COMPUTATIONAL HOMOGENIZATION OF ONE-DIMENSIONAL TEXTILE STRUCTURES

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Abstract. In this contribution a computational homogenization method for one-dimensional technical textiles, i.e. ropes or cables, is developed. Regarding the large length-to-thickness ratio that characterizes rope-like textiles on the macro level, a discretization with beam elements is numerically efficient. The homogeneous macro level is influenced by the heterogeneous micro structure involving contact between the fibers. Therefore, the fibers are modeled explicitly a representative volume element to capture the contact within interactions. For the connection of the two scales a specific homogenization method is introduced involving theoretical aspects and numerical examples for periodic rope-like structures.

1 INTRODUCTION

To determine the behavior of one-dimensional textiles, i.e. ropes or cables, a specific computational homogenization method is presented. Thereby, ropes are arranged of several fibers mainly in length direction oriented. For the macroscopic behavior, beside the fiber material, the arrangement of the fibers plays a significant role. As the macroscopic constitutive behavior is a priori unknown, it is obtained from the micro structure by a two scale approach.

Due to the scale separation the heterogeneities are assigned to the micro level and the macro level is considered to be homogeneous. To transfer information across the length scales suitable scale transitions have to be applied [1]. Here, a beam specific homogenization scheme is introduced. The homogenization scheme couples the micro and the macro level directly by requiring the equality of the internal macroscopic and microscopic power densities known as the Hill-Mandel condition. Thereby, the micro level is modeled by an appropriate representative volume element (RVE) consisting of fibers to capture
the contact properly. For the macro to micro scale transition the macro deformation is applied to the RVE. From the resulting microscopic stress field the macroscopic stress resultants are computed and constitutive relations can be derived. As beam elements are numerically more efficient to simulate rope-like structures, they are applied on the macro scale [5, 6, 7].

For several rope-like structures this scheme was applied in [8]. In [9, 10] a related approach for the application of two-dimensional shell structures is given. A computational homogenization scheme for macroscopic Bernoulli beams without contact on the micro scale can be found in [11]. An asymptotic homogenization scheme for beams consisting of parallel, laminated fibers neglecting contact is introduced in [12]. Another approach in [13] investigates sandwich structures that are modeled as continua on the macro level and are composed of one- and two-dimensional structural elements on the micro level.

This contribution is organized as follows: As the macro level is considered to be a homogeneous beam, beam kinematics are introduced and work-conjugate stress quantities are defined. For the heterogeneous, volumetric micro structure the boundary value problem is characterized for the RVE. A beam specific Hill-Mandel condition is developed based on the equality of the macroscopic and microscopic averaged power density. Both aspects of the macro to micro and the micro to macro scale transition are discussed. Selected numerical examples of a cable rope, i.e. a Warrington structure, are given and discussed.

2 MACROSCOPIC PROBLEM

2.1 Beam kinematics

A body $B$ is a collection of material points $\mathcal{P}$ with $X$ denoting the positions of $\mathcal{P}$ in the material configuration $B_0$ at time $t_0$ in the three-dimensional Euclidean vector space $\mathbb{E}^3$. The deformation map to the spatial configuration $B_t$ is defined by the nonlinear deformation map

$$x = \varphi(X, t),$$

where $x$ indicates the position of the material point $\mathcal{P}$ at the time $t$.

The geometry and deformation of a beam is described by curvilinear convective coordinates $\theta^i$. In the sequel Latin indices range from 1 to 3 and Greek indices range from 2 to 3. The middle axis of the beam $L_\tau$ with $\tau \in [0, t]$ is defined by $\theta^a = 0$. The material position and the nonlinear deformation map of a finitely deforming beam are then specified as

$$X(\theta^i) = X_L(\theta^1) + \theta^a D_\alpha(\theta^1) \quad \text{and} \quad \varphi(\theta^i) = \varphi_L(\theta^1) + \theta^a d_\alpha(\theta^1).$$

The material and spatial vectors $X_L$ and $\varphi_L$ provide a parametric representation of the middle axis of the beam in the material and the spatial configuration. $D_\alpha$ and $d_\alpha$ denote the corresponding directors. The parameters $\theta^a \in [-\frac{1}{2}H_{0a}, \frac{1}{2}H_{0a}]$ determine the position of a point normal to the middle axis in the material configuration. Hence, the possibly
extensible directors \( d_\alpha \) capture changes in the cross sectional direction. The covariant basis vectors of the beam space are

\[
G_1 = \frac{\partial X}{\partial \theta^1} = \frac{\partial X}{\partial \theta^1} + \theta^\alpha \frac{\partial D_\alpha}{\partial \theta^1} = A_1 + \theta^\alpha D_{\alpha,1} , \quad G_\alpha = \frac{\partial X}{\partial \theta^\alpha} = D_\alpha ,
\]

\[
g_1 = \frac{\partial \varphi}{\partial \theta^1} = \frac{\partial \varphi}{\partial \theta^1} + \theta^\alpha \frac{\partial d_\alpha}{\partial \theta^1} = a_1 + \theta^\alpha d_{\alpha,1} , \quad g_\alpha = \frac{\partial \varphi}{\partial \theta^\alpha} = d_\alpha ,
\]

where the covariant basis vectors on the middle axis \( A_1 \) and \( a_1 \) denote the derivatives of the position with respect to the curvilinear coordinate \( \theta^1 \) in the material and the spatial configuration. The corresponding contravariant basis vectors result from the relations \( G_i \cdot G_j = \delta_{ij} \) and \( g_i \cdot g_j = \delta_{ij} \) with the Kronecker delta \( \delta_{ij} \). The covariant basis vectors are tangents to the coordinate lines and the contravariant basis vectors are normal to the coordinate surfaces. Finally, the deformation gradient \( F \) is defined as the material gradient of \( \varphi \) and results with the covariant basis vectors of the beam in equation (3) in

\[
F = \frac{\partial \varphi}{\partial X} = g_1 \otimes G^1 = [a_1 + \theta^\alpha d_{\alpha,1}] \otimes G^1 + d_\alpha \otimes G^\alpha .
\]

The deformation gradient can be partitioned by the power of the thickness coordinates as

\[
F = F^0 + \theta^\alpha F^1 \quad \text{with} \quad F^0 = a_1 \otimes G^1 + d_\alpha \otimes G^\alpha , \quad F^1 = d_{\alpha,1} \otimes G^1.
\]

### 2.2 Internal power expressed in terms of stress resultants

The internal virtual power \( U^{INT} \) is captured as the volume integral of the internal virtual power density \( \psi^{INT} \) per unit volume in \( B_0 \) and can be specified in work conjugate quantities e.g. the Piola stress \( P \) and the variation of the deformation gradient \( F \) as

\[
U^{INT} = \int_{B_0} \psi^{INT} dV = \int_{B_0} P : \delta F dV .
\]

For a beam, the volume integral is transferred to the parameter space of the curvilinear coordinates

\[
U^{INT} = \iiint_{\theta^1 \theta^2 \theta^3} \sqrt{G} P : \delta F d\theta^3 d\theta^2 d\theta^1 ,
\]

where the material volume element \( dV \) of \( B_0 \) is defined as \( dV = \sqrt{G} d\theta^1 d\theta^2 d\theta^3 \) with \( \sqrt{G} = [G_1 \times G_2] \cdot G_3 \). An insertion of the beam specific deformation gradient and a pre-integration of the resultung tractions \( t^i := \sqrt{G} P \cdot G^i \) over the cross section \( A \) in the parameter space leads to an integral over the middle axis

\[
U^{INT} = \int_{\theta^1} [\delta a_1 \cdot N^1 + \delta d_{\alpha,1} \cdot M^\alpha + \delta d_\alpha \cdot N^\alpha] d\theta^1 ,
\]
with the stress resultants for forces and moments, respectively

\[ N^i := \int \int_{\theta^2 \theta^3} t^i d\theta^3 d\theta^2 \quad \text{and} \quad M^\alpha := \int \int_{\theta^2 \theta^3} \theta^\alpha t^i d\theta^3 d\theta^2. \]  

(9)

The virtual power density \( \wp^{\text{INT}} \) per unit length in \( \theta^1 \) parameter space reads

\[ \wp^{\text{INT}} := \int \int_{\theta^2 \theta^3} \sqrt{G} \wp^{\text{INT}} d\theta^3 d\theta^2. \]  

(10)

3 THE REPRESENTATIVE VOLUME ELEMENT

A requirement for the applicability of a homogenization scheme is the separation of length scales, i.e. the characteristic macroscopic length is much larger than the microscopic length. If beams are considered this condition is relaxed for the cross section. The macroscopic and microscopic length is subjected to the separation of length scales but the cross sections \( A_\tau \) and \( \bar{A}_\tau \) are considered to be equal.

The micro level is represented by a suitable RVE that characterizes the constitutive behavior of the micro structure, see Figure 1. Microscopic variables are denote by an overbar. The RVE of a textile consists of different parts, a material part \( \bar{B}_0 \) and a material free part \( \bar{G}_0 \) such that \( \bar{V}_0 = \bar{B}_0 \cup \bar{G}_0 \), with the total RVE volume \( \bar{V}_0 \). The middle axis \( \bar{L}_0 \) is defined normal to the cross section in the material configuration. A microscopic RVE is related to each cross section, i.e. to each point along the middle axis on the macro level.

The boundary \( \partial \bar{B}_0 \) of the material part of the RVE can be split into three non-overlapping sections \( \partial \bar{B}_{0,N}, \partial \bar{B}_{0,D} \) and \( \partial \bar{B}_{0,C} \) with \( \partial \bar{B}_{0,N} \cup \partial \bar{B}_{0,D} \cup \partial \bar{B}_{0,C} = \partial \bar{B}_0 \) and \( \partial \bar{B}_{0,N} \cap \partial \bar{B}_{0,D} \cap \partial \bar{B}_{0,C} = \emptyset \). The RVE is subjected to Neumann conditions on \( \partial \bar{B}_{0,N} \), to Dirichlet conditions on \( \partial \bar{B}_{0,D} \) and to unilateral contact conditions with sticking and Coulomb friction between the fibers on \( \partial \bar{B}_{0,C} \). The Neumann boundary is composed of the micro fibers’ external surfaces, where zero traction conditions are applied to model plain stress in the thickness direction. The cut surfaces at the end of the middle axis compose the Dirichlet boundary, which is used to impose the macro to micro scale transition via boundary conditions.

The heterogeneous material within the RVE is considered as a first order continuum, with the microscopic deformation gradient

\[ \bar{F} = \frac{\partial \bar{\varphi}}{\partial \bar{X}}. \]  

(11)

To obtain the homogenized stress resultants, the micro problem has to be solved first. Therefore, a quasi static continuum mechanics framework is considered. For the RVE the equilibrium condition in the absence of volume forces takes the format

\[ \text{Div}\, \bar{P} = 0. \]  

(12)
Figure 1: Determination of a unit cell in the material configuration. The fiber structured RVE is related to a cross section on the homogeneous macro level at a point along the middle axis.

The material behavior of each fiber in the micro structure is described by an isotropic hyper-elastic St. Vernant material and the stress strain relation is therefore given by

\[
\bar{S} = \lambda \text{tr} \bar{E} + 2\mu \bar{E},
\]

where \( \bar{S} \) is the Piola-Kirchhoff stress, \( \bar{E} \) is the Green-Lagrange strain and \( \lambda \) and \( \mu \) are the Lamé constants.

Between the fibers contact interactions occur. For the unilateral contact problem a kinematic impenetrability condition is formulated with the gap function \( g \geq 0 \) between two fibers i.e. if \( g > 0 \) no contact appears and for \( g = 0 \) contact interactions appear. The contact tractions \( \bar{t}_c \) can be classified into tractions \( \bar{t}_n \) normal to the contact plane and tangential tractions \( \bar{t}_t \) in the contact plane by the relation

\[
\bar{t}_c = \bar{t}_n n + \bar{t}_t,
\]

with the contact normal vector \( n \). The normal contact is unilateral and therefore leads to the inequality constraint

\[
\bar{t}_n \geq 0.
\]

In the case of frictional contact tractions tangential to the contact plane \( \bar{t}_t \) appear. If the tangential tractions \( \bar{t}_t \) exceed a certain threshold, an irreversible relative motion of the contact partners follows. An isotropic contact law for the tangential tractions is considered with the sticking condition

\[
\Phi := ||\bar{t}_t|| - t_{max} \leq 0,
\]
that determines the maximum value \( t_{\text{max}} = \mu_s \bar{t}_n \) of the tangential tractions in relation to the current normal tractions, where \( \mu_s \) is the sticking coefficient. If \( \Phi = 0 \), irreversible sliding between the fibers occurs. In this case the relation between normal and tangential stress is defined by Coulomb’s law of friction

\[
||\bar{t}_t|| = \mu_f \bar{t}_n ,
\]

with the friction coefficient \( \mu_f \leq \mu_s \) and the sliding resistance \( \mu_f \bar{t}_n \). The relative slip rate \([v_t]\) is then given by the evolution law

\[
[v_t] = \dot{\gamma} \frac{\bar{t}_t}{||\bar{t}_t||} \Leftrightarrow \bar{t}_t = \mu_f \bar{t}_n \frac{[v_t]}{||[v_t]||}
\]

proportional to the slip parameter \( \dot{\gamma} \) and in direction of the tangential traction [15].

4 HOMOGENIZATION FOR BEAMS

The geometry \( \bar{X}(\bar{\theta}^i; \theta^1) \) of the beam RVE that captures the micro structure at the macroscopic position \( \theta^1 \) of the middle axis \( \bar{X}_L(\theta^1) \) is characterized by

\[
\bar{X}(\bar{\theta}^i; \theta^1) = \bar{X}_L(\theta^1) + \bar{\theta}^i A_i(\theta^1) ,
\]

which means that the RVE is spanned by the macroscopic covariant basis vectors \( A_i \) of the middle axis in terms of rectilinear coordinates \( \bar{\theta}^i \) whereby the cross sectional coordinates coincide for the micro and macro scale \( \theta^\alpha = \bar{\theta}^\alpha \). Therewith, the microscopic covariant basis vectors and the contravariant basis vectors render

\[
\bar{G}_i = \frac{\partial \bar{X}}{\partial \bar{\theta}^i} = A_i \quad \text{and} \quad \bar{G}^i = \frac{\partial \bar{\theta}^i}{\partial \bar{X}} = A^i .
\]

The position \( \bar{\varphi}(\bar{\theta}^i; \theta^1) \) of a point in the spatial configuration on the micro scale is given by an affine part and a fluctuation

\[
\bar{\varphi}(\bar{\theta}^i; \theta^1) = \bar{F} \cdot \theta(\theta^1) = 0) \cdot \bar{X}(\bar{\theta}^i; \theta^1) + \bar{w}(\bar{\theta}^i) ,
\]

where \( \bar{F}(\theta^1; \theta^\alpha = 0) \) is the macroscopic deformation gradient evaluated on the middle axis position \( \bar{X}_L(\theta^1) \) and \( \bar{w}(\bar{\theta}^i) \) denotes the microscopic fluctuation field. By applying the gradient with respect to the material position \( \nabla_{\bar{X}} \) the microscopic deformation gradient \( \bar{F} \) is given by

\[
\bar{F} = \bar{F} + \nabla_{\bar{X}} \bar{w} .
\]

The fluctuation field \( \bar{w} \) is expressed in a beam specific form in analogy to (2) as

\[
\bar{w} = \bar{w}_L(\bar{\theta}^i) + \bar{\theta}^\alpha d^\alpha(\bar{\theta}^1) ,
\]
with the fluctuation of the microscopic middle axis \( w_L(\vec{\theta}^1) \) and the microscopic director fluctuation \( d_\theta^w(\vec{\theta}^1) \). Thus, the material gradient of the fluctuation results as
\[
\nabla \bar{X} w = [w_L,1 + \bar{\theta}^\alpha \bar{d}_\alpha^w] \otimes A^1 + d_\alpha^w \otimes A^\alpha .
\] (24)

For the homogenization scheme a power averaging theorem is introduced, which is a special format of the known Hill-Mandel condition. A basic concept of the scale transition is the equality of the macroscopic and the averaged microscopic internal power density. The macroscopic internal power has to be equal to the microscopic internal power averaged over the middle axis \( \tilde{L}_0 \) of the RVE
\[
\bar{q}^{INT}_L = \frac{1}{\tilde{L}_0} \int_{\tilde{B}_0} \bar{q}^{INT} dV ,
\] (25)

where \( \tilde{L}_0 \) is the length measure of the middle axis in the parameter space \( \vec{\theta}^1 \in [0, \tilde{L}_0] \).

A characteristic of the beam specific homogenization scheme is that the microscopic representative volume element is related to a flat cross section area that is normal to the middle axis. This is a result of the beam assumption and the pre-integration over the cross section of the RVE to refer all quantities to the middle axis. The left hand side of the Hill-Mandel condition is replaced by the integrand of equation (8) and the right hand side is expressed with work conjugated quantities like the microscopic deformation gradient and the microscopic Piola stress
\[
\delta a_1 \cdot N^1 + \delta d_{\alpha,1} \cdot M^\alpha + \delta d_\alpha \cdot N^\alpha = \frac{1}{\tilde{L}_0} \int_{\tilde{B}_0} \bar{P} : \delta \bar{F} dV .
\] (26)

The microscopic internal power can be expressed in terms of micro stress resultants
\[
\bar{N}^i := \int \int \sqrt{\tilde{A}} \bar{P} \cdot A^i d\bar{\theta}^3 d\bar{\theta}^2 ,
\]
\[
\bar{M}^\alpha := \int \int \sqrt{\tilde{A}} \bar{P} \cdot A^1 d\bar{\theta}^3 d\bar{\theta}^2 ,
\] (27)

with \( \sqrt{\tilde{A}} = [A_1 \times A_2] \cdot A_3 = 1 \). Since the micro deformation gradient consists of the macroscopic deformation gradient and the gradient of the fluctuations, two terms result if the variation of the microscopic deformation gradient (22), using equations (4) and (24) and (27) are introduced into (26). Thus the Hill-Mandel condition reads
\[
\delta a_1 \cdot N^1 + \delta d_{\alpha,1} \cdot M^\alpha + \delta d_\alpha \cdot N^\alpha = \frac{1}{\tilde{L}_0} \int_{\tilde{B}_1} [\delta a_1 \cdot \tilde{N}^1 + \delta d_{\alpha,1} \cdot \tilde{M}^\alpha + \delta d_\alpha \cdot \tilde{N}^\alpha] d\tilde{\theta}^1 \\
+ \frac{1}{\tilde{L}_0} \int_{\tilde{B}_1} [\delta w_L \cdot \tilde{N}^1 + \delta d_{\alpha,1} \cdot \tilde{M}^\alpha + \delta d_\alpha \cdot \tilde{N}^\alpha] d\tilde{\theta}^1 .
\] (28)
Here \( \mathbf{\ddot{N}}_i \) denotes the partial derivative with respect to the microscopic coordinate \( \dot{\theta}^1 \). The following macroscopic quantities (or stress resultants) can be identified as

\[
N_i := \frac{1}{L_0} \int \mathbf{\ddot{N}}_i d\dot{\theta}^1, \quad M^\alpha := \frac{1}{L_0} \int \mathbf{\ddot{M}}^\alpha d\dot{\theta}^1. \tag{29}
\]

By inserting equations (29) into (28) it follows that the part of the internal power related to the fluctuations averaged over the length of the RVE has to vanish

\[
0 = \int_{\dot{\theta}^1} \left[ \delta w_{L,1} \cdot \mathbf{\ddot{N}}_i + \delta d^\omega_{\alpha,1} \cdot \mathbf{\ddot{M}}^\alpha + \delta d^\omega_{\alpha} \cdot \mathbf{\ddot{N}}^\alpha \right] d\dot{\theta}^1. \tag{30}
\]

As shown in [8], the fluctuation term vanishes for the chosen periodic boundary conditions with periodic fluctuations \( \mathbf{w}^+ = \mathbf{w}^- \) and anti-periodic tractions \( \mathbf{t}^+ = -\mathbf{t}^- \) and thus the Hill-Mandel condition is satisfied.

5 NUMERICAL RESULTS

The calculations are performed with the commercial finite element code MSC.Marc\textsuperscript{©}. A Newton-Raphson equation-solving strategy is applied to solve the boundary value problems, which are discretized by 20-noded hexahedral elements with quadratic ansatz functions to capture the contact behavior in a proper way. The FE-meshes are built up in a structured way i.e. the nodes on the opposite boundaries \( \partial \mathcal{B}_D \) and \( \partial \mathcal{B}_D \) of the RVE are at equal positions to apply the periodic boundary conditions. The macroscopic deformation gradient is applied by a Fortran subroutine to eight additionally introduced dummy nodes that are positioned at the corner points of a cuboid, limiting the RVE space. All other boundary nodes move in relation to them. The set of equations that impose the relative displacements to the nodes on \( \partial \mathcal{B}_D \) are applied according to [17] via a Python script. Periodic boundary conditions only constrain rotations and prescribe the displacements on the dummy nodes directly. Thus, the system has no unique solution and shows a singularity of the order of the translatory degrees of freedom. For a unique solution a zero displacement constraint is applied to a node at the center of the RVE. Nodes in contact are detected by a node-to-surface strategy and the contact is realized in normal and tangential direction by a penalty method.

The RVE has a length of 10mm and represents Warrington structure. A Warrington strand is composed of 19 fibers. The fibers are drilled around a center fiber in two layers with 6 and 12 fibers in the same direction. The diameter of the center fiber and the fibers of the first layer is 1mm. In layer two the fiber diameter alternates between 1mm and 0.8mm. The rotation angle is 60 degrees. The RVE is composed of about 25000 elements. The Lamé constants are chosen as \( \lambda = 230 \text{ MPa} \) and \( \mu = 910 \text{ MPa} \), which is characteristic for the properties of polyamide. Between the fibers contact is applied with constant Coulomb sticking and friction coefficients of \( \mu_s = \mu_f = 0.2 \). Plane stress in thickness direction is assumed.
5.1 Tensile deformation of a drilled strand

In the first example the RVE is loaded via a tensile deformation $F_{11}^0$ in axial direction. All other deformations are set to zero, such that no shear, drilling or bending occurs.

![Von Mises stress plot](image)

**Figure 2**: Von Mises stress plot in the deformed state of a rope under tensile deformation (left) and a diagram of the normal force over tensile deformation (right).

In Figure 2 the strand is plotted in the deformed state (left). The von Mises stress shows a radial symmetry, where the maximum stress occurs at the center fiber. This results from the arrangement of the fibers. The center fiber coincides with the loading direction and shows consequently the highest tensile deformation. The outer fibers are drilled and winding up is possible when tension is applied. This results in less tension deformation and lower axial stresses. The outer fiber’s drilling causes a compression of the middle fiber perpendicular to the applied tensile loading, which results in normal contact between the outer and the inner fibers. The macroscopic normal force $N_{11}^1$ is calculated for different tensile deformations, see Figure 2 (right). It shows a slightly progressive behavior resulting from the increasing contact zones between the fibers.

5.2 Twisting deformation of a Warrington strand

In the second example a twisting deformation is considered. The RVE is loaded by a pure torsional deformation $F_{32}^1 = -F_{23}^1$ around the middle axis. All other deformations are set to zero and plane stress is assumed in the transverse direction.

In Figure 3 the strand is plotted in the deformed state with the drilling direction (left) and the von Mises stress state is depicted. By loading the RVE with a torsional deformation which coincides with the drilling direction i.e. $F_{32}^1 > 0$, the outer fibers are compressed on each other such that high contact pressures occur. A radial symmetric stress state appears, see Figure 3 (left). Further the stress resultant moment over the
deformation is captured, see Figure 3 (right), and it shows a progressive behavior. This is a result from contact between the fibers, as they are compressed on each other due to drilling.

6 CONCLUSION

For the modeling of rope-like textiles with a beam specific two scale approach beam kinematics were introduced and work-conjugate stress quantities were defined. The micro level as modeled volumetric to capture the contact interactions of the fibers. The coupling of the two scales was introduced. Selected numerical examples of a cable rope i.e. a Warrington structure were given and discussed. It was found that due to contact the effective behavior of the rope is nonlinear. These nonlinearities can be captured properly with the presented scheme and transferred to the macroscopic scale.

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