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INVARIANT ALGORITHMS OF SPATIAL CONSTRUCTIONS ELEMENTS FORMING AND CUTTING

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Summary. This report clearly demonstrates the capabilities of an advanced research area of applied mathematics, i.e., computational geometry to be applied for shaping dimensional structures. Vector-matrix models are provided to cover the composite membrane piecewise-smooth structures composed of surface elements with zero Gaussian curvature.

General algorithms are presented for finding cutting lines for cylindrically and conically-shaped elements, into which the curves contained in the developable surfaces transform. A dome-like peak-shaped structure comprising components of both cylindrical and conical shape is given as an example to present equations describing the cutting lines explicitly, which makes it possible to implement a high-precision technique for producing such a structure.

1 INTRODUCTION

Developable surfaces as shaping structural components have found wide application in construction technology practices. Because of their large variety, developable surface structures make it possible to realize a wide spectrum of original solutions in architectural design to meet various demands of artistic, aesthetic, and constructional preferences, including the possibility of creating free-form designs ranging from Pseudo-Russian style to a high-tech development. Moreover, having a straight-line directrix affords a facility for creating various dimensional structures of fanciful shapes using straight-line supporting members only.

In particular, developable surfaces are in great demand when it comes down to designing tent structures to be made of composite vinyl fabrics, because using developable surface technique makes it possible to cut out planar pieces which are then bent as required and are joined along the cutting lines.

2 METHODS FOR PROJECTING A DIRECTING CURVE IN STRUCTURAL DESIGN APPLICAITONS

One of the available -- and quite simple -- methods for shaping structural components of sheet materials as developable surfaces can be realized analytically on the basis of a procedure for either parallel or central projection of a free-form directing curve on a given plane (Fig. 1). In the former case, the component to be obtained is of a cylindrical shape, while in the latter case the component is to be of a conical shape.

For the purposes of analytical representation of components of cylindrical or conical shape, the following should be specified:

– an equation of directing curve $\vec{r}_{н} = \vec{r}_{н}(u),\ u_1 \le u \le u_2$;

– a unit vector of a normal $\vec{n}$ to the projection plane;

– a position of a random point C in the projection plane $\vec{r}_C$; and

– a unit vector $\vec{l}$ of the directing curve for a cylinder surface or of the projecting center $\vec{r}_S$ for a conical surface.

Then the required components of shaping surfaces are described with the following equations:

$$\vec{r} = v\vec{r}_{н}(u) + (1-v)\vec{r}_{п}(u),\ \ 0 \le v \le 1,\ u_1 \le u \le u_2, \qquad (1)$$

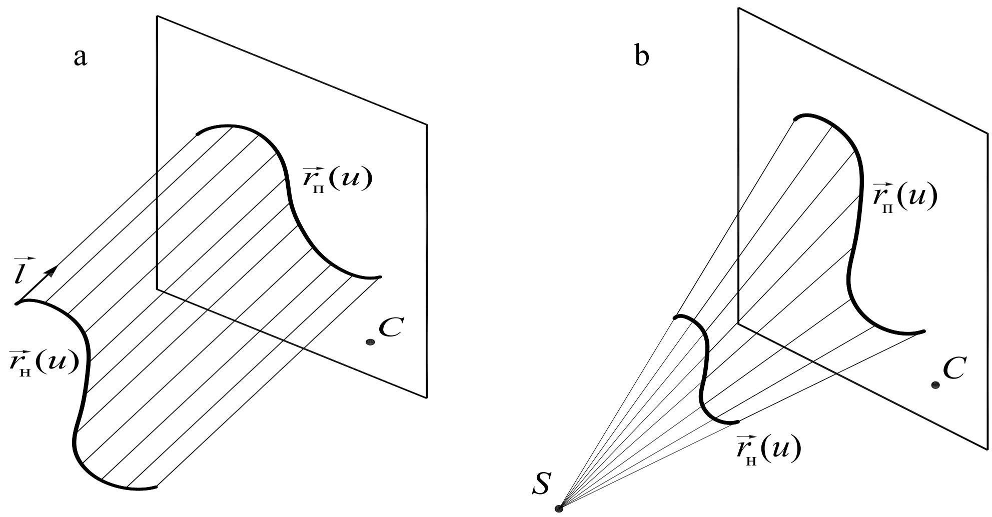

Figure 1: Projecting arrangements: a – parallel projection; b – central projection

where $\vec{r}_{п}(u)$ is a position vector of points contained in the curve projected on a given plane. For a cylindrical surface, the equation for this curve in a vector notation takes the form

$$\vec{r}_{п} = \vec{r}_{н}(u) + \frac{(\vec{r}_C - \vec{r}_{н}(u))\cdot\vec{n}}{\vec{l}\cdot\vec{n}}\vec{l}\,, \tag{2}$$

and for a conical surface

$$\vec{r}_{п} = \vec{r}_{S} + \frac{(\vec{r}_C - \vec{r}_S)\cdot\vec{n}}{(\vec{r}_{н}(u) - \vec{r}_S)\cdot\vec{n}}(\vec{r}_{н}(u) - \vec{r}_S) \tag{3}$$

3 ANALYTICAL METHODS FOR CUTTING OUT RULED COMPONENTS OF TENT AND SHEET STRUCTURES

Let us consider a general algorithm for constructing on an involute plane a cutting line into which curves contained in the cylindrical or conical surfaces transform.

Assume that a smooth curve is specified as $\vec{r}_{н} = \{x(u), y(u), z(u)\}$ $(u_1 \le u \le u_2)$ on a cylindrical surface so that one of the vectors of the coordinate base $(\vec{i}, \vec{j}, \vec{k})$ coincides with the vector $\vec{l}$ and $\dot{\vec{r}}_{н} \times \vec{l} \ne 0$ (a doted symbol is to designate a parameter derivative). For a certainty, let us assume $\vec{k} = \vec{l}$. Now let us find the equation of the curve into which the curve $\vec{r}_{н}(u)$ is transforming in the course of the cylindrical surface development. Let us introduce a Cartesian involute plane of projection (ξ, η). Then one of the coordinates for the resultant curve is determined as the length of the projection of the given directing curve on the plane perpendicular to the *generatrix* of the cylindrical surface, and the other coordinate coincides with a spatial coordinate along the z axis. It means that

$$\begin{cases} \xi = \int\limits_{u_1}^{u_2} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du, \\ \eta = z(u), \end{cases} \qquad u_1 \le u \le u_2 \,. \tag{4}$$

If the curve is determined on a conical surface, then the transformed curve to be resulted from the conical surface development can be found easier in polar coordinates:

$$\begin{cases} \rho = \rho(u), \\ \psi = \psi(u), \end{cases} \qquad u_1 \le u \le u_2 \,. \tag{5}$$

At the same time, the elemental polar angle $d\psi$ can be found as a ratio of the “reduced” elemental arc $ds' = \sqrt{ds^2 - d\,\rho^2}$ to the distance ρ from a random point on the curve to the apex $\vec{r}_S = \{x_S, y_S, z_S\}$ of the conical surface

$$d\psi = \frac{\sqrt{ds^2 - d\rho^2}}{\rho} \quad \text{or} \quad d\psi = \frac{\sqrt{(\vec{r}_н - \vec{r}_S)^2\,\dot{\vec{r}}_н^{\,2} - \left((\vec{r}_н - \vec{r}_S)\cdot\dot{\vec{r}}_н\right)^2}}{(\vec{r}_н - \vec{r}_S)^2}\,du\,. \tag{6}$$

The equations of the required curve on the projection of the conical surface development in the parametric representation take the form

$$\begin{cases} \rho = |\vec{r}_и - \vec{r}_S|, \\ \psi = \int\limits_{u_1}^{u} \frac{\sqrt{(\vec{r}_и - \vec{r}_S)^2 \left(\frac{d\vec{r}}{du}\right)^2 - \left((\vec{r}_и - \vec{r}_S)\cdot\frac{d\vec{r}_и}{du}\right)^2}}{(\vec{r}_и - \vec{r}_S)^2} du, \end{cases} \quad u_1 \le u \le u_2 . \tag{7}$$

4 CUTTING THE COMPONENTS OUT OF TENT AND SHEET STRUCTURES

Let us consider an algorithm for constructing mathematical models of surfaces and involute surfaces by example of a four-wedge peak-shaped tent model depicted in Fig. 2.

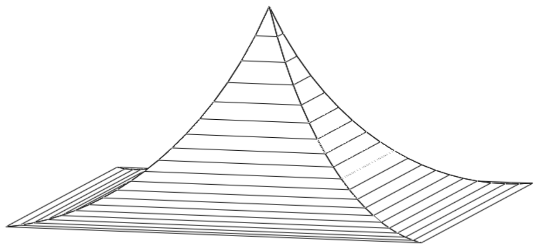

Figure 2: A four-wedge peak-shaped tent model

With the skin made of tent cloth materials, this peak-shaped tent is a tightly stretched structure supported by a polygonal base, which is stretched using a central pole. In general, such peak-shaped tent is characterized with three structural parameters: a peak height h, a circumscribed circle base radius R, and a number of base edges n. In this example, the parameters are $h = 3.5$ м, $R = 5$ м, and $n = 4$. The tent surface is made of wedge-shaped pieces, with each wedge being surface-modeled with an elliptic cylinder surface.

The first wedge is determined with an elliptic cylinder equation

$$\left(\frac{x - R\cos\alpha}{a}\right)^2 + \left(\frac{z-b}{b}\right)^2 = 1, \tag{8}$$

where a and b are the ellipse semiaxes; bounded with the planes

$$z = 0, \; y = \pm x tg\alpha , \tag{9}$$

where $\alpha = \pi/n$ is an internal half-angle of the tent sector (Fig. 3).

The ellipse halfaxes (Fig. 4) can be described via the tent's structural parameters, substituting the point A coordinates into Equation (8) at $a = kb$:

$$\left(\frac{R\cos\alpha}{kb}\right)^2 + \left(\frac{h-b}{b}\right)^2 = 1 . \tag{10}$$

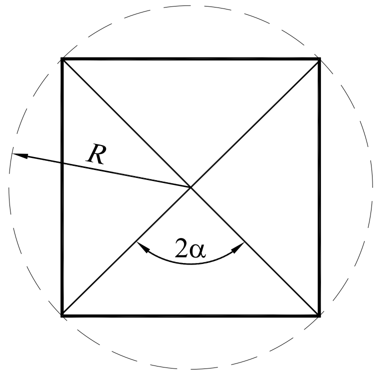

Figure 3: A pick-tent base schematic drawing

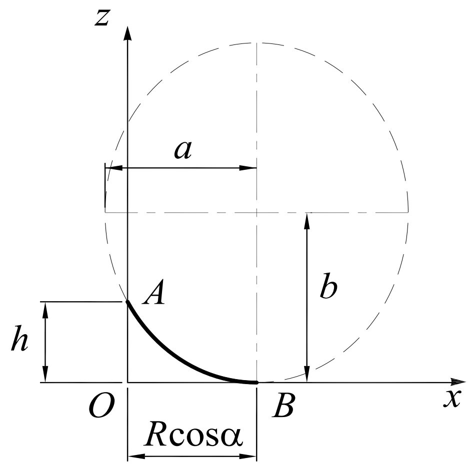

Figure 4: Elliptic cylinder section

Hence,

$$b = \frac{(R\cos\alpha)^2 + h^2k^2}{2hk^2} \text{ and } a = \frac{(R\cos\alpha)^2 + h^2k^2}{2hk}. \tag{11}$$

The mathematical model for a wedge as a shape of cylindrical surface element is described with an equation

$$\vec{r}_1 = v\vec{r}_1(t) + (1-v)\vec{r}_2(t),\ 0 \le v \le 1, \tag{12}$$

$$\vec{r}_1(t) = \begin{pmatrix} t \\ t\,\mathrm{tg}\alpha \\ b - \dfrac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2} \end{pmatrix},\quad \vec{r}_2(t) = \begin{pmatrix} t \\ -t\,\mathrm{tg}\alpha \\ b - \dfrac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2} \end{pmatrix},\quad 0 \le t \le R\cos\alpha \tag{13}$$

The functions (13) determine the directing curves of an element of the ruled surface.

Elements of wedge pieces of the peak-shaped tent are obtained by rotating the first wedge piece around the *Oz* axis by angles of $\pi/2$, π and $3\pi/2$ with the use of corresponding affine transformation matrixes.

As the peak-shaped tent is composed of the elements of cylindrical surface, the method described above can be used for cutting out the tent elements. Let us put down an equation for the directrix of elliptical cylinder in the parametric form:

$$\begin{cases} \left(\dfrac{x - R\cos\alpha}{a}\right)^2 = \sin^2\varphi, \\ \left(\dfrac{z-b}{b}\right)^2 = \cos^2\varphi; \end{cases} \quad 0 \le \varphi \le \arcsin\frac{R\cos\alpha}{a}. \tag{14}$$

For the part of ellipse depicted in Fig. 4 we find

$$\begin{cases} x = R\cos\alpha - a\sin\varphi, \\ z = b - b\cos\varphi, \end{cases} \quad 0 \le \varphi \le \arcsin\frac{R\cos\alpha}{a}. \tag{15}$$

Then the equation for the cylindrical surface directrix takes the form

$$\vec{r}_2(\varphi) = \begin{pmatrix} R\cos\alpha - a\sin\varphi \\ -(R\cos\alpha - a\sin\varphi)\mathrm{tg}\alpha \\ b - b\cos\varphi \end{pmatrix}, \ 0 \le \varphi \le \arcsin\frac{R\cos\alpha}{a}. \tag{16}$$

The curve bounding an element of the cylindrical surface determined with an equation $\vec{r}_2 = \vec{r}_2(t)$, upon the development of the cylindrical surface in the plane of $O\xi\eta$ according to Equation (4), is described with the expressions

$$\begin{cases} \xi = \int_s ds, \\ \eta = y(\varphi); \end{cases} \quad , \ ds = \sqrt{dx^2 + dz^2}. \tag{17}$$

As $dx = -a\cos\varphi d\varphi$, and $dz = b\sin\varphi d\varphi$, then

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}d\varphi = \sqrt{(a^2 - b^2)\cos^2\varphi + b^2}d\varphi. \tag{18}$$

Subject to the found expressions, we obtain the equation for a cutting line as

$$\begin{cases} \xi = \int_0^{\varphi} \sqrt{(a^2 - b^2)\cos^2\varphi + b^2}d\varphi, \\ \eta = -(R\cos\alpha - a\sin\varphi)\mathrm{tg}\alpha. \end{cases} \quad 0 \le \varphi \le \arcsin\frac{R\cos\alpha}{a}. \tag{19}$$

Similarly, one can find an equation for a cutting line, which corresponds to the tent edge $\vec{r}_1 = \vec{r}_1(t)$. A pattern of a wedge piece cut-out of the tent to be obtained using the involutes of $\vec{r}_1(t)$ and $\vec{r}_2(t)$ is displayed in Fig. 5.

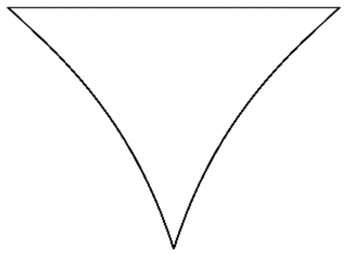

Figure 5: A pattern of a wedge piece of the tent

The considered tent model can be modified somewhat (Fig. 6) to render visual smoothness to the area of transition from one wedge to another. To this end, let us divide each wedge surface into three elements, with the middle element on each wedge surface being modeled with elliptic cylinder surface and the side elements being modeled with conical surfaces.

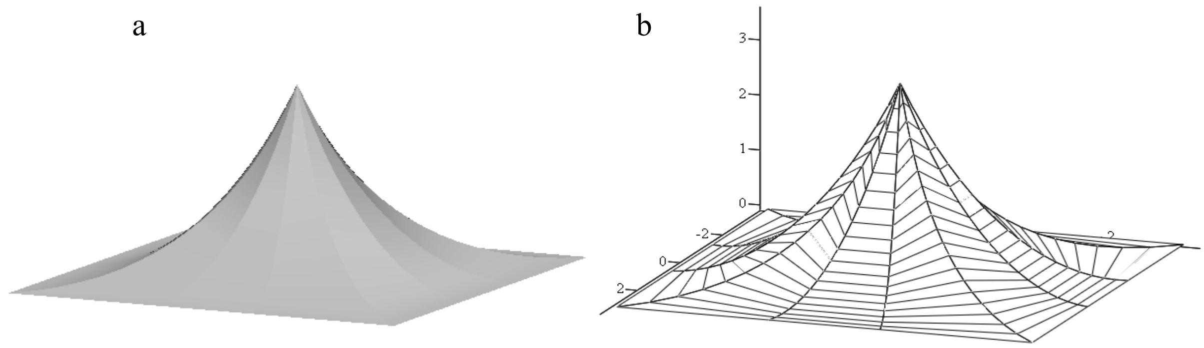

Figure 6: A peak-shaped tent model: a – a general view of the model; b – a schematic arrangement of cylindrical and conical elements

The middle element of the front surface of the tent (Fig. 7) is determined with Equation (8) of an elliptic cylinder bounded with the planes

$$z = 0, \; y = \pm x \operatorname{tg} \frac{\alpha}{3} \quad , \tag{20}$$

using the same model parameters as for the four-wedge tent considered above.

The left side element of the tent (Fig. 8) on the front wedge surface is modeled with an equation

$$\vec{r}_{II} = v\vec{r}_2(t) + (1-v)\vec{r}_3(t), \; 0 \le v \le 1, \; 0 \le t \le R\cos\alpha, \tag{21}$$

where the directing curve $\vec{r}_3(t)$ of the conical surface is determined with a function that can be obtained by Equation (2) central projecting of the directing curve $\vec{r}_2(t)$ on the plane $y = -x\text{tg}\alpha$, with the projection center being located in the point with the position vector

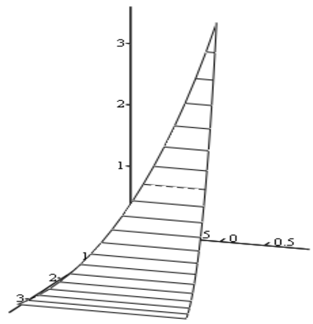

Figure 7: A model of the middle element of a wedge surface

$\vec{r}_S = \{R\cos\alpha; d; 0\}$. The function $\vec{r}_3(t)$ has the form determined by the expression

$$\vec{r}_3(t) = \vec{r}_S + \frac{(\vec{r}_0 - \vec{r}_S)\cdot\vec{n}}{(\vec{r}_2(t) - \vec{r}_S)\cdot\vec{n}}(\vec{r}_2(t) - \vec{r}_S), \tag{22}$$

where $\vec{n} = \{\sin\alpha; \cos\alpha; 0\}$ is the normal to the projection plane, $\vec{r}_0 = \{0; 0; 0\}$ is the position vector of a point contained in the projection center.

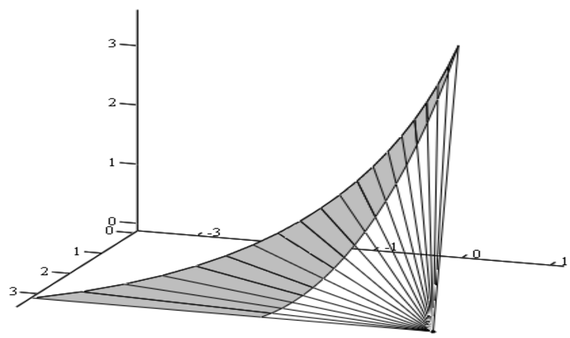

Figure 8: A model of the left-side element of the wedge surface

To obtain an equation for the curve $\vec{r}_3(t)$, let us perform the following calculations:

$$(\vec{r}_0 - \vec{r}_S)\cdot\vec{n} = -R\cos\alpha\sin\alpha - d\cos\alpha,$$

$$(\vec{r}_2(t) - \vec{r}_S) = \left\{(t - R\cos\alpha), -\left(t\,tg\,{}^{\alpha}\!/_{3} + d\right), b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right\}, \tag{23}$$

$$(\vec{r}_2(t) - \vec{r}_S)\cdot\vec{n} = (t - R\cos\alpha)\sin\alpha - \left(t\,tg\,{}^{\alpha}\!/_{3} + d\right)\cos\alpha.$$

For convenience, let us introduce the following designations:

$$f = \frac{(\vec{r}_0 - \vec{r}_S)\cdot\vec{n}}{(\vec{r}_2(t) - \vec{r}_S)\cdot\vec{n}} = \frac{-\cos\alpha(R\sin\alpha + d)}{(t - R\cos\alpha)\sin\alpha - \left(t\, tg\,{}^{\alpha}\!/_{3} + d\right)\cos\alpha} =$$
$$= \frac{-\cos\alpha(R\sin\alpha + d)}{t\left(\sin\alpha - tg\,{}^{\alpha}\!/_{3}\cos\alpha\right) - \cos\alpha(R\sin\alpha + d)} = -\frac{v}{tu - v}, \tag{24}$$
$$v = \cos\alpha(R\sin\alpha + d),\ u = \sin\alpha - tg\,{}^{\alpha}\!/_{3}\cos\alpha .$$

Considering the designations introduced above, the function determining the curve $\vec{r}_3(t)$ takes the form

$$\vec{r}_3(t) = \begin{pmatrix} R\cos\alpha + \dfrac{v}{v - tu}(t - R\cos\alpha) \\ d - \dfrac{v}{v - tu}\left(t\,\mathrm{tg}\,{}^{\alpha}\!/_{3} + d\right) \\ \dfrac{v}{v - tu}\left(b - \dfrac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right) \end{pmatrix}. \tag{25}$$

Figure 6 shows a tent surface constructed with the use of the mathematical model described above. The middle and the left-side elements of the front surface of the wedge were determined with Equation (1). The right-side element of the front wedge surface can be found using mirroring transformation of the left-side element relative to the plane *Ozx*. The elements of other wedge surfaces are obtained by rotation of the corresponding elements of the front wedge surface relative to the axis *Oz* by the angles of $\pi/2$, π and $3\pi/2$, respectively.

To cut out patterns for the tent wedges, development of the cylindrical and conical surface elements is performed according to the algorithms set forth above.

Development of the cylindrical surface is carried out similarly to what has been considered earlier for the four-wedge peak-shaped tent having wedges in the form of cylindrical surface elements. Equations for the cutting lines are put down in the following form

$$\begin{cases} \xi = \displaystyle\int_0^{\varphi} \sqrt{(a^2 - b^2)\cos^2\varphi + b^2}\, d\varphi, \\ \eta = \pm(R\cos\alpha - a\sin\varphi)\,\mathrm{tg}\,\dfrac{\alpha}{3}, \end{cases} \qquad 0 \le \varphi \le \arcsin\frac{R\cos\alpha}{a}. \tag{26}$$

To construct an evolvent for a conical element of the test, a position vector of the conical surface apex is required to be determined as well as the corresponding directrix line.

The involute of $\vec{r}_2(t)$ curve on the plane of the conical surface development can be found according to the relationships similar to Equation (7):

$$\begin{cases} \rho = \left|\vec{r}_2(t) - \vec{r}_S\right|, \\ \psi = \int_s \frac{\sqrt{(ds)^2 - (d\rho)^2}}{\rho}, \end{cases} \qquad 0 \le t \le R\cos\alpha, \tag{27}$$

where ρ is the polar radius, *ds* is an elementary arc of the developed curve, and ψ is the polar angle.

Subject to the conditions of the problem being considered, the values in Equation (7) are determined with the following expressions:

$$\rho = \sqrt{(t - R\cos\alpha)^2 + \left(t\, tg\, \alpha/3 + d\right)^2 + \left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right)^2},$$

$$d\rho = \frac{(t - R\cos\alpha) + \left(t\, tg\, \alpha/3 + d\right) tg\, \alpha/3}{\sqrt{(t - R\cos\alpha)^2 + \left(t tg\, \alpha/3 + d\right)^2 + \left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right)^2}} +$$

$$+ \frac{\left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right) \frac{b(t - R\cos\alpha)}{a\sqrt{a^2 - (t - R\cos\alpha)^2}}}{\sqrt{(t - R\cos\alpha)^2 + \left(t\, tg\, \alpha/3 + d\right)^2 + \left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right)^2}}, \tag{28}$$

$$ds = dt\sqrt{1 + \left(\mathrm{tg}\, \alpha/3\right)^2 + \frac{b^2 (t - R\cos\alpha)^2}{a^2\left(a^2 - (t - R\cos\alpha)^2\right)}}.$$

Substituting these values into Equation (27), one of the cutting lines for a tent conical surface element can be found.

To find the other cutting line, development of the curve $\vec{r}_3(t)$ needs to be carried out, which is determined with Equation (25) and is contained in the conical surface. The polar radius and its increment and the elemental arc of the curved developed, which are included in Equation (27) for the cutting line, in this case can be determined with the following expressions:

$$\rho = f\sqrt{(t - R\cos\alpha)^2 + \left(t\, tg\, \alpha/3 + d\right)^2 + \left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right)^2},$$

$$d\rho = f'\sqrt{(t - R\cos\alpha)^2 + \left(t\, tg\, \alpha/3 + d\right)^2 + \left(b - \frac{b}{a}\sqrt{a^2 - (t - R\cos\alpha)^2}\right)^2} + \tag{29}$$

$$+\frac{f\left((t-R\cos\alpha)+\left(t\,tg\,\alpha/_{3}+d\right)tg\,\alpha/_{3}\right)}{\sqrt{(t-R\cos\alpha)^2+\left(t\,tg\,\alpha/_{3}+d\right)^2+\left(b-\frac{b}{a}\sqrt{a^2-(t-R\cos\alpha)^2}\right)^2}}+$$

$$+\frac{f\left(\left(b-\frac{b}{a}\sqrt{a^2-(t-R\cos\alpha)^2}\right)\frac{b(t-R\cos\alpha)}{a\sqrt{a^2-(t-R\cos\alpha)^2}}\right)}{\sqrt{(t-R\cos\alpha)^2+\left(t\,tg\,\alpha/_{3}+d\right)^2+\left(b-\frac{b}{a}\sqrt{a^2-(t-R\cos\alpha)^2}\right)^2}},$$

$$ds=\sqrt{dx^2+dy^2+dz^2}\ ,$$

где $dx=\left(f'(t-R\cos\alpha)+f\right)dt$, $dy=\left(f'\left(t\,tg\,\alpha/_{3}+d\right)+f\ tg\,\alpha/_{3}\right)dt$,

$$dz=\left(\left(b-\frac{b}{a}\sqrt{a^2-(t-R\cos\alpha)^2}\right)f'+\frac{f\,b(t-R\cos\alpha)}{a\sqrt{a^2-(t-R\cos\alpha)^2}}\right)dt\ ,\quad f'=\frac{vu}{(tu-v)^2}=-f\,\frac{u}{tu-v}.$$

Substituting the expressions found for ρ, *ds* and *d*ρ into Equation (27), the equations for the second cutting line of a tent conical element are obtained.

The evolvents of cylindrical and conical elements are presented in Figs. 9 and 10.

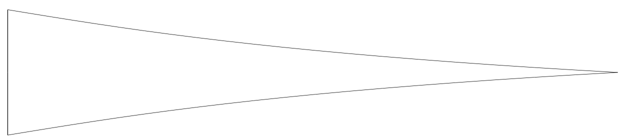

Figure 9: Involute surface of a cylindrical element of a tent

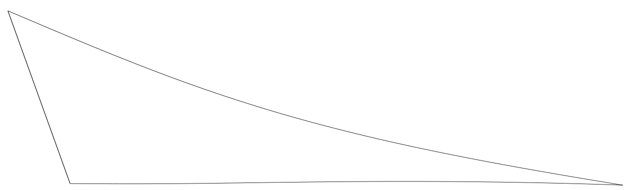

Figure 10: Involute surface of a conical element of a tent

Figures 11-13 demonstrate of the algorithm for designing the described patterns implemented in actual structures.

Figure 11: A tent structure as an awning over a boat deck on the Iset river

Figure 12: A tent structure serving as a café on Malysheva Street, Yekaterinburg

Figure 13: Tents serving as exhibition pavilions in Nyzhnii Tagil