Output Regulation Problem for Differentiable Families of Linear Systems

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Abstract. Given a family of linear systems depending on a parameter varying in a differentiable manifold, we obtain sufficient conditions for the existence of a (global or local) differentiable family of controllers solving the output regulation problem for the given family. Moreover, we construct it when these conditions hold.

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INTRODUCTION

The output regulation problem arose as one of the main research topics in linear control theory in the 1970s. This problem considers controlling a given plant such that its output tracks a reference signal or rejects a disturbance. The reference and disturbance signals are typically generated by an external system called exosystem. The output regulation problem for linear time-invariant (LTI) systems has been well studied by many authors like, for example, [1, 2, 3, 4, 5, 6, 7] and [8]. Usually, these LTI systems are obtained by linearizing around an operating point or by using system identification techniques. However, in some cases this approach is restrictive and it would be better to allow the system to depend differentiably on a specific parameter. We think that this situation is interesting to consider and, in this case, the controller must be able to maintain the properties of closed-loop stability and output regulation when it is modelled as a global or a local differentiable family.

Thus, we lead to the quite general question of whether pointwise solvability implies the existence of a nicely parameterized solution [9]. See, for example, [10] for a general introduction to families of linear systems (in the more general case of bundles over the space of parameters) or [9] for a survey mainly centered on control and stabilization problems in the also more general context of systems with entries in a commutative ring.

In this work we deal with the existence of differentiable families of controllers following the pattern in [6] for the constant case. In particular, we assume that the hypotheses there hold. We will denote them by (F1)-(F7) (see Theorem 1), and we add (') for the natural generalization to parameterized families (see Theorems 4 and 6). We remark (see [6]) that (F1)-(F6) either involve no loss of generality or are necessary for the existence of the synthesis.

We also remark that among the above assumptions, those concerning controllability or detectability of a system are easily transferred from the pointwise case to parameterized families (see, for example, [11]). Obstructions arise when subsystems are considered and in general for geometrical conditions. In both cases, the key tools are the techniques in [12]. They are valid when the manifold of parameters is contractible and the dimensions of the involved vectorial subspaces are constant. Moreover, we will see that these hypotheses can be weakened in the local case thanks to the results in [13], based on the Arnold's techniques about versal deformations, introduced in [14] for square matrices and generalized, for example, in [15].

The paper is organized as follows: first we present, as a reminder, the main result in [6], emphasizing the pointwise hypotheses that will be considered later. In the following sections we deal with the global and the local cases.

Notations $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. $\overline{\mathbb{C}^+}$ is the closed right-half complex plane and $\mathbb{C}^-$ is the open left-half one. If $v_1,\ldots,v_s$ are vectors of $\mathbb{R}^n$, then $[v_1,\ldots,v_s]$ denotes the subspace spanned by them. We write $M_{n,m}(\mathbb{R})$ for the vector space of matrices with n rows and m columns with entries in $\mathbb{R}$. We identify $M_{n,n+m}(\mathbb{R})$ with the pairs of matrices $M_{n,n}(\mathbb{R})\times M_{n,m}(\mathbb{R})$. If A is a matrix, A^t is its transpose matrix. I denotes the identity matrix and I_k the identity k-matrix. $\sigma(\cdot)$ denotes the spectrum of the corresponding matrix or pairs of matrices, each eigenvalue being repeated as many times as its multiplicity. For linear spaces $\mathscr{R}$ and $\mathscr{S}$, $\mathscr{R}\cong\mathscr{S}$ means $\mathscr{R}$ and $\mathscr{S}$ are isomorphic and $\mathrm{Hom}(\mathscr{R},\mathscr{S})$ is the linear space of all linear maps $\mathscr{R}\longrightarrow\mathscr{S}$.

PRELIMINARIES: THE NON-PARAMETERIZED CASE

In this section, we summarize the problem and the results stated in [6], which have been the basis of our work. This reference deals with the regulation of the linear system represented by

$$\dot{x}_1 = A_1x_1 + A_3x_2 + B_1u, \quad \dot{x}_2 = A_2x_2,$$
$$y = C_1x_1 + C_2x_2, \quad z = D_1x_1 + D_2x_2,$$

where x_1 is the plant state vector, u the control input, x_2 the vector of exogenous signals, y the vector of measurements available for control, and z the output to be regulated. These vectors belong to fixed finite-dimensional real linear spaces $\mathscr{X}_1$, $\mathscr{U}$, $\mathscr{X}_2$, $\mathscr{Y}$, $\mathscr{Z}$, of dimensions n_1, m, n_2, n_y, n_z, respectively; the linear maps or real matrices A_1,A_2, A_3, B_1, C_1, C_2, D_1, D_2 are time-invariant.

The objective is to construct a controller modelled by the equations

$$\dot{x}_c = A_cx_c + B_cy, \quad u = F_cx_c + G_cy,$$

where x_c is the compensator state vector, which belongs to a finite-dimensional real linear space $\mathscr{X}_c$, and the linear maps A_c, B_c, F_c, G_c are time-invariant. This system must achieve two properties: closed loop stability and output regulation of the associated closed loop defined as

$$\dot{x}_L = A_Lx_L + B_Lx_2, \quad z = D_Lx_L + D_2x_2,$$

where

$$x_L = \begin{pmatrix} x_1 \\ x_c \end{pmatrix}, \quad \mathscr{X}_L = \mathscr{X}_1 \oplus \mathscr{X}_c,$$
$$A_L = \begin{pmatrix} A_1 + B_1G_cC_1 & B_1F_c \\ B_cC_1 & A_c \end{pmatrix}, \quad B_L = \begin{pmatrix} A_3 + B_1G_cC_2 \\ B_cC_2 \end{pmatrix}, \quad D_L = \begin{pmatrix} D_1 & 0 \end{pmatrix}.$$

Closed-loop stability means that A_L is stable, that is, $\sigma(A_L) \subset \mathbb{C}^-$, and *output regulation* means that $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for all $x_L(0)$ and $x_2(0)$.

We next introduce the mathematical setting used in [6]. For any linear space $\mathscr{R}$, bring in the linear space $\underline{\mathscr{R}} = \mathrm{Hom}(\mathscr{X}_2, \mathscr{R})$. Given maps $A : \mathscr{X} \longrightarrow \mathscr{X}$ and $C : \mathscr{X} \longrightarrow \mathscr{Y}$, define, respectively, the linear maps $\underline{A} : \underline{\mathscr{X}} \longrightarrow \underline{\mathscr{X}}$ and $\underline{C} : \underline{\mathscr{X}} \longrightarrow \underline{\mathscr{Y}}$ by

$$\underline{A}X = AX - XA_2, \quad \underline{C}X = CX, \quad X \in \underline{\mathscr{X}}.$$

Analogous notation will be used for the subsystems involved in the construction.

Now we state the main result obtained in [6].

Theorem 1 *[6] Assume that:*

(F1) $\sigma(A_2) \subset \overline{\mathbb{C}^+}$,

(F2) $\mathrm{Im}\, C_1 + \mathrm{Im}\, C_2 = \mathscr{Y}$,

(F3) $\mathrm{Im}\, D_1 = \mathscr{Z}$,

(F4) (A_1, B_1) *is stabilizable*,

(F5) (C_1, A_1) *is detectable*,

(F6) (C, A) *is detectable, where* $A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$ *and* $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$.

Then, a controller exists iff

(F7) $\begin{pmatrix} A_3 \\ D_2 \end{pmatrix} \in \mathrm{Im} \begin{pmatrix} \underline{A}_1 & \underline{B}_1 \\ \underline{D}_1 & 0 \end{pmatrix}$.

Following [6] again, an algorithm to compute a controller can be organized into the following four steps.

Step 1. Select B_c so that $A - B_cC$ is stable.

Step 2. Select F_1 so that $A_1 + B_1F_1$ is stable.

Step 3. Select F_2 so that

$$\begin{pmatrix} A_3 + B_1F_2 \\ D_2 \end{pmatrix} \in \mathrm{Im} \begin{pmatrix} \underline{A}_1 + \underline{B}_1\underline{F}_1 \\ \underline{D}_1 \end{pmatrix}.$$

Step 4. Set $F_c = \begin{pmatrix} F_1 & F_2 \end{pmatrix}$, $A_c = A - B_cC + BF_c$, $G_c = 0$.

THE GLOBALLY PARAMETERIZED CASE

Our aim is to generalize the above results when the involved matrices depend on a parameter varying in a differentiable manifold. That is to say, we consider a family of linear systems depending differentiably on a parameter and we study sufficient conditions to ensure the existence of a differentiable solution of the output regulation problem for the given family.

Thus, we consider a differentiable family of linear systems modelled by the equations

$$\dot{x}_1 = A_1(\tau)x_1 + A_3(\tau)x_2 + B_1(\tau)u, \quad \dot{x}_2 = A_2(\tau)x_2,$$
$$y = C_1(\tau)x_1 + C_2(\tau)x_2, \quad z = D_1(\tau)x_1 + D_2(\tau)x_2,$$

or, equivalently, it will be represented by a differentiable family of matrices $(A_1(\tau), A_2(\tau), A_3(\tau), B_1(\tau), C_1(\tau), C_2(\tau), D_1(\tau), D_2(\tau))$.

To this end, basic tools are introduced in [16]. More specifically, we will need the following result.

Lemma 2 *Let M be a contractible manifold. If $A(\tau)$, $\tau \in M$, is a differentiable family of $q \times p$ real matrices having constant rank, and $v(\tau)$, $\tau \in M$, a differentiable family of vectors in $\operatorname{Im} A(\tau) \subset \mathbb{R}^q$, then there exists a differentiable family of vectors $u(\tau) \in \mathbb{R}^p$ such that*

$$A(\tau)u(\tau) = v(\tau)$$

for any $\tau \in M$.

In addition, we will use the following result from [12], where these tools are applied to prove the existence of a global parameterized pole assignment. Recall that if M is a contractible manifold and $(A(\tau), B(\tau))$, $\tau \in M$, is a differentiable family of pairs of matrices of $M_{n,n+m}(\mathbb{R})$, the family $(A(\tau), B(\tau))$ has *constant Brunovsky type* if

(i) the controllability indices are constant,

(ii) the Jordan invariants have constant type, that is to say, the number of distinct eigenvalues and the list of sizes of the Jordan blocks corresponding to different eigenvalues are independent of τ.

Theorem 3 *[12] Let M be a contractible manifold, $(A(\tau), B(\tau))$, $\tau \in M$, a differentiable family of pairs of matrices of $M_{n,n+m}(\mathbb{R})$ having constant Brunovsky type, $\lambda_1(\tau), \dots, \lambda_q(\tau) \in \mathbb{C}$ giving the distinct eigenvalues of $(A(\tau), B(\tau))$, and $m_1, \dots, m_q$ their respective algebraic multiplicities. If $\mu_i(\tau) \in \mathbb{C}$, $1 \leq i \leq s$, is a set of maps closed under conjugation, then there exists a differentiable family of matrices $F(\tau) \in M_{m,n}(\mathbb{R})$ such that the eigenvalues of $A(\tau) + B(\tau)F(\tau)$ are $\mu_1(\tau), \dots, \mu_s(\tau)$, $\lambda_1(\tau), \dots, \lambda_q(\tau)$, the latter having multiplicities $m_1, \dots, m_q$.*

This results allow us to study the existence of a global differentiable family of controllers. In the next section we address the local case.

Theorem 4 *Let M be a contractible manifold and $(A_1(\tau), A_2(\tau), A_3(\tau), B_1(\tau), C_1(\tau), C_2(\tau), D_1(\tau), D_2(\tau))$, $\tau \in M$, a differentiable family of linear systems verifying that for any $\tau \in M$:*

(F1') $\sigma(A_2(\tau)) \subset \overline{\mathbb{C}^+}$,

(F2') $\operatorname{Im} C_1(\tau) + \operatorname{Im} C_2(\tau) = \mathscr{Y}$,

(F3') $\operatorname{Im} D_1(\tau) = \mathscr{Z}$,

(F4') $(A_1(\tau), B_1(\tau))$ *is stabilizable*,

(F5') $(C_1(\tau), A_1(\tau))$ *is detectable*,

(F6') $(C(\tau), A(\tau))$ *is detectable*,

(F7') $\begin{pmatrix} A_3(\tau) \\ D_2(\tau) \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} \underline{A}_1(\tau) & \underline{B}_1(\tau) \\ \underline{D}_1(\tau) & 0 \end{pmatrix}$.

Then, there exists a global differentiable family of controllers $(A_c(\tau), B_c(\tau), F_c(\tau), G_c(\tau))$ if, when τ varies in M, we have:

(i) $(C(\tau), A(\tau))$ *has constant Brunovsky type*,

(ii) $(A_1(\tau), B_1(\tau))$ *has constant Brunovsky type*,

(iii) $\operatorname{rank} \begin{pmatrix} I_{n_2} \otimes A_1(\tau) - A_2(\tau)^t \otimes I_{n_1} & I_{n_2} \otimes B_1(\tau) \\ I_{n_2} \otimes D_1(\tau) & 0 \end{pmatrix}$ *is constant.*

THE LOCALLY PARAMETERIZED CASE

In this section, we tackle the local case, that is, we deal with *local differentiable families*. This means that the parameter varies only in an open neighborhood of the origin of $\mathbb{R}^k$.

Similarly to the global case, we will use the existence of a local differentiable family of pole assignments for a local differentiable family of stabilizable pairs. As above, a pointwise construction does not guarantee the differentiability of the family of feedbacks. It is so by means of the following result, based in the Arnold's techniques about versal deformations:

Theorem 5 *[13] Let $(A,B) \in M_{n,n+m}(\mathbb{R})$ be a stabilizable pair and $(A(\tau),B(\tau))$, $\tau \in V$, a local differentiable family of pairs with $(A(0),B(0)) = (A,B)$. Then there is a local differentiable family of feedbacks $F(\tau) \in M_{m,n}(\mathbb{R})$, defined in some open neighborhood of the origin $W \subset V$, such that $\sigma(A(\tau)+B(\tau)F(\tau)) \subset \mathbb{C}^-$.*

Finally, we apply this result to our output regulation problem.

Theorem 6 *Let $(A_1(\tau),A_2(\tau),A_3(\tau),B_1(\tau),C_1(\tau),C_2(\tau),\ D_1(\tau),D_2(\tau))$ be a differentiable family of linear systems defined in some open neighborhood of the origin $V \subset \mathbb{R}^k$. Assume that, for any $\tau \in V$:*

(F1') $\sigma(A_2(\tau)) \subset \overline{\mathbb{C}^+}$,

(F2) $\operatorname{Im} C_1 + \operatorname{Im} C_2 = \mathscr{Y}$,

(F3) $\operatorname{Im} D_1 = \mathscr{X}$,

(F4) (A_1,B_1) *is stabilizable*,

(F5) (C_1,A_1) *is detectable*,

(F6) (C,A) *is detectable*,

(F7') $\begin{pmatrix} A_3(\tau) \\ D_2(\tau) \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} \underline{A}_1(\tau) & \underline{B}_1(\tau) \\ \underline{D}_1(\tau) & 0 \end{pmatrix}$.

Then, there exists a local differentiable family of controllers $(A_c(\tau),B_c(\tau),F_c(\tau),G_c(\tau))$ in some open neighborhood of the origin $W \subset V$ if, when τ varies in V, we have:

(iii) $\operatorname{rank} \begin{pmatrix} I_{n_2} \otimes A_1(\tau) - A_2(\tau)^t \otimes I_{n_1} & I_{n_2} \otimes B_1(\tau) \\ I_{n_2} \otimes D_1(\tau) & 0 \end{pmatrix}$ *is constant.*

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