UNIVERSITY OF ALBERTA

Influence of Statistical Distributions on Availability and Inspection Interval of Protective Devices

by

David Sánchez González

Supervised by

Prof. Ming J. Zuo
Dr. Xihui Liang

A thesis written at University of Alberta and submitted to Universitat Politècnica de Catalunya in partial fulfillment of the requirements for the degree of Master of Science

in

Industrial Engineering

Edmonton, Alberta
January 2017
Abstract

Protective devices are designed to protect people, the environment and material assets under emergency situations. If protective devices do not work well, serious consequences may be resulted. It is critical to pay special attention to their maintenance. For this reason, many availability models have been developed to obtain an optimal inspection interval and to maximize their availability. However, few attention have been paid to the relationship between the statistical distributions used to describe the lifetime of protective devices and their optimal inspection interval and maximum availability. Furthermore, the problem that might occur when the normal distribution takes negative values has not been considered yet in protective device maintenance. This thesis aims to calculate the optimal inspection interval and maximum availability for the Weibull, normal, truncated normal and exponential distributions. Also, the relationship between these statistical distributions, and the availability and the inspection interval is studied. Finally, this thesis intends to study the problem that arises when the normal distribution might take negative values. To meet these objectives, an existing availability model, which considers constant time between inspections, is adapted to the Weibull, normal, truncated and exponential distributions. After adapting the model to each distribution, the effects of each distribution’s parameters on the optimal inspection interval and maximum availability are analyzed. It is not recommended to use the normal distribution if it has a large number of negative values while the truncated normal distribution is suggested as a possible approach to replace the normal distribution. This analysis help us to have a understanding on what is the performance and limitations of each of the four distributions.
Acknowledgements

There are so many people to thank for helping me during my five-months exchange at University of Alberta. In one way or another they have made my short stay a lot easier than I thought it was going to be.

First, I would like to express my deepest gratitude to my supervisor Prof. Ming J. Zuo for accepting me in his research group. During five months, I have been surrounded by brilliant and talented students and professors. I also would like to thank him for introducing me to the topic I have been working on, the useful comments and support throughout my exchange.

I would also like to thank Dr. Xihui Liang, for his guidance, encouragement, and specially, for his infinite patience with me. Thank you so much for all the comments, because undoubtedly, they have helped me to look at my thesis in different ways and they have opened my mind.

Many thanks to my colleagues from the Reliability Research Lab and friends from Edmonton for their various forms of help and support.

Finally, I must express my very profound gratitude to my family and friends from Spain for providing me with unfailing support and continuous encouragement not only during the process of writing this thesis, but also during my journey as a student.
# Table of Contents

Abstract .......................................................................................................................... ii
Acknowledgements ........................................................................................................ iii
List of Tables .................................................................................................................. v
List of Figures ................................................................................................................ vi
List of Symbols and Abbreviations ............................................................................... viii

Chapter 1: Introduction ................................................................................................. 1
  1.1 Background and motivation .................................................................................... 1
  1.2 Literature review .................................................................................................... 4
    1.2.1 Availability ...................................................................................................... 4
    1.2.2 Availability models .......................................................................................... 5
    1.2.3 Statistical distributions used in detective maintenance ................................. 8
  1.3 Research objective ................................................................................................ 10
  1.4 Organization of thesis ........................................................................................... 11

Chapter 2: Distributions used to model protective devices’ lifetime ......................... 12
  2.1 The Weibull distribution ....................................................................................... 13
  2.2 The exponential distribution ............................................................................... 16
  2.3 The normal distribution ....................................................................................... 17
  2.4 The truncated normal distribution ....................................................................... 20
  2.5 Conclusions .......................................................................................................... 22

Chapter 3: An Availability Model for a Protective Device ........................................ 24
  3.1 Availability model description ............................................................................ 24
  3.2 Numerical example ............................................................................................... 26
  3.3 Conclusions .......................................................................................................... 28

Chapter 4: Analysis of the Statistical Distributions .................................................... 30
  4.1 Weibull distribution analysis ............................................................................... 30
    4.1.1 Analysis of the shape parameter .................................................................. 32
    4.1.2 Analysis of the scale parameter .................................................................. 34
  4.2 Exponential distribution analysis ......................................................................... 36
  4.3 Normal distribution analysis ............................................................................... 39
  4.4 Truncated normal distribution analysis ............................................................... 44
  4.5 Comparison between the normal and the truncated normal distributions ........... 50
  4.6 Conclusions .......................................................................................................... 52

Chapter 5: Future Work ............................................................................................... 54

References ..................................................................................................................... 55
List of Tables

Table 2.1 Functions of the Weibull distribution................................................................. 13
Table 2.2 Functions of the exponential distribution.......................................................... 16
Table 2.3 Functions of the normal distribution ............................................................... 19
Table 2.4 Functions of the truncated normal distribution ........................................... 22
List of Figures

Fig. 2.1: Effect of β on the failure density with scale parameter fixed at 2 years .......... 15
Fig. 2.2: Effect of η on the failure density with shape parameter fixed at 2.5 .......... 15
Fig. 2.3: Effect of λ on the probability density function ........................................ 17
Fig. 2.4: Effect of μ on the failure density with σ fixed at 1.5 years....................... 18
Fig. 2.5: Effect of σ on the failure density with μ fixed at 6 years.......................... 19
Fig. 2.6: a) Normal distributions with σ = 0.05, 0.1 and 0.20, μ fixed at 0.1. b) Truncated distributions from a) .............................................................. 21
Fig. 3.1: Cycle length of the availability model selected ........................................ 26
Fig. 3.2: Steady-stat availability as a function of the inspection interval ............... 28
Fig. 4.1: Relationship between the optimal inspection interval and the scale parameter. (Scale fixed at 1 year, and shapes range from 0.5 to 10). .............................. 33
Fig. 4.2: Relationship between the maximum availability and the shape parameter. (Scale fixed at 1 year and shapes range from 0.5 to 10). .............................. 33
Fig. 4.3: Relationship between the optimal inspection interval and the scale parameter. (Scale ranges from 1.5 to 10 years. Shapes fixed at: 0.5, 0.8, 1, 2.2, 5 and 10). ............... 35
Fig. 4.4 Relationship between the maximum availability and the scale parameter. (Scale ranges from 1.5 to 10 years. Shapes fixed at: 0.5, 0.8, 1, 2.2, 5 and 10) ................... 36
Fig. 4.5: Relationship between the optimal inspection interval and the rate parameter. (Rate parameter ranges from 0.01 to 4 failures/years) ................................. 38
Fig. 4.6: Relationship between the maximum availability and the rate parameter. (Rate parameter ranges from 0.01 to 4 failures/years) ................................. 39
Fig. 4.7: Relationship between the optimal inspection interval and the mean. (Mean fixed at 2029.4 days Standard deviation ranges from 73 days to 1047.5 days). ........ 41
Fig. 4.8: Relationship between the optimal inspection interval and the standard deviation. (Mean fixed at 2029.4 days. Standard deviation ranges from 73 days to 2029.4 days) ................................................................. 42
Fig. 4.9: Relationship between maximum availability and standard deviation. (Mean fixed at 2029.4 days. Standard deviation ranges from 73 days to 2029.4 days) ........ 43
Fig. 4.10: Normal distribution with different levels of truncation: (a) 5%, (b) 50% (c) 0.05% ................................................................. 45

Fig. 4.11: Probability density function of a normal distribution (μ=400 days and σ=250 days) and its truncated distribution ................................................................. 46

Fig. 4.12: Relationship between optimal inspection interval and truncated area. (Mean fixed at 2029.4 days. Truncated area ranges from 0 % to 15.86%) ................................. 48

Fig. 4.13: Probability density function. Mean fixed at 2029.4 days, truncated areas: 4.54%, 10.23% and 15.51% ................................................................. 49

Fig. 4.14: Relationship between maximum availability and truncated area .................. 50

Fig. 4.15: Relationship optimal inspection interval and truncated area. Normal and truncated normal distributions. (Mean fixed at 2029.4 days. Truncated area ranges from 0 % to 15.86%) ................................................................. 51
### List of Symbols and Abbreviations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Truncation point</td>
</tr>
<tr>
<td>$A(t)$</td>
<td>Instantaneous availability of the system at time $t$</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>Steady-state availability</td>
</tr>
<tr>
<td>$f(t)$</td>
<td>Probability density function of the system</td>
</tr>
<tr>
<td>$F(t)$</td>
<td>Cumulative density function of the system</td>
</tr>
<tr>
<td>FFI</td>
<td>Failure-Finding Interval</td>
</tr>
<tr>
<td>$i$</td>
<td>Number of inspections carried out</td>
</tr>
<tr>
<td>$I$</td>
<td>Fixed inspection interval length</td>
</tr>
<tr>
<td>$p$</td>
<td>Truncated area</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>Reliability function of the system</td>
</tr>
<tr>
<td>RCM</td>
<td>Reliability Centered Maintenance</td>
</tr>
<tr>
<td>$T_F$</td>
<td>Total downtime when a failure is found</td>
</tr>
<tr>
<td>$T_W$</td>
<td>Downtime due to inspection</td>
</tr>
<tr>
<td>$v_i$</td>
<td>Working time since the last repair</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Truncation parameter</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Shape parameter of the weibull distribution</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Gamma distribution</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Scale parameter of the weibull distribution</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Number of failures per unit of measurement used in the exponential distribution</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Mean lifetime of the system</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Standard deviation of the normal distribution</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Background and motivation

Protective devices are our last line of protection when things do not function as expected. They are designed to protect people, the environment and material assets under emergency situations [1]. These kind of devices are used in many fields; ranging from respiratory protections in underground coal mines to hotbox detectors on railway cars [2]. Due to high impacts that protective devices may trigger, their maintenance represents crucial concerns for all kind of companies.

Tasks designed to ensure all protective devices work as expected are called detective tasks [3]. Detective maintenance was defined by Campbell and Reyes-Picknell [4] as a form of proactive work done to detect failures that have already occurred but remain undetected because the functionality that has been lost is normally not used or dormant.

It is important to note the main difference between preventive maintenance and detective maintenance. The first is based on overhauling items or replacing some components, while the second is designed to check whether the device or equipment still works.

A parameter that allows to measure the performance of detective maintenance and is widely used [4]–[6] is availability, which is defined as the ratio of uptime to total time [7].
The particular maintenance challenge of a protective device is that failures are not detected until it is called into use. These failures are called hidden or unrevealed failures [2]. Therefore, in order to check whether a protective device still works it is necessary to schedule functional inspections. Such inspections are called failure-finding tasks or functional tasks [3]. The optimal interval between inspections is known as failure-finding interval (FFI). FFI solely refers to equipment which may suffer from hidden failures [2].

Moubray [3] points out that even if Reliability Centered Maintenance (RCM) is correctly applied to almost all modern, complex industrial systems, it is common to find that up to 40% of failures correspond to hidden failures. Moreover, up to 80% of these failures require a Failure Finding, which means that around one third of the tasks generated by comprehensive, correctly applied maintenance strategy development programs are failure finding tasks.

Inspection policies to guarantee a satisfactory level of availability have been deeply studied [2][5][8]. Although many factors may affect these policies for finding the optimal interval, almost all of them aim to maximize availability or minimize expected cost. If the interval between inspections is very short the maintenance costs will rise due to the increase of times of inspection. On the other hand, if the interval is very long, the failure may not be detected timely.

Current research explains that the frequency of failure-finding tasks depends on two variables: the desired availability and the lifetime distribution of a device [3]. Although cost can also be considered as a factor to determine the frequency, to narrow down the scope of this thesis it is not considered. For a protective device, it may be more useful to determine frequency based on the availability rather than the cost. This is due to the
consequences a failed protective device may trigger. The common used lifetime distributions are exponential, Weibull, log-normal, log-logistic and gamma [9]. Nevertheless, the Weibull distribution is the most widely used for protective devices. Some examples can be found in [10] and [11]. Although normal distribution is widely used in many fields, in maintenance it is not very common since it might take negative numbers, and obviously, this is not realistic to model the lifetimes. Left-truncated normal distribution (truncating normal distribution at zero) is one way to avoid the negative numbers in the normal distribution [12].

If protective devices are poorly maintained, their failure risks increase. Consequences of failure in protective devices may be catastrophic. For instance, a clear evidence of the importance of protective devices’ maintenance is the blackout of August 14th, 2003, which affected Northeast United States and Canada, inconveniencing around 50 million people, 11 people died and $6 billion damage cost [13]. One of the main causes of this blackout was due to a hidden failure of a relay protection [14].

The author’s interest in writing this thesis stems from the fact that in recent decades, maintenance studies are gaining importance. At universities, maintenance is becoming one of the most essential courses for every engineering faculty, while all companies that aim to be competitive have their own maintenance department or programs. Nevertheless, protective devices’ maintenance has been considerably neglected [5]. Due to the potential consequences that they can trigger, it is worth to pay more attention to them. Moubray [3] stated that protective device’s maintenance would become a bigger maintenance strategy issue in the following decade than predicative maintenance.

The lifetime distribution of protective devices may follow different distributions, for example, the Weibull distribution and exponential distribution. This study will
investigate the relationships between lifetime distribution models, machine availability and inspection interval. It is necessary to have a clear understanding of effect of distribution models for the selection of a protective device. In addition, left-truncated normal distribution has not been previously considered for calculating the availability in a protective device or its inspection interval. We will investigate the possibility of using it as an alternative to the normal distribution to address the issue of negative values of the normal distribution.

1.2 Literature review

Protective devices are used in a wide variety of fields, for example, protection relays in the electrical distribution field and fire suppression systems in the vehicle field [2]. These devices protect people, environment and materials from hazardous situations. One main challenge in maintenance is that protective devices lie dormant for most of their life and consequently they may hide failures until they are called into use. In this section, literature related to protective devices is reviewed, paying particular attention to availability, availability models and statistical distributions used to model lifetime of protective devices.

1.2.1 Availability

Availability is an important parameter to measure the performance of devices. It can represent reliability\(^1\) as well as maintainability\(^2\)[7].

---

\(^1\) The probability that a unit will perform a required function under stated conditions for a stated period of time [15].

\(^2\) The ability of an item, under stated conditions of use, to be retained in, or restored to, a state in which it can perform its required functions, when maintenance is performed under stated conditions and using prescribed procedures and resources [1].
Sandler [16] gave three definitions of availability depending on the time interval considered.

1. **Instantaneous availability, \( A(t) \):** defined as the probability that the system is operational at any random time \( t \). Several researchers used instantaneous availability in their models since it can tell the reliability at any given time [11][16].

2. **Average uptime availability, \( A(T) \):** is the proportion of time in a specified interval \((O, T)\) that the system is available for use. It is expressed:
   \[
   A(T) = \frac{1}{T} \int_{0}^{T} A(t) \, dt \tag{1.1}
   \]
   Based on the literature review carried out for this thesis, no reference of the use of average uptime availability in protective devices could be found.

3. **Steady-state availability, \( A(\infty) \):** when the time interval considered is very large and is given by:
   \[
   A(\infty) = \lim_{T \to \infty} A(T) \tag{1.2}
   \]
   Steady-state availability is more widely used to determine an optimal interval according to Martínez et al. [17], Pascual et al. [18] and Pak et al. [19]. Tang [5] carried out a comparison between instantaneous and steady-state availabilities.

### 1.2.2 Availability models

Several models have been proposed to maximize the availability of protective devices. This section will review the most recent and relevant literature.
Sarkar and Sarkar [20] developed an availability model, which could calculate both instantaneous availability and steady-state availability, considering periodic inspections and a perfect repair policy with constant repair time. In their model two assumptions are considered. In the first one, a working component is treated as good as new upon a perfect repair. If a component is found failed at an inspection, a perfect repair, which takes a constant amount of time, will be carried out immediately. In the second one, a component found working well at an inspection is not intervened. If an inspection reveals a component failure, a perfect repair is carried out instantaneously and the time cost used for the repair is considered to negligible.

Later, Klutke and Yang [21] considered failures caused by random shocks which occurred following a Poisson process, and graceful degradation at a constant rate. Inspections were scheduled periodically. Steady-state availability was studied.

The effect of imperfect repairs was studied by Biswas, Sarkar and Sarkar [22]. They consider that systems are maintained through a fixed number of imperfect repairs before being replaced or perfectly repaired, and a repaired system is restored to operation at the next scheduled inspection time. The lifetime distribution of the system in its new and imperfectly repaired states is arbitrary. The times required for imperfect repairs and for perfect repair are randomly distributed.

Jardine and Tsang [2] considered times to make a repair or replacement and to carry out an inspection. All downtimes considered in their model are constant. They assumed that if a system is found working well after an inspection, it is considered as good as new condition. After each repair or replacement, the system is also considered as good as new. Average availability is studied in their model.
In contrast to previous models, Pak et al. [19] did not consider periodic intervals. They proposed a model where at any decision point a dynamic program calculated the optimal time to the next inspection, and the type of action to be undertaken, depending on the observed state of a device. When an inspection is carried out, if the system is found failed, a repair or replacement is performed and optimal time to the next inspection is determined. If it is found in operating conditions, optimal time to the next action is calculated too. As in Biswas et al. [22], repairs are assumed to be imperfect and they change device’s failure distribution. They consider inspection time and repair or replacements times to be constant.

Tang [5] proposed three availability models. To formulate more realistic and more generalizable models, he claimed that inspection and repair (or replacement) times could not be neglected, and both could be either constant or random. Two assumptions are made in their model, like in Sarkar and Sarkar [20]. In the first, after an inspection, the system is always restored to as good as new condition. In the second, at an inspection, the system found working is returned to operation without intervention. In addition, two types of inspection policies are studied. The first type is called calendar-based, and inspections are scheduled at a fixed calendar interval, for instance once a week. Downtime due to inspection and repair is included in the interval between inspections. The second type, which is called age-based, schedules inspections at a fixed age interval. Downtime is not included in the inspection interval. Fig.1 shows the differences between both policies. Instantaneous and steady-state availabilities are studied. More details of this model will be given in Chapter 3.

Among almost all researchers, periodic inspections are preferred rather than sequential as illustrated in the previous models. For this thesis, an availability model is needed. As it has been explained, there is a wide range of considerations that can be assumed when
the optimal inspection interval or the availability is calculated. Some authors do not consider the downtime due to repair or inspection; however, the model to be used in this considers both of them. In some situations, a system or plant shutdown is needed, and this time cannot be neglected. Moreover, in this thesis, the device is only considered as good as new after a repair or a replacement. Finally, the interval between inspections is considered fixed, this is due to the easiness of scheduling periodic inspections in factories. All these considerations lead to the use of one model developed by Tang et al. [6].

1.2.3 Statistical distributions used in detective maintenance

A protective device accumulates damage due to random processes and natural degradation until it fails. Consequently, the time to failure, also called lifetime, of a protective device depends on these random processes. Due to its random nature, probabilistic distributions are used to model lifetime, which is an essential parameter for calculating availability.

Many distributions have been used in literature to model lifetime. A deep review of all distributions used in maintained systems was carried out by Lie, Hwang and Tillman [7]. This section reviews distributions used to model lifetime of protective devices.

Tippachon et al. [10] carried out a failure analysis of protective devices in power distribution systems for reliability purposes. Such protective devices included breakers, reclosers, fuses and switches. Through a Kolmogorov-Smirnov Test on failure data, it was determined that these devices’ lifetime could be described by the Weibull distribution. Wang et al. [11] also carried out a failure analysis and proved that lifetime of protective capacitors also followed the Weibull distribution. As in Nakagawa and
Yasui [23], Jiang and Jardine [24] used a data set of which the lifetime was described by the Weibull distribution to compare the accuracy and robustness of two optimization models. In another comparison carried out by Kaio and Osaki [25], not only did they studied the Weibull distribution, but also the gamma distribution. Tang [5] proposed two cases of studies in order to put in practice his availability models. In the first, Tang analysed ten years of data of safety valves installed in a thermal power plant, and found that the failure times followed the Weibull distribution. In the second, historical data of safety valves in a mining and refining company was studied, and it was also proved that the failure time of these safety valves followed the Weibull distribution.

The normal distribution has also been studied in literature. Hauge [26] worked with the failure history data of a gaseous nitrogen dome regulator. The data fitted the normal distribution well. Jiang and Jardine [24], inspired by Barlow et al. [8], used another data set in their comparison, where the time to failure was normally distributed. However, normal distribution can take negative values and this can lead to errors.

An approach to solve the problems caused by the normal distribution can be the use of the truncated normal distribution. Although this distribution has not been used in detective maintenance, it has been widely used in other fields. For instance, Johnson and Thomopoulos [27] used the truncated normal distribution to model demand to determine safety stocks. As a result, they improved the accuracy of the demand. Liu and Ding [28] used the truncated normal distribution to describe the annual dry-bulb temperature. Song et al. [29] studied the static reliability of a pantograph, and they found the same problem that have been mentioned previously. For the stress and strength problems, the distribution mostly used is the normal distribution. However, there is a conflict between the values of stress or strength and the values of the normal distribution. The stress or strength is only defined for positive values, as lifetime does,
and normal distribution can take negative values. In order to avoid this problem, they
[29] proposed an analysis based on the truncated normal distribution. More applications
of the truncated normal distribution can be found in [30]–[33].

Some authors, such as Sim [34], considered that the lifetime of a protective device
followed the exponential distribution. Badía, Berrade and Campos [35] studied the
behaviour of an inspection policy using the exponential and the Pareto lifetime
models when the time to failure followed the exponential distribution.

1.3 Research objective
As it has been mentioned before, inspections play a decisive role to detect and fix
failures in detective devices. For this reason, this thesis aims to calculate the optimal
inspection interval for maximizing the availability of a protective device when its
lifetime follows a specific probabilistic distribution. To meet this objective, an existing
model reported in [6] is used. This model will be described in Chapter 3.

The literature review has exposed that the lifetime of the vast majority of protective
devices is represented by the following distributions: the normal distribution, the
Weibull distribution, the exponential distribution. Therefore, it is necessary to
investigate the relationships between distribution models, inspection interval and
maximum availability. Furthermore, it will be investigated how the parameters of each
distribution affect the optimal inspection interval and the maximum availability.

This thesis will also investigate the problem that arises for the normal distribution
taking negative values. The truncated normal distribution will be studied as an approach
to solve this problem. This study will let us know whether it is admissible to use the normal distribution when it may take a large number of negative values.

1.4 Organization of thesis

The thesis is structured as follows. In Chapter 1, existing literature related to availability, availability models and lifetime distributions used in detective maintenance is reviewed. Chapter 2 describes all the distributions this thesis will use: the Weibull, exponential, normal and truncated normal distributions. In Chapter 3, an availability model to be used in this thesis is presented. In Chapter 4, the Weibull, exponential and normal distributions are analysed respectively in making maintenance decisions. Later, the truncated normal distribution is presented as a solution to the negative value problem that arises when the normal distribution is used. A comparison between the normal and truncated normal distributions is carried out too. Finally, future work is presented in Chapter 5.
Chapter 2

Statistical Distributions Used to Model Lifetime of Protective Devices

Due to the importance of statistical distributions for calculating the optimal inspection interval and the availability of protective device, Chapter 2 describes briefly the most relevant distributions. These statistical distributions are used to model the lifetime of protective devices, and based on the literature review the most used are: the Weibull, exponential and normal. Although the truncated normal distribution has not been widely used so far, it is possible a solution to the issue of negative values of the normal distribution.

Generally, the lifetime of a device is not predetermined or fixed; thus, we can claim that it is a random variable. For this reason, the approach used in this study is in the sense of probability theory. The following functions are presented for the Weibull, exponential, normal and truncated normal distributions, respectively: the failure density, the failure distribution, the hazard rate and the reliability functions. The effect of their parameters is discussed. These functions will be used in Chapter 4 to adapt the availability model that will be presented in Chapter 3 to calculate the maximum availability and optimal inspection interval of a protective device.
2.1 The Weibull distribution

Regarding the literature review, the Weibull distribution is one of the most widely used distributions in detective maintenance. Together with the normal and exponential is the most popular distribution in modern statistics. It is one of the most interesting distributions since it is able to fit data from various fields, ranging from life data to weather data or economics [36].

Although originally the Weibull distribution depends on three parameters: \( \beta \) (the shape parameter), \( \gamma \) (the location parameter) and \( \eta \) (the scale parameter) [2], this thesis only considers the two parameters version (\( \beta - \eta \)), which is the most used one in maintenance [36].

In Table 2.1, the most relevant functions of the Weibull distribution for this research can be found. It is important to note that the time \( t \) is only defined for positive values (\( t \geq 0 \)).

<table>
<thead>
<tr>
<th>Table 2.1: Functions of the Weibull distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Probability density</strong></td>
</tr>
<tr>
<td><strong>Cumulative density</strong></td>
</tr>
<tr>
<td><strong>Hazard rate</strong></td>
</tr>
<tr>
<td><strong>Reliability</strong></td>
</tr>
</tbody>
</table>
The mean is expressed as:

$$\bar{T} = \eta \Gamma (1/\beta + 1),$$

(2.1)

where $\Gamma$ is the gamma function.

**Weibull distribution parameters**

The $\beta$ parameter determines the shape of the distribution. Different values of $\beta$ may have an important effect on the behavior of this distribution:

- When $0 < \beta < 1$, it indicates that the failure rate decreases over time. In this situation, it has a hyperbolic shape with $f(0) = \infty$.
- When $\beta = 1$, the failure rate is constant over time. It becomes an exponential function.
- When $\beta > 1$ indicates that the failure rate increases with the time. It is a unimodal function in which skewness changes from left to right as the value of $\beta$ increases [2].

Fig. 2.1 shows the behaviour of the probability density function for the Weibull distribution when the shape parameter takes different values and the scale parameter is fixed at 2 years. The values selected for the shape parameter are: 0.5, 1, 2.5 and 5 (years). As we can see, depending on the $\beta$ value the shape is completely different.
The scale parameter, $\eta$, determines the spread of the distribution. If $\eta$ is increased, while $\beta$ is fixed, the distribution’s height decreases and it spread increases.

Fig. 2.2 shows the effects of the scale parameter on the shape of the Weibull distribution. In this figure, the probability density function for scale parameters of 2, 4, 6 and 8 years are plot. The shape parameter is fixed at 2.5 years. Here we can see the phenomenon described previously.
2.2 The exponential distribution

The exponential distribution is used when the probability of failure occurring in the next small time interval does not vary through time [37]. It is also used when an equipment is subjected to random failures [2].

Besides maintenance applications, the exponential distribution has many other applications. For instance: the time to decay of a radioactive atom, the time taken for an ambulance to arrive at an accident or the time to answer a telephone inquiry [37].

The parameter that defines this distribution is the rate parameter, $\lambda$, which represents the arrival rate of failures.

Table 2.2 presents the most relevant exponential expressions for its analysis.

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability density</td>
<td>$f(t) = \lambda \exp[-\lambda t]$</td>
</tr>
<tr>
<td>Cumulative density</td>
<td>$F(t) = 1 - \exp[-\lambda t]$</td>
</tr>
<tr>
<td>Hazard rate</td>
<td>$h(t) = \lambda$</td>
</tr>
<tr>
<td>Reliability</td>
<td>$R(t) = \exp[-\lambda t]$</td>
</tr>
</tbody>
</table>

Fig. 2.3 shows the effects of the rate parameter on the probability density function for values of 0.5, 1 and 1.5 failures/year. When $\lambda$ increases, the shape of the probability density becomes wider.
2.3 The normal distribution

The normal distribution, also called the Gaussian distribution, is one of the most widely used in statistics due to its intuitive obviousness. It is applicable to a broad range of phenomena [37]. This distribution is described by two parameters. The first one is the mean, $\mu$, which describes the location. The second is the standard deviation, $\sigma$, which describes the spread of the distribution. The effects of these parameters are shown in Fig. 2.4 and Fig. 2.5.
In Fig. 2.4, we can find the probability density function of four different normal distributions. All of them share the same standard deviation, $\sigma = 1.5$ years, but the mean is different for each of them: 0, 3, 6 and 9 years. From this picture, we can see that if the mean increases, the probability density function displaces to the right. If the mean decreases, it displaces to the left.

Fig. 2.4: Effect of $\mu$ on the failure density with $\sigma$ fixed at 1.5 years

Fig. 2.5 shows four different probability density functions of normal distributions. The standard deviations of the distributions are: 1.5, 3, 4.5 and 6 years. The mean is fixed at 6 years for all the distributions. As we can see, the higher the standard deviation is, the wider the distribution is.
Table 2.3 shows the most important functions for this study:

Table 2.3: Functions of the normal distribution

<table>
<thead>
<tr>
<th><strong>Probability density</strong></th>
<th>( f(t) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{t - \mu}{\sigma} \right)^2 \right] )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cumulative density</strong></td>
<td>( F(t) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{t} \exp \left[ -\frac{(t - \mu)^2}{2\sigma^2} \right] dt )</td>
</tr>
<tr>
<td><strong>Hazard rate</strong></td>
<td>( h(t) = \frac{\exp[(t - \mu)^2/2\sigma^2]}{\int_{t}^{\infty} \exp[(t - \mu)^2/2\sigma^2] dt} )</td>
</tr>
<tr>
<td><strong>Reliability</strong></td>
<td>( R(t) = \frac{1}{\sigma \sqrt{2\pi}} \int_{t}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{t - \mu}{\sigma} \right)^2 \right] dt )</td>
</tr>
</tbody>
</table>

Fig. 2.5: Effect of \( \sigma \) on the failure density with \( \mu \) fixed at 6 years
It is really important to remark that $t$ ranges from $-\infty$ to $+\infty$. For this reason, it might take negative values. Although this distribution allows negative values with positive probability, it is sometimes used as a lifetime distribution [1].

This thesis considers that the distribution does not take negative values if the mean $\mu$ is greater than $3.5\sigma$, this consideration is based on the fact that 99.7% of the values are within $[\mu - 3\sigma, \mu + 3\sigma]$. This assumption leads to a 1 in 4000 change of the distribution giving negative failure time [2].

### 2.4 The truncated normal distribution

As it has been explained in Section 2.3, the normal distribution is defined from $-\infty$ to $+\infty$. In maintenance, when we work with the lifetime of a device, the lifetime has to be considered to be positive values. Thus, we need to consider the distribution that describe the lifetime over the domain 0 to $+\infty$. To guarantee the non-negative nature of the lifetime of a device, the left-truncated normal distribution is used.

When a normal distribution is singly truncated on the left or the right, the truncated normal distribution is expressed as [38]:

$$
 f_T(t) = \frac{f(t)}{\int_A f(t) \, dt} \tag{2.2}
$$

where $A$ represents the range of interest, and $f(t)$ is the probability density function of the function that is intended to be truncated.

Similar to the normal distribution, the truncated normal distribution is described by two parameters, the mean and the variance. These two parameters are different from the
mean and variance of the original normal distribution. They can be expressed as follows:

\[
\mu_T = \int_A tf_T(t) \, dt
\]  \hspace{1cm} (2.3)

\[
\sigma_T^2 = \int_A t^2 f_T(t) \, dx - \left( \int_A tf_T(t) \, dt \right)^2
\]  \hspace{1cm} (2.4)

The mean of a truncated distribution is larger than that of the original distribution. However, the variance is smaller [39].

Fig. 2.6 a) shows the probability density function of three normal distributions. The mean is fixed at 0.1, but the standard deviation is: 0.05, 0.1 and 0.2. Fig. 2.6 b) shows the truncated probability density functions (truncated at 0) of the three previous distributions. In this figure, when the standard deviation increases, the probability of taking negative number also does. We can see that the bigger the truncation area is, the wider the truncated distribution becomes.

Fig. 2.6: a) Normal distributions with \( \sigma = 0.05, 0.1 \) and 0.20, \( \mu \) fixed at 0.1. b) Truncated distributions from a)
Table 2.4 presents the expressions which are more important for the analysis of this distribution [1].

<table>
<thead>
<tr>
<th>Probability density</th>
<th>$f(t) = \frac{1}{\sigma} \phi \left( \frac{t - \mu}{\sigma} \right) \frac{1}{1 - \Phi(\alpha)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reliability</td>
<td>$R(t) = \frac{\Phi \left( \frac{\mu - t}{\sigma} \right)}{\Phi \left( \frac{\mu}{\sigma} \right)}$</td>
</tr>
<tr>
<td>Hazard rate</td>
<td>$h(t) = \frac{1}{\sigma} \frac{\Phi \left( \frac{t - \mu}{\sigma} \right)}{1 - \Phi \left( \frac{t - \mu}{\sigma} \right)}$</td>
</tr>
</tbody>
</table>

where $\phi(\cdot)$ is the probability density of the standard normal distribution and $\Phi(\cdot)$ is the standard normal cumulative function. The expression that gives $\alpha$ is:

$$\alpha = \frac{a - \mu}{\sigma} \quad (2.5)$$

where $a$ is the truncation point.

It is important to note that all the expressions from table 2.3 are only defined when $t \geq 0$.

2.5 Conclusions

In this chapter, four statistical distributions: the Weibull, exponential, normal and truncated normal, are described. The Weibull distribution is described by two parameters: the shape and scale parameter. The influence of the shape parameter is significant since depending on this parameter the performance of the Weibull
distribution is quite different. The exponential distribution is just described by one parameter, the rate parameter. The normal distribution is described by two parameters, the mean and the standard deviation. The main problem of this distribution is that if no considerations are taken into account, it might take negative values. Finally, the truncated normal distribution is presented which may be able to solve the negative value problem of the normal distribution.
Chapter 3

An Availability Model for Protective Devices

3.1 Availability model description

As it was briefly mentioned in Section 1.2.2, it is important to guarantee the availability of a protective device and a mathematical model is needed to maximize its availability. This study focuses on maximizing a device’s availability by optimizing the inspection interval. Many availability models are available according to the literature review in Section 1.2.2. The model proposed by Tang [6] is used directly in this study. In the following, a detailed description of this model is given.

The aforementioned model makes the following assumptions:

(1) An inspection is carried out at a fixed time interval. In other words, the time between inspections does not vary. Periodic inspections are easier to schedule than sequential inspections. Furthermore, to calculate the interval in sequential inspections a dynamic program is needed [19] and it increases the difficulty to solve the problem.

(2) If a device is found in a failed state at an inspection, it is repaired or replaced immediately. Both inspection and repair times are considered non-negligible and constant.

(3) The performance of inspections and repairs or replacements is considered perfect. It means that inspections always detect a failure device and do not
degrade the device, and when a repair or replacements is carried out, it always restores the device to as good as new condition.

(4) Inspection and repair or replacement downtimes are included in the inspection interval. See Fig. 3.1.

(5) When a system is found failed, the total downtime, due to inspection and repair or replacement, is smaller than the inspection time interval length.

Under all the above assumptions, the steady-state availability can be calculated as [6]:

$$\bar{A} = \int_{0}^{\infty} R(t)dt = \frac{\mu}{I \sum_{i=1}^{\infty} iq_i} = \frac{\mu}{I \sum_{i=0}^{\infty} R(v_i)}$$

(3.1)

where

$$q_i = R(v_{i-1}) - R(v_i)$$

$$v_i = \begin{cases} 0, & i = 0 \\ i(I - T_w) + T_w - T_F, & i = 1, 2, ... \end{cases}$$

(3.2)

where, $i$ is the number of inspections carried out in one cycle, $I$ is the length of the inspection interval, $T_w$ and $T_F$ are respectively the downtime due to inspection, and total downtime when a failure is found, $T_F$ includes the time for inspection and the time for repair or replacement, $v_i$ represents the time that a device has been working since the last repair or replacement and $\mu$ is the mean lifetime of a system.

To narrow down the scope of the thesis along all the thesis $T_w$ and $T_F$ are respectively considered to be 8h and 16h.

In this model, a cycle is defined as the time interval between two inspections in which failures have been detected, see Fig. 3.1. The expected time in one cycle is calculated as the length of the inspection interval, $I$, multiplied by the integration of the reliability
function of the device, \( R(v_i) \) with \( i \) going from zero to infinite. The reliability function depends on the parameter \( v_i \). Basically, the availability is calculated as a device’s uptime in one cycle divided by the total time used in one cycle.

Fig. 3.1: Cycle length of the availability model selected

### 3.2 Numerical example

To demonstrate how this availability model works and what considerations have to be taken into account, a numerical example is given.

The maximum availability and the optimal inspection interval are calculated for a pressure safety valve in an oil and gas field [2]. It is assumed that it fails following an exponential distribution, and its mean time to failure is 10 years. For meeting the requirements of the model studied, the downtime due to inspection is assumed to be 8 hours and the total downtime when a fail is found is fixed at 16 hours.

The exponential reliability function for this device is given by:

\[
R(t) = e^{-\lambda t}
\]  
(3.3)

where \( \lambda \) is the number of failures per unit of measurement; in this example, \( \lambda = 1/10 \) (failures/year); \( t \) is the point when the reliability want to be known; \( t \) units are years too.
From Eq. 3.2 and Eq. 3.3, the reliability function expressed as a function of the parameter \( v_i \) and substituting the values from the numerical example is given by:

\[
R(v_i) = \begin{cases} 
0, & i = 0 \\
\exp\left(-\frac{1}{10 \cdot 365}\right)\left(i \left(I - \frac{8}{24}\right) + \frac{8}{24} - \frac{16}{24}\right), & i = 1, 2, \ldots
\end{cases}
\]

The different parameters are expressed in days. \( \lambda \) was originally in years while \( T_w \) and \( T_F \) were in hours.

The optimal inspection interval, \( I \), is calculated through an optimization process with Matlab. \( I \) units are days. The number of inspections, \( i \), has to be limited in order to carry out this optimization. Otherwise, this process would never end since there is a summation from zero to infinite that depends on \( i \) in the availability model. To approximate this values it is considered that the Eq. 3.3 reaches a value near to zero, \( 10^{-8} \). From this equation, \( t \) is considered to be the limit of \( i \). For this example, the limit is 67.235.

The mean lifetime of the system, \( \mu \), is given directly by \( 1/\lambda \). Expressed in days too.

From Eq. 3.1, Eq. 3.2 and Eq. 3.3, the steady-state average availability general expression for this numerical example is:

\[
\bar{A} = \frac{\mu}{I \sum_{i=0}^{\infty} R(v_i)} = \frac{1/\lambda}{I + I \sum_{i=1}^{\infty} \exp(-\lambda(i(I - T_w) + T_w - T_F))}
\]  (3.4)

The results are given in Fig. 3.2, which shows the steady-state availability as a function of the inspection interval. This model gives a maximum availability of 0.9865 when the inspection interval is 49.5 days. This figure proves that if the inspection interval is increased over or decreased below the optimal interval, the availability decreases.
Chapter 3 introduces an availability model that is used along this thesis to meet the objectives. In Section 3.1, the considerations that this model takes into account are described and the model expression is given. Three of the most relevant assumptions that this model makes are the following: the time between inspections is constant, the inspection and repair time are constant and non-negligible and the performance of inspections and repairs are perfect. From the expression that describes this model, we see that it can be adapted quite easily to different statistical distributions. When using different distributions, only the mean, $\mu$, and the reliability function of the working time since the last repair, $R(u_i)$, need to be adapted. Section 3.2 presents a brief example where the optimal inspection interval and the maximum availability are calculated for a
pressure safety valve in an oil and gas field whose lifetime is described by the exponential distribution.
Chapter 4

Analysis of the Statistical Distributions

This chapter is devoted to acquiring a thorough understanding of how distribution models may influence on the availability and the inspection interval of a protective device. Four distributions are investigated: the Weibull, exponential, normal and truncated normal. For each distribution, their respective parameters are studied. To carry out this analysis, the availability model presented in Chapter 3 is directly used.

Chapter 4 is organized as follows. From Sections 4.1 to 4.4, the Weibull, exponential, normal and truncated normal distributions are analysed, respectively. Later, in Section 4.5 a comparison between the normal and truncated normal distributions, when the first one might take negative values, is carried out. Finally, conclusions are drawn in Section 4.6.

4.1 Weibull distribution analysis

This section analyzes how the scale and shape parameters of the Weibull distribution may affect the maximum availability and optimum inspection interval of a protective device. First, the scale parameter is fixed at a constant value, and the optimal interval and maximum availability are determined for a range of different shape parameters. Then, the shape parameter is fixed and the maximum availability and optimal inspection interval are calculated for a range of different scale parameters.
To carry out this study, the availability model presented in Chapter 3 is adapted, which is given in equation (4.1).

\[
A = \frac{\int_0^\infty R(t)dt}{I \sum_{i=1}^\infty iq_i} = \frac{\mu}{I \sum_{i=0}^\infty R(v_i)}
\]  
(4.1)

According to this availability expression, \(\mu\) and \(R(v_i)\) need to be determined. From Chapter 2 (see Table 2.1), we know the reliability function of the Weibull distribution is given as follows:

\[
R(t) = \exp \left[ -\left( \frac{t}{\eta} \right)^\beta \right]
\]  
(4.2)

From Chapter 3 (see equation (3.2)), the expression of \(v_i\) is written as follows:

\[
v_i = \begin{cases} 
0, & i = 0 \\
(i(I - T_W) + T_w - T_F), & i = 1, 2, \ldots
\end{cases}
\]  
(4.3)

Substituting equation (4.3) into (4.2), the expression for \(R(v_i)\) can be obtained:

\[
R(v_i) = \begin{cases} 
1, & i = 0 \\
\exp -((i(I - T_W) + T_w - T_F)/\eta)^\beta, & i = 1, 2, \ldots
\end{cases}
\]  
(4.4)

Also, we know from Chapter 2 (see equation (2.1)), \(\mu\) can be expressed by:

\[
\mu = \eta \Gamma (1/\beta + 1)
\]  
(4.5)

Consequently, the availability expression for the Weibull distribution is given by:

\[
\bar{A} = \frac{\eta \Gamma (1/\beta + 1)}{I + I \sum_{i=1}^\infty \exp -((i(I - T_W) + T_w - T_F)/\eta)^\beta}
\]  
(4.6)
In the numerical calculations, the integration upper bound in the denominator is calculated using equation (4.7), as did in Chapter 3. It corresponds to $10^{-8}$ of the Weibull reliability function (see equation (4.2)).

$$n = -(\ln 10^{-8})^{-\beta} \eta \quad (4.7)$$

### 4.1.1 Analysis of the shape parameter

As explained in Section 2.1, the Weibull distribution is described by two parameters, which are the scale and the shape. This section is intended to analyze the effect of the shape parameter on the maximum availability and the optimal inspection interval of a protective device.

To study the effect of the shape parameter, $\beta$, different values have been selected. According to the explanation of the distribution parameters in Section 2.1, values from 0.5 to 10 have been chosen. This selection is intended to study the effect of $\beta$ when: $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. The interval of the shape parameter is every 0.1. In this analysis, the scale parameter is fixed at 1 year.

First, for a shape of 0.5 and a scale of 1 year, its optimal inspection interval and maximum availability have been calculated through the availability model presented in equation (4.6). Then, with an analogous procedure, the same results are calculated for shapes parameters from 0.6 to 10, the scale is always fixed at 1 year.

Fig 4.1 shows the relationship between the optimal inspection interval and the shape parameter. This figure proves that for shape parameters between 0.5 and 2.2, the optimal interval falls significantly, from 21 days to 15 days. When the shape is 2.2, a minimum point is reached. If the shape parameter is bigger than 2.2, the optimal interval increases slowly.
The relationship between maximum availability and the shape parameter is shown in Fig. 4.2. Its trend is fairly similar to the relationship between the optimal inspection interval and the shape parameter. For shape parameters from 0.5 to 2.2, the optimal interval falls sharply from 0.9687 to 0.9551. Then, it increases slowly until a value of 0.9567 is reached. Also, the minimum point is found when the shape parameter is 2.2.
After analyzing the optimal inspection interval and the maximum availability for shapes parameters from 0.5 to 10 years and a scale parameter fixed at 1 year, for the same shape range (0.5 to 10 years) the following scales are tested: 2, 4, 6, 8 and 10 years.

The relationship between the shape and the optimal inspection interval, and the shape and the maximum availability is fairly similar to all the scales tested. For this reason, results are not presented. For both the optimal inspection interval and the maximum availability, from a shape parameter 0.5 there is a sharply fall until 2.2. Then, there is a gradually growth. All the scales tested present a minimum point when the shape parameter is 2.2

**4.1.2 Analysis of the scale parameter**

This section is devoted to analyzing the effect of the scale parameter on the maximum availability and the optimal inspection interval of a protective device.

To study the effect of this parameter, the shape parameter is fixed at different constant values and the optimal inspection interval and maximum availability are calculated for a range of scale parameters.

For the shape parameter, different values have been fixed to study the effect of $\beta$ when: $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$. The values selected for the shape parameter are: 0.5, 0.8, 1, 2.2, 5 and 10. The scale parameter range is selected to be from 1.5 to 10 years, the interval between values is every 0.1 years.

First, the maximum availability and optimal inspection interval are calculated for a shape parameter of 0.5 and a scale parameter that ranges from 1.5 to 10 years. The
model presented in equation (4.6) is used. Then, the same is done for the remaining shapes parameters: 0.8, 1, 2.2, 5 and 10.

Fig. 4.3 presents the relationship between the optimal inspection interval and the scale parameter. Each curve corresponds to a fixed shape parameters at 0.5, 0.8, 1, 2.2, 5 or 10. When the scale parameter increases, the optimal inspection interval increases too. The higher optimal inspection interval for a scale is obtained when the shape parameter is 0.5. Whereas the lowest intervals are obtained when the shape is 2.2. For all the shape parameters studied, the relationship between the optimal inspection interval and the scale follows an upward trend.

In this analysis, the maximum availability also keeps a relationship with the optimal inspection interval. Fig 4.4 shows the relationship between the maximum availability and the shape parameter. As we can see the trend is fairly similar to Fig. 4.3. Each curve from Fig. 4.4 represent the different shapes parameters selected. When the scale
parameter increases, the maximum availability also does. For all the shape parameters studied, the results obtained follows an upward trend. Like in the previous analysis, the maximum availability is reached for a shape parameter of 0.5. While the lowest maximum availability is reached when the shape is 2.2. The maximum availability obtained is 0.9906 when the shape parameter is 0.5 and the scale parameter 3650.

Fig. 4.4: Relationship between the maximum availability and the scale parameter. (Scale ranges from 1.5 to 10 years. Shapes fixed at: 0.5, 0.8, 1, 2.2, 5 and 10)

### 4.2 Exponential distribution analysis

Section 4.2 studies the relationship between the exponential rate parameter and the optimal inspection interval and the maximum availability of a protective device. The exponential distribution is only described by one parameter, the rate parameter, $\lambda$. Therefore, to carry out this analysis different values of this parameter are studied.

It is important to note that the exponential distribution is a particular case of the Weibull distribution when the shape parameter is 1. The relationship between both distributions is given in equation (4.8).
\[ \lambda = 1/\eta \]  

(4.8)

where \(\eta\) is the scale parameter of the Weibull distribution. When the shape parameter \(\beta\) of the Weibull distribution is 1, the results from the Weibull distribution and the exponential distribution should be the same. This section is intended to study the exponential distribution. Moreover, it will be useful to validate the results of the Weibull distribution when \(\beta = 1\) obtained in Section 4.1.

To carry out this analysis, the availability model, which is given in equation (4.1), needs to be adapted to the exponential distribution. According to equation (4.1), \(R(v_i)\) needs to be determined.

From Chapter 2 (see table 2.2) the reliability function of the exponential distribution is given as follows:

\[ R(t) = \exp[-\lambda t] \]  

(4.9)

Substituting equation (4.3) into (4.9), \(R(v_i)\) is obtained.

\[ R(v_i) = \begin{cases} 1, & i = 0 \\ \exp[-\lambda(i(I - T_w) + T_w - T_f)], & i = 1, 2, ... \end{cases} \]  

(4.10)

Combining equation (4.1) and (4.10) the availability model for a device which lifetime

\[ \bar{A} = \frac{1/\lambda}{1 + \lambda \sum_{i=1}^{\infty} \exp(-\lambda(i(I - T_w) + T_w - T_f))} \]  

(4.11)

follows an exponential distribution is given as follows:

To study the performance of this distribution, values from 0.01 to 4 fails/year for the rate parameter are studied. The step between values is every 0.05. These values are selected based on reference [35].
Using the model presented in equation (4.11), the maximum availability and optimal inspection interval are calculated for \( \lambda \) ranging from 0.01 to 4 fails/year.

Fig. 4.5 shows the relationship between the optimal inspection interval and the rate parameter. As we can see, when the rate parameter increases, the optimal inspection interval falls. When the rate parameter goes from 0.01 to around 0.5, the optimal interval drops dramatically. However, it remains fairly constant for values from 0.5 to 4.

Fig. 4.5: Relationship between the optimal inspection interval and the rate parameter.
(Rate parameter ranges from 0.01 to 4 failures/years)

The relationship between the maximum availability and the rate parameter is presented in Fig. 4.6. The trend followed between these two parameters is slightly different to the trend followed between the optimal inspection interval and the rate parameter. Fig. 4.6 shows that when number of failures per year increases, the maximum availability drops for the whole range studied.
If we compare the optimal inspection interval and the maximum availability obtained for the rate parameter from 0.1 to 0.66 failures/year with their respective results (according to equation (4.8)) for the Weibull distribution when $\beta = 1$, we can see that the results are analogous.

### 4.3 Normal distribution analysis

As it has been explained in Chapter 2, the normal distribution is described by two parameters: the mean and the standard deviation. Section 4.3 studies how changing the standard deviation affect the maximum availability and optimum inspection interval of a protective device. Changing the standard deviation may lead to have a higher or lower probability of taking negative numbers. To achieve this objective, the mean is fixed at a value, and the effect of a range of standard deviations is studied.
To study the effect of the standard deviation, the availability function, which is given in equation (4.1), needs to be adapted to the normal distribution. According to the expression described by equation (4.1), $R(v_i)$ needs to be determined.

From Chapter 2 (see table 2.3) the reliability function of the normal distribution is given as follows:

$$R(t) = \frac{1}{\sigma \sqrt{2\pi}} \int_{t}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{t - \mu}{\sigma} \right)^2 \right] dt \quad (4.12)$$

Substituting equation (4.3) into (4.12), the expression for $R(v_i)$ can be obtained.

$$R(v_i) = \begin{cases} \frac{1}{\sigma \sqrt{2\pi}} \int_{i(T_w) + T_w - T_f}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{(i(T_w) + T_w - T_f - \mu)}{\sigma} \right)^2 \right], & i = 0 \\ \frac{1}{\sigma \sqrt{2\pi}} \int_{i(T_w) + T_w - T_f}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{(i(T_w) + T_w - T_f - \mu)}{\sigma} \right)^2 \right], & i = 1, 2 \ldots \quad (4.13) \end{cases}$$

Combining equation (4.1) and (4.13), the availability model for a device that follows a normal distribution is obtained:

$$\bar{A} = \frac{\mu}{I + I \sum_{i=1}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \int_{i(T_w) + T_w - T_f}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{(i(T_w) + T_w - T_f - \mu)}{\sigma} \right)^2 \right]}$$

$\quad (4.14)$

Hauge [26] used a normal distribution with mean $\mu = 5.56$ years (2029.4 days) and standard deviation $\sigma = 2.87$ years (1047.5 days) to model the time to failure of a Gaseous Nitrogen (GN$_2$) Dome Regulator. The probability of this distribution taking values under zero is 2.64%. In our study, we fix the mean at 2029.4 days, while the standard deviation ranges from 73 days to 1047.5 days to analyze the effect of standard deviation on the optimal inspection interval and the maximum availability.

Standard deviation is only defined positive, for this reason a minimum value of 73 is selected. The interval between the different standard deviation is every 36.5 days. For
instance, some of the standard deviations studied are: 73, 109.5, 146 days…This decision intends to have a significant number of results to analyze.

To carry out this analysis, first, the optimal inspection interval and the maximum availability are calculated for a standard deviation of 73 days and a mean of 2029.4 days. Then, the same is done for all the standard deviations proposed. The mean is always fixed at 2029.4 days.

Fig. 4.7 shows how the optimal inspection interval varies according to the standard deviation. When the standard deviation varies from 73 days to a value near to 473 days, the optimal interval is 37.1 days and it remains almost unchanged. However, when this parameter is above 473 days, the optimal interval decreases sharply until 36.7 days. The change of trend in the relationship between the optimal inspection interval and the standard deviation may be due to the number of negative values the normal distribution is taking when the standard deviation increases.

Fig. 4.7: Relationship between the optimal inspection interval and the mean. (Mean fixed at 2029.4 days Standard deviation ranges from 73 days to 1047.5 days)
When standard deviations that imply taking a significant amount of negative values (in the previous data set $\sigma \geq 0.23\mu$) are studied, a noticeable error is introduced to the availability model and it causes calculation errors. These errors lead to untruthful optimal inspection intervals. If $\sigma < 0.23\mu$, an error is also introduced but it is not significant. For the data set studied, when $\sigma \geq 0.23\mu$ the interval calculated is named the so-called “optimal inspection interval”.

To acquire a deeper theoretical understanding, the range of the standard deviation is extended. We intend to study if it follows the same downward trend. The studied range is extended until a value of $\sigma = \mu$ is reached. Therefore, standard deviations from 73 to 2029.4 days are analyzed. The probability of taking negative values when $\sigma = 2029.4$ and $\mu = 2029.4$ days is 15.86%.

Fig. 4.8 shows the relationship between the optimal inspection interval and the standard deviation when the latter ranges from 73 days to 2029.4 days. The so-called “optimal inspection interval”, in the extended studied interval, follows a downward trend as well.

![Fig. 4.8: Relationship between the optimal inspection interval and the standard deviation. (Mean fixed at 2029.4 days. Standard deviation ranges from 73 days to 2029.4 days)](image-url)
To study the relationship between the maximum availability and the standard deviation, the mean is fixed at 2029.4 days, and the range for the standard deviation selected is from 73 to 2029.4 years.

The relationship followed by the maximum availability and the standard deviation is fairly similar to the relationship followed by the optimal inspection interval and the standard deviation. Fig. 4.9 represents the relationship between the maximum availability and the standard deviation. It shows that when the number of negative values the distribution is not noticeable, when the standard deviation ranges from 73 to 473 days, the maximum availability remains almost constant at a value of 0.9826. But, when the percentage of negative values increases, standard deviation over 473 days, the maximum availability falls considerably until 0.9078.

![Fig. 4.9: Relationship between maximum availability and standard deviation. (Mean fixed at 2029.4 days. Standard deviation ranges from 73 days to 2029.4 days)](image)
From the previous analysis, we can see that when the availability model presented in equation (4.1) is used, and a normal distribution describes the lifetime of the protective device and the probability of taking negative values is significant, neither the maximum availability nor the optimal inspection interval are trustful. Therefore, in this situation is not recommended using this model. In the data set studied, the effect of the negative values is noticeable when $\sigma \leq 0.23\mu$. In other data sets this relationship may vary.

### 4.4 Truncated normal distribution analysis

As we have seen in Section 4.3, when the normal distribution is used to describe the lifetime of a protective device, it is possible that it takes negative values. Obviously, the lifetime of a device cannot be considered negative. One approach to solve this problem is using the truncated normal distribution instead of the normal distribution. By using this distribution truncated at zero, negative values are restricted.

Section 4.4 analyzes what are the effects of truncating a lifetime normal distribution on the maximum availability and inspection optimal interval using the model presented in equation (4.1).

To carry out this analysis, the first step is to calculate the truncated normal distribution from the original normal distribution. Fig. 4.10 shows the effects of truncating a normal distribution at zero on the probability density function. When the probability of taking negative values is low, for example around 5%, the truncated distribution looks quite similar to the original distribution, see Fig. 4.10 (a). However, when this probability increases, for instance when it is 50%, the shapes of both distributions are fairly different, see Fig. 4.10 (b). The probability of taking values around zero increases notably for the truncated distribution. Finally, when probability of taking negative
values is not noticeable, for example when it is 0.05%, as Fig. 4.10 (c) shows, the truncated distribution may be approximated to the original normal distribution.

![Graphs showing normal distribution with different truncation levels](image)

Fig. 4.10: Normal distribution with different levels of truncation: (a) 5%, (b) 50% (c) 0.05%

To study the effect of using a truncated normal distribution on the maximum availability and the optimal inspection interval, we study several scenarios where the truncated area from the original normal distribution is different. This truncation area is given by:

\[ p = \Phi(\alpha), \]  \hspace{1cm} (4.15)

where \( \Phi(\cdot) \) is the standard normal cumulative function. From Chapter 2 (see equation (2.5)) \( \alpha \) is given as follows:
\[ \alpha = \frac{a - \mu}{\sigma} \]  

(4.16)

where \( a \) is the truncation point, in this study it is 0. The other parameters, \( \mu \) and \( \sigma \) are the mean and the standard deviation of the normal distribution, respectively.

To illustrate the meaning of the parameter \( p \), Fig. 4.11 is used. This figure presents the probability density function of a normal distribution with \( \mu=400 \) days and \( \sigma=250 \) days. It also presents its truncated normal distribution at zero. To obtain the truncated normal distribution, the area below zero, which is \( p \), is eliminated. Then, to obey the property of equation (4.17) every point of the probability density function is divided by \( (1 - \Phi(\alpha)) \).

\[ \int_{-\infty}^{\infty} f(t) \, dt = 1 \]  

(4.17)

As for the previous distributions studied, to analyze the effect of the truncated normal distribution, the availability model given in equation (4.1) needs to be adapted. According to equation (4.1), \( R(v_i) \) and \( \mu \) need to be determined.
From Chapter 2 (see Table 2.3) the reliability function of the truncated normal distribution is given as follows:

\[ R(t) = \frac{\Phi\left(\frac{\mu - t}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)} \]  

(4.18)

Substituting equation (4.3) into (4.18), the expression for \( R(v_i) \) is:

\[ R(v_i) = \begin{cases} 1, & i = 0 \\ \frac{\Phi\left(\frac{\mu - (i(I - T_w) + T_w - T_f)}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)}, & i = 1, 2, \ldots \end{cases} \]  

(4.19)

Also, we know from Chapter 2 (see equation (2.3)) that \( \mu \) can be expressed by:

\[ \mu_T = \int_0^\infty t f_T(t) \, dt \]  

(4.20)

Consequently, combining equations (4.1), (4.19) and (4.20), the availability expression for a truncated normal distribution is given by:

\[ A = \frac{\int_0^\infty t f_T(t) \, dt}{I + I \sum_{i=1}^\infty \frac{\Phi\left(\frac{\mu - (i(I - T_w) + T_w - T_f)}{\sigma}\right)}{\Phi\left(\frac{\mu}{\sigma}\right)}} \]  

(4.21)

The same data sets from Section 4.4 are used in this study. They are based on a distribution used by Hauge [27]. He used a normal distribution to describe the time to failure of a Gaseous Nitrogen (GN\(_2\)) Dome Regulator, where the mean \( \mu = 2029.4 \) days and the standard deviation \( \sigma = 1047.5 \) days. These data sets are used because later in Section 4.4 a comparison between the normal and the truncated normal distribution is carried out.
To carry out this study, the mean of the original normal distribution is fixed at 2029.4 days, and its standard deviation ranges from 73 to 2029.4 days. Correspondingly, the parameter $p$ ranges from 0 to 15.86%.

After calculating the truncated normal distribution, the maximum availability and optimal inspection interval are calculated. It is important to note these two parameters should be fairly similar for the normal and truncated normal distributions when the probability of taking negative values is not significant.

Fig. 4.12 shows the relationship between the truncated area, $p$, and the optimal inspection interval obtained. The optimal interval increases rapidly when the truncated area increases. When the truncated area is about 0, the optimal interval is 37.1 days, but when the truncated area is 15.86% it is 42 days. The variation between the intervals obtained from 0% and 15.86% truncation area is around 13.2%. To understand why the optimal inspection interval increases when the truncated area increases, the probability density function of three different data sets are studied.

![Graph showing the relationship between optimal inspection interval and truncated area. Mean fixed at 2029.4 days. Truncated area ranges from 0% to 15.86%](image)

Fig. 4.12: Relationship between optimal inspection interval and truncated area. (Mean fixed at 2029.4 days. Truncated area ranges from 0% to 15.86%)
Fig 4.13 shows three probability density functions with $\mu = 2029.4$ days and truncations areas, $p$, of 5%, 10% and 15%, respectively. As we can see, the higher the truncated area is, the wider the spread of the distribution is. When the spread of the distributions is wider, it means that the probability of a device to fail later is higher. Consequently, this phenomenon leads to a bigger optimal inspection interval.

![Probability density function](image)

The results obtained for the maximum availability are fairly similar to the ones obtained for the optimal inspection interval. Fig. 4.14 shows the relationship between the maximum availability and the proportion of truncated area. As we can see, it follows an upward trend. When the truncated area is 0 %, the maximum availability is 0.9819. The maximum availability reaches 0.9840 when the truncated area is 15.86 %. The variation between the maximum and minimum values is about 0.21 %.
4.5 **Comparison between the normal and the truncated normal distributions**

In Section 4.3 we have seen that when the normal distribution might take negative values, the optimal inspection interval and the maximum availability calculated are not accurate. An approach to obtain trustful results may be using the truncated normal distribution, its analysis has been previously carried out in Section 4.4. This distribution restricts negative lifetimes.

This section intends to compare the normal and the truncated normal distributions. To compare both distributions, the results for the maximum availability and optimal inspection interval from Sections 4.3 and 4.4 are gathered in Fig. 4.15 and Fig. 4.16.

Fig. 4.15 shows the relationship between the optimal inspection interval and the truncated area for the normal and the truncated normal distributions. For the normal
distribution, the truncated area can be understood as the percentage of negative values that it might take. The mean of the distribution studied is 2029.4 days. When the truncated area is zero, the optimal inspection interval is exactly the same for both distributions. This is due to the fact that the normal and the truncated distribution are fairly similar when area to truncate is zero. If the truncated area increases, the optimal inspection interval for the truncated normal grows, while the so-called “optimal inspection interval” for the normal distributions decreases. Therefore, for the truncation area studied from 0% to 15.86%, the bigger the truncated area is, the bigger the difference between both values is.

The relationship that exists between both distributions for the maximum availability is similar to the relationship existing for the optimal inspection interval. Fig. 4.16 presents the relationship between the truncated area and the maximum availability. When the truncated area increases, the maximum availability growth of the truncated normal
distribution is almost not notable, while that of the normal distribution decreases very fast. As in Fig. 4.15, the higher the truncated area is, the bigger the difference between the two distributions is.

![Graph showing the relationship between maximum availability and truncated area for normal and truncated normal distributions.](image)

Fig. 4.16: Relationship between maximum availability and truncated area. Normal and truncated normal distributions. (Mean fixed at 2029.4 days. Truncated area ranges from 0 % to 15.86%)

From the previous analysis, we can see that if inspections are scheduled based on the results obtained using the normal distribution, in the long run, more inspections than needed are carried out.

### 4.6 Conclusions

In this chapter, the model presented in Chapter 3 is used to calculate the maximum availability and optimal inspection interval for protective devices. I studied the relationship between the optimal inspection interval and maximum availability calculated by the aforementioned model with the following statistical distributions: the Weibull, exponential, normal and truncated normal.
After analyzing the four distributions, the following conclusions are drawn:

- When the lifetime of a protective devices follows the Weibull distribution and the scale parameter is: 1, 2, 4, 6, 8 or 10 years, the shortest optimal inspection interval and lowest maximum availability are found when the shape parameter is 2.2.

- For exponential distribution, the lower the failure rate is, the higher the optimal inspection interval is. Consequently, the maximum availability is also higher.

- It is not recommended using the normal distribution to describe the lifetime of a protective device when the negatives values that it is taking is noticeable. If the probability of taking negative values is high it may lead to a wrong optimal inspection interval, and consequently to a wrong maximum availability.

- The truncated normal distribution is possible an approach to solve the negative value problem presented by the normal distribution. If the truncated area of the distribution increases, in a range from 0% to 15.86%, the optimal inspection interval and the maximum availability also do.
Chapter 5

Future Work

In this thesis, only four distributions are analyzed: the Weibull, exponential, normal and truncated normal distributions. And only one availability model has been used to carry out the analysis. Based on the scope of this thesis, the following three perspectives are suggested for future considerations.

- Other availability models should be used to analyze the statistical distributions. The literature review showed that there are several availability models, each of them take into account different considerations. This thesis has used a model in which the time between inspections is considered constant. It could be interesting to carry out the same study but using a model which considers that the time between inspections can vary.

- Other distributions, such as the gamma, the log-logistic or the log-normal distributions could be studied. According to the literature review, they are one of the most used in maintenance.

- This thesis has focused only on the truncated normal distribution to solve the problem of taking negative lifetimes presented by the normal distribution. Other distributions or methods could be found to solve this problem.
References


