AN IMPROVED HEURISTIC FOR CAPACITATED LOCATION PROBLEMS: SOME THEORETICAL CONSIDERATIONS ABOUT ITS PERFORMANCE.

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Abstract

This paper describes a dual greedy heuristic for capacitated plant location problems and studies its computational behaviour. The good computational results obtained are explained from a theorical viewpoint by means of an extension of the analysis of Wolsey about maximising real-valued, nondecreasing, piecewise linear, concave submodular functions subjet to a knapsack constraint, to the case when fixed costs are included.

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Resumen

Este trabajo describe una heurística dual de tipo greedy para problemas de localización de plantas con restricciones de capacidad y estudia su comportamiento computacional. Los buenos resultados computacionales obtenidos son explicados desde un punto de vista teórico mediante una extensión del análisis de Wolsey sobre la maximización de funciones reales, lineales poligonales, no decrecientes, cóncavas y submodulares, sometidas a una restricción knapsack, al caso en que se incluyen además costes fijos.

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1. INTRODUCTION

Given the capacitated plant location problem (P) defined by

$$[MIN] \quad \sum_{i\in I} \sum_{j\in J} c_j \, x_{ij} + \sum_{j\in J} f_j \, y_j \tag{1}$$

s.t.

$$\sum_{j\in J} x_{ij} = 1 \quad , \forall i \in I \tag{2}$$

$$\sum_{j\in J} y_j \leqslant K \tag{3}$$

$$\sum_{i\in I} d_i \, x_{ij} \leqslant b_j \, y_j , \forall j \quad J \tag{4}$$

$$y_j \in \{0,1\}, \ \forall j \in J \tag{5}$$

where, as usual, c_{ij} is the cost of satisfying the demand of the i-th center from the j-th plant and f_j is the fixed cost of opening the j-th plant. The decision variable $x_{ij} = 1$ represents the decision of supplying the center at i from the plant at j and $x_{ij} = 0$ otherwise; and $y_i = 1$ that of opening a plant a j, and $y_j = 0$ otherwise. d_i is the demand level of the i-th center while b_j is the capacity level of the j-th plant. Constraints (2) guarantee that the demand of every center is satisfied, constraints (3) limit the number of plants to be opened to a maximum of K, and the set of constraints (4) avoids trying to satisfy the demand of the i-th center from a plant not previously open, given that

$$y_j = 0 \Rightarrow x_{ij} = 0, \ \forall i \in I$$

In this paper we consider two alternative complementary constraints: Either

$$x_{ij} >= 0, \ \forall i \in I, \ \forall j \in J \tag{6}$$

and then the problem is a mixed integer one, and the x_{ij} are the fraction of the demand at center i-th satisfyed from plant j-th, or

$$x_{ij} \in \{0,1\} \quad ,\forall i \in I, \ \forall j \in J \tag{7}$$

and then we deal with a pure integer problem arising when we ask that the demands of centers be satisfied only from one plant.

Obviously the problem (P) can have feasible solutions if and only if the following condition holds:

$$\sum_{i \in J^*} b_j \geq \sum_{i \in I} d_i \tag{8}$$

where J^*, the set of opened plants is such that $J^* \subseteq J$ and $|J^*| <= K$. However it should be noticed that in the pure integer case this condition is necessary but not sufficient and is not difficult to build counter examples for which the condition holds but no feasible solution exist.

In a former paper, /1/, we proposed a heuristic for problem (P) consisting of the following two step procedure:

1st step: Plant Selection Step.

Relaxing constraints (5) to: $y_i <= 0$, the analysis of the dual of the continuous LP relaxation of problem (P), allows us to calculate the multipliers u_i associated with the assignment constraints (2), that is the lagrangean multipliers of the lagrangean relaxation (RL1) of problem (P) with respect to the constraints (2):

$$\text{(RL1):} \quad L(u) = \text{MIN} \sum_{j \in J} \left(\sum_{i \in I} (c_{ij} - u_i) x_{ij} + f_j y_j \right) + \sum_{i \in I} u_i$$

s.t. (3), (4), (5) and (6)

The resulting values for the multipliers are

$$u_i = c_{ij} + d_i f_i / b_j \tag{9}$$

from which we can define the amounts

$$\rho_j = \sum_{i \in I} \left(c_{ij} + \frac{d_i f_j}{b_j} - c_{ik} - \frac{d_i f_k}{b_j} \right), \forall j \in J - J^* \tag{10}$$

where J^* is the set of the already selected plants, and k is the last opened plant. These amounts can be interpreted as a measure of the advantage of opening the j-th plant. With these amounts the decision rule:

Find

$$\rho_k = \underset{j \in J - J^*}{\mathrm{MIN}} \{\rho_j\}$$

set

$$J^* = J^* \cup \{k\}$$

used as incremental step defines a greedy heuristic, /2/, /8/, /9/, that selects the plants to be opened.

2^{nd} Step: Assignment

Once defined the set J^* of opened plants, the assignment of the centers to the opened plants can be stated as the following generalized assignment problem

$$[\mathrm{MIN}] \quad \sum_{i \in I} \sum_{j \in J^*} c_{ij} \, x_{ij}$$

s.t.

$$\sum_{j \in J^*} x_{ij} = 1, \quad \forall i \in I$$

$$\sum_{i \in I} d_i \, x_{ij} \leq b_j \quad , \forall j \in J^*$$

and

$$x_{ij} \geq 0 \quad \text{or} \quad x_{ij} \in \{0,1\}, \forall i \in I \text{ and } \forall j \in J^*$$

2. AN IMPROVED HEURISTIC

The constraint (3) alone does not always guarantee the existence of feasible solutions at least condition (8) holds, that means that in practice (and it is not difficult to find counterexamples) working only with constraint (3) restrict us to work only with subsets $J^* \subseteq J$ that meet condition (8), but this can not be a useful way of stating the problem. From this viewpoint it is more useful to replace constraint (3) by

$$\sum_{j \in J} b_j \, y_j \geqslant D \qquad (3')$$

where

$$D = \sum_{i \in I} d_i$$

Constraint (3') guarantees always that the subset J^* of opened plants meets condition (8).

On the other hand the amounts (10) can be interpreted as derived from the relaxation of problem (P) that considers it as an uncapacitated problem. In fact to calculate (9) we calculate the differences between each not yet opened plant and the last opened plant but we did it for all the centers as if the j-th plant taken into account was able of supplying of all them, as in the uncapacitated case. Consequently a most realistic approach must include considerations about the capacities of the plants when we examine them as candidates to be opened.

To do this we propose a new amount as measure of the advantage of opening the j-th plant. Let us define $\forall j \in J - J^*$, the amount ρ_j as:

$$\rho_j(u^k) = \text{MIN} \sum_{i \in I} \left[c_{ij} + \frac{d_j \, f_k}{b_k} - c_{ik} - \frac{d_i \, f_k}{b_k} \right] z_i$$

s.t.

$$\sum_{i \in I} d_i \, z_i \leqslant b_j \qquad (11)$$

and either

$$0 \leqslant z_i \leqslant 1 \;, \; \forall i \in I \qquad (12)$$

for the mixed integer problem or

$$z_i \in \{0,1\}, \forall i \in I \qquad (13)$$

for the pure integer. Where, as in the former case, k is the index of the last opened plant:

Marking:

$$\sigma_i = c_{ij} + \frac{d_i \, f_j}{b_j} - c_{ik} - \frac{d_i \, f_k}{b_k}, \forall i \in I, \forall j \in J - J^* \qquad (14)$$

then the amount ρ_j that is the value of the incremental step for the greedy heuristic, is the solution to the following

Knapsack problem

$$\rho_j(u^k) = \text{MIN} \sum_{i\in I} \sigma_i z_i \qquad \text{(KP)}$$

s.t.

$$\sum_{i\in I} d_i z_i \leqslant b_j$$

and, either

$$0\leqslant z_i\leqslant 1, \ \forall i \in I, \text{ or } z_i \in\{0,1\} \quad , \forall i \in I$$

case a) All the σ coefficients are negative: $\sigma_i >=0$, $\forall i \in I$, then multiplying by-1, we get the ordinary knapsack problem:

$$\rho_j(u^k) = \text{MIN} \sum_{i\in I} (-\sigma_i) z_i \qquad (-\sigma\geqslant 0)$$

$$\sum_{i\in I} d_i z_i \leqslant b_j$$

$$\text{either } 0 \leqslant z_i \leqslant 1 \quad \text{or} \quad z_i \in \{0,1\} , \forall i \in I$$

case b) some σ coefficients are negative: $\sigma_i \leqslant 0$ and Let us define in that case the sets of indexes:

$$I^- = \{i \mid \sigma_i \leqslant 0\} \text{ and } I^+ = \{i \mid \sigma_i >0\}$$

and setting $z_i = 0$, $\forall i \in I^+$, the problem becomes

$$\rho_j(u^k) = \text{MIN} \sum_{i\in I^-} \sigma_i z_i$$

$$\sum_{i\in I^-} d_i z_i \leqslant b_j$$

either $0<=z_i<=1$, or, z_i $\{0,1\}$, $\forall i \in I^-$

with all the $\sigma_i <=0$, as in the case a.

case c) All the σ coefficients are positive: $\sigma_i >=0$,

$\forall\, i \in I$, then $z_i = 0$, $\forall\, i \in I$ and $\rho_j(u^k)=0$

With this amounts we can redefine the heuristic proposed in /1/, for selecting the plants to be opened. The improved heuristic is:

Step 0 (Initialization)

Set $k = 0$, $J^* = \emptyset$

Define: $u_i^k = \max\limits_{j \in J} \{c_{ij} + \dfrac{d_i f_i}{b_j}\}$, $\forall\, i \in I$

Go to step 1.

Step 1 (Plant selection)

Calculate: $\sigma_i = c_{ij} + \dfrac{d_i f_i}{b_j} - u_i^k$, $\forall\, i \in I$

Solve the (KP) problem $\forall\, j \in J - J^*$

$$\rho_j(u^k) = \text{MIN} \sum_{i \in I} \sigma_i z_i$$

$$\sum_{i \in I} d_i z_i \leqslant b_j$$

either $0 <= z_i \leqslant 1$, or, $z_i \equiv \{0,1\}$, $\forall\, i \in I$

Find

$$\rho_k = \rho_{jk}(u^k) = \min_{j \in J - J^*} \{\rho_j(u^k)\}$$

Set $J^* = J^* \cup \{j_k\}$

If $\sum\limits_{j \in J} b_j \geqslant \sum\limits_{i \in I} d$: stop. The set J^* is the solution.

Otherwise go to step 2.

Step 2 (Improvement of the multipliers)

Set $k=k+1$

Calculate:

$$u_i^k = u_i^{k-1} + \text{MIN}\ \{0,\ c_{ij_{k-1}} + \frac{d_i f_{k-1}}{b_{j_{k-1}}} - u_i^{k-1}\}\ \forall i \in I$$

Go to step 1

Once the heuristic has obtained a set J^* of plants to be opened we proceed as in /1/, to solve the remaining generalized assignment subproblem and to trying to improve the solutions through an interchange heuristic.

2.1. COMPUTATIOINAL EXPERIENCE

The set of problems reported in /1/, was solved again with the improved heuristics, which proved to be very much better than the former one given that in more than the 90% of the cases the new heuristic provided the optimal solution, that means the optimal set of plants to be opened, directly without need of using the complementary interchange heuristic.

3. A THEORIC INSIGHT INTO THE BEHAVIOUR OF THE HEURISTIC

The computing behaviour of both heuristics, the one proposed in /1/, and that described above, arises the theorical question of why all they have these properties. For the uncapacitated case Cornuejols et al. /2/, Nemhauser et al /9/ and Fisher et al. /4/, have justified analytically the behaviour of the heuristics defined by the same Cornuejols, /2/, in a study that also includes some other heuristics like the one of Erlenkotter, /3/. The results of Cornuejols et al. /2/, show that a greedy heuristic for the simple k-location problem formulated as:

$$\max_{S\subseteq J} \{v(s) \mid |S| \leq K, \text{ where } v(S)=\sum_{i\in I} \max_{j\in J} c_{ij}, \text{ and } c_{ij} \geq 0\} \qquad (15)$$

achieves at least $1-e^{-1} \simeq 63\%$ of the optimal value of a "strong" linear programming relaxation of (15). Nemhauser et al. generalise in /8/, this result to the problem

$$\max_{S\subseteq J} \{Z(s) \mid |S| \leq K, \text{ z submodular and nondecreasing, } z(\emptyset)=0\} \qquad (16)$$

The considerable practical succes with dual heuristics for the capacitated plant location problem, reported by Geoffrion and McBride in /6/ and by Guignard and Spielberg in /7/, raised the question of whether or not the former results could be extended to the capacitated plant location problems.

Wolsey answers in /12/ this question for the case without fixed costs. To do this he considers the following formulation of the capacitated plant location problem (with a knapsack constraint replacing the cardinality constraint):

$$Z_b = \text{MAX}\{\ w(y)\ |\ \sum_{j\in J} b_j\ y_j \quad b,\ y_j \in \{0,1\}\ ,\ \forall\, j\in J\} \tag{17}$$

where:

$$w(y) = \text{MAX} \sum_{i\in I} \sum_{j\in J} c_{ij}\ x_{ij}$$

$$\text{s.t.}$$

$$\sum_{j\in J} x_{ij} \leqslant d_i \ ,\ \forall\, i \in I$$

$$\sum_{i\in I} x_{ij} \leqslant b_i\ y_j,\ \forall\, j \in J$$

$$x_{ij} \leqslant d_i\ y_j,\ \forall\, i \in I,\ \forall\, j \in J$$

$$x_{ij} \geqslant 0$$

Problem (17) has a linear programming relaxation

$$Z_b^{LP} = \text{MAX} \sum_{i\in I} \sum_{j\in J} c_{ij}\ x_{ij}$$

$$\sum_{j\in J} x_{ij} \leqslant 1,\ \forall\, i \in I$$

$$\sum_{i\in I} x_{ij} \leqslant b_j\ y_j,\ \forall\, j \in J$$

$$x_{ij} \leqslant d_i y_j,\ \forall\, i \in I,\ \forall\, j \in J$$

$$\sum_{j\in J} b_j y_j \leqslant b$$

$$0 \leqslant y_j \leqslant 1,\ \forall\, j \in J$$

$$x_{ij} \geqslant 0,\ \forall\, i \in I,\ \forall\, j \in J$$

where each center of demand d_i has been replaced by d_i centers of capacity 1. And from the definition of w(y) the following proposition can be proven:

Proposition 3.1.:

a) w(y) is a piecewise rational function on R^m_+, (m=|J|)

b) $w(y) = \underset{u\geq 0}{\text{MIN}} \; L(u,y) = \sum_{i\in I} u_i + \sum_{j\in J} \sigma_j(u) y_j$

and $\sigma_j(u)$ is the sum of the b_j largest terms of $(c_{ij}-u_i)^+$, $\forall i \in I$.

c) L(u,y) is submodular.

d) w(y) is submodular

An outline of the proof (see /12/ for details) is as follows: By linear programming duality if y>=0

$$w(y) = \text{MIN} \left\{ \sum_{i\in I} u_i + \sum_{j\in J} b_j y_j v_j + \sum_{j\in J} \sum_{i\in I} y_j w_{ij} \mid (u,v,w) \in X \right\}$$

where

$$X = \{ (u,v,w) >= 0 \mid u_i + v_j + w_{ij} >= c_{ij}, \; \forall i \in I, \; \forall j \in J \}$$

Then

$$w(y) = \underset{s\in S}{\text{MIN}} \sum_{i\in I} \hat{u}_i^s + \sum_{j\in J} (b_j \hat{v}_j^s + \sum \hat{w}_{ij}^s) y_j \; b_j - \hat{v}_j^s$$

where $(\hat{u}_i, \hat{u}_j, \hat{w}_{ij})$ are the extreme points of X, and (a) is proved.

Dualising with respect to the constraints $\sum_{j\in J} x_{ij} \leq 1$, $w(y) = \underset{u\geq 0}{\text{MIN}} \; L(u,y)$, where

$$L(u,y) = \sum_{i\in I} u_i + \sum_{j\in J} \{\text{MAX} \sum_{i\in I} (c_i - u_{ij}) x_{ij} \mid \sum_{i\in I} x_{ij} \leq b_j y_j, \; 0 \leq x_{ij} \leq y_j, \forall i \in I\}$$

and then

$$L(u,y) = \sum_{i\in I} u_i + \sum_{j\in J} \sigma_j(u)y_j$$

and (b) is proved. Parts (c) and (d) follow from the properties of the submodular function proved in /2/, /4/, and /9/. Consequently the general continuous greedy heuristic can be applied to problem (17), and then the results proved in /2/, and /9/ for problems (15) and (16) can be extended to problem (17).

The question now is what happens when we add fixed costs to problem (17). Let us return to problem (P), stated at the beginning of this paper, and let us reformulate the problem in the following way:

$$[\text{MIN}] \quad \sum_{i\in I} \sum_{j\in J} c_{ij}\, x_{ij} + \sum_{j\in J} f_j y_j \tag{1}$$

s.t.

$$\sum_{j\in J} x_{ij} = 1, \ \forall i \in I \tag{2}$$

$$\sum_{j\in J} b_j y_j \geq D \quad (D = \sum_{i\in I} d_i) \tag{3'}$$

$$\sum_{i\in I} d_i\, x_{ij} \leq b_j y_j, \ \forall j \in J \tag{4}$$

$$x_{ij} \leq y_j \ , \ \forall i \in I, \ \forall j \in J \tag{18}$$

$$0 \leq x_{ij} \leq 1, \quad y_j \in \{0,1\}, \ \forall i \in I, \ \forall j \in J \tag{19}$$

This problem can be formulated as a maximization problem making the transformation:

Let t_i be: $\quad t_i = \text{MAX}_{j\in J} \{C_{ij}\}\ , \ \forall i \in I$

then, let us define

$$p_{ij} = t_i - c_{ij}$$

and thus

$$c_{ij} = t_i - p_{ij} \quad \text{and} \quad T = \sum_{i\in I} t_i$$

Then:

$$\sum_{i\in I} \sum_{j\in J} c_{ij}\, x_{ij} = \sum_{i\in I} \sum_{j\in J} (t_i - p_{ij}) x_{ij} = \sum_{i\in I} t_i (\sum_{j\in J} x_{ij}) - \sum_{i\in I} \sum_{j\in J} p_{ij} x_{ij} =$$

$$= T - \sum_{i\in I} \sum_{j\in J} p_{ij}\, x_{ij} \quad \text{(by (2))}$$

Thus

$$\text{MIN} \sum_{i\in I} \sum_{j\in J} c_{ij}\, x_{ij} + \sum_{j\in J} f_j\, y_j =$$

$$= \text{MIN } T - \sum_{i\in I} \sum_{j\in J} p_{ij} x_{ij} + \sum_{j\in J} f_j y_j$$

and the objective function (1) becomes equivalent to

$$\text{MAX} \sum_{i\in I} \sum_{j\in J} p_{ij}\, x_{ij} - \sum_{j\in J} f_j y_j - T \qquad (20)$$

and introducing the change $y_j = 1 - z_j$ constraint (3') becomes

$$\sum_{j\in J} b_j\, z_j \leq b \quad (\text{where } b = \sum_{j\in J} b_j - D) \qquad (21)$$

and constraints (4) and (19) are transformed respectively in:

$$\sum_{i\in I} d_i\, x_{ij} + b_j\, z_j \leq b_j, \quad \forall j \in J \qquad (22)$$

and

$$x_{ij} + z_j \leq 1, \quad \forall i \in I, \quad \forall j \in J \qquad (23)$$

Thus, the resulting problem is:

$$\text{MAX} \sum_{i\in I} \sum_{j\in J} p_{ij}\, x_{ij} + \sum_{j\in J} f_j\, z_j - (T+F), \quad (F= \sum_{j\in J} f_j) \tag{24}$$

s.t.

$$\sum_{j\in J} x_{ij} = 1, \quad \forall i \in I \tag{2}$$

$$\sum_{j\in J} b_j\, z_j \leqslant b \tag{21}$$

$$\sum_{i\in I} d_i\, x_{ij} + b_j\, z_j \leqslant b_j, \quad \forall j \in J \tag{22}$$

$$x_{ij} + z_j \leqslant 1, \quad \forall i \in I, \quad \forall j \in J \tag{23}$$

$$0 \leqslant x_{ij} \leqslant 1, \quad z_j \in 0,1, \quad \forall i \in I, \quad \forall j \in J \tag{25}$$

We can rewrite this problem, in a form that is the generalization of problem (17) including fixed costs, as:

$$\text{MAX} \left\{ w(z) \mid \sum_{j\in J} b_j\, z_j \leqslant b, \quad z_j \in 0,1, \quad \forall j \in J \right\} \tag{26}$$

where:

$$w(z) = \text{MAX} \sum_{i\in I} \sum_{j\in J} p_{ij}\, x_{ij} + \sum_{j\in J} f_j\, z_j \tag{27}$$

s.t.

$$\sum_{j\in J} x_{ij} \leqslant 1, \quad \forall i \in I \tag{28}$$

$$\sum_{i\in I} d_i\, x_{ij} + b_j\, z_j \leqslant b_j, \quad \forall j \in J \tag{22}$$

$$x_{ij} + z_j \leqslant 1, \quad \forall i \in I, \quad \forall j \in J \tag{23}$$

$$x_{ij} \geqslant 0 \tag{29}$$

the linear programming problem given by (26), (27), (28), (22), (23) and (29) allows us to extend the results in proposition 3.1 to the case with fixed costs.

Theorem 3.1

$w(z)$ is a piecewise, linear, rational function on R_+^m, $(m=|J|)$

Proof:

By linear programming duality if $z>=0$, then

$$w(z) = \mathrm{MIN}\Big\{\sum_{i\in I} u_i + \sum_{j\in J} b_j v_j (1-z_j) + \sum_{i\in I}\sum_{j\in J} w_{ij}(1-z_j) + \sum_{j\in J} f_j z_j \;|$$

$$(u,v,w) \in X\Big\}$$

where

$$X = \{ (u,v,w) \geq 0 | \; u_i+d_i v_i+w_{ij} \geq p_{ij}, \; \forall i \in I, \; \forall j \in J\}$$

and thus

$$w(z) = \mathrm{MIN}_{k\in K}\Big\{ \sum_{i\in I} \hat{u}_i^k \quad \sum_{j\in J} (b_j \hat{v}_j^k + \sum_{i\in I} \hat{w}_{ij}^k)\; (1-z_j) + \sum_{j\in J} f_j z_j \Big\}$$

being K the set of extreme points of X.

Theorem 3.2

$$w(z) = \mathrm{MIN}_{u}\; L(u,z)$$

where

$$L(u,z) = \sum_{i\in I} u_i + \sum_{j\in J} (\sigma_j(u)+f_j) z_j$$

and $\sigma_j(u)$ is the value of the optimal solution to the knapsack problem

$$\sigma_j(u) = \mathrm{MAX} \sum_{i\in I} (p_{ij}-u_i)\; x_{ij}$$

$$\sum_{i\in I} d_i x_{ij} \leq b_j \tag{30}$$

$$0 \leq x_{ij} \leq 1$$

Proof:

Dualising in a lagrangean fashion, with respect to the constraints

$\sum_{j\in J} x_{ij} \leqslant 1$:

$$w(z) = \underset{u}{\mathrm{MIN}} \quad L(u,z)$$

where:

$$L(u,z) = \sum_{i\in I} u_i \sum_{j\in J} \mathrm{MAX}\left\{ \sum_{i\in I} (p_{ij}-u_i)x_{ij} \mid \sum_{i\in I} d_i x_{ij} + b_j z_j \leqslant b_j, \forall j\in J, \text{ and } x_{ij} + z_j \leqslant 1, \forall i\in I, \forall j\in J, x_{ij} \geqslant 0 \right\} + \sum_{j\in J} f_j z_j$$

but:

$$\mathrm{MAX} \quad \sum_{i\in I} (p_{ij}-u_i)x_{ij}$$

s.t.

$$\sum_{i\in I} d_i x_{ij} + b_j z_j \leqslant b_j \qquad \forall j\in J$$

$$\left.\begin{array}{l} x_{ij} + z_j \leqslant 1 \\ x_{ij} \geqslant 0 \end{array}\right\} \forall i\in I, \quad \forall j\in J$$

is equivalent to $\sigma_j(u)z_j$ where $\sigma_j(u)$ is given by (30)

Theorem 3.3

$L(u,z)$ is submodular

Proof:

a) $\sigma_j(u)$ is submodular.
The function $\sigma_j(u)$ defined by (30) finds its maximum in a vertex of the knapsack polyhedron and as it is a linear function, is piecewise linear, rational and nondecreasing, on the other hand, dualising with respect to the knapsack constraint

we get

$$LK_j(s,x) = b_j s + MAX \left\{ \sum_{i\in I} (p_{ij}-u_i-d_i s) x_{ij} \mid 0 \leq x_{ij} \leq 1 \right\}$$

and defining

$$I^+ = \left\{ i \mid p_{ij}-u_i-d_i s > 0 \right\}$$

$$\alpha = \sum_{i\in I^+} (p_{ij}-u_i-d_i s)$$

α is then the sum of the $|I^+|$ largest coordinates of the cost vector and then by proposition 4 in /12/, α is submodular, and like the sum of submodular functions is submodular, then $LK_j(s,x)$ is submodular. By the properties of lagrangean relaxations, /5/, /10/,:

$$\sigma_j(u) = \underset{S\geq 0}{MIN}\ LK_j(s,x)$$

and then by proposition 6a in /12/ (this proposition, proved by Topkis in /11/, states that if L(u,y) is submodular on $R^m_+ \times R^n_+$, $w(y) = \underset{u\geq 0}{MIN}$ L(u,y), and the set of optimal solutions U^y = u>=0 w(y)=L(u,y) is a nonempty polyhedron for all $y \in R^n_+$, then w(y) is submodular), $\sigma_j(u)$ is submodular.

b) Given that $\sigma_j(u)$ is submodular, and given that the product of a submodular function by a positive real number is also a submodular function (proposition 5 in /12/), and that the sum of submodular functions is also submodular, then L(u,z) is submodular.

Theorem 3.4

$$w(z) = \underset{u>=0}{MIN}\ L(u,z) \text{ is submodular}$$

The proof follows inmediately from theorem 3.3 using proposition 6a in /12/. These results extend the results in /12/ to the capacitated plant location problem with fixed costs, and explains why the dual greedy heuristics perform so well in that case. As an extension, from the dual of (26), (27), (28), (22) and (29) we get:

$$w(z) = MIN \left\{ \sum_{i\in I} u_i + \sum_{j\in J} b_j v_j \mid (u,v) \in K \right\}$$

where

$$K = \left\{ (u,v) \geq 0 \mid u_i + d_i v_j \geq p_{ij},\ \forall i\in I,\ \forall j\in J, \text{ and } b_j v_j \geq f_j,\ \forall j \in J \right\}$$

then

$$u_i \geq p_{ij} - d_i v_j \quad \text{and} \quad v_j \geq f_j/b_j$$

consequently

$$u_i \geq p_{ij} - d_i f_j/b_j \quad \forall i \in I$$

and replacing p_{ij} by its value $p_{ij} = t_i - c_{ij}$

$$u_i \geq t_i - (c_{ij} + \frac{d_i \; f_i}{b_j}), \quad \forall i \in I$$

and noting that t_i is a constant value for each i, this result is equivalent to that obtained in (10), used in the greedy heuristic of paragraph 2.

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