On the Analysis of the Quasi-Static Regime [⋆]

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Abstract: Hysteresis is a nonlinear phenomenon exhibited by systems stemming from various science and engineering areas. To detect experimentally the presence of hysteresis in a system, the graph output versus input of the system is plotted for different frequencies of the input. For hysteresis systems, these graphs converge to a quasi-static limit set when the frequency goes to zero. Moreover, the quasi-static graph approaches asymptotically a periodic orbit. Thus, hysteresis is nonlinear phenomenon that can be detected only in the quasi-static regime, that is when the frequency content of the input goes to zero. The relevance of hysteresis in applications and the fact that it is essentially a quasi-static phenomenon makes it important to characterize mathematically the quasi-static regime, which is the purpose of this paper. Although this work is motivated by hysteresis systems, the tools that are presented are not limited to this class of systems. For this reason, the systems that we consider are seen as operators that map an input signal and initial condition to an output signal, all of them belonging to some specified sets. The main result of this paper is a new criterion for the existence, uniqueness and mathematical characterization of the quasi-static regime. The tools presented are illustrated using the semilinear hysteresis Duhem model.

Keywords: Input/output, operator, quasi-static regime.

1. INTRODUCTION

Hysteresis is a nonlinear behavior encountered in a wide variety of processes in which the input-output dynamic relations between variables involve memory effects Macki et al. (1993); Visintin (1994); Brokate and Sprekels (1996); Krasnosel'skii and Pokrovskii (1989); Ikhouane and Rodellar (2007); Mayergoyz (2003). The way hysteresis is detected experimentally in physical systems is by doing the following. Take as input of the hysteresis system the signal $u_\omega = U_0 \sin(\omega t)$ with amplitude U_0 and frequency ω, and let y_ω be the corresponding system output. When y_ω is plotted against u_ω, we get a curve G_ω parametrized with time. When this experiment is repeated with different frequencies, and when we take $\omega \to 0$, we observe that the curves G_ω converge to some curve $G^\star$. It is observed that the quasi-static curve $G^\star$ converges asymptotically to a periodic orbit $G^\diamond$ which is commonly called hysteresis loop. The curve G_ω is independent of ω for the so-called rate-independent hysteresis systems like the ones described in Macki et al. (1993), and G_ω depends on ω for the rate-dependent hysteresis systems like the ones described in Fuzi and Ivanyi (2001); Enachescu (2006); Dong et al. (2008). For linear systems $G^\diamond$ is a line, and for hysteresis systems $G^\diamond$ is a non-trivial curve Ghost (2007). Thus, hysteresis is a nonlinear phenomenon that can be detected only in the quasi-static regime, that is when the frequency content of the input goes to zero. The relevance of hysteresis in applications and the fact that it is essentially a quasi-static phenomenon makes it important to characterize mathematically the quasi-static regime.

The objective of this paper is to present a rigorous mathematical framework for the analysis of the quasi-static regime. Although this work is motivated by hysteresis systems, the tools that are presented are not limited to this class of systems. For this reason, the systems that we consider are seen as operators that map an input signal and initial condition to an output signal, all of them belonging to some specified sets. The inputs and output are taken to be finite dimensional; however, if the system (or operator) has a state description, the state may be finite or infinite dimensional. The system (or operator) may be continuous or discontinuous, and if it is a hysteresis it may be rate-dependent or rate-independent. One of the main results of this paper is that, when the input signal is such that its total variation is increasing, the existence, uniqueness and mathematical description of the quasi-static regime can be done in a very general framework in which the only assumption on the operator that describes the system is causality. When the input signal can be constant on some interval (or intervals) so that its total variation is only nondecreasing, an additional assumption has to made on the operator in order to characterize its quasi-static regime.

The focus, objective and results of this paper are new. The author couldn't find a precedent for this systematic study in the literature although some contributions have been done for some particular systems with a limited class of

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inputs Oh and Bernstein (2005). The quasi-static regime of dynamical systems is discussed in numerous papers with a different meaning for what a quasi-static regime is Mielke et al. (2008); F. Li (2008); Rimon et al. (2008). For example, in singular perturbations theory, the systems are considered as two coupled subsystems, one describing the slow motion and the other describing the fast motion. The last subsystem is parametrized with a real parameter ε, and the main issue of singular perturbations techniques is to describe the slow motion (and in some extent the fast one) of the system in the quasi-static regime that is when $\varepsilon \to 0$. The focus, objectives and assumptions that are usual in singular perturbation theory differ from ours as explained at the end of the paper. However, we have preferred to keep the words "quasi-static regime" because of their widespread use in experimental works, and since this concept is defined formally in this paper.

The paper is organized as follows. Section 2 presents some mathematical tools that are useful in the rest of the paper. Section 3 gives the class of operators under consideration. In Section 4 explores the quasi-static regime of operators that verify a constant input constant output assumption. Finally, Section 4.4 considers the case-study of the semi-linear Duhem model. The conclusion is given in Section 5. Due to space limitation, the proofs have been eliminated. They are given in Ikhouane (2009).

2. MATHEMATICAL PRELIMINARIES

In this section, we present some mathematical tools that will be useful in the next sections.

The Lebesgue measure on $\mathbb{R}$ is denoted μ. We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Consider a function $f : I \subset \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^m$; we say that f is measurable when f is (M, B)-measurable where B is the class of Borel sets of $\mathbb{R}^m$ and M is the class of measurable sets of $\mathbb{R}_+$. For a measurable function $f : I \subset \mathbb{R}_+ \to \mathbb{R}^m$, $\|f\|_{\infty,I}$ denotes the essential supremum of the function $|f|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. When $I = \mathbb{R}_+$, it will be denoted simply $\|f\|_\infty$.

We consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u : \mathbb{R}_+ \to \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative $\dot{u}$ is defined a.e. and is equal a.e. to the weak derivative of u. Moreover, we have $\|u\|_\infty < \infty$ and $\|\dot{u}\|_\infty < \infty$. Endowed with the norm $\|u\|_{1,\infty} = \max(\|u\|_\infty, \|\dot{u}\|_\infty)$, $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space Adams and Fournier (2003).

For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$.

$$\rho_u(t) = \int_0^t |\dot{u}(\tau)| \, \mathrm{d}\tau \in \mathbb{R}_+$$

The function $\rho_u(t)$ is well defined as $\dot{u} \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$.[1] It is nondecreasing and absolutely continuous. Denote $\rho_{u,\max} = \lim_{t\to\infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ is necessarily finite).
- $I_u = [0, \rho_{u,\max})$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ may be finite or infinite).

Now, given some value $\varrho \in I_u$, there exists at least some $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ due to the continuity of ρ_u. The value t_ϱ may not be unique as the function ρ_u is not necessarily increasing.

Lemma 1. $u(t_\varrho)$ is independent of the particular choice of t_ϱ. It depends solely on ϱ.

Lemma 1 shows that we can define the following function

$$\begin{aligned} \psi_u : I_u &\to \mathbb{R}^n \\ \varrho &\to u(t_\varrho) \end{aligned}$$

The function ψ_u depends only on the function u, and we have Dom $(\psi_u) = I_u$. Note that we have $\psi_u \circ \rho_u = u$.

Lemma 2. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant. Then, $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$, $\|\psi_u\|_{\infty,I_u} = \|u\|_\infty$ and $\|\dot{\psi}_u\|_{\infty,I_u} = 1$. Moreover, $\mu\left[\left\{\varrho \in I_u / \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right] = 0$.

We consider the linear time scale change $s_\gamma(t) = \frac{t}{\gamma}$ for any $\gamma > 0$.

Lemma 3. For any $\gamma > 0$, we have $I_{u\circ s_\gamma} = I_u$ and $\psi_{u\circ s_\gamma} = \psi_u$.

Definition 1. Suppose that the function $w : \mathbb{R}_+ \to \mathbb{R}$ is continuous and periodic with the period $T > 0$. Furthermore we assume that there exists a scalar $0 < T^+ < T$ such that the function w is increasing and C^1 on the interval $(0, T^+)$, and decreasing and C^1 on the interval (T^+, T). We denote $w_{\min} = w(0)$ and $w_{\max} = w(T^+) > w_{\min}$ the minimal and maximal values of the function w respectively. Due to the particular shape of w, we call it wave-periodic.

Lemma 4. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a non-constant T-periodic input. Then, $I_u = \mathbb{R}_+$ and ψ_u is periodic of period $\rho_u(T) > 0$. Moreover, if u is wave-periodic as in Definition 1, then ψ_u is also wave-periodic and we have $\dot{\psi}_u(\varrho) = 1$ for $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for $\varrho \in (\rho_u(T^+), \rho_u(T))$.

3. CLASS OF OPERATORS

Let Ξ be a set of initial conditions. Let $\mathcal{H}$ be an operator that maps the input function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ to an output in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ where m is a positive integer. So we have $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

In this paper we consider only causal operators. This means that, for any function $y = \mathcal{H}(u, \xi^0)$, the value $y(t)$ may depend on the input values $u(\tau)$ for $\tau \le t$, but cannot depend on any value $u(\tau)$ for $\tau > t$. This is an intrinsic property of all physical systems so that it is a natural assumption for an operator that represents a physical system. Mathematically, this can be written as (Visintin, 1994, p.60): $\forall (u_1, \xi^0), (u_2, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, if $u_1 = u_2$ in $[0, \alpha]$, then $\mathcal{H}(u_1, \xi^0) = \mathcal{H}(u_2, \xi^0)$ in $[0, \alpha]$.

4. CONSTANT INPUT CONSTANT OUTPUT OPERATORS

In this section, we consider that the operator $\mathcal{H}$ of Section 3 satisfies the following.

[1] $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of locally measurable functions $\mathbb{R}_+ \to \mathbb{R}^n$.

Assumption 1. Let $(u,\xi^0) \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n) \times \Xi$; if there exists a time instant $\theta \in \mathbb{R}_+$ such that u is constant in $[\theta,\infty)$, then the corresponding output $\mathcal{H}(u,\xi^0)$ is constant in $[\theta,\infty)$.

Assumption 1 is satisfied by all causal and rate-independent hysteresis operators (see for example (H. Logemann, 2008, Proposition 2.1) for a proof). This includes relay hysteresis, Ishlinskii model, Preisach model, Krasnosel'skii and Pokrovskii hysteron and generalized play Macki et al. (1993). Assumption 1 is also satisfied by some causal and rate-dependent hysteresis models like the generalized Duhem model Oh and Bernstein (2005). It is also obviously satisfied by static nonlinearities like dead-zone, saturation, etc.

Note that, when the input u is constant on $\mathbb{R}_+$, the corresponding output is constant on $\mathbb{R}_+$ by Assumption 1 so that the graph output versus input $\{(u(t),y(t)),\forall t \geq 0\}$ is reduced to a single point which makes the analysis of the quasi-static regime irrelevant. For this reason, unless otherwise specified, we will consider inputs that are not constant on $\mathbb{R}_+$ which, due to the absolute continuity of u, implies that $\|\dot{u}\|_\infty \neq 0$ and $\overset{\circ}{I}_u \neq \emptyset$.

4.1 Some properties of the operator $\mathcal{H}$

Now, let $(u,\xi^0) \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n) \times \Xi$ and let $y = \mathcal{H}\left(u,\xi^0\right) \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ where the causal operator $\mathcal{H}$ satisfies Assumption 1.

Lemma 5. For any $0 \leq t_1 < t_2$, if u is constant on the interval $[t_1,t_2]$, then y is constant on $[t_1,t_2]$.

Lemma 6. Let $\varrho \in I_u$ be given and let $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$. Then, $y(t_\varrho)$ is independent of t_ϱ. It depends solely on ϱ.

Lemma 6 shows that we can define a function

$$\begin{aligned} \varphi_u : I_u &\to \mathbb{R}^m \\ \varrho &\to y(t_\varrho) \end{aligned}$$

The function φ_u depends only on the input u and the initial condition ξ^0, and we have $\mathrm{Dom}\left(\varphi_u\right) = I_u$. Note that we have $\varphi_u \circ \rho_u = y$.

Lemma 7. $\varphi_u \in L^\infty(I_u,\mathbb{R}^m)$ and $\|\varphi_u\|_{\infty,I_u} \leq \|y\|_\infty$. If y is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty,I_u} = \|y\|_\infty$.

4.2 Characterization of the quasi-static regime

The objective of this section is to present a criterion for the existence of a quasi-static regime. To this end, we denote G_u the graph output versus input defined as $G_u = \{(u(t),y(t)),\forall t \in \mathbb{R}_+\} \subset \mathbb{R}^n \times \mathbb{R}^m$. Define the function

$$\begin{aligned} \phi_u : I_u &\to \mathbb{R}^n \times \mathbb{R}^m \\ \varrho &\to (\psi_u(\varrho),\varphi_u(\varrho)) \end{aligned}$$

Then, we have $\mathrm{Dom}(\phi_u) = I_u$ and $\mathrm{Range}(\phi_u) = \{(\psi_u(\varrho),\varphi_u(\varrho)),\forall \varrho \in I_u\} \subset \mathbb{R}^n \times \mathbb{R}^m$.

Lemma 8. $\mathrm{Range}(\phi_u) = G_u$.

Now, our objective is to illustrate the relevance of Lemma 8 to our purpose and introduce the definition of a quasi-static regime. To this end, let $M \in \mathbb{R}^n \times \mathbb{R}^m$ be some point of the set G_u, then there exists $t \in \mathbb{R}_+$ such that $M = (u(t),y(t))$. Let $\varrho = \rho_u(t)$, then $M = (\psi_u(\varrho),\varphi_u(\varrho)) = \phi(\varrho)$. The functions ϕ and (u,y) seen as parametrized curves have the same range which is the set G_u by Lemma 8. Also they have the same orientation when this orientation exists due to the fact that $\phi \circ \rho_u = (u,y)$ with $\dot{\rho}_u \geq 0$. If instead of u the input is $u \circ s_\gamma$, the range of $\left(u \circ s_\gamma(t),[\mathcal{H}(u \circ s_\gamma,\xi^0)](t)\right)$, $t \in \mathbb{R}_+$ is $G_{u\circ s_\gamma} = \mathrm{Range}(\phi_{u\circ s_\gamma}) = \mathrm{Range}\left(\{(\psi_{u\circ s_\gamma}(\varrho),\varphi_{u\circ s_\gamma}(\varrho)),\varrho \in I_{u\circ s_\gamma}\}\right)$. Using Lemma 3 it follows that

$$G_{u\circ s_\gamma} = \mathrm{Range}\left(\{(\psi_u(\varrho),\varphi_{u\circ s_\gamma}(\varrho)),\varrho \in I_u\}\right) \tag{1}$$

On the other hand, for a given γ, the derivative of $u \circ s_\gamma$ with respect to time is $\overbrace{(u \circ s_\gamma)}^{\cdot}(t) = \frac{1}{\gamma}\dot{u}\left(\frac{t}{\gamma}\right)$ a.e., so that $\|\overbrace{(u \circ s_\gamma)}^{\cdot}\|_\infty = \frac{1}{\gamma}\|\dot{u}\|_\infty$. For large values of γ, $\|\overbrace{(u \circ s_\gamma)}^{\cdot}\|_\infty$ is small which means that the input signal $u \circ s_\gamma$ contains only low frequencies. This means that taking $\gamma \to \infty$ corresponds to having an input signal frequency content that goes to zero. Thus, it would be natural to characterize the quasi-static regime by the convergence of the series of sets $\{G_{u\circ s_\gamma}\}_{\gamma>0}$ to some fixed set when $\gamma \to \infty$, with respect to some metric. However, as shown in Example 1, a definition of the quasi-static regime based on the convergence of the sets $G_{u\circ s_\gamma}$ with respect to some distance is not adequate because a distance between sets does not take into account information on the orientation (when it exists) of the trajectories $\left(u \circ s_\gamma(t),[\mathcal{H}(u \circ s_\gamma,\xi^0)](t)\right)$, $t \in \mathbb{R}_+$. For this reason, we propose the following definition which is based on Equation (1).

Definition 2. Let $\mathcal{H}$ be an operator as in Section 3 that satisfies Assumption 1. Let $u \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ be given. The operator $\mathcal{H}$ is said to have a quasi-static regime with respect to the input u and initial condition ξ^0 if and only if the series of functions $\{\varphi_{u\circ s_\gamma}\}_{\gamma>0}$ converges in $L^\infty(I_u,\mathbb{R}^m)$.

Note that $\mathrm{Dom}(\varphi_{u\circ s_\gamma}) = I_u$ for all $\gamma > 0$ by Lemma 3. Definition 2 implies that there exists a function $\varphi_u^\star \in L^\infty(I_u,\mathbb{R}^m)$ such that $\lim_{\gamma\to\infty}\|\varphi_{u\circ s_\gamma} - \varphi_u^\star\|_{\infty,I_u} = 0$. Define the function

$$\begin{aligned} \phi_u^\star : I_u &\to \mathbb{R}^n \times \mathbb{R}^m \\ \varrho &\to (\psi_u(\varrho),\varphi_u^\star(\varrho)) \end{aligned}$$

Then, we have $\mathrm{Dom}(\phi_u^\star) = I_u$ and $\mathrm{Range}(\phi_u^\star) = \{(\psi_u(\varrho),\varphi_u^\star(\varrho)),\forall \varrho \in I_u\} \subset \mathbb{R}^n \times \mathbb{R}^m$. The function $\phi_u^\star$ describes completely the quasi-static behavior of the operator $\mathcal{H}$ with respect to (u,ξ^0). This function is unique due to the uniqueness of the limit $\varphi_u^\star$ in $L^\infty(I_u,\mathbb{R}^m)$.

Now, for any two nonempty compact sets S_1 and S_2 in $\mathbb{R}^{n+m}$, define the Hausdorff metric

$$d(S_1,S_2) = \max\left\{ \sup_{\eta_1\in S_1}\left(\inf_{\eta_2\in S_2}|\eta_1-\eta_2|\right), \right.$$
$$\left. \sup_{\eta_2\in S_2}\left(\inf_{\eta_1\in S_1}|\eta_1-\eta_2|\right)\right\}$$

Then, the collection of all nonempty compact subsets of $\mathbb{R}^{n+m}$ is a complete metric space with respect to the Hausdorff metric d (Edgar, 1990, p.67).

Lemma 9. Suppose that the operator $\mathcal{H}$ has a quasi-static regime with respect to (u,ξ^0). Then, the series of sets $\{\text{Closure}(G_{u\circ s_\gamma})\}_{\gamma>0}$ converges to Closure $(\text{Range}(\phi_u^\star))$ with respect to the metric d.

When $\mathcal{H}$ has a quasi-static regime with respect to (u,ξ^0), Lemma 9 shows that the series of graphs Closure $(G_{u\circ s_\gamma})$ converges to a fixed set Closure $(\text{Range}(\phi_u^\star))$ with respect to the metric d as $\gamma\to\infty$. The converse is not true. That is, the fact that the series of sets $\{\text{Closure}\left(G_{u\circ s_\gamma}\right)\}_{\gamma>0}$ converges to some fixed set with respect to the distance d does not imply that the operator $\mathcal{H}$ has a quasi-static regime with respect to the input u and initial condition ξ^0. The following example illustrates this fact.

Example 1. Consider the left-derivative operator Δ_- defined on $W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ by

$$[\Delta_-(u)](t) = \lim_{\tau\uparrow t}\frac{u(\tau)-u(t)}{\tau-t}$$

The operator Δ_- is causal as $[\Delta_-(u)](t)$ depends only on values of $u(\tau)$ for $\tau\le t$, and we have $\Delta_-(u)=\dot{u}$ a.e. as $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ so that $\Delta_-(u)\in L^\infty(\mathbb{R}_+,\mathbb{R})$. Furthermore, it is clear that Δ_- satisfies Assumption 1. Now, consider the periodic input u with period 4 defined as:

$$\begin{cases} u(t) = t-1 \text{ for } 0\le t\le 2\\ u(t) = -t+3 \text{ for } 2\le t\le 4\end{cases}$$

Then, for every $\gamma>0$ we have, $[\Delta_-(u\circ s_\gamma)](t) = \pm 1/\gamma$ depending on the time interval. On the other hand, consider the function Θ defined on $\mathbb{R}$ as follows

$$\begin{cases} \Theta(w) = 1 \text{ if } |w|>1\\ \Theta(w) = (-1)^n \text{ if } \dfrac{1}{n+1} < w \le \dfrac{1}{n},\ n\in\mathbb{N}, n>0\\ \Theta(w) = (-1)^{n+1} \text{ if } \dfrac{-1}{n} \le w < \dfrac{-1}{n+1},\ n\in\mathbb{N}, n>0\end{cases}$$

Now, consider the operator $\Gamma=\Theta\circ\Delta_- : W^{1,\infty}(\mathbb{R}_+,\mathbb{R})\to L^\infty(\mathbb{R}_+,\mathbb{R})$. This operator is causal and satisfies Assumption 1. For $\gamma>1$, the corresponding sets $G_{u\circ s_\gamma}=[-1,1]\times\{-1,1\}$ are independent of γ which shows that the series $\{G_{u\circ s_\gamma}\}_{\gamma>0}$ converges to the set $S=[-1,1]\times\{-1,1\}$ with respect to the distance d as $\gamma\to\infty$. However, when $\frac{1}{n+1}<\frac{1}{\gamma}\le\frac{1}{n}$ with n odd, $G_{u\circ s_\gamma}$ is counterclockwise; and when n is even, $G_{u\circ s_\gamma}$ is clockwise. This means that when $\gamma\to\infty$, the points $(u\circ s_\gamma(t),[\Gamma(u\circ s_\gamma)](t)), t\in\mathbb{R}_+$ cover the set $G_{u\circ s_\gamma}=S$ clockwise and counterclockwise infinitely many times. Thus, we have to expect that the operator Γ does not have a quasi-static regime with respect to the input u. This can be demonstrated using Definition 2.

Indeed, using Lemma 4 it can be checked that $I_u=\mathbb{R}_+$ and $\psi_u=u$. On the other hand, since $\rho_{u\circ s_\gamma}$ is invertible, we have $\varphi_{u\circ s_\gamma}=\Gamma(u\circ s_\gamma)\circ\rho^{-1}_{u\circ s_\gamma}$. A simple calculation shows that $\varphi_{u\circ s_\gamma}$ is periodic of period 4 and is defined as

$$\begin{cases}\varphi_{u\circ s_\gamma}(\varrho) = (-1)^n \text{ for } 0<\varrho<2\\ \varphi_{u\circ s_\gamma}(\varrho) = (-1)^{n+1} \text{ for } 2<\varrho<4\end{cases}$$

where $\frac{1}{n+1}<\frac{1}{\gamma}\le\frac{1}{n}$, $n\in\mathbb{N}, n>0$. This implies that $\|\varphi_{u\circ s_{2k}}-\varphi_{u\circ s_{2k+1}}\|_\infty = 2$ for any positive integer k. Thus, $\{\varphi_{u\circ s_\gamma}\}_{\gamma>0}$ is not a Cauchy sequence so it does not converge. This implies by Definition 2 that the operator Γ does not have a quasi-static regime with respect to the input u.

4.3 The case of periodic inputs

In this section we consider input functions $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n)$ that are T-periodic. In this case, Lemmas 2 and 4 show that the function $\psi_u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^n)$ is $\rho_u(T)$-periodic. The quasi-static regime is characterized by the function $\phi^\star=(\psi_u,\varphi_u^\star)$. Define the functions $\varphi^\star_{u,k}\in L^\infty([0,\rho_u(T)],\mathbb{R}^m)$, $k\in\mathbb{N}$ by the relation $\varphi^\star_{u,k}(\varrho)=\varphi^\star_u(k\rho_u(T)+\varrho)$, $\forall\varrho\in[0,\rho_u(T)]$.

Definition 3. The quasi-static regime of the operator $\mathcal{H}$ defined by the function $\phi^\star$ is said to have a steady-state if and only if the series of functions $\{\varphi^\star_{u,k}\}_{k\in\mathbb{N}}$ converges with respect to the norm $\|\cdot\|_{\infty,[0,\rho_u(T)]}$ in $L^\infty([0,\rho_u(T)],\mathbb{R}^m)$.

Let $L^\infty([0,\rho_u(T)],\mathbb{R}^m)\ni\varphi_u^\diamond=\lim_{k\to\infty}\varphi^\star_{u,k}$, and let $\psi_u|_{[0,\rho_u(T)]}$ be the restriction of the $\rho_u(T)$-periodic function ψ_u to the interval $[0,\rho_u(T)]$. The steady-state is characterized completely by the function $\phi^\diamond=(\psi_u|_{[0,\rho_u(T)]},\varphi_u^\diamond)$ defined on the interval $[0,\rho_u(T)]$. The range of the function $\phi^\diamond$, that is the set $\{(\psi_u(\varrho),\varphi_u^\diamond(\varrho)),\varrho\in[0,\rho_u(T)]\}$ is called the steady-state graph, and it satisfies the following.

Lemma 10. Suppose that the quasi-static regime of the operator $\mathcal{H}$ has a steady-state as in Definition 3. Then, given $z\in\text{Range}(\phi^\diamond)$, there exists an increasing divergent sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty,\gamma\to\infty}|(u\circ s_\gamma(\gamma t_i),[\mathcal{H}(u\circ s_\gamma,\xi^0)](\gamma t_i))-z|=0$.

4.4 Case-study: Application to the hysteresis semi-linear Duhem model

In this section, the tools developed in the previous sections are applied to the problem of analyzing the quasi-static regime of the semi-linear Duhem model defined as follows Oh and Bernstein (2005).

$$\dot{x}(t) = [g_+(\dot{u}(t))\,I_n, g_-(\dot{u}(t))\,I_n]\times \left(\begin{bmatrix}A_+\\A_-\end{bmatrix}x(t)+\begin{bmatrix}B_+\\B_-\end{bmatrix}u(t)+\begin{bmatrix}E_+\\E_-\end{bmatrix}\right) \tag{2}$$

$$x(0)=x_0 \tag{3}$$

$$y(t)=Cx(t)+Du(t) \tag{4}$$

where $A_+\in\mathbb{R}^{n\times n}$, $A_-\in\mathbb{R}^{n\times n}$, $B_+\in\mathbb{R}^n$, $B_-\in\mathbb{R}^n$, $E_+\in\mathbb{R}^n$, $E_-\in\mathbb{R}^n$, $C\in\mathbb{R}^{1\times n}$, $D\in\mathbb{R}$, and where $g_+:\mathbb{R}\to\mathbb{R}$, $g_-:\mathbb{R}\to\mathbb{R}$, are continuous and satisfy $g_+(w)=0$ for $w\le 0$, $g_-(w)=0$ for $w\ge 0$. Define

$$\bar{g}_+(w) = \frac{g_+(w)}{|w|}, \forall w \neq 0 \tag{5}$$

$$\bar{g}_-(w) = \frac{g_-(w)}{|w|}, \forall w \neq 0 \tag{6}$$

As in Oh and Bernstein (2005), we assume that [2]

$$\lim_{w\downarrow 0} \bar{g}_+(w) = 1 \text{ and } \lim_{w\uparrow 0} \bar{g}_-(w) = -1 \tag{7}$$

This problem is solved in Oh and Bernstein (2005) for wave-periodic inputs u. The analysis is done in two steps. First, it is assumed that $g_+(w) = \max\{0, w\}$ and $g_-(w) = \min\{0, w\}$ which corresponds to a rate-independent semi-linear Duhem model. In this case, Equations (2)-(4) are re-parameterized in terms of u in each interval where u is monotone, which leads to two linear systems whose explicit time solutions are determined. Then, the periodicity of u is exploited to obtain the sufficient condition $\rho\left(e^{\beta A_+}e^{-\beta A_-}\right) < 1$ so that the existence and uniqueness of a periodic solution are guaranteed. In the second step, the relation $g_+(w) = \max\{0, w\}$ or $g_-(w) = \min\{0, w\}$ is not assumed so that the semi-linear Duhem model may be rate-dependent. In this case, it is shown that the graph output versus input of the rate-dependent semi-linear Duhem model converges with respect to the Hausdorff metric to the graph output versus input of the rate-independent semi-linear Duhem model.

The proof of these results reposes heavily on the wave-periodic nature of the input u since it is based on the piecewise monotonicity of the input. However, wave-periodic inputs represent a limited class of input excitations for a physical system, thus it is of interest to generalize the results of Oh and Bernstein (2005) to inputs that are not necessarily piecewise monotone or periodic. Our objective in this section is to find a sufficient condition that insures the existence of a quasi-static regime with respect to any $u \in W^{1,\infty}(\mathbb{R}_+)$ and any initial condition $x_0 \in \mathbb{R}^n$. Furthermore, when the input u is periodic (not necessarily wave-periodic), we show that the quasi-static regime has a steady-state.

Theorem 1. Consider the semi-linear Duhem model (2)-(4) where $u \in W^{1,\infty}(\mathbb{R}_+)$. Assume that the matrices A_+ and $-A_-$ are both Hurwitz [3] and have a common Lyapunov matrix $P = P^T$, that is such that $A_+^T P + PA_+ < 0$ and $-A_-^T P - PA_- < 0$. Then, the semi-linear Duhem model has a quasi-static regime with respect to (u, x_0) given by the equations

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\varrho}(\varrho) = \left[\frac{\dot{\psi}_u(\varrho)+1}{2} I_n, \frac{\dot{\psi}_u(\varrho)-1}{2} I_n\right] \times \left(\begin{bmatrix} A_+ \\ A_- \end{bmatrix} \zeta(\varrho) + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} \psi_u(\varrho) + \begin{bmatrix} E_+ \\ E_- \end{bmatrix}\right)$$

a.e. in I_u (8)

$$\zeta(0) = x_0 \tag{9}$$

$$\varphi_u^\star(\varrho) = C\zeta(\varrho) + D\psi_u(\varrho), \forall \varrho \in I_u \tag{10}$$

Furthermore, if u is periodic, then the quasi-static regime has a steady-state given by Equations (8) and (10) where I_u is replaced by $[0, \rho_u(T)]$ and $\varphi_u^\star$ is replaced by $\varphi_u^\diamond$. In this case, the initial condition is not necessarily x_0.

Comment: Relationship with singular perturbations. Consider the nonlinear system

$$\dot{x} = f(x, u) \tag{11}$$

$$y = h(x, u) \tag{12}$$

$$x(0) = x_0 \tag{13}$$

where $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, f and h are such that a unique bounded solution exists on $\mathbb{R}_+$. The following relation can be obtained for all $\sigma \in I_u$

$$\frac{1}{\gamma}\frac{\mathrm{d}\eta_\gamma}{\mathrm{d}\sigma}(\sigma) = \frac{f\left[\eta_\gamma(\sigma), \psi_{u_\gamma}(\sigma)\right]}{\dot{\rho}_{u_\gamma}\left(\rho_{u_\gamma}^{-1}(\sigma)\right)} \tag{14}$$

which can be written as a singularly perturbed system

$$\varepsilon\frac{\mathrm{d}z}{\mathrm{d}\sigma}(\sigma, \varepsilon) = g(\sigma, z(\sigma, \varepsilon), \varepsilon) \tag{15}$$

with $\varepsilon = \frac{1}{\gamma}$ and $z(\sigma, \varepsilon) = \eta_\gamma(\sigma)$. The dependence of the function g on σ and ε comes from the presence in the right-hand side of Equation (14) of the terms $\psi_{u_\gamma}(\sigma)$ and $\dot{\rho}_{u_\gamma}\left(\rho_{u_\gamma}^{-1}(\sigma)\right)$. The problem of finding the quasi-static regime of (11)-(13) is equivalent to finding $\bar{z}$ such that the solutions $z(\cdot, \varepsilon)$ converge uniformly to $\bar{z}$ when $\varepsilon \to 0$. In singular perturbations techniques, the standard model is given as (Khalil, 2000, p.430)

$$\frac{\mathrm{d}w}{\mathrm{d}\sigma}(\sigma, \varepsilon) = \ell\left[\sigma, w(\sigma, \varepsilon), z(\sigma, \varepsilon), \varepsilon\right] \tag{16}$$

$$\varepsilon\frac{\mathrm{d}z}{\mathrm{d}\sigma}(\sigma, \varepsilon) = g(\sigma, w(\sigma, \varepsilon), z(\sigma, \varepsilon), \varepsilon) \tag{17}$$

and the issue is to approximate z and w by their quasi-static solutions. Tikhonov's theorem gives sufficient conditions under which such approximation is valid (Khalil, 2000, Theorem 11.1, p.434). The focus and objective of singular perturbations techniques differ from ours in several issues so that the conditions of applicability of these techniques as presented in Tikhonov's theorem do not allow to solve the problem of finding the quasi-static regime as exposed in this paper.

- The main assumption in Tikhonov's theorem is that in Equation (16), $g(\sigma, w(\sigma, 0), z(\sigma, 0), 0) = 0$ has a finite number of isolated roots $\bar{z}(\sigma) = h(\sigma, \bar{w}(\sigma))$. The main objective of our paper is precisely to present a general criterion that guarantees the uniform convergence of the solutions $z(\cdot, \varepsilon)$ to $\bar{z}$ when $\varepsilon \to 0$.
- In our case, putting $\varepsilon = 0$ in Equation (15) may be a nonsense because this assumes that the series of functions $\{\rho_{u_\gamma}^{-1}\}_{\gamma>0}$ has a limit when $\gamma \to \infty$. This is not the case in general because the function ρ_u^{-1} is not defined as ρ_u is not necessarily increasing.

[2] If $\lim_{w\downarrow 0} \bar{g}_+(w) = a_+ \neq 0$ and $\lim_{w\uparrow 0} \bar{g}_-(w) = -a_- \neq 0$, the constants a_+ and a_- are incorporated into the matrices A_+ and A_- respectively.

[3] A matrix is Hurwitz if and only if all its eigenvalues have negative real part.

- The state z is seen as a fast dynamic that disappears rapidly and the focus of singular perturbations techniques is to characterize the slow dynamic of the system which is captured by the state w. In this sense, Thikonov's theorem approximates well the fast dynamic z only on an interval $[t_1, t_2]$ where $t_1 > 0$. The difference between the quasi-static solution $\bar{z}$ and the solution z may be large in the interval $[0, t_1)$ even if ε is infinitely small. This is an important difference from our objective which is the exact characterization of the quasi-static regime.

5. CONCLUSION

The objective of the paper was to provide a mathematical framework as general as possible for the characterization of the quasi-static regime of operators mapping an input and initial condition to an output, all of them belonging to some specified sets. When the total variation of the input is increasing, the quasi-static regime can be characterized under the only assumption the the operator is causal. If the input presents an interval (or intervals) in which it is constant, then two classes of operators can be been considered: those that verify a constant input constant output assumption (Section 4), and those that verify a smoothness assumption Ikhouane (2009). In both cases, the quasi-static regime can been defined appropriately. The case-study of the semi-linear Duhem model illustrates the concepts.

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