{1\},\ S_2 = \{2k : k = 1, \ldots, \lfloor n/2 \rfloor\},\ S_3 = \{2k+1 : k = 1, \ldots, \lfloor n/2 \rfloor\},$$

$$S_1' = \{1, 2\},\ S_2' = \{2k : k = 2, \ldots, \lfloor n/2 \rfloor\}.$$

Take the partitions $\Pi = \{S_1, S_2, S_3\}$ and $\Pi' = \{S_1', S_2', S_3\}$. It is straightforward to check that Π (resp. Π') is an MLD-partition of both P_n and C_n, if n is odd (resp. C_n, if n is even). □

Next, we show some results relating the partition MLD-number η_p to other parameters such as the MLD-number η, the partition dimension β_p, the order and the diameter.

Proposition 7. *Let G be a graph of order $n \geq 2$. Then, $\eta_p(G) \leq \eta(G) + 1$.*

Proof. This inequality is a consequence of the fact that if $\eta(G) = k$ and $S = \{u_1, \ldots, u_k\}$ is an MLD-set of G, then $\Pi = \{\{u_1\}, \ldots, \{u_k\}, V \setminus S\}$ is an MLD-partition of G. □

Proposition 8. *Given a pair of integers a, b such that $3 \leq a \leq b + 1$, there exists a graph G with $\eta_p(G) = a$ y $\eta(G) = b$.*

Proof. Let $j = a - 2$ and $k = b - a + 2$. Take the caterpillar displayed in Figure 1. Consider the set $W = \{w_1, \ldots, w_j, u_1\}$ of leaves hanging from vertex v_1. If Π is an ML-partition of G, then notice that no two vertices of W belong to the same part of Π. Observe also that any vertex of W and vertex v_1 must belong to a same part of Π. Thus, $\eta_p(G) \geq j + 2$. On the other hand, take the $j + 2$-partition $\Pi = \{\{u_1, \ldots, u_k\}, \{v_1, \ldots, v_k\}, \{w_1\}, \ldots, \{w_j\}\}$. Clearly, Π is both dominating and an ML-partition. Hence, $\eta_p(G) = j + 2 = a$.

To prove that $\eta(G) = b$, note first that every MLD-set S must contain all vertices from W except at most one. Observe also that for every $i \in \{1, \ldots, k\}$, either u_i or v_i must belong to S. Thus, $\eta(G) \geq j + k = b$. On the other hand, take the set $S = \{w_1, \ldots, w_j, u_1, \ldots, u_k\}$. Clearly, S is both dominating and metric-locating. Hence, $\eta(G) = j + k = b$. □

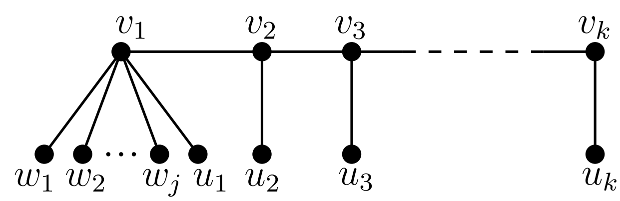


Figure 1: Caterpillar G of order $n = 2k + j$, $\eta_p(G) = j + 2$ and $\eta(G) = j + k$.

Next, an important double inequality relating both the partition dimension and the partition MLD-number is shown.

Theorem 9. *Let G be a graph of order $n \geq 2$. Then, $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1$.*

Proof. The first inequality follows directly from the definition of MLD-partition. Let $\beta_p(G) = r$ and $\Pi = \{P_1, \ldots, P_r\}$ be an ML- partition of G. If Π is a dominating partition, then $\eta_p(G) = \beta_p(G)$. Suppose that Π is not a dominating partition. Let $S = \{u \in V(G) : N[u] \subseteq P_i \text{ for some } i \in \{1, \ldots, r\}\}$. Note that $S \neq \emptyset$ and $P_i \setminus S \neq \emptyset$ for every $i \in \{1, \ldots, r\}$, since G is connected. In order to show that $\eta_p(G) \leq \beta_p(G) + 1$, we construct an MLD-partition of cardinality $r + 1$.

Let $C_1, \ldots, C_s$ be the connected components of the subgraph $G[S]$ induced by S. Clearly, for every $i \in \{1, \ldots, s\}$, all vertices of C_i belong to the same part of Π. Next, we define a subset $S' \subseteq S$ as follows. If $|V(C_i)| = 1$, then add the unique vertex of C_i to S'. If $|V(C_i)| \geq 2$, then consider a 2-coloring of a spanning tree of C_i, choose one color and add all vertices having this color to S'. Note that, if $V(C_k) \subseteq P_{i_k}$ and a pair of vertices $x, y \in C_k$ are adjacent, then one endpoint of xy is in S' and the other one belongs to $P_{i_k} \setminus S'$. Let $\Pi' = \{P'_1, \ldots, P'_r, S'\}$, where $P'_i = P_i \setminus S' \subseteq P_i$ for every $1 \leq i \leq r$. We claim that Π' is an MLD-partition.

On the one hand, observe that the sets $P_1 \setminus S', \ldots, P_r \setminus S', S'$ are nonempty by construction. On the other hand, observe that for every $u \in P_i$, $d(u, P_j) = d(u, w)$ for some vertex $w \in P_j \setminus S$ whenever $i \neq j$. Indeed, assume to the contrary that $d(u, P_j) = d(u, w)$ and $w \in P_j \cap S$. Since $w \in S$, we have $N[w] \subseteq P_j$. Thus, the vertex w' adjacent to w in a shortest (u, w)-path is also in P_j, implying that $d(u, w') < d(u, w) = d(u, P_j)$ with $w' \in Pj$, a contradiction. From this last observation, we conclude that $d(u, P_j) = d(u, P'_j)$ if $u \in P_i$ and $j \neq i$.

Next, we show that Π' is a dominating partition, i.e., that for any $u \in V(G)$, the vector $r(u|\Pi')$ has at least one component equal to 1. We distinguish two cases.

Case 1: $u \in S'$. Assume that $u \in P_i$, for some $i \in \{1, \ldots, r\}$. If u belongs to a trivial connected component of $G[S]$, then every neighbor of u is in P'_i. So, $d(u, P'_i) = 1$. If u belongs to a non-trivial connected component C_k of $G[S]$, then any neighbor of u with different color in the spanning tree of C_k considered in the construction of S' belongs to P'_i. So, $d(u, P'_i) = 1$.

Case 2: $u \in P'_i$, for some $i \in \{1, \ldots, r\}$. If $u \notin S$, as $P'_i \setminus S = P_i \setminus S$, then u has a neighbor v in some P_j with $j \neq i$. Therefore, $d(u, P'_j) = 1$, if $v \in P'_j$; and $d(u, S') = 1$, if $v \in S'$. If $u \in S$, then u belongs to a non-trivial connected component of $G[S]$ and, by construction of S', u has a neighbor in S'. Thus, $d(u, S') = 1$.

Finally, we show that Π' is an ML-partition, i.e., that $r(u|\Pi') \neq r(v|\Pi')$ for every pair of distinct vertices $u, v \in V(G)$ belonging to the same part of Π'. We distinguish two cases.

Case 1: $u, v \in P'_i$ for some $i \in \{1, \ldots, r\}$. In such a case, $u, v \in P_i$. Since Π is a metric-locating partition, $d(u,P_j) \neq d(v,P_j)$ for some $j \neq i$. Using the observation above, we have that $d(u,P'_j) = d(u,P_j) \neq d(v,P_j) = d(v,P'_j)$ for some $\neq i$. Therefore, $r(u|\Pi') \neq r(v|\Pi')$.

Case 2: $u,v \in S'$. If $u,v \in P_i$ for some $i \in \{1,\ldots,r\}$, then proceeding as in the previous case, we have that $r(u|\Pi') \neq r(v|\Pi')$. Suppose thus that $u \in P_i$ and $v \in P_j$ with $i \neq j$. Notice that $d(u,P'_i) = 1$ and $N[v] \subseteq P_j$ because $v \in P_j$ and $v \in S' \subseteq S$. Thus, $d(v,P_i) \geq 2$, and so $d(v,P'_i) = d(v,P_i) \geq 2$. Finally, from $d(u,P'_i) \neq d(v,P'_i)$ we get that $r(u|\Pi') \neq r(v|\Pi')$. □

The following result is a direct consequence of Theorem 1 and Theorem 9.

Corollary 10. *Let G be a graph of order $n \geq 3$ and diameter d. Then, $\eta_p(G) \leq n-d+2$. Moreover, this bound is sharp, and is attained, among others, by P_n and $K_{1,n}$.*

Proposition 11. *Let G be a graph of order $n \geq 2$ and diameter d such that $\eta_p(G) = k$. Then, $n \leq k\,(d^{k-1} - (d-1)^{k-1})$.*

Proof. Let $\Pi = \{S_1,\ldots,S_k\}$ be an MLD-partition. If $u \in S_i$, then the i-th component of $r(u|\Pi)$ is 0, any other component is a value from $\{1,2,\ldots,d\}$ and at least one component must be 1. There are $d^{k-1} - (d-1)^{k-1}$ such k-tuples. Hence, $|S_i| \leq d^{k-1} - (d-1)^{k-1}$, and therefore, $n \leq \sum_{i=1}^{k} |S_i| \leq k(d^{k-1} - (d-1)^{k-1})$. □

3 Extremal graphs

In [3, 15], all graphs of order $n \geq 9$ satisfying $\beta_p(G) = n$, $\beta_p(G) = n-1$ and $\beta_p(G) = n-2$ were characterized (see Theorem 1 and Remark 24). This section is devoted to approach the same problem for the partition MLD-number $\eta_p(G)$.

To this end, we show a pair of technical lemmas.

Lemma 12. *Let $k \geq 2$ be an integer. If G is a graph of order $n \geq 2k+1$ containing a vertex u such that $k \leq \deg(u) \leq n-k-1$, then $\eta_p(G) \leq n-k$.*

Proof. Let $\deg(u) = d \geq k$ and $N(u) = \{x_1,\ldots,x_k,\ldots x_d\}$. Let A be the set containing all leaves at distance 2 from u and let $B = (V \setminus N[u]) \setminus A$ (i.e., B contains all non-leaves at distance 2 and all vertices at distance at least 3 from u). Let $|A| = \alpha$ and $|B| = \beta$. Observe that $\alpha+\beta = n-d-1 \geq k$. If $\beta \geq k$, then take $y_1,\ldots,y_k \in B$, and notice that

$$\Pi = \{\{x_1,y_1\},\ldots,\{x_k,y_k\}\} \cup \{\{z\} : z \notin \{x_1,\ldots,x_k,y_1,\ldots,y_k\}\}$$

is an MLD-partition. Indeed, $\{u\}$ resolves every pair x_i, y_i, for $i = 1,\ldots,k$, because $d(u,x_i) = 1 < 2 \leq d(u,y_i)$. Furthermore, vertices x_i, $i = 1,\ldots,k$ are adjacent to u; and for $i \in \{1,\ldots,k\}$, vertex $y_i \in B$ is adjacent to a vertex different from x_i, because in the case y_i has degree 1, its neighbor does not belong to $N(u)$ by definition of B. Thus, Π is a dominating partition. Therefore, $\eta_p(G) \leq n-k$.

Now, assume $\beta < k$. Note that $\alpha \geq k-\beta$. Let $\alpha' = k-\beta$. Observe that $\alpha \geq \alpha' \geq 1$ and $\alpha' \leq k$. First, we seek if it is possible to pair α' vertices of A with α' vertices of $N(u)$ satisfying that each pair is formed by non-adjacent vertices. Observe that this is equivalent to finding a matching M that saturates a subset $A' \subseteq A$ of size α' in the bipartite graph H defined as follows: $N(u)$ and A are its stable sets, and if $x_i \in N(u)$ and $z \in A$, then $x_iz \in E(H)$ if and only if $x_iz \notin E(G)$. So, the degree in H of a vertex $z \in A$ is $\deg_H(z) = d-1$. For every nonempty set $W \subseteq A$ with $|W| \leq k-1$, we have $|W| \leq k-1 \leq d-1 \leq |N_H(W)|$, and for $W \subseteq A$ with $|W| = k$ we have $|W| \leq |N_H(W)|$

whenever $d \geq k+1$ or $|N_H(W)| \geq k$. Therefore, by Hall's Theorem, there exists a matching M saturating a subset of A of size α', except for the case $\alpha' = k = d$, provided that $|N_H(W)| < k$ for every subset $W \subseteq A$ with $|W| = k$. Let M be such a matching, whenever it exists. We distinguish two cases.

Case 1: $\alpha' < k$. Consider the partition Π formed by the α' pairs of the matching M, $\beta(= k - \alpha')$ pairs formed by pairing the vertices in B with β vertices in $N(u)$ not used in the matching M, and a part for each one of the remaining vertices formed only by the vertex itself. Part $\{u\}$ resolves each pair of vertices of parts of size 2 and, by construction, Π is dominating. Thus, Π is an MLD-partition, implying that $\eta_p(G) \leq n-k$.

Case 2: $\alpha' = k$. In such a case, $\beta = 0$ (i.e., $A = V \setminus N[u]$). If $d > k$, then consider the partition Π formed by the $k(= \alpha')$ pairs of the matching M and a part for each one of the remaining vertices formed only by the vertex itself. As in the preceding case, it can be shown that Π is an MLD-partition, and so $\eta_p(G) \leq n-k$. If $d = k$ and there is a subset $W \subseteq A$ of size k with $|N_H(W)| \geq k$, then there exists a matching M between the vertices of W and the vertices of $N(u)$. Consider the partition Π formed by the k pairs of the matching M and a part for each one of the remaining vertices formed only by the vertex itself. As in the preceding case, it can be shown that Π is an MLD-partition, and so $\eta_p(G) \leq n-k$.

Finally, if $d = k$ and there is no subset $W \subseteq A$ of size k with $|N_H(W)| \geq k$, then all vertices of A are leaves hanging from the same vertex of $N(u)$. We may assume without loss of generality that all vertices in A are adjacent to x_1. Let $y_1, \ldots, y_k \in A$ (they exist because $n \geq 2k+1$). In such a case,

$$\Pi = \{\{u, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}\} \cup \{\{z\} : z \notin \{u, x_2, \ldots, x_k, y_1, \ldots, y_k\}\}$$

is an MLD-partition (see Figure 2). Indeed, for $i = 2, \ldots, k$, $P_1 = \{u, y_1\}$ resolves every pair of vertices x_i and y_i because $d(x_i, P_1) = d(x_i, u) = 1 < 2 = d(y_i, P_1)$; and $P_2 = \{x_2, y_2\}$ resolves the pair u and y_1, because $d(u, P_2) = d(u, x_2) = 1 < 2 = d(y_1, P_2)$. Besides, every vertex has a neighbor in another part by construction. Thus, $\eta_p(G) \leq n-k$. □

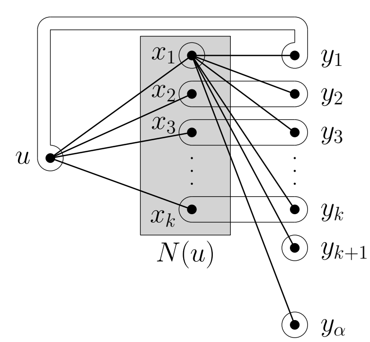


Figure 2: An MLD-partition of size $n-k$. There may be edges joining vertices of $N(u)$.

Lemma 13. *Let $G = (V, E)$ be a graph order n and diameter d.*

(1) *If $n \geq 5$ and $d \geq 3$, then $\eta_p(G) \leq n-2$.*

(2) *If $n \geq 7$ and $d \geq 4$, then $\eta_p(G) \leq n-3$.*

Proof.

(1) If $d \geq 4$, then according to Corollary 10, $\eta_p(G) \leq n-d+2 \leq n-2$. Assume thus that $d=3$ and take a vertex u of eccentricity $ecc(u)=3$. If u is not a leaf, then $2 \leq \deg(u) \leq n-3$ and, by Lemma 12, $\eta_p(G) \leq n-2$. If u is a leaf let $D_i = \{v \mid d(u,v)=i\}$, $i \in \{1,2,3\}$. Take $x_i \in D_i$, $i \in \{1,2,3\}$ such that $ux_1, x_1x_2, x_2x_3 \in E(G)$. We distinguish cases depending on the cardinality of D_2.

Case 1: $|D_2| \geq 2$. Take a vertex $y_2 \in D_2 - x_2$. Note that $x_1y_2 \in E(G)$, as u is a leaf. Take the partition:

$$\Pi = \{\{x_1,x_2\},\{x_3,y_2\} \cup \{\{z\} : z \neq x_1,x_2,x_3,y_2\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-2$. Thus, $\eta_p(G) \leq n-2$.

Case 2: $|D_2| = 1$. Notice that $|D_3| \geq 2$ since $n \geq 5$. Take a vertex $y_3 \in D_3 - x_3$. Observe that $x_2y_3 \in E(G)$. Take the partition:

$$\Pi = \{\{x_1,x_2\},\{u,y_3\} \cup \{\{z\} : z \neq u,x_1,x_2,y_3\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-2$. Thus, $\eta_p(G) \leq n-2$.

(2) If $d \geq 5$, then according to Corollary 10, $\eta_p(G) \leq n-d+2 \leq n-3$. Assume thus that $d=4$ and take a vertex u of eccentricity of $ecc(u)=4$. Notice that $\deg(u) \leq n-4$ and hence, according to Lemma 12 (case $k=3$), $\eta_p(G) \leq n-3$ whenever $\deg(u) \geq 3$. Suppose finally that $1 \leq \deg(u) \leq 2$ and let $D_i = \{v \mid d(u,v)=i\}$, $i \in \{1,2,3,4\}$. Take $x_i \in D_i$, $i \in \{1,2,3,4\}$ such that $ux_1, x_1x_2, x_2x_3, x_3x_4 \in E(G)$. We distinguish cases depending on the cardinality of D_1 and D_2.

Case 1: $|D_1| = 2$. Take a vertex $y_1 \in D_1 - x_1$. Take the partition:

$$\Pi = \{\{u,x_1\},\{x_2,x_3\},\{x_4,y_1\} \cup \{\{z\} : z \neq u,x_1,x_2,x_3,x_4,y_1\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 2: $|D_1| = 1$ and $|D_2| \geq 2$. Take a vertex $y_2 \in D_2 - x_2$. Take the partition:

$$\Pi = \{\{u,x_4\},\{x_1,x_2\},\{x_3,y_2\} \cup \{\{z\} : z \neq u,x_1,x_2,x_3,x_4,y_2\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 3: $|D_1| = 1$, $|D_2| = 1$ and $|D_3| \geq 2$. Take a pair of vertices $y_3, w \in D_3 \cup D_4 \setminus \{x_3,x_4\}$ such that $y_3 \in D_3$. Take the partition:

$$\Pi = \{\{x_1,w\},\{x_2,x_3\},\{x_4,y_3\} \cup \{\{z\} : z \neq x_1,x_2,x_3,x_4,y_3,w\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$.

Case 4: $|D_1| = 1$, $|D_2| = 1$ and $|D_3| = 1$. Take a pair of vertices $y_4, w_4 \in D_4 - x_4$. Note that $x_3y_4, x_3w_4 \in E(G)$. Take the partition:

$$\Pi = \{\{u,y_4\},\{x_1w_4\},\{x_2,x_3\} \cup \{\{z\} : z \neq u,x_1,x_2,x_3,x_4,y_4,w_4\}.$$

Clearly, Π is an MLD-partition of G of cardinality $n-3$. Thus, $\eta_p(G) \leq n-3$. □

In [3], all graphs of order n satisfying $n-1 \leq \beta_p \leq n$ were characterized (see Theorem 1). We display a similar result for the partition metric-location-domination number η_p.

$H_1 \cong K_{n-3} \vee (K_2 + K_1)$	$H_2 \cong K_{n-3} \vee \overline{K_3}$	$H_3 \cong K_{n-4} \vee C_4$
$H_4 \cong K_{n-4} \vee P_4$	$H_5 \cong K_{n-4} \vee 2\,K_2$	$H_6 \cong K_{2,n-2}$
$H_7 \cong \overline{K_{n-2}} \vee K_2$	$H_8 \cong (K_{n-3} + K_1) \vee K_2$	$H_9 \cong (K_{n-3} + K_1) \vee \overline{K_2}$
$H_{10} \cong (K_{n-3} + K_2) \vee K_1$	$H_{11} \cong (K_{n-4} + K_1) \vee P_3 - e'$	$H_{12} \cong (\overline{K_{n-3}} + K_2) \vee K_1$
$H_{13} \cong (K_{n-3} + \overline{K_2}) \vee K_1$	$H_{14} \cong H_{11} - e_4$	$H_{15} \cong H_9 - e_1$
$H_{16} \cong H_{10} - e_2$	$H_{17} \cong H_{12} - e_3$	

Table 1: All graphs of order $n \geq 7$ such that $\eta_p(G) = n - 2$ (see Figures 5, 6 and 7).

Theorem 14. *Let G be a graph of order $n \geq 6$. Then,*

(1) *$\eta_p(G) = n$ if and only if G is isomorphic to either the complete graph K_n or the star $K_{1,n-1}$.*

(2) *$\eta_p(G) = n - 1$ if and only if G is isomorphic to either the complete split graph $K_{n-2} \vee \overline{K_2}$, or the graph $K_1 \vee (K_1 + K_{n-2})$.*

Proof. (1) According to Theorem 9, if $\eta_p(G) = n$ then $n - 1 \leq \beta_p(G) \leq n$. By direct inspection on graphs with $\beta_p(G) = n$ and $\beta_p(G) = n - 1$ (see Theorem 1) the stated result is derived.

(2) It is a routine exercise to check that $\eta_p(K_{n-2} \vee \overline{K_2}) = \eta_p(K_1 \vee (K_1 + K_{n-2}) = n - 1$. Conversely, let G be a graph such that $\eta_p(G) = n - 1$. By Lemma 13(1), $\mathrm{diam}(G) = 2$. Take a pair of vertices u, v such that $d(u, v) = 2$. By Lemma 12 (case $k = 2$), $\deg(u), \deg(v) \in \{1, n - 2\}$. We distinguish three cases.

Case 1: $\deg(u) = \deg(v) = 1$. Let w the vertex such that $N(u) = N(v) = \{w\}$. Clearly, the rest of vertices of G have degree 1, as they are not adjacent neither to u nor to v. Hence, all vertices of G other than vertex w are leaves hanging from w, i.e., $G \cong K_{1,n-1}$, a contradiction.

Case 2: $\deg(u) = \deg(v) = n - 2$. In this case, $N(u) = N(v) = V \setminus \{u, v\} = W$ and for all vertex $z \in W$, $\deg(z) \geq 2$. Then, by Lemma 12 (case $k = 2$), $\deg(z) \in \{n - 2, n - 1\}$.

If for all $z \in W$ $\deg(z) = n - 1$, then G is isomorphic to the complete split graph $K_{n-2} \vee \overline{K_2}$.

If there is a vertex $t \in W$ such that $\deg(t) = n - 2$, then let $s \in W$ be the vertex that is not adjacent to t. Observe that both t and s are adjacent to any other vertex of W. If $a, b \in W \setminus \{s, t\}$, then $\Pi = \{\{u, a\}, \{s, b\}\} \cup \{\{z\} : z \neq a, b, u, s\}$ is an MLD-partition, and thus $\eta_p(G) \leq n - 2$.

Case 3: $\deg(u) = 1$ and $\deg(u) = n - 2$. Let t be the adjacent vertex to u. As the diameter is 2, every vertex $w \notin \{u, t, v\}$ is adjacent both to t and v. In particular, for all vertex $w \notin \{u, t, v\}$, $\deg(w) \geq 2$ and, by Lemma 12 (case $k = 2$), $\deg(w) = n - 2$ and then G is isomorphic to the graph $K_1 \vee (K_1 + K_{n-2})$. □

Next, we characterize those graphs with $\eta_p(G) = n - 2$. Concretely, we prove that, for every integer $n \geq 7$, a graph of order n satisfies $\eta_p(G) = n - 2$ if and only if it belongs to the family $\Lambda_n = \{H_1, \ldots, H_{17}\}$ (see Table 1 and Figures 5, 6 and 7).

To this end, we present a technical lemma.

Lemma 15. *Let W be a twin set of cardinality $|W| = k$ of a graph G. If G is a non-complete graph, then $\beta(G) \geq k$. Moreover,*

1. *If all vertices of W are leaves hanging from the same vertex u, then $\eta_p(G) \geq k+1$.*

2. *If $G[W]$ is a complete subgraph of G, then $\eta_p(G) \geq \beta_p(G) \geq k+1$.*

Proof. Notice that if $w, z \in W$, then they must belong to different parts of any ML-partition Π, since $d(w,v) = d(z,v)$ for all $v \in V(G) \setminus \{w,z\}$. Thus, $\beta_p(G) \geq k$.

Suppose next that G is a graph with k leaves hanging from a vertex u. On the one hand, two leaves hanging from u must belong to different parts of Π. On the other hand, any of these leaves and u must belong to distinct elements of Π since otherwise Π would not be a dominating partition. Thus, $\eta_p(G) \geq k+1$.

Finally assume that W induces a complete graph. Take a vertex v adjacent to w, for all $w \in W$. On the one hand, no two vertices of W belong to a same part of Π. On the other hand, one of these vertices of W and v belong to distinct elements of Π since otherwise Π would not be an ML-partition. Then, $\eta_p(G) \geq \beta_p(G) \geq k+1$. □

Proposition 16. *Let If $G \in \Lambda_n = \{H_1, \ldots, H_{17}\}$, then $\eta_p(G) = n-2$. Moreover, if $G \notin \{H_{12}, H_{17}\}$, then $\beta_p(G) = n-2$.*

Proof. According to Theorem 14, for every graph $H_i \in \Lambda_n$, $\beta_p(G) \leq \eta_p(G) \leq n-2$. Thus, it is enough to check that, for every graph $H_i \in \Lambda_n$, $\eta_p(H_i) \geq n-2$, and also that if $i \notin \{12, 17\}$, then $\beta_p(H_i) \geq n-2$.

Case 1: If $G \in \{H_6, H_7\}$, then it contains a twin set W of cardinality $n-2$ and thus, by Lemma 15, $\eta_p(G) \geq \beta_p(G) \geq n-2$.

Case 2: If $G \in \{H_1, H_2, H_8, H_9, H_{10}, H_{13}, H_{15}, H_{16}\}$, then there exists a set of vertices W of $n-3$ vertices of G such that W induces a complete graph, and thus, by Lemma 15, $\eta_p(G) \geq \beta_p(G) \geq (n-3)+1 = n-2$.

Case 3: If $G \in \{H_3, H_4, H_5, H_{11}, H_{14}\}$, then for all these graphs $\operatorname{diam}(G) = 2$, and there exists a set of vertices W of $n-4$ vertices of G such that W induces a complete graph, and thus, by Lemma 15, $\eta_p(G) \geq \beta_p(G) \geq (n-4)+1 = n-3$. Suppose that there exists an ML-partition $\Pi = \{S_1, \ldots, S_{n-3}\}$ of cardinality $n-3$. If $W = \{w_1, \ldots, w_{n-4}\}$, assume that, for every $i \in \{1, \ldots, n-4\}$, $w_i \in S_i$. We distinguish two cases.

Case 3.1: $G \in \{H_3, H_4, H_5\}$. Note that $N(W) = V(G)$ and in all cases there is a labelling $V(G) \setminus W = \{a_1, a_2, b_1, b_2\}$ such that $d(a_1, a_2) = 1$, $d(b_1, b_2) = 1$, $d(a_1, b_1) = 2$ and $d(a_2, b_2) = 2$ (see Figure 3(a)).

Clearly, $|S_{n-3}| = 1$, as $r(z, \Pi) = (1, \ldots, 1, 0)$ for every $z \in \{a_1, a_2, b_1, b_2\} \cap S_{n-3}$. Notice also that $|S_i| \leq 2$ for $i \in \{1, \ldots, n-4\}$, as for every $x \in S_i$, we have $r(x, \Pi) = (1, \ldots, 1, \overset{i)}{0}, 1, \ldots, 1, h)$, with $h \in \{1, 2\}$. Hence, there are exactly three sets of Π of cardinality 2. We can suppose without loss of generality that $S_1 = \{w_1, x\}$, $S_2 = \{w_2, y\}$, $S_3 = \{w_3, z\}$ and $S_{n-3} = \{t\}$, where $\{x, y, z, t\} = \{a_1, a_2, b_1, b_2\}$. Hence, $d(t, x) = d(t, y) = d(t, z) = 2$, a contradiction.

Case 3.2: $G \in \{H_{11}, H_{14}\}$. Note that $|N(W) \setminus W| = 3$ and that there is a labelling $V(G) \setminus W = \{a, b, c, z\}$ such that $N(W) \setminus W = \{a, b, c\}$, $d(a, b) = d(b, c) = d(b, z) = 1$, $d(c, a) = d(c, z) = 2$ and $d(a, z) \in \{1, 2\}$ (see Figure 3(b)).

Notice that $|S_{n-3}| \leq 2$, since for every $x \in \{a,b,c\} \cap S_{n-3}$, $r(x,\Pi) = (1,\ldots,1,0)$. Moreover, $b \notin S_{n-3}$, otherwise a and c do not belong to S_{n-3} and we would have $r(a,\Pi) = r(c,\Pi) = (1,\ldots,1,1)$. So, we can assume without loss of generality that $\{w_1, b\} \subseteq S_1$. If $\{a,c\} \cap S_{n-3} \neq \emptyset$, then $r(w_1,\Pi) = r(b,\Pi) = (1,\ldots,1,1)$. Consequently, $r(w_i,\Pi) = r(c,\Pi) = (1,\ldots,1,2)$ for every $i \in \{1,\ldots,n-4\}$, a contradiction.

Case 4: If $G \in \{H_{12}, H_{17}\}$, then G is a graph with $n-3$ leaves hanging from a vertex u and then, by Lemma 15, $\beta_p(G) \geq n-3$ and $\eta_p(G) \geq (n-3)k+1 = n-2$. □

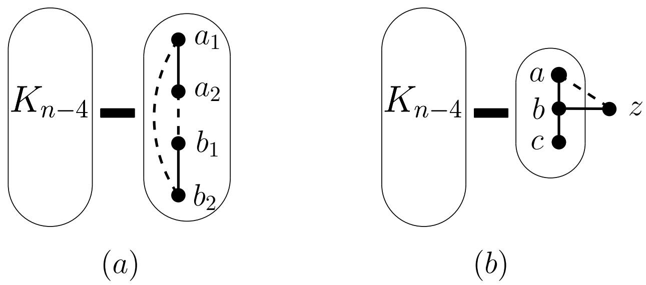


Figure 3: Graphs containing a set W on $n-4$ vertices such that $G[W]$ is a complete graph. Solid lines hold for adjacent vertices meanwhile dashed lines are optional.

The remainder of this section is devoted to showing that these 17 graph families are the only ones satisfying $\eta_p(G) = n-2$.

Firstly, note that as a direct consequence of Lemma 13(2) the following result is derived.

Corollary 17. *If G is a graph with $\eta_p(G) = n-2$, then $2 \leq \text{diam}(G) \leq 3$.*

3.1 Case diameter 2

We consider the case $\eta_p(G) = n-2$ and $\text{diam}(G) = 2$. We distinguish two cases depending whether $\delta(G) \geq n-3$ or $\delta(G) \leq n-4$. To approach the first case (notice that the restriction $\text{diam}(G) = 2$ is redundant) we need the following technical lemma.

Lemma 18. *If $G = (V,E)$ is a graph of order $n \geq 7$, minimum degree $\delta(G) \geq n-3$ and containing at most $n-5$ vertices of degree $n-1$, then $\eta_p(G) \leq n-3$.*

Proof. Observe that the complement $\overline{G}$ of G is a graph with vertices of degree 0, 1 or 2. Thus, the components of $\overline{G}$ are either isolated vertices, or paths of order at least 2, or cycles of order at least 3. By hypothesis, G has at most $n-5$ vertices of degree $n-1$, therefore $\overline{G}$ has at least 5 vertices of degree 1 or 2. We distinguish three cases.

Case 1: $\overline{G}$ has only one non-trivial component. In such a case, $\overline{G}$ has al least a (not necessarily induced) subgraph isomorphic to P_5. Let x_1, x_2, x_3, x_4 and x_5 be the vertices of this path, where $x_i x_{i+1} \in E(\overline{G})$ for $i = 1,2,3,4$. Let $z \notin \{x_1,x_2,x_3,x_4,x_5\}$. Consider the partition:

$$\Pi = \{\{x_2\},\{x_4\},\{x_1,x_3,x_5,z\}\} \cup \{\{v\} : v \notin \{x_1,x_2,x_3,x_4,x_5,z\}\}.$$

We claim that Π is an MLD-partition of G (see Figure 4 (a)). Indeed, if $S_1 = \{x_2\}$ and $S_2 = \{x_4\}$, then $r(x_1, \Pi) = (2, 1, \dots)$, $r(x_3, \Pi) = (2, 2, \dots)$, $r(x_5, \Pi) = (1, 2, \dots)$, $r(z, \Pi) = (1, 1, \dots)$. Moreover, x_3 is adjacent in G to any vertex $w \notin \{x_1, x_2, x_3, x_4, x_5, z\}$, that exists because the order of G is at least 7. Therefore, Π is an MLD-partition of G. Thus, $\eta_p(G) \leq n - 3$.

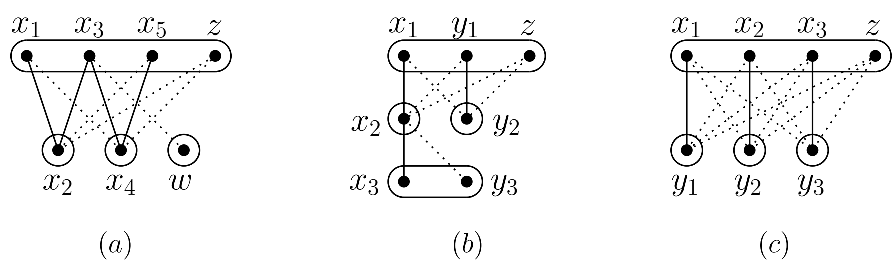


Figure 4: Solid (resp. dotted) lines mean non-adjacent (resp. adjacent) vertices in G.

Case 2: $\overline{G}$ has at least two non-trivial components and one of them has order ≥ 3. If there is only one component of order ≥ 3, say C_1, then there is at least a component of order 2, say C_2. Otherwise, there are two components, say C_1 and C_2, of order ≥ 3. In both cases, let x_1, x_2, x_3 be vertices of C_1 and y_1, y_2, y_3 be vertices of C_2, such that $x_1x_2 \in E(\overline{G})$, $x_2x_3 \in E(\overline{G})$, $y_1y_2 \in E(\overline{G})$ and finally, if C_2 has order ≥ 3, then $y_2y_3 \in E(\overline{G})$. Consider the partition:

$$\Pi = \{\{x_2\}, \{y_2\}, \{x_1, y_1, z\}, \{x_3, y_3\}\} \cup \{\{v\} : v \notin \{x_1, x_2, x_3, y_1, y_2, y_3, z\}\}.$$

We claim that Π is an MLD-partition of G (see Figure 4 (b)). Indeed, if $S_1 = \{x_2\}$ and $S_2 = \{y_2\}$, then $r(x_1, \Pi) = (2, 1, \dots)$, $r(y_1, \Pi) = (1, 2, \dots)$ and $r(z, \Pi) = (1, 1, \dots)$; $r(x_3, \Pi) = (2, 1, \dots)$, and $r(y_3, \Pi) = (1, \dots)$. Therefore, Π is an MLD-partition of G. Thus, $\eta_p(G) \leq n - 3$.

Case 3: all non-trivial components of $\overline{G}$ have order 2. Then, $\overline{G}$ has at least 3 components that are copies of K_2. Let $\{x_i, y_i\}$, for $i = 1, 2, 3$, be the vertices of three of these copies, and let z be a vertex not belonging to them. Then,

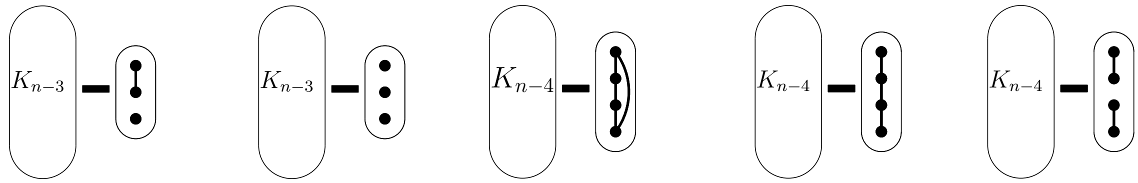
$$\Pi = \{\{y_1\}, \{y_2\}, \{y_3\}, \{x_1, x_2, x_3, z\}\} \cup \{\{v\} : v \neq x_1, x_2, x_3, y_1, y_2, y_3, z\}$$

is an MLD-partition of G (see Figure 4 (c)). Indeed, if $S_1 = \{y_1\}$, $S_2 = \{y_2\}$ and $S_3 = \{y_3\}$, then $r(x_1, \Pi) = (2, 1, 1, \dots)$, $r(x_2, \Pi) = (1, 2, 1, \dots)$, $r(x_3, \Pi) = (1, 1, 2, \dots)$ and $r(z, \Pi) = (1, 1, 1, \dots)$. Therefore, $\eta_p(G) \leq n - 3$. □

Proposition 19. *Let G be a graph of order $n \geq 7$, diameter* $\text{diam}(G) = 2$ *and minimum degree $\delta(G) \geq n - 3$ such that $\eta_p(G) = n - 2$. Then, $G \in \{H_1, H_2, H_3, H_4, H_5\}$ (see Figure 5).*

Proof. Let $\Omega \subset V$ be the set of vertices of G of degree $n - 1$, which according to Lemma 18 contains at least $n - 4$ vertices. We distinguish cases depending on the cardinality of Ω.
Case 1: $|\Omega| \geq n - 2$. If $|\Omega| \geq n$, then $G \cong K_n$ and thus $\eta_p(G) = n$. Case $|\Omega| = n - 1$ is not possible. If $|\Omega| = n - 2$, then $G \cong K_{n-2} \vee \overline{K_2}$, and according to Theorem 14(3), $\eta_p(G) = n - 1$.

Case 2: $|\Omega| = n - 3$. Let H be the subgraph of order 3 induced by $V \setminus \Omega$, i.e., $H = G[V \setminus \Omega]$. Notice that $|E(H)| \leq 1$. If $|E(H)| = 1$, then $G \cong H_1$. Otherwise, if $|E(H)| = 0$, then $G \cong H_2$.



$H_1 \cong K_{n-3} \vee (K_2 + K_1)$ $\quad H_2 \cong K_{n-3} \vee \overline{K_3}$ $\quad H_3 \cong K_{n-4} \vee C_4$ $\quad H_4 \cong K_{n-4} \vee P_4$ $\quad H_5 \cong K_{n-4} \vee 2K_2$

Figure 5: Graphs of order $n \geq 7$, diameter $\operatorname{diam}(G) = 2$ and minimum degree $\delta(G) \geq n-3$ such that $\eta(G) = n-2$.

Case 3: $|\Omega| = n-4$. Consider the graph of order 4, $H = G[V \setminus \Omega]$. Note that all vertices of H have degree either 1 or 2. There are thus three possibilities. If $H \cong C_4$, then $G \cong H_3$. If $H \cong P_4$, then $G \cong H_4$. If $H \cong 2K_2$, then $G \cong H_5$. □

Proposition 20. *Let $G = (V, E)$ be a graph of order $n \geq 7$, diameter $\operatorname{diam}(G) = 2$ and minimum degree $\delta(G) \leq n-4$. If $\eta_p(G) = n-2$, then $G \in \{H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}, H_{13}, H_{14}\}$ (see Figure 6).*

Proof. By Lemma 12 for $k = 3$, we have that $\deg(w) \in \{1, 2, n-3, n-2, n-1\}$, for every vertex $w \in V(G)$. Hence, $\delta(G) \leq 2$. We distinguish two cases.

Case 1: *There exists a vertex u of degree* 2. Consider the subsets $D_1 = N(u) = \{x_1, x_2\}$ and $D_2 = \{v \in V : d(u, v) = 2\}$, so that $|D_2| = n-3$. Assume that $\deg(x_1) \leq \deg(x_2)$.

(1.1): *$G[D_2]$ is neither complete nor empty.* Then, there exist three different vertices $r, s, t \in D_2$ such that $rs \in E(G)$ and $rt \notin E(G)$. Let $y \in D_2 \setminus \{r, s, t\}$. We distinguish cases taking into account whether or not y and t are leaves.

- *Both y and t are leaves hanging from the same vertex.* Assume that they hang from x_1. Let $S_1 = \{u, y\}$ and $S_2 = \{x_2, s, t\}$. In such a case, r resolves the pair $\{s, t\}$; S_2 resolves S_1; and S_1 resolves the pairs $\{x_2, s\}$ and $\{x_2, t\}$. Therefore, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is an ML-partition. It can be easily checked that Π is also dominating. Hence, $\eta_p(G) \leq n-3$, a contradiction.

- *Both y and t are leaves but not hanging from the same vertex, or neither y nor t are leaves.* If both y and t are leaves but not hanging from the same vertex, assume $x_1 y \in E$ and $x_2 t \in E$. Let $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. If neither y nor t are leaves and $N(t) \neq \{s, x_1\}$, let $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. If neither y nor t are leaves and $N(t) = \{s, x_1\}$, let $S_1 = \{x_1, y\}$ and $S_2 = \{x_2, s, t\}$. In all these cases, u resolves the vertices in S_1; r resolves the pair $\{s, t\}$ and u resolves any other pair from S_2. Hence, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is an ML-partition of G. It can be easily checked that Π is a dominating partition. Thus, $\eta_p(G) \leq n-3$, a contradiction.

- *Exactly one of the vertices y or t is a leaf.* We may assume that it hangs from x_1. If t is a leaf, then take $S_1 = \{x_1, y\}$ and $S_2 = \{x_2, s, t\}$. If y is a leaf and $N(t) \neq \{x_1, s\}$ then take $S_1 = \{x_2, y\}$ and $S_2 = \{x_1, s, t\}$. In both cases, $\{r\}$ resolves the pair $\{s, t\}$ and $\{u\}$ resolves any other pair in S_1 or S_2. If y is a leaf and $N(t) = \{x_1, s\}$ then take $S_1 = \{u, y\}$ and $S_2 = \{x_2, s, t\}$. Then, r resolves the pair $\{s, t\}$, S_1 resolves the other pairs from S_2; and S_2

resolves S_1. In all cases, $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is dominating partition. Thus, $\eta_p(G) \leq n-3$, a contradiction.

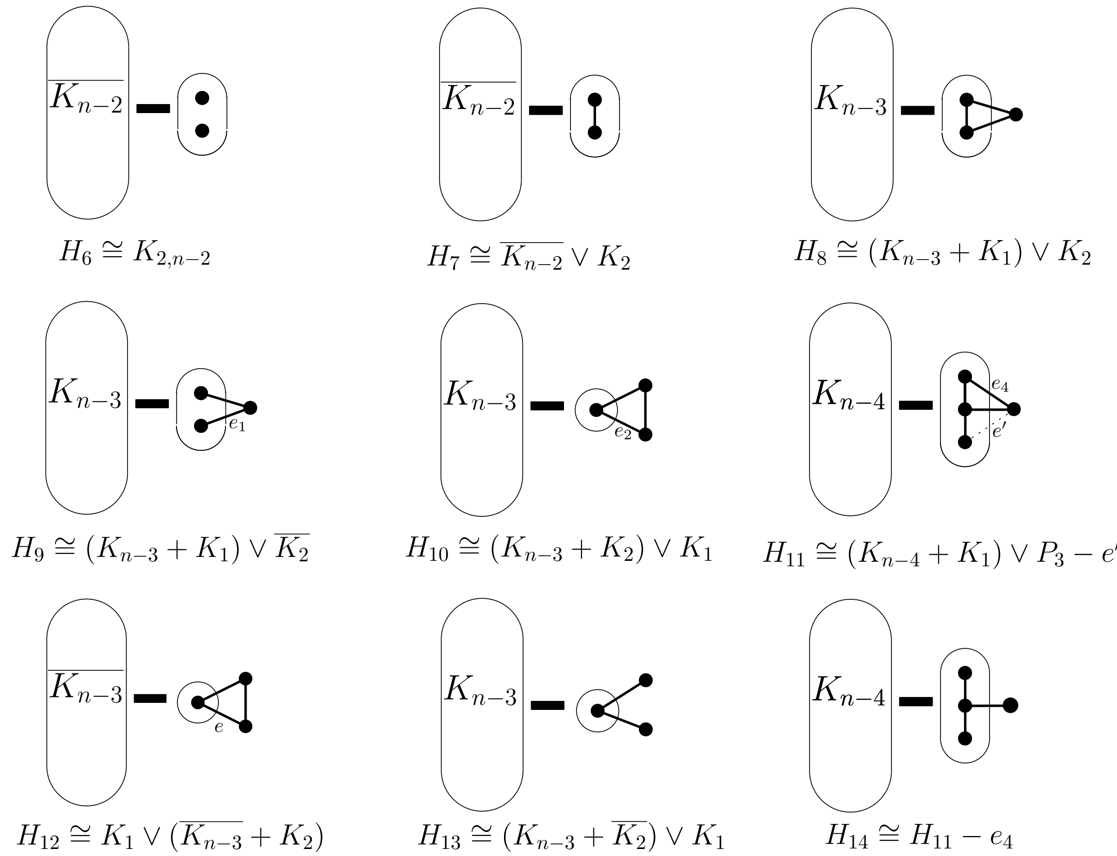


Figure 6: Graphs of order $n \geq 7$, diameter $\mathrm{diam}(G) = 2$ and minimum degree $1 \leq \delta(G) \geq 2$ such that $\eta(G) = n-2$.

(1.2): *$G[D_2]$ is either complete or empty.* Consider the subsets $N_1 = N(x_1) \cap D_2$ and $N_2 = N(x_2) \cap D_2$. Observe that $N_1 \cup N_2 = D_2$, and the sets $N_1 \setminus N_2$, $N_1 \cap N_2$ and $N_2 \setminus N_1$ are pairwise disjoint. Besides, $|N_2 \setminus N_1| \geq |N_1 \setminus N_2|$ because we have assumed $\deg(x_2) \geq \deg(x_1)$. Notice also that $\deg(x_2) \geq deg(x_1) \geq 2$, as otherwise $\mathrm{diam}(G) \geq 3$. We distinguish two cases.

(1.2.1): $\deg(x_1) = 2$. Thus, $\deg(x_2) \geq (|D_2| - 1) + 1 \geq n-3$.

- If $x_1x_2 \in E$, then $N_1 = \emptyset$ and $D_2 = N_2$. If $G[D_2] \cong K_{n-3}$, then $G \cong H_{10}$. If $G[D_2] \cong \overline{K_{n-3}}$, then $G \cong H_{12}$.
- If $x_1x_2 \notin E$, then $|N_1| = 1$ and $|N_2 \setminus N_1| = n-4 \geq 3$. If $G[D_2] \cong \overline{K_{n-3}}$, then $\mathrm{diam}(G) = 3$. Hence, $G[D_2] \cong K_{n-3}$. Consider $y \in N_1$ and $z_1, z_2 \in N_2 \setminus N_1$. Let $S_1 = \{u, x_1\}$, $S_2 = \{x_2, z_2\}$ and $S_3 = \{y, z_1\}$. and consider the partition $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$. Then, S_1 resolves S_2 and S_3; and S_3 resolves S_1. Moreover, Π is a dominating partition of G. Thus, $\eta_p(G) \leq n-3$, a contradiction.

(1.2.2): $\deg(x_1) \geq n-3$. Hence, $\deg(x_2) \geq \deg(x_1) \geq n-3$. In such a case, $|N_1| \geq n-5$ and $|N_2| \geq n-5$, and so $n-7 \leq |N_1 \cap N_2| \leq n-3$. We distinguish cases depending on the cardinality of $|N_1 \cap N_2|$.

- $|N_1 \cap N_2| = n-3$. If $x_1x_2 \in E$, then $G \cong H_8$ if $G[D_2] \cong K_{n-3}$, and $G \cong H_7$ if $G[D_2] \cong \overline{K_{n-3}}$. If $x_1x_2 \notin E$, then $G \cong H_9$ if $G[D_2] \cong K_{n-3}$, and $G \cong H_6$ if $G[D_2] \cong \overline{K_{n-3}}$.

- $|N_1 \cap N_2| = n-4$. Then, $|N_2 \setminus N_1| + |N_1 \setminus N_2| = 1$. Thus, $|N_2 \setminus N_1| = 1$, $|N_1 \setminus N_2| = 0$ and $|N_1 \cap N_2| \geq 3$. If $G[D_2] \cong \overline{K_{n-3}}$, then $diam(G) \geq 3$, a contradiction. If $G[D_2] \cong K_{n-3}$ and $x_1x_2 \in E$, then $G \cong H_{11}$. If $G[D_2] \cong K_{n-3}$ and $x_1x_2 \notin E$, then let $y_1, y_2, y_3 \in N_1 \cap N_2$ and let $z \in N_2 \setminus N_1$. Consider $S_1 = \{u, y_1\}$, $S_2 = \{x_2, y_2\}$, $S_3 = \{z, y_3\}$ and let $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$. Then, $\{x_1\}$ resolves S_2 and S_3, and S_3 resolves S_1. It is easy to check that it is a dominating partition. Therefore, $\eta_p(G) \leq n-3$, a contradiction.

- $|N_1 \cap N_2| = n-5$. Then $|N_2 \setminus N_1| + |N_1 \setminus N_2| = 2$ and $|N_1 \cap N_2| \geq 2$. Let $y_1, y_2 \in (N_2 \setminus N_1) \cup (N_1 \setminus N_2)$ and $z_1, z_2 \in N_1 \cap N_2$, and let $S_1 = \{y_1, z_1\}$, $S_2 = \{y_2, z_2\}$ and $S_3 = \{u, x_1\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$ is an MLD-partition of G. Indeed, S_1 resolves S_3, and for $i \in \{1, 2\}$, S_i is resolved by S_1 if $y_i \in N_2 \setminus N_1$ and S_i is resolved by $\{x_2\}$ if $y_i \in N_1 \setminus N_2$. Besides, Π is dominating. Hence, $\eta_p(G) \leq n-3$, a contradiction.

- $|N_1 \cap N_2| \in \{n-6, n-7\}$. In such a case, $|N_2 \setminus N_1| + |N_1 \setminus N_2| \in \{3, 4\}$. Since $|N_2 \setminus N_1| \geq |N_1 \setminus N_2|$, we have $|N_2 \setminus N_1| \geq 2$. Since $\deg(x_1) \geq n-3$, we have $|N_1| \geq n-5 \geq 2$. Let $y_1, y_2 \in N_1$ and $z_1, z_2 \in N_2 \setminus N_1$. If $S_1 = \{u, x_1\}$, $S_2 = \{y_1, z_1\}$ and $S_3 = \{y_2, z_2\}$, and $\Pi = \{S_1, S_2, S_3\} \cup \{\{w\} : w \notin S_1 \cup S_2 \cup S_3\}$, then S_1 resolves S_2 and S_3; and S_2 resolves S_1. Moreover, Π is a dominating partition. Therefore, $\eta_p(G) \leq n-3$, a contradiction.

Case 2: *There exists at least one vertex u of degree 1 and there is no vertex of degree 2.* Since $\text{diam}(G) = 2$, the neighbor v of u satisfies $\deg(v) = n-1$. Let Ω be the set of vertices different from v that are not leaves. Notice that there are at most two vertices of degree 1 in G, as otherwise all vertices in Ω would have degree between 3 and $n-4$, contradicting the assumption made at the beginning of the proof.

If there are exactly two vertices of degree 1, then $|\Omega| = n-3$. In such a case, Ω induces a complete graph in G, since otherwise the non-universal vertices in $G[\Omega]$ would have degree at most $n-4$. So, in this case $G \cong H_{13}$.

Suppose thus that u is the only vertex of degree 1, which means that Ω contains $n-2$ vertices, all of them of degree $n-3$ or $n-2$. Consider the graph $J = \overline{G[\Omega]}$. Certainly, J has $n-2$ vertices, all of them of degree 0 or 1. Let L denote the set of vertices of degree 1 in J. Observe that the cardinality of L must be even. We distinguish three cases.

- If $|L| = 0$, then $G \cong K_1 \vee (K_1 + K_{n-2})$, and by Theorem 14 we have $\eta_p(G) = n-1$, a contradiction.

- If $|L| = 2$, then $G \cong H_{14}$.

- If $|L| \geq 4$, let $\{x_1, x_2, x_3, x_4\} \subseteq L$ such that x_1x_2 and x_3x_4 are edges of J, and let $y \in \Omega \setminus \{x_1, x_2, x_3, x_4\}$. Consider the partition $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$, where $S_1 = \{v, x_1\}$, $S_2 = \{u, x_3, y\}$. Observe that $\{x_2\}$ resolves the vertices of S_1, and the pairs x_3, u and y, u of S_2; and $\{x_4\}$ resolves the pair x_3, y of S_2. Besides, Π is dominating partition. Therefore, $\eta_p(G) \leq n-3$, a contradiction. □

3.2 Case diameter 3

We consider the case $\eta_p(G) = n-2$ and $\mathrm{diam}(G) = 3$.

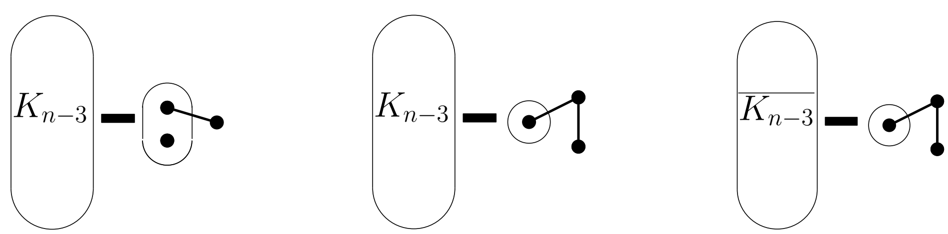
Proposition 21. *Let G be a graph of order $n \geq 7$, diameter $\mathrm{diam}(G) = 3$ such that $\eta_p(G) = n-2$. Then, $G \in \{H_{15}, H_{16}, H_{17}\}$ (see Figure 7).*

Proof. By Corollary 12 (case $k = 3$), every vertex has degree 1, 2, $n-3$, $n-2$ or $n-1$. Let u and v be two antipodal vertices, that is, such that $d(u, v) = 3$. In such a case, both u and v have degree at most $n-3$.

Notice that on the one hand, it is not possible to have neither $\{\deg(u), \deg(v)\} = \{2, n-3\}$ nor $\{\deg(u), \deg(v)\} = \{n-3\}$, as otherwise we would have more than n vertices because $N(u) \cap N(v) = \emptyset$, a contradiction.

On the other hand, if $\deg(u) = \deg(v) = 2$, then $\eta_p(G) \leq n-3$. Indeed, let ux_1x_2v be a (u, v)-path and let $D_i = \{z : d(u, z) = i\}$, for $i \in \{1, 2, 3\}$. Since $|D_1| = 2$, we may assume that $D_1 = \{x_1, y_1\}$. If $|D_2| \geq 2$, let $y_2 \in D_2 \setminus \{x_2\}$. If $x_1y_2 \in E$, let $S_1 = \{x_1, x_2\}$ and $S_2 = \{y_1, y_2, v\}$. If $x_1y_2 \notin E$, then $y_1y_2 \in E$, and consider $S_1 = \{y_1, x_2\}$ and $S_2 = \{x_1, y_2, v\}$. If $|D_2| = 1$, then v has a neighbor $z \in D_3$, so that z must be also adjacent to x_2. Let $S_1 = \{x_1, x_2, v\}$ and $S_2 = \{y_1, z\}$. In all cases $\Pi = \{S_1, S_2\} \cup \{\{w\} : w \notin S_1 \cup S_2\}$ is an MLD-partition, because it is dominating and $\{u\}$ resolves S_1 and S_2. Hence, $\eta_p(G) \leq n-3$, a contradiction.

Therefore, we may assume that u and v are antipodal vertices with $\deg(u) = 1$ and every vertex at distance 3 from u has degree 1, 2 or $n-3$. Let $D_i = \{x \in V : d(u, x) = i\}$, for $i = 1, 2, 3$. Thus, $|D_1| = 1$. Let $D_1 = \{w\}$. We distinguish cases, depending on de cardinality of D_3.



$H_{15} \cong H_9 - e_1$ $\quad$ $H_{16} \cong H_{10} - e_2$ $\quad$ $H_{17} \cong H_{12} - e_3$

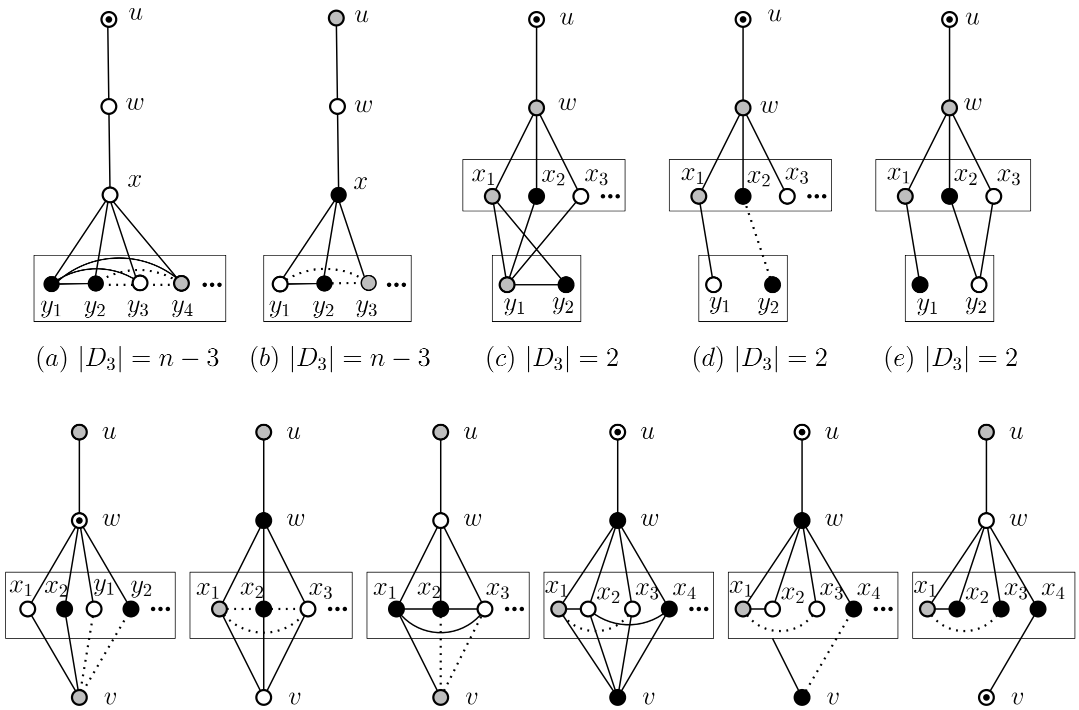
Figure 7: Graphs of order $n \geq 7$ and diameter 3 such that $\eta(G) = n-2$.

Case 1: $|D_3| \geq 3$. Then, $\deg(w) \leq n-4$, and therefore, $\deg(w) = 2$, $|D_1| = |D_2| = 1$ and $|D_3| = n-3 \geq 4$. Let x be the only vertex in D_2. Notice that every vertex of D_3 is adjacent to x. We distinguish cases taking into account the degree of the vertices in D_3.

- *There is a vertex of degree $n-3$ in D_3.* A vertex in D_3 of degree $n-3$ must be adjacent to all the other vertices of D_3. Therefore, there is exactly one vertex of degree $n-3$ in D_3 or every vertex in D_3 has degree $n-3$. In the last case, that is, if every vertex in D_3 has degree $n-3$, then D_3 is a clique and $G \cong H_{16}$. Otherwise, let y_1 be the only vertex in D_3 of degree $n-3$. Any other vertex in D_3 has degree 2, since it is adjacent to x and to y_1. Let $y_2, y_3, y_4 \in D_3 \setminus \{y_1\}$. Consider $S_1 = \{y_1, y_2\}$ and $S_2 = \{w, x, y_3\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is an MLD-partition of G. Indeed, it is dominating partition, $\{u\}$ resolves S_2 and $\{y_4\}$ resolves S_1 (see Figure 8(a)). Thus, $\eta_p(G) \leq n-3$, a contradiction.

- *Every vertex in D_3 has degree 1 or 2, and at least one of them has degree 2.* Then, $G[D_3]$ contains at least a copy of K_2. Let y_1 and y_2 be the vertices of such a copy of K_2, and take $y_3 \in D_3 \setminus \{y_1, y_2\}$. Consider $S_1 = \{w, y_1\}$, $S_2 = \{x, y_2\}$ and $S_3 = \{u, y_3\}$. It is straightforward to prove that $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is an MLD-partition of G (see Figure 8(b)), and thus $\eta_p(G) \leq n - 3$, a contradiction.

- *Every vertex in D_3 has degree 1.* Then, D_3 induces an empty graph and $G \cong H_{17}$.



(a) $|D_3| = n-3$ (b) $|D_3| = n-3$ (c) $|D_3| = 2$ (d) $|D_3| = 2$ (e) $|D_3| = 2$

(f) $|D_3| = 1$ (g) $|D_3| = 1$ (h) $|D_3| = 1$ (i) $|D_3| = 1$ (j) $|D_3| = 1$ (k) $|D_3| = 1$

Figure 8: Solid (resp. dotted) lines mean adjacent (resp. non-adjacent) vertices. Vertices with the same "color" belong to the same part.

Case 2: $|D_3| = 2$. Then, $|D_2| = n - 4$. Let $D_3 = \{y_1, y_2\}$. Recall that both y_1 and y_2 have at least a neighbor in D_2. We distinguish cases taking into account the degree of the vertices in D_3.

- *There is a vertex of degree $n - 3$ in D_3.* We may assume that this vertex is y_1, and it must be adjacent to y_2 and to all vertices in D_2. So, there is a vertex $x_1 \in D_2$ adjacent to both y_1 and y_2. Let $x_2 \in D_2 \setminus \{x_1\}$ and consider $S_1 = \{w, x_1, y_1\}$ and $S_2 = \{x_2, y_2\}$ Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition, and $\{u\}$ resolves both S_1 and S_2 (see Figure 8(c)). Hence, $\eta_p(G) \leq n - 3$, a contradiction.

- *Both vertices in D_3 have degree 1 or 2.* Let $x_1 \in D_2$ be a neighbor of y_1.

  If there exists a vertex $x_2 \in D_2 \setminus \{x_1\}$ not adjacent to y_2, let $x_3 \in D_2 \setminus \{x_1, x_2\}$. Consider $S_1 = \{w, x_1\}$, $S_2 = \{x_2, y_2\}$ and $S_3 = \{x_3, y_1\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$

is a dominating partition and $\{u\}$ resolves S_1, S_2 and S_3 (see Figure 8(d)). Therefore, $\eta_p(G) \leq n-3$, a contradiction.

If all vertices in $D_2 \setminus \{x_1\}$ are adjacent to y_2, then $\deg(y_1) \leq n-4$, with means that $\deg(y_1)$ and $|D_2| = 3 = n-4$, and thus $n = 7$. Let $D_2 = \{x_1, x_2, x_3\}$ and consider $S_1 = \{w, x_1, \}$, $S_2 = \{x_2, y_1\}$ and $S_3 = \{x_3, y_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition and $\{u\}$ resolves S_1, S_2 and S_3 (see Figure 8(e)). Therefore, $\eta_p(G) \leq n-3$, a contradiction.

Case 3: $|D_3| = 1$. Then, $D_3 = \{v\}$ and $|D_2| = n-3$. We distinguish cases taking into account the degree of v and the subgraph induced by D_2.

- $\deg(v) = 2$. Let x_1 and x_2 be the two neighbors of v, and take $y_1, y_2 \in D_2 \setminus \{x_1, x_2\}$. Let $S_1 = \{u, v\}$, $S_2 = \{x_1, y_1\}$ and $S_3 = \{x_2, y_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is dominating partition such that $\{w\}$ resolves S_1, and S_1 resolves both S_2 and S_3 (see Figure 8(f)), implying that $\eta_p(G) \leq n-3$, a contradiction.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces an empty graph.*

  If $\deg(v) = n-3$, let $x_1, x_2, x_3 \in D_2$ and let $S_1 = \{u, x_1\}$, $S_2 = \{w, x_2\}$ and $S_3 = \{v, x_3\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition such that S_1 resolves both S_2 and S_3, and S_3 resolves S_1. (see Figure 8(g)), implying that $\eta_p(G) \leq n-3$, a contradiction.

  If $\deg(v) = 1$, then $G \cong H_{17}$.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces a complete graph.*

  If $\deg(v) = n-3$, then $G \cong H_{15}$.

  If $\deg(v) = 1$, let $x_1 \in D_2$ be the neighbor of v and $x_2, x_3 \in D_2 \setminus \{x_1\}$. Consider $S_1 = \{u, v\}$, $S_2 = \{w, x_3\}$ and $S_3 = \{x_1, x_2\}$. Then, $\Pi = \{S_1, S_2, S_3\} \cup \{\{z\} : z \notin S_1 \cup S_2 \cup S_3\}$ is a dominating partition such that S_1 resolves both S_2 and S_3, and S_3 resolves S_1 (see Figure 8(h)), implying that $\eta_p(G) \leq n-3$, a contradiction.

- $\deg(v) \in \{1, n-3\}$ *and* D_2 *induces neither a complete, nor an empty graph.*

  In that case, there exist vertices $x_1, x_2, x_3 \in D_2$ such that $x_1x_2 \in E(G)$ and $x_1x_3 \notin E(G)$.

  If $\deg(v) = n-3$, then $\deg(x_1) \geq 3$, and thus, $\deg(x_1) \geq n-3$. Hence, x_1 must be adjacent to any other vertex in D_2 different from x_3. Let $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$ and consider $S_1 = \{w, x_4, v\}$ and $S_2 = \{x_2, x_3\}$ Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{u\}$ resolves S_1 and $\{x_1\}$ resolves S_2 (see Figure 8(i)), implying that $\eta_p(G) \leq n-3$, a contradiction.

  Finally, suppose that $\deg(v) = 1$. If there is a leaf x in D_2, then both u and x are antipodal vertices of v. In such a case, interchanging the role of the vertices u and v, the preceding cases for $|D_3| \geq 2$ apply and we are done. So, we can assume that any vertex in D_2 has degree at least 2. Let $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$ not adjacent to v. Notice that such a vertex exists whenever $n \geq 8$, because D_2 has at least 5 vertices. Let $S_1 = \{w, x_4, v\}$ and $S_2 = \{x_2, x_3\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{u\}$ resolves S_1 and $\{x_1\}$ resolves S_2. Therefore, Π is an MLD-partition of G (see Figure 8(j)), and so $\eta_p(G) \leq n-3$, a contradiction.

If $n = 7$ and the only vertex $x_4 \in D_2 \setminus \{x_1, x_2, x_3\}$ is adjacent to v, take $S_1 = \{x_2, x_3, x_4\}$ and $S_2 = \{u, x_1\}$. Then, $\Pi = \{S_1, S_2\} \cup \{\{z\} : z \notin S_1 \cup S_2\}$ is a dominating partition such that $\{v\}$ resolves $\{x_2, x_4\}$ and $\{x_3, x_4\}$; S_2 resolves $\{x_2, x_3\}$; and S_1 resolves S_2. Therefore, Π is an MLD-partition of G (see Figure 8(k)), and so $\eta_p(G) \leq n - 3$, a contradiction. □

As a straight consequence of Propositions 16, 19, 20 and 21, the following result is obtained.

Theorem 22. *Let G be a graph of order $n \geq 7$.*
Then, $\eta_p(G) = n - 2$ if and only if $G \in \Lambda_n = \{H_1, \ldots, H_{17}\}$ (see Figures 5, 6 and 7).

The solution for $\beta_p(G) = n - 2$ is also almost immediately derived.

Theorem 23. *Let G be a graph of order $n \geq 7$.*
Then, $\beta_p(G) = n - 2$ if and only if $G \in \Lambda_n \setminus \{H_{12}, H_{17}\}$.

Proof. Let $G \in \Lambda_n \setminus \{H_{12}, H_{17}\}$. Then, according to Proposition 16, $\beta_p(G) = n - 2$.
Conversely, let G be a graph of order $n \geq 7$ such that $\beta_p(G) = n - 2$. Thus, $\eta_p(G) = n - 2$, since by Theorem 1 and Theorem 14 we know that $\beta_p(G) \geq n - 1$ if and only if $\eta_p(G) \geq n - 1$. Hence, by Theorem 22, we derive that $G \in \Lambda_n$. Finally, it is a routine exercise to check that $\beta_p(H_{12}) = n - 3$ and $\beta_p(H_{17}) = n - 3$ (see Figure 9). □

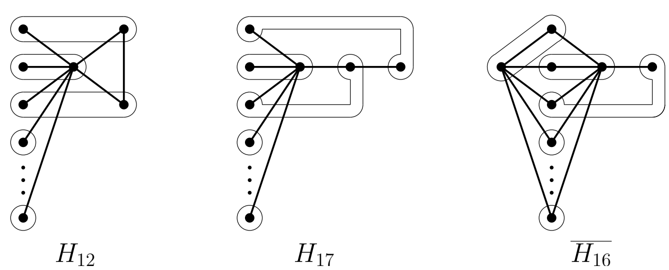


Figure 9: ML-partitions of cardinality $n - 3$ of H_{12}, H_{17} and $\overline{H_{16}}$.

Remark 24. *Theorem 23 corrects a wrong result shown in [15] (Theorem 3.2).*

4 Nordhaus-Gaddum bounds

A Nordhaus-Gaddum bound is a tight lower or upper bound on the sum of a parameter of a graph G and its complement G. In this last section, we present some tight Nordhaus-Gaddum bounds for both the partition dimension $\beta_p(G)$ and the partition metric-location-domination number $\eta_p(G)$.

Theorem 25. *Let G be a graph of order $n \geq 2$. Then, $4 \leq \eta_p(G) + \eta_p(\overline{G}) \leq 2n$. Moreover,*

(1) $\eta_p(G) + \eta_p(\overline{G}) = 4$ if and only if $\{G, \overline{G}\} = \{K_2, \overline{K_2}\}$.

(2) $\eta_p(G) + \eta_p(\overline{G}) = 2n$ if and only if either $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$ or $\{G, \overline{G}\} = \{K_{1,n-1}, K_{n-1} + K_1\}$.

Proof. The lower bound and item (1) are a straight consequence of Proposition 5. The upper bound and item (2) are immediately derived from Theorem 14. □

Theorem 26. *Let G be a graph of order $n \geq 2$. Then, $4 \leq \beta_p(G) + \beta_p(\overline{G}) \leq 2n$. Moreover,*

(1) $\beta_p(G) + \beta_p(\overline{G}) = 4$ if and only if either $\{G, \overline{G}\} = \{K_2, \overline{K_2}\}$ or $G = \overline{G} = P_4$.

(2) $\beta_p(G) + \beta_p(\overline{G}) = 2n$ if and only if $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$.

Proof. The lower bound and item (1) are a straight consequence of Theorem 1(3). The upper bound and item (2) are immediately derived from Theorem 1(4). □

A graph G is called *doubly-connected* if both G and its complement $\overline{G}$ are connected.

Theorem 27. *Let G be a doubly-connected graph of order $n \geq 3$. Then, $6 \leq \eta_p(G)+\eta_p(\overline{G}) \leq 2n-4$. Moreover,*

- *The equality $\eta_p(G) + \eta_p(\overline{G}) = 6$ is attained, among others, by P_4 and C_5.*
- *If $n \geq 7$, then $\eta_p(G) + \eta_p(\overline{G}) = 2n - 4$ if and only if $\{G, \overline{G}\} = \{H_{15}, H_{17}\}$.*

Proof. Observe that if $\eta_p(G) \geq n - 1$, then $\overline{G}$ is not connected. Thus, $\eta_p(G) + \eta_p(\overline{G}) \leq 2n - 4$. Certainly, the equality holds if and only if $\{G, \overline{G}\} \subset \{H_1, \ldots, H_{17}\}$. Finally, it is easy to check that the only two cases where it really happens are H_{15} and H_{17}, since $\overline{H_{15}} = H_{17}$. □

Theorem 28. *Let G be a doubly-connected graph of order $n \geq 3$. Then, $4 \leq \beta_p(G)+\eta_p(\overline{G}) \leq 2n-5$. Moreover,*

- *$\beta_p(G) + \beta_p(\overline{G}) = 4$ if and only if $G = P_4$.*
- *If $n \geq 7$, then $\beta_p(G) + \beta_p(\overline{G}) = 2n - 5$ if and only if either $\{G, \overline{G}\} = \{H_{15}, H_{17}\}$ or $\{G, \overline{G}\} = \{H_{15}, \overline{H_{16}}\}$.*

Proof. Note that if $\beta_p(G) \geq n - 1$, then $\overline{G}$ is not connected. It is a routine exercise to check the following facts: *(i)* the only two graphs belonging to $\Lambda_n \setminus \{H_{12}, H_{17}\}$ whose complement is connected are H_{15} and H_{16}, *(ii)* $\overline{H_{15}} = H_{17}$, and *(iii)* $\beta_p(H_{17}) = n - 3$ and $\beta_p(\overline{H_{16}}) = n - 3$ (see Figure 9). □

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