EXAMPLES OF CF-BI-INMUNE AND CF-LEVELABLE SETS IN LOGSPACE

José Luis BALCÁZAR
Josep DÍAZ
Joaquim GABARRÓ

RR84/08

EXAMPLES OF CF-BI-INMUNE AND CF-LEVELABLE SETS IN LOGSPACE

J.L. BALCÁZAR, J. DÍAZ, J. GABARRÓ

Facultat d'Informàtica de Barcelona
c/Jordi Girona Salgado, 31.
08034 Barcelona
ESPANYA

ABSTRACT.

We find sets in the class LOGSPACE (on-line) that do not have good respectively optimal approximations by context-free languages. For that we introduce the languages CF-bi-inmune and CF-levelable. We prove the results by defining directly the sets and obtaining the properties via pumping arguments.

RESUM.

Trobem conjunts en la classe LOGSPACE (on-line) que no tenen bones respectivament òptimas aproximacions per llenguatges incontextuals. Per això introduïm els llenguatges CF-bi-inmune i CF-nivellable. Demostrem els resultats directament, definim els llenguatges i demostren les propietats mitjançant el lema d'iteracció.

EXAMPLES OF CF-BI-IMMUNE AND CF-LEVELABLE SETS IN LOGSPACE

J.L.Balcázar, J.Díaz, J.Gabarró
Facultat d'Informática de Barcelona

Our aim in this note is to find sets in the class LOGSPACE (on-line) that do not have good, respectively optimal approximations by context-free languages. Our "good approximations" are in the sense that, given a set L in LOGSPACE, we say that it is well approximated if there are context-free grammars which generate infinite subsets of L. The idea is related to the idea of polynomial complexity core for a set; see [2], [4].

We prove two results; the first one is inspired by a result of [1], and presents a set in LOGSPACE which is CF-bi-immune. A set is CF-bi-immune if neither L nor its complement $\bar{L}$ have an infinite subset in CF, the class of the context-free languages. Thus, this results exhibits a set which can not be approximated by "pieces" that are infinite context-free languages.

The second result presents a set in LOGSPACE which is CF-levelable. The concept of CF-levelability is taken in the sense of [5]. Assume the class LOGSPACE ordered by inclusion modulo finite sets; then L is CF-levelable if it has no maximal context-free subset for this partial ordering. This second result means that often it is not possible to obtain a "best" infinite approximation to sets in LOGSPACE via the use of CF languages.

We prove the results by defining directly the set and obtaining the desired properties via pumping arguments. For the basic definitions and results see e.g. [3].

Theorem 1. Define the set A as follows:

$$A = \{ w \mid \exists p,q \ |w| = p+q^2,\ 0 \leq p < q \}$$

Then A is recognizable by an on-line logspace machine, and is CF-bi-immune.

Proof. A is easily recognized in logarithmic space by a search over all possible values of p,q. As only the length of the input is needed, the computation can be made on-line.

Assume L infinite, context-free, and included in A. By the pumping lemma, there are integers x,y and a family of words $\{ w_n \mid n \in N \}$, all of them in L and with lengths $|w_n| = x+ny$. Let q be an integer greater than both x and y; let m be the greatest integer such that $x+my<q^2+q$. Then

$$x+(m+1)y \geq q^2+q$$

and

$$x+(m+1)y = x+y+my < y+q^2+q \leq q^2+2q$$

so that $|w_{m+1}| = q^2+p$ with $q \leq p < 2q$. Hence $w_{m+1} \notin A$.

Now assume that L is infinite, context-free, and included in $\bar{A}$. As before, we have x, y and the words w_n. Let q be greater than both x and y, and m be the greatest integer such that $x+my<q^2$. Then

$$x+(m+1)y \geq q^2$$

and

$$x+(m+1)y = x+y+my<y+q^2$$

so that $|w_{m+1}| = q^2+p$ with $p<q$. Hence $w_{m+1} \in A$. □

<u>Theorem 2.</u> Let P be the set of the prime integers. The set

$$B = \{ a^p\# \ / \ p \in P \}^*$$

is CF-levelable (a is any symbol of the alphabet).

<u>Proof.</u> As prime numbers are written down in tally, the set B is easily recognized by a on-line machine within logarithmic space. We infer the levelability from the following fact:

<u>Fact</u>. If L is a context-free language which is included in B, then there is a finite set of primes F such that L is included in

$$\{ a^p\# \ / \ p \in F \}^*.$$

<u>Proof of fact.</u> Let k be the constant given by Ogden's lemma for L. Let F be the smallest set of primes such that L is included in $\{ a^p\# \ / \ p \in F \}^*$. If F is infinite then there exists a word in L such that

$$w = u\#a^m\#v, \qquad u,v \in \{ a,\# \}^*, \ m>k.$$

By marking the block of a^m letters in w we can obtain a word w' with

$$w' = u'\#a^q\#v' \qquad u,v \in \{ a,\# \}^*, \; q \in P$$

and hence w $\in$ L.

(Note that F finite does not imply L finite, because of the star in the definition of B.)

Now any infinite context-free subset of B can be extended by adding to it an infinite regular set disjoint to it: take $\{ a^p\# \}^*$ for some prime p higher than the ones which occur in the words of L; hence there is no maximal CF subset of B, which implies that B is CF-levelable.

□

The CF-levelability of several other languages can be proved in a similar way. Also, if we consider the concept of levelability respect to regular languages, it can be proved using the same ideas that the set

$$\{ a^nb^n \;/\; n \geq 0 \}^*$$

is context-free, recognizable on-line within LOGSPACE, and is regular-levelable.

REFERENCES

[1] J.L. Balcázar, U. Schöning: "Bi-immune sets for complexity classes", Math. Syst. Theory (to appear).

[2] S. Even, Y. Yacobi: "A 'hard core' theorem for randomized algorithms", Bulletin of the EATCS, 13, 30-36 (1981).

[3] J. Hopcroft, J. Ullman: Introduction to automata theory, languages, and computation, Addison-Wesley, Reading, 1979.

[4] N. Lynch: "On reducibility to complex or sparse sets", J. ACM, 22, 341-345 (1975).

[5] P. Orponen, D. Russo, U. Schöning: "Optimal approximations and polynomially levelable sets", unpublished preprint (1984).