

# Optimal Power Flow for resistive DC Networks: a Port-Hamiltonian approach<sup>\*</sup>

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**Abstract:** This paper studies the optimal power flow problem for resistive DC networks. The gradient method algorithm is written in a port-Hamiltonian form and the stability of the resulting dynamics is studied. Stability conditions are provided for general cyclic networks and a solution, when these conditions fail, is proposed. In addition, the results are exemplified by means of numerical simulations.

*Keywords:* Optimal power flow, port-Hamiltonian systems, gradient method, DC networks, cyclic networks

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## 1. INTRODUCTION

The DC networks emerged as reliable energy transmission system for both low voltage systems (such as household DC networks, Shivakumar et al. (2015)) and high voltage applications (HVDC transmission systems, van Hertem and Ghandhari (2010)). The Optimal Power Flow (OPF) problem in electrical networks consists on finding an optimal working point of the system ensuring a set of constraints in terms of power, current and/or voltages (Gavrilita et al. (2015)). Usually, the optimization implies the minimization of a cost (loss) function that, expressed in terms of voltages, depends on the weighted Laplacian matrix. Since this matrix is positive semidefinite, the function turns to be *just* convex. The use of the gradient method to find the optimal point has been extensively used, but stability problems could appear when the cost function is no strictly convex, Arrow et al. (1958). Alternatively, modification of the problem statement could skip this requirement (see Cherukuri and Cortés (2015) or Feijer and Paganini (2010)).

Recently, some papers proposed a port-Hamiltonian description of the gradient method T.W. Stegink et al. (2015) and Stegink et al. (In press). The advantages of casting the algorithm in the port-Hamiltonian form are twofold: firstly, passivity-based properties can be used for the stability analysis and, secondly, the optimization algorithm can be easily interconnected with the network, providing stability of the whole dynamics. The optimization problem of DC networks using the gradient method in a port-Hamiltonian form was studied in Bedito et al. (2016).

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The main contribution there was to propose a change of variables that renders the cost (loss) function in a strictly convex function that guarantees the stabilization in the minimum point assuming that the network is acyclic, as many papers studying OPF (see for example Zhang and Papachristodoulou (2015)). However, the interest of cyclic networks is clear since they commonly appear in electrical networks. Additionally, it is well known following the Rosen's Theorem (or nodal-mesh transformation) that any network with internal nodes can be represented by an equivalent meshed circuit, Rosen (1924). Moreover, the inverse transformation (mesh-to-nodal) is subjected to certain conditions, see necessary and sufficient conditions for a mesh to star conversion in Wang and Tokad (1961).

## 2. PRELIMINARIES

### 2.1 Definitions

In this paper we consider a resistive DC network: a undirected, connected, and weighted graph,  $\mathcal{G}$ , with  $n$  nodes (vertices) and  $m$  branches (edges). Following results are obtained from classical graph books (Biggs (1974)).

*Definition 1.*  $\mathbf{1} \in \mathbb{R}^n$  is the vector consisting in only ones.

*Definition 2.* (Incidence matrix). Consider an arbitrary orientation of the edges. The (vertex-edge) *incidence matrix*,  $\mathbf{B} \in \mathbb{R}^{n \times m}$ , is defined by the  $(k, l)$ -th elements as

$$\mathbf{b}_{kl} = \begin{cases} 1 & \text{if the vertex } k \text{ is the head of edge } l \\ -1 & \text{if the vertex } k \text{ is the tail of edge } l \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

*Definition 3.* (Laplacian matrix). The weighted Laplacian matrix can be defined as

$$\mathbf{W} = \mathbf{B}\mathbf{G}\mathbf{B}^T, \quad (2)$$

where  $\mathbf{G}$  is the  $m \times m$  diagonal matrix with the weights of each edge (van der Schaft (2010)).

From the definitions above, the following properties are satisfied:

**P1.**  $\ker(\mathbf{B}^T) = \{\alpha \mathbf{1} | \alpha \in \mathbb{R}\}$ , then  $\mathbf{B}^T \mathbf{1} = 0$ .  $\text{rank}(\mathbf{B}^T) = n - 1$ .

**P2.** Consider a  $n - 1$  column matrix of  $\mathbf{B}^T$ , say  $\mathbf{B}_1^T$ , then:

- (1)  $\ker(\mathbf{B}_1^T) = \{0\}$  and  $\text{rank}(\mathbf{B}_1^T) = n - 1$ .
- (2)  $\mathbf{B}_1^T$  is a square matrix if the graph is a tree (the circuit do not contain any cycle).
- (3)  $(\mathbf{B}_1 \mathbf{B}_1^T)^{-1} \mathbf{B}_1$  is the pseudoinverse matrix of  $\mathbf{B}_1^T$ .

**P3.** The Laplacian matrix,  $\mathbf{W}$ , has zero row-sum:  $\sum_l \mathbf{w}_{kl} = 0$ ,  $k = 1, \dots, n$

**P4.**  $\mathbf{1}$  is a right eigenvector of  $\mathbf{W}$  with eigenvalue 0, i.e.,  $\mathbf{W} \mathbf{1} = 0$ .

*Proposition 1.* Let  $\mathbf{B}_1^T$  be a matrix with  $n - 1$  columns of  $\mathbf{B}^T$  and let  $\mathbf{S}$  be the pseudoinverse matrix of  $\mathbf{B}_1^T$ . Then,

- The set of eigenvalues of the matrix  $\mathbf{B}_1^T \mathbf{S}$  is  $\{0, 1\}$ ,
- The column vectors of  $\mathbf{B}^T$  are eigenvectors of  $\mathbf{B}_1^T \mathbf{S}$  with eigenvalue 1.
- The vectors of  $\ker(\mathbf{B})$  are eigenvectors of  $\mathbf{B}_1^T \mathbf{S}$  with eigenvalue 0.

**Proof.** Let  $c_1, \dots, c_{n-1}$ ,  $n - 1$  column vectors of the matrix  $\mathbf{B}^T$ , and let  $k_1, \dots, k_{m-n+1}$  a basis of  $\ker(\mathbf{B})$ . Then,  $c_1, \dots, c_{n-1}, k_1, \dots, k_{m-n+1}$  is a basis of  $\mathbb{R}^m$  as they are linearly independent,  $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{B}^T) = n - 1$  and  $\dim(\ker(\mathbf{B}^T)) + \text{rank}(\mathbf{B}^T) = m$ .

On the one hand,  $\mathbf{B}_1^T \mathbf{S} k_j = 0, \forall j$ , as  $\ker(\mathbf{B}^T) \subseteq \ker(\mathbf{B}_1^T)$ . On the other hand we claim that  $\mathbf{B}_1^T \mathbf{S} c_i = c_i, \forall i$ . As  $\mathbf{B}_1^T \mathbf{S} \mathbf{B}_1^T = \mathbf{B}_1^T$ , if  $c_i$  is a column vector of  $\mathbf{B}_1^T$ , the claim is proved. If not, from Property P1,  $c_i = -\mathbf{B}_1^T \mathbf{1}$  and then,  $\mathbf{B}_1^T \mathbf{S} c_i = -\mathbf{B}_1^T \mathbf{1} = c_i$ .

*Corollary 1.* The matrix  $\mathbf{B}_1^T \mathbf{S}$  does not depend on the selected  $\mathbf{B}_1^T$  matrix.

*Corollary 2.* The set

$$\mathcal{B} := \{\psi \in \mathbb{R}^m | (\mathbf{I} - \mathbf{B}_1^T \mathbf{S})\psi = 0\} \quad (3)$$

is the span of the column vectors of  $\mathbf{B}^T$  and  $\dim(\mathcal{B}) = n - 1$ .

*Remark 1.* The Kirchhoff's Current Law (KCL) in a circuit with external current sources, natural arises from the incidence matrix as

$$\mathbf{B} \mathbf{i} = \mathbf{i}_{\text{ext}}, \quad (4)$$

where  $\mathbf{i} \in \mathbb{R}^n$  are the current through the branches (edges), and  $\mathbf{i}_{\text{ext}} \in \mathbb{R}^m$  are the injected currents at the nodes (vertices).

## 2.2 Port-Hamiltonian representation of the gradient method algorithm

The stability of the gradient method, for strictly convex function was already studied in Arrow et al. (1958). Recently, the stability analysis has been done using passive systems properties in T.W. Stegink et al. (2015) and Stegink et al. (In press), which entails a different perspective that becomes very useful for interconnecting systems, see Benedito et al. (2016) as example. In this subsection, the port Hamiltonian representation of the gradient method algorithm presented in T.W. Stegink et al. (2015) and Stegink et al. (In press) is revised.

Consider the minimization problem defined by

$$\min_x f(x) \quad (5)$$

$$\text{s.t. } \mathbf{A}x - b = 0, \quad (6)$$

where  $x \in \mathbb{R}^n$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{A} \in \mathbb{R}^{p \times n}$  and  $b \in \mathbb{R}^p$ . The optimal value of (5)-(6) can be obtained finding the saddle-point of the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T (\mathbf{A}x - b) \quad (7)$$

where  $\lambda \in \mathbb{R}^p$ . The gradient method for finding the saddle-point of (7) is the following system of differential equations:

$$\dot{x} = -\nabla f(x) - \mathbf{A}^T \lambda \quad (8)$$

$$\dot{\lambda} = \mathbf{A}x - b \quad (9)$$

and the port-Hamiltonian representation of the gradient method is given by

$$\dot{z} = \begin{pmatrix} 0 & -\mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix} \nabla H - \begin{pmatrix} \nabla f(x) \\ b \end{pmatrix} \quad (10)$$

where  $z = (\tau_x x, \tau_\lambda \lambda)$  and  $\tau_x, \tau_\lambda > 0$  are symmetric positive definite matrices. The Hamiltonian function is given by

$$H = \frac{1}{2} z^T \tau^{-1} z \quad (11)$$

where  $\tau = \text{blockdiag}(\tau_x, \tau_\lambda)$ , and the  $\nabla(\cdot)$  operator is used for the gradient (as a column vector).

Let us define  $z^* = (\tau_x x^*, \tau_\lambda \lambda^*)$  as the (unique) equilibrium point of (10) and the shifted Hamiltonian by

$$H^* = \frac{1}{2} (z - z^*)^T \tau^{-1} (z - z^*), \quad (12)$$

and (10) is equivalent to

$$\dot{z} = \begin{pmatrix} 0 & -\mathbf{A}^T \\ \mathbf{A} & 0 \end{pmatrix} \nabla H^* - \begin{pmatrix} \nabla f(x) - \nabla f(x^*) \\ 0 \end{pmatrix}. \quad (13)$$

The asymptotic stability of (10) can be proved under the following conditions.

*Proposition 2.* Assume that  $z^*$  is an (unique) equilibrium point of (10),  $\ker(\mathbf{A}^T) = \{0\}$  and  $f(x)$  is strictly convex. Then, the dynamics in (10) will converge asymptotically to  $z^*$ , i.e.,  $(x, \lambda) \rightarrow (x^*, \lambda^*)$ .

**Proof.** (From T.W. Stegink et al. (2015)) The time derivative of the shifted Hamiltonian

$$\dot{H}^* = -(x - x^*)^T (\nabla f(x) - \nabla f(x^*)) \leq 0 \quad (14)$$

since  $f(x)$  is convex, and the equality holds if and only if  $x = x^*$  since  $f(x)$  is strictly convex. Using LaSalle's invariant principle, on the largest invariant set where  $\dot{H}^* = 0$ , we have that  $\lambda = \lambda^*$  as  $\mathbf{A}^T (\lambda - \lambda^*) = 0$ , that proves the Proposition above.

## 3. MAIN RESULT

### 3.1 Problem statement

Consider a resistive DC network with  $n$  nodes and  $m$  branches, with a resistor  $r_{kl} > 0$  (denoting the resistance of the branch connecting nodes  $k$  and  $l$ ) associated in each branch and one voltage source in each node,  $v_k$  where  $k = 1, \dots, n$ .

From the Kirchhoff laws, the voltages (at each node) are related with the currents (through each resistor) by

$$\mathbf{B}^T v = \mathbf{R} i \quad (15)$$

where  $v \in \mathbb{R}^n$  and  $i \in \mathbb{R}^m$ , are the voltage and current vectors, respectively,  $\mathbf{B}$  is the incidence matrix of the network, and  $\mathbf{R} = \text{diag}(r_{kl}) > 0$ .

The control problem consists in to find an optimal voltage vector  $v^{\text{opt}}$  that minimizes the losses by Joule's effect, when some voltages or currents at the nodes are already set. The network losses function is the sum of the losses in all resistors,  $P(i) = \sum r_{kl} i_{kl}^2$ , where  $k, l = 1, \dots, n$ , in a matrix form

$$P(i) = i^T \mathbf{R} i. \quad (16)$$

From the conductance of the  $kl$ -branch,  $g_{kl} = \frac{1}{r_{kl}}$ , the conductance matrix can be defined as  $\mathbf{G} = \mathbf{R}^{-1}$ , and using (15), the cost function yields in terms of the weighted Laplacian as

$$P(v) = v^T \mathbf{W} v, \quad (17)$$

where (2) has been used.

*Remark 2.* Note that from Property P3, the weighted Laplacian,  $\mathbf{W}$ , is positive semidefinite but it is not positive definite. Then, the loss function  $P(v)$  in (17) is not strictly convex.

The OPF problem can be defined as

$$\min_v P(v) = v^T \mathbf{W} v \quad (18)$$

$$\text{s.t. } \mathbf{T} v - v^d = 0 \quad (19)$$

$$\mathbf{U} \mathbf{W} v - i^d = 0 \quad (20)$$

where  $v^d \in \mathbb{R}^p$  and  $i^d \in \mathbb{R}^q$  are, respectively, the voltage references for certain nodes and the current references injected in some other nodes,  $\mathbf{T} \in \mathbb{R}^{p \times n}$ ,  $\mathbf{U} \in \mathbb{R}^{q \times n}$  with  $p + q \leq n$ , and the matrix

$$\mathbf{A}_v := \begin{pmatrix} \mathbf{T} \\ \mathbf{U} \mathbf{W} \end{pmatrix} \quad (21)$$

is full rank.

### 3.2 Gradient method in DC networks

The port-Hamiltonian representation of the gradient method applied to the problem (18)-(20) is given by

$$\dot{z} = \begin{pmatrix} \mathbf{W} & -\mathbf{T}^T & -\mathbf{W} \mathbf{U}^T \\ \mathbf{T} & 0 & 0 \\ \mathbf{U} \mathbf{W} & 0 & 0 \end{pmatrix} \nabla H - \begin{pmatrix} 0 \\ v^d \\ i^d \end{pmatrix} \quad (22)$$

where  $z = (\tau_v v, \tau_T \lambda_T, \tau_U \lambda_U)$  and  $\tau_v, \tau_T, \tau_U > 0$  are symmetric positive definite matrices,  $\lambda_T \in \mathbb{R}^p$ , and  $\lambda_U \in \mathbb{R}^q$ . The Hamiltonian function is given by

$$H = \frac{1}{2} z^T \tau^{-1} z \quad (23)$$

where  $\tau = \text{blockdiag}(\tau_v, \tau_T, \tau_U)$ .

Since  $P(v)$  is not strictly convex, the stability of (22) can not be guaranteed using Proposition 2. However, the stability of (22) can be proved when  $\mathbf{T}$  meet certain conditions, as is stated in the next proposition.

Let us define  $z^* = (\tau_v v^*, \tau_T \lambda_T^*, \tau_U \lambda_U^*)$  as an equilibrium point of (22) and the shifted Hamiltonian by

$$H^* = \frac{1}{2} (z - z^*)^T \tau^{-1} (z - z^*), \quad (24)$$

and (22) is equivalent to

$$\dot{z} = \begin{pmatrix} \mathbf{W} & -\mathbf{T}^T & -\mathbf{W} \mathbf{U}^T \\ \mathbf{T} & 0 & 0 \\ \mathbf{U} \mathbf{W} & 0 & 0 \end{pmatrix} \nabla H^* \quad (25)$$

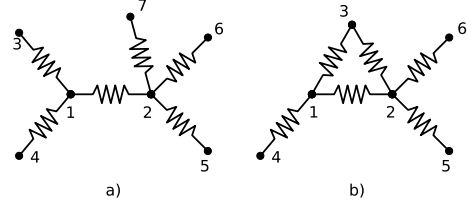


Fig. 1. Resistive circuits examples: a) acyclic resistive network, and b) cyclic resistive network.

The asymptotic stability of (22) can be proved under the following conditions.

*Proposition 3.* Assume that  $z^*$  is an equilibrium point of (22), and  $\mathbf{1}$  is not an eigenvector of  $\tau_v^{-1} \mathbf{T}^T \tau_T^{-1} \mathbf{T}$ . Then, the dynamics in (22) will converge asymptotically to  $z^*$ , i.e.,  $(v, \lambda_T, \lambda_U) \rightarrow (v^*, \lambda_T^*, \lambda_U^*)$ .

**Proof.** The time derivative of the shifted Hamiltonian

$$\dot{H}^* = -(v - v^*)^T \mathbf{W} (v - v^*) \leq 0 \quad (26)$$

since  $\mathbf{W}$  is positive semidefinite, and the equality holds if and only if  $v - v^* \in \ker(\mathbf{W})$ , i. e.  $v - v^* = a \mathbf{1}$  with  $a \in \mathbb{R}$ .

On the largest invariant set where  $\dot{H}^* = 0$ , we have that

$$\dot{a} \mathbf{1} = -a \tau_v^{-1} \mathbf{T}^T \tau_T^{-1} \mathbf{T} \mathbf{1}, \quad (27)$$

where we used that  $\mathbf{U} \mathbf{W} \mathbf{1} = 0$  as  $\mathbf{1} \in \ker(\mathbf{W})$ . Then  $a = 0$  as  $\tau_v^{-1} \mathbf{T}^T \tau_T^{-1} \mathbf{T} \mathbf{1} \neq \alpha \mathbf{1}$ , and  $v = v^*$ . Using LaSalle's invariant principle, in the set we have that  $\lambda_T = \lambda_T^*$  and  $\lambda_U = \lambda_U^*$  as

$$\mathbf{A}_v \begin{pmatrix} \lambda_T - \lambda_T^* \\ \lambda_U - \lambda_U^* \end{pmatrix} = 0 \quad (28)$$

and  $\ker(\mathbf{A}_v) = \{0\}$ , that proves the Proposition above.

### 3.3 Change of coordinates

As pointed out in Section 3.2, the gradient method (22) can oscillate when  $\mathbf{1}$  is an eigenvector of  $\tau_v^{-1} \mathbf{T}^T \tau_T^{-1} \mathbf{T}$ . Tailored solutions to this stability problem could be setting appropriated values to  $\tau_v$  and  $\tau_T$ , or premultiplying in (19) any row of  $\mathbf{T}$  by a factor different than one. Alternatively, in this subsection we suggest a change of coordinates that solves the stability problem independently on  $\mathbf{T}$ ,  $\tau_T$ , since  $\tau_T > 0$  is diagonal.

Similarly to Benedito et al. (2016), the following new set of variables,  $\psi \in \mathbb{R}^m$ , are defined

$$\begin{pmatrix} \psi \\ v_0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1^T & \mathbf{B}_0^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_0 \end{pmatrix}, \quad (29)$$

$v_0$  is the voltage at an arbitrary node,  $\mathbf{B}_0^T$  is the column of  $\mathbf{B}^T$  corresponding to the arbitrary node, and  $\mathbf{B}_1^T$  is a  $n - 1$  column matrix with the rest of columns of  $\mathbf{B}^T$ .

*Proposition 4.* The proposed map (29) defines the bijective linear map:

$$\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathcal{B} \times \mathbb{R} \\ (v_1, v_0) \rightarrow (\mathbf{B}_1^T v_1 + \mathbf{B}_0^T v_0, v_0)$$

where  $\mathcal{B} := \{\psi \in \mathbb{R}^m | (\mathbf{I} - \mathbf{B}_1^T \mathbf{S}) \psi = 0\}$  and  $\mathbf{S}$  is the pseudoinverse matrix of  $\mathbf{B}_1^T$ .

**Proof.** From Property P2.1,  $\phi$  is injective and from Corollary 2,  $\phi$  is surjective.

From Proposition 4, the original voltages,  $v$ , can be obtained from potentials  $\psi$  and  $v_0$ , using

$$v = \begin{pmatrix} v_1 \\ v_0 \end{pmatrix} = \begin{pmatrix} \mathbf{S} & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \psi \\ v_0 \end{pmatrix}, \quad (30)$$

where  $\mathbf{S} = (\mathbf{B}_1 \mathbf{B}_1^T)^{-1} \mathbf{B}_1$ , if the following condition is fulfilled

$$(\mathbf{I} - \mathbf{B}_1^T \mathbf{S}) \psi = 0. \quad (31)$$

As  $(\mathbf{I} - \mathbf{B}_1^T \mathbf{S})$  is not full rank, condition (31) can be written as

$$\mathbf{D} \psi = 0, \quad (32)$$

where  $\mathbf{D} \in \mathbb{R}^{(m-n+1) \times m}$  is a full rank reduced matrix of  $(\mathbf{I} - \mathbf{B}_1^T \mathbf{S})$ . Additionally, from Corollary 1, this result is the same independently on the selected node.

Then, by using (2) and (30) in (17), the losses are now given by the function

$$P(\psi) = \psi^T \mathbf{S}^T \mathbf{B}_1 \mathbf{R}^{-1} \mathbf{B}_1^T \mathbf{S} \psi, \quad (33)$$

and the set of constrains yields

$$\begin{pmatrix} \mathbf{T}_1 \mathbf{S} & \mathbf{T}_1 \\ \mathbf{U} \mathbf{W}_1 \mathbf{S} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ v_0 \end{pmatrix} = \begin{pmatrix} v^d \\ i^d \end{pmatrix} \quad (34)$$

where matrices  $\mathbf{T}$  and  $\mathbf{W}$  has been split as follows

$$\mathbf{T} = (\mathbf{T}_1 \ \mathbf{T}_0), \quad \mathbf{W} = (\mathbf{W}_1 \ \mathbf{W}_0) \quad (35)$$

with  $\mathbf{T}_1 \in \mathbb{R}^{p \times (n-1)}$ ,  $\mathbf{T}_0 \in \mathbb{R}^p$ ,  $\mathbf{W}_1 \in \mathbb{R}^{n \times (n-1)}$ ,  $\mathbf{W}_0 \in \mathbb{R}^n$ .

Finally, the problem defined in (18)-(20) is redefined in terms of  $\psi, v_0$  as follows

$$\min_{\psi} P(\psi) = \psi^T \mathbf{R}^{-1} \psi \quad (36)$$

$$\text{s.t. } (\mathbf{T}_1 \mathbf{S} \ \mathbf{T}_1) \begin{pmatrix} \psi \\ v_0 \end{pmatrix} - v^d = 0 \quad (37)$$

$$\mathbf{U} \mathbf{W}_1 \mathbf{S} \psi = i^d \quad (38)$$

$$\mathbf{D} \psi = 0, \quad (39)$$

where the fact  $\mathbf{B}_1^T \mathbf{S} \psi = \psi$  imposed by (31) has been included in (36) and the matrix

$$\mathbf{A}_{\psi} := \begin{pmatrix} \mathbf{T}_1 \mathbf{S} & \mathbf{T}_1 \\ \mathbf{U} \mathbf{W}_1 \mathbf{S} & 0 \\ \mathbf{D} & 0 \end{pmatrix} \quad (40)$$

is full rank.

Based on the port-Hamiltonian representation of the gradient method in (22), the optimal point of the problem stated in (36)-(39) can be obtained from

$$\dot{z} = \mathbf{F} \nabla H - \mathbf{G} b \quad (41)$$

where

$$\mathbf{F} := \begin{pmatrix} -\mathbf{R}^{-1} & 0 & -\mathbf{S}^T \mathbf{T}_1^T & -\mathbf{S}^T \mathbf{W}_1^T \mathbf{U}^T & -\mathbf{D}^T \\ 0 & 0 & -\mathbf{1}^T \mathbf{T}_1^T & 0 & 0 \\ \mathbf{T}_1 \mathbf{S} & \mathbf{T}_1 & 0 & 0 & 0 \\ \mathbf{U} \mathbf{W}_1 \mathbf{S} & 0 & 0 & 0 & 0 \\ \mathbf{D} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (42)$$

$$\mathbf{G}^T := (0 \ 0 \ \mathbf{I} \ \mathbf{I} \ 0), \quad b^T := (0 \ 0 \ v^d \ i^d \ 0), \quad (43)$$

the states are  $z = (\tau_{\psi} \psi, \tau_0 v_0, \tau_T \lambda_T, \tau_U \lambda_U, \tau_D \lambda_D)$ , where  $\lambda_D \in \mathbb{R}^{m-n+1}$ ,  $\tau_{\psi}, \tau_T, \tau_U, \tau_D > 0$  are symmetric matrices,  $\tau_0 > 0$ , and the Hamiltonian (11) with  $\tau = \text{blockdiag}(\tau_{\psi}, \tau_0, \tau_T, \tau_U, \tau_D)$ .

The stability of the gradient method (41) is guaranteed as stated the following Proposition.

*Proposition 5.* Assume that

$$z^* = (\tau_{\psi} \psi^*, \tau_0 v_0^*, \tau_T \lambda_T^*, \tau_U \lambda_U^*, \tau_D \lambda_D^*)$$

is an equilibrium point of (41). Then, the dynamics in (41) with (42)-(43) will converge asymptotically to  $z^*$ , for any selected node  $v_0$  such that  $\mathbf{1} \notin \ker(\mathbf{T}_1^T \tau_T^{-1} \mathbf{T})$ .

**Proof.** The result is obtained similarly to Proposition 2, defining a shifted Hamiltonian,  $H^*$ , that satisfies

$$\dot{H}^* = -(\psi - \psi^*)^T \mathbf{R}^{-1} (\psi - \psi^*) \leq 0, \quad (44)$$

where the equality holds if and only if  $\psi = \psi^*$ . On the largest invariant set where  $\dot{H}^* = 0$ , we have that

$$0 = -(v_0 - v_0^*) \mathbf{S}^T \mathbf{T}_1^T \tau_T^{-1} \mathbf{T} \mathbf{1} \quad (45)$$

and then  $v_0 = v_0^*$  as  $\mathbf{1} \notin \ker(\mathbf{T}_1^T \tau_T^{-1} \mathbf{T})$  and  $\ker(\mathbf{S}^T) = \{0\}$  from Property P2.1. With this result we obtain that

$$\mathbf{A}_{\psi} \begin{pmatrix} \lambda_T - \lambda_T^* \\ \lambda_U - \lambda_U^* \\ \lambda_D - \lambda_D^* \end{pmatrix} = 0 \quad (46)$$

and then  $\lambda_T = \lambda_T^*, \lambda_U = \lambda_U^*, \lambda_D = \lambda_D^*$  as  $\ker(\mathbf{A}_{\psi}^T) = \{0\}$  (as  $\mathbf{A}_{\psi}$  is full rank). Invoking LaSalle's invariance principle, we have that  $(v_0, \lambda_T, \lambda_U, \lambda_D) \rightarrow (v_0^*, \lambda_T^*, \lambda_U^*, \lambda_D^*)$  as  $t \rightarrow \infty$ .

*Corollary 3.* The gradient method in (41) with (42)-(43), and  $\mathbf{T}$  such that  $\tau^{-1} \mathbf{T}^T \tau_T^{-1} \mathbf{T} \mathbf{1} = \mathbf{1}$  with  $\tau = \text{blockdiag}(\tau_1, \dots, \tau_k)$ ,  $k > 1$  and  $\tau_i \neq 0$

- has an unique equilibrium point,  $z^*$ ,
- and is globally asymptotically stable,

for any selected node  $v_0$ .

## 4. EXAMPLE

The circuit in Figure 1b has been used to test the proposed approach, where its weighted graph,  $\mathcal{G}$ , has  $n = 6$  nodes (vertices) and  $m = 6$  branches (edges). For simplicity, all the resistance has been set at  $r_{kl} = 1\Omega$ , which implies  $\mathbf{R} = \mathbf{I}$ . Also, a power flow were assigned arbitrary, namely, nodes N3, N4, N5, and N6 are voltage or current independent sources and nodes N1 and N2 are points of common coupling or interconnection nodes. In order to present the paper contributions, three cases are simulated and performed using Matlab. For all cases, the simulation starts with the null initial conditions:  $v_k(0) = 0$  V,  $\psi_k(0) = 0$  V, and  $i_{kl}(0) = 0$  A.

*4.1 Case A: Gradient method with the node voltages  $v$  and convergencing responses.*

Considering the dynamics of (22) with

$$\mathbf{T} = (1 \ 0 \ 0 \ 0 \ 0 \ 0), \quad \mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (47)$$

$\tau_v = 0.05 \cdot \mathbf{I}_6$ ,  $\tau_T = 0.5$ ,  $\tau_U = 0.5 \cdot \mathbf{I}_2$ ,  $v^d = 5$  and  $i^d = (1, 2)$ , with initial conditions  $\lambda_T(0) = 0$ ,  $\lambda_U(0) = 0 \in \mathbb{R}^2$ . According to matrices (47),  $p = 1$ ,  $q = 2$ , the node voltage reference for  $v_1$  is 5 V, and injected current references for  $i_4$ , and  $i_6$  are 1 and 2 A, respectively. Although, the speed of convergence depends on the design parameters  $\tau_i$ , for

simplicity all the enters are equals. At small values of  $\tau_i$ , the convergence is relatively fast, but there are oscillations in the transient response.

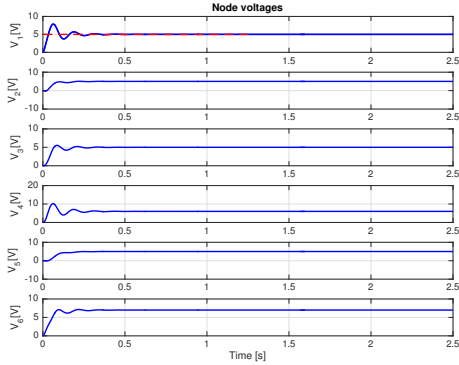


Fig. 2. Case A: Time responses of the node voltages,  $v_k(t)$  (solid blue), and the desired reference  $v_1^d$  (dash red).

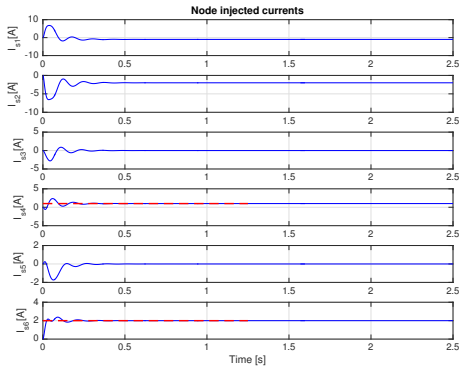


Fig. 3. Case A: Time responses of the node injected currents,  $i_k(t)$  (solid blue), and the desired references  $i_4^d$  and  $i_6^d$  (dash red).

Figures 2 and 3 show the simulation results. We can notice that the final values of the node voltage and injected currents, i.e.,  $v_1$ ,  $i_4$ , and  $i_6$ , reach the desired values in both cases.

#### 4.2 Case B: Gradient method with the node voltages $v$ and oscillating responses.

For this case, the node voltage desired reference is changed, therefore for the dynamics of (22), the following  $\mathbf{T}$  and  $\mathbf{U}$  are selected:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (48)$$

and the rest of parameters are similar to Case A except  $\tau_T = 0.5 \cdot \mathbf{I}_2$ ,  $v^d = (2, 5)$  and  $\lambda_T(0) = 0 \in \mathbb{R}^2$ . According to (48),  $p = 2$  and the constraints for the desired node voltages are  $v_1 + v_2 + v_3 = 2$  and  $v_4 + v_5 + v_6 = 5$ .

Since the selected matrices  $\mathbf{T}$ ,  $\tau_T$ ,  $\tau_U$ , (48) does not satisfy the sufficient condition of Proposition 3, it renders the node voltage responses with sustainable oscillations, see Fig. 4. However, we can notice that the final values of the node injected currents  $i_4$ , and  $i_6$  reach the desired values, see Fig. 5.

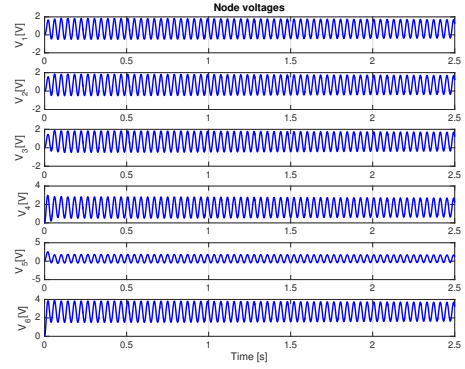


Fig. 4. Case B: Time responses of the node voltages,  $v_k(t)$  (solid blue), and the desired reference  $v_1^d = v_1 + v_2 + v_3$ , and  $v_2^d = v_4 + v_5 + v_6$  (dash red).

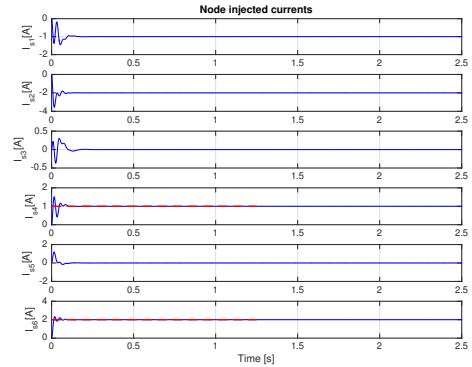


Fig. 5. Case B: Time responses of the node injected currents,  $i_k(t)$  (solid blue), and the desired references  $i_4^d$  and  $i_6^d$  (dash red)

#### 4.3 Case C: Gradient method with the difference of potentials $\psi$ .

The proposed change of coordinates in Section 3.3 is applied to improve the results of previous case with (48). From the incidence matrix  $\mathbf{B}$ , we have

$$\mathbf{B}_1^T = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_0^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \quad (49)$$

where the selected node is N6. Consequently, by condition (31), the matrix  $\mathbf{D}$  yields  $\mathbf{D} = (1 \ 0 \ 0 \ 0 \ 0 \ 0)$ .

With the same parameters and initial condition values used in Case B, the proposed change of coordinates renders the node voltage responses with asymptotic trajectories, see Fig. 6. Moreover, the node voltage constraints,  $v_1^d(t) = v_1 + v_2 + v_3$  and  $v_2^d(t) = v_4 + v_5 + v_6$  are satisfied, see the comparison between Case B and C in Fig. 7. In the meantime, the final values of the node injected currents  $i_4$ , and  $i_6$  reach the desired values, see Fig. 8.



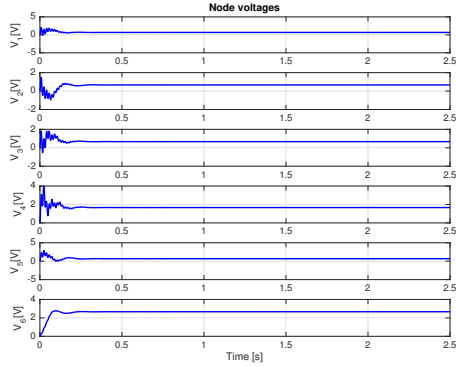


Fig. 6. Case C: Time responses of the node voltages,  $v_k(t)$ .

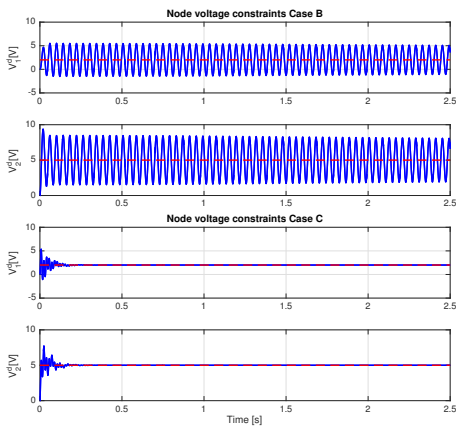


Fig. 7. Cases B and C: Time responses of the node voltage constraints,  $v_1^d(t) = v_1 + v_2 + v_3$  and  $v_2^d(t) = v_4 + v_5 + v_6$  (solid blue) and the desired reference values  $v^d = (2, 5)$  (dash red).

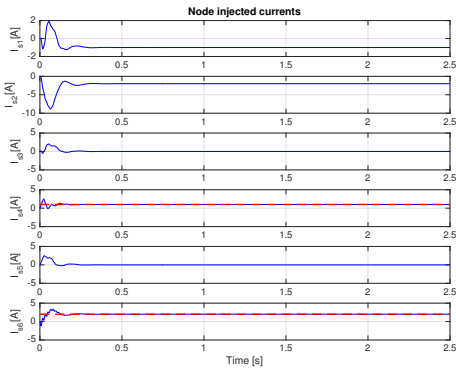


Fig. 8. Case C: Time responses of the node injected currents,  $i_k(t)$  (solid blue), and the desired references  $i_4^d$  and  $i_6^d$  (dash red)

## 5. CONCLUSIONS

The OPF problem for a DC network has been written using the port-Hamiltonian formalism. The main feature of this description is the ability of interconnecting dynamics preserving the stability properties.

In this paper has been show that the gradient method applied to the OPF problem for minimizing losses in DC networks is stable under a certain condition on the used matrices. Additionally, the paper provides a change of coordinates that modifies the problem statement to avoid stability problems when such condition fails. The change of coordinates results in writing the problem in terms of the electric potential difference with respect to one node.

Future works include: i) to interconnect the OPF to a DC network with dynamics using the port-Hamiltonian description and, ii) to extend the study to a more realistic problems that involve inequality constraints and nonlinearities such as power limits.

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