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# A family of stacked central configurations in the planar five-body problem

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**Abstract** We study planar central configurations of the five-body problem where three bodies,  $m_1$ ,  $m_2$  and  $m_3$  are collinear and ordered from left to right, while the other two,  $m_4$  and  $m_5$  are placed symmetrically with respect to the line containing the three collinear bodies. We prove that when the collinear bodies form an Euler central configuration of the three-body problem with  $m_1 = m_3$ , there exists a new family, missed by Gidea and Llibre (Celestial Mech. Dynam. Astronom. 106: 89-107, 2010), of stacked five-body central configuration where the segments  $m_4m_5$  and  $m_1m_3$  do not intersect.

**Keywords** Planar five-body problem · Central configurations · Stacked central configurations

## 1 Introduction

A configuration of the  $n$ -body problem is called *central* if the acceleration vector of every body is proportional (common scalar) to its position vector (with respect to the center of mass). When  $n = 2, 3$ , the number of planar

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central configurations is completely known. When  $n > 3$ , there are only partial results.

The question about finiteness is an open problem. Recently, Albouy and Kaloshin (2012) proved that, for almost all choices of masses, there exists a finite number of equivalence classes of central configuration in the planar five-body problem.

Hampton (2005) provides a new family of planar central configurations in the five-body problem, called *stacked* central configuration, that is, a subset of the points also forms a central configuration. In the same paper, the following question is posed: In addition to the symmetric collinear configuration and the square with a mass at its center, are there any planar five-body central configurations with a subset forming a four-body central configuration? An answer was provided in (Fernandes and Mello 2013); however the proof is erroneous on page 302, as claimed in a footnote in (Chen and Hsiao, to appear in Trans. Amer. Math. Soc.).

In (Gidea and Llibre 2010), the authors studied the case where three bodies,  $m_1$ ,  $m_2$  and  $m_3$ , are in a symmetric Euler configuration (the two bodies located at the extremes have the same mass, that is,  $m_1 = m_3$ ), and the other two,  $m_4$  and  $m_5$  are placed symmetrically with respect to the line containing the first three bodies. They claim that there are no central configurations when the segments  $m_4m_5$  and  $m_1m_3$  do not intersect. However, there is an error in the proof on page 97 of (Gidea and Llibre 2010), where the authors assume that  $(1-s)t((s-1)^2+t^2)^{-3/2} - (1+s)t((1+s)^2+t^2)^{-3/2} + 2st(2t)^{-3}$  is equal to  $g(t, s)$ , defined on page 94 of (Gidea and Llibre 2010), which is not true.

The goal of this paper is to show the existence of the missing family in the paper (Gidea and Llibre 2010) for case (iii) in the proof of part (a) of Theorem 1.

The structure of the present paper is as follows. In section 2, we study central configurations of the planar five-body problem that possess two symmetries. We find, numerically, the set of admissible mutual distances and prove that the positive mass vector associated to each double symmetric configuration is unique **except in one case**. In section 3, assuming that the collinear bodies are in an Euler central configuration of the three-body problem, we prove, analytically, the existence of a new family of stacked central configurations in the planar five-body problem.

## 2 Double symmetric central configurations

Consider five bodies in the plane, subject to their mutual Newtonian gravitational attraction, with mass and position given by  $m_i$  and  $\mathbf{q}_i \in \mathbb{R}^2$ , respectively, for  $i = 1, \dots, 5$ . We denote by  $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$  the distance between the  $i$ th and  $j$ th bodies and by  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_5) \in \mathbb{R}^{10}$ , the position vector.

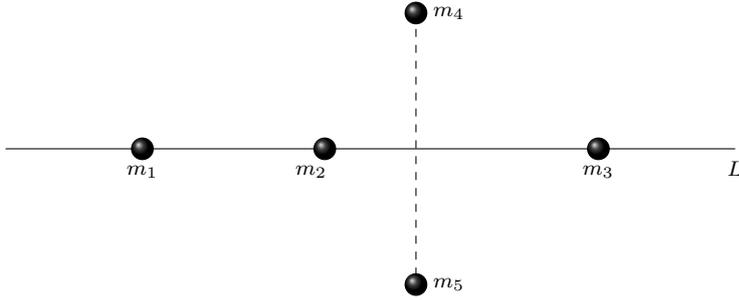
The equations for central configurations in terms of the mutual distances  $r_{ij}$ , named the Dziobek/Laura/Andoyer equations (see page 241 (Hagihara

1970)), are given by the following ten equations

$$f_{ij} = \sum_{\substack{k=1 \\ k \neq i, j}}^5 m_k (R_{ik} - R_{jk}) \Delta_{ijk} = 0, \quad (1)$$

for  $1 \leq i < j \leq 5$ . Here,  $R_{ij} = 1/r_{ij}^3$  and  $\Delta_{ijk} = (\mathbf{q}_i - \mathbf{q}_j) \wedge (\mathbf{q}_i - \mathbf{q}_k)$ . Thus,  $\Delta_{ijk}$  gives twice the signed area of the triangle with vertices  $\mathbf{q}_i$ ,  $\mathbf{q}_j$  and  $\mathbf{q}_k$ .

Next, we suppose that the configuration has an axis of symmetry containing three bodies. That is, assume that  $m_1$ ,  $m_2$  and  $m_3$ , ordered from left to right, lie on a straight line  $L$ , and the other two bodies  $m_4$  and  $m_5$  are placed symmetrically with respect to  $L$  (see Figure 1).



**Fig. 1** Symmetric configuration of the planar five-body problem.

Using this symmetry, from (1) we get that  $m_4 = m_5$ . Then, by a suitable scaling, we may assume that  $r_{12} = 1$  and  $m_4 = m_5 = 1$ . We can also consider, without loss of generality, that the line connecting  $m_4$  and  $m_5$  crosses  $L$  to the right of  $m_2$ . The case where the line connecting  $m_4$  and  $m_5$  goes through  $m_2$  was already studied by Roberts (1999). This is the 1+rhombus relative equilibria, which consists of four bodies at the vertices of a rhombus, with opposite vertices having the same mass, and a central body of arbitrary mass. Therefore, this configuration we will be excluded from our work.

Given our setup, system (1) is reduced to the following three equations:

$$\begin{aligned} f_{14} &= m_2(1 - R_{24})\Delta_{142} + m_3(R_{13} - R_{34})\Delta_{143} + (R_{14} - R_{45})\Delta_{145} = 0, \\ f_{24} &= m_1(1 - R_{14})\Delta_{241} + m_3(R_{23} - R_{34})\Delta_{243} + (R_{24} - R_{45})\Delta_{245} = 0, \\ f_{34} &= m_1(R_{13} - R_{14})\Delta_{341} + m_2(R_{23} - R_{24})\Delta_{342} + (R_{34} - R_{45})\Delta_{345} = 0. \end{aligned} \quad (2)$$

Let us introduce an additional symmetry where the three collinear bodies are also symmetrical with respect to the middle mass  $m_2$ . Then  $r_{12} = r_{23} = 1$ , and  $r_{13} = 2$ . Moreover,  $\Delta_{142} = \Delta_{243} = -\Delta_{342} = -\Delta_{241}$ , and  $\Delta_{143} = -\Delta_{341} = -2\Delta_{241}$ .

Assuming the two symmetries, every configuration  $\mathbf{q} \in \mathbb{R}^{10}$  is completely determined by two distances; namely  $c > 0$ , the distance between  $m_2$  and the

line connecting  $m_4$  and  $m_5$ , and  $d > 0$ , half the distance of the segment joining  $m_4$  and  $m_5$ . Notice that  $d = 0$  corresponds to collision between  $m_4$  and  $m_5$ . All mutual distances and the signed area of the triangles can be written in terms of  $c$  and  $d$  as follows:

$$\begin{aligned} r_{14} &= \sqrt{(1+c)^2 + d^2}, & r_{24} &= \sqrt{c^2 + d^2}, \\ r_{34} &= \sqrt{(1-c)^2 + d^2}, & r_{45} &= 2d, \\ \Delta_{145} &= -2d(1+c), & \Delta_{245} &= -2cd, & \Delta_{345} &= 2d(1-c), & \Delta_{241} &= d. \end{aligned}$$

When  $\Delta_{345} = 0$  ( $c = 1$ ), masses  $m_3$ ,  $m_4$  and  $m_5$  are also collinear, but such configuration violates the Perpendicular Bisector Theorem, see (Moeckel 1990). Thus,  $c \in (0, 1) \cup (1, \infty)$ .

## 2.1 The positive mass region

Let  $\mathcal{M}_i$  be the set of points in the  $(c, d)$ -plane for which  $m_i > 0$  for  $i = 1, 2, 3$ . The positive mass region is the union of the two disjoint sets:  $\mathcal{N}_1 = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3$  when  $0 < c < 1$  and  $\mathcal{N}_2 = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3$  when  $c > 1$ . Let us define the operator  $\overline{\Delta}$  by  $\overline{\Delta}(c, d) = \Delta(-c, d)$ . From (2) we can express the masses  $m_1$ ,  $m_2$  and  $m_3$  in terms of  $c$  and  $d$  as follows:

$$\begin{aligned} m_1 &= \frac{m_3 \overline{\gamma} + \alpha(\varepsilon - \overline{\varepsilon})}{\gamma}, \\ m_2 &= \frac{-2m_3 \overline{\beta} - 2\delta\varepsilon}{\gamma + \delta - \alpha}, \\ m_3 &= \frac{(\overline{\varepsilon} - \varepsilon)\alpha\beta + \gamma(\delta\varepsilon - \overline{\delta}\overline{\varepsilon})}{-\frac{7}{8}(\overline{\gamma} - \gamma)}, \end{aligned} \tag{3}$$

where

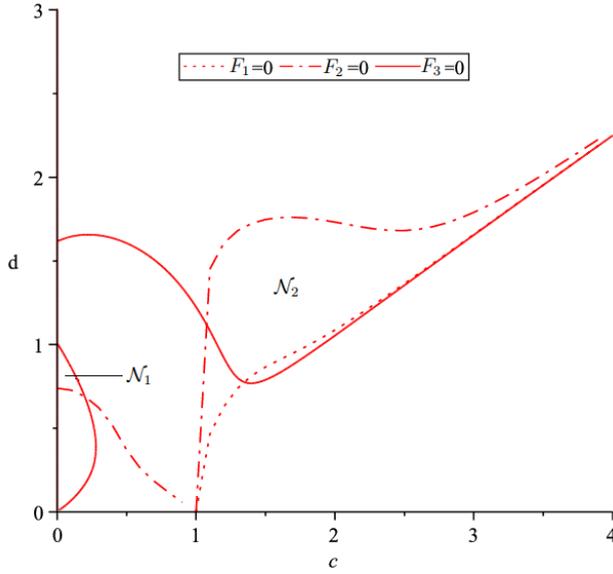
$$\begin{aligned} \alpha &= \frac{1}{(c^2 + d^2)^{\frac{3}{2}}} - \frac{1}{8d^3}, & \beta &= \frac{1}{8} - \frac{1}{((1+c)^2 + d^2)^{\frac{3}{2}}}, \\ \gamma &= 1 - \frac{1}{((1+c)^2 + d^2)^{\frac{3}{2}}}, & \delta &= \frac{1}{((1+c)^2 + d^2)^{\frac{3}{2}}} - \frac{1}{8d^3}, \end{aligned}$$

and  $\varepsilon = 1 + c$ .

Let  $F_i$ ,  $i = 1, 2, 3$  be the numerator of  $m_i$ ,  $i = 1, 2, 3$ , respectively. The boundary of  $\mathcal{N}_1 \cup \mathcal{N}_2$  is given by the following equations

$$\begin{aligned} F_1 &= (\overline{\varepsilon} - \varepsilon)\alpha\overline{\beta} + \overline{\gamma}(\delta\varepsilon - \overline{\delta}\overline{\varepsilon}) = 0, \\ F_2 &= (\varepsilon - \overline{\varepsilon})\alpha\beta\overline{\beta} + \overline{\delta}\overline{\gamma}\overline{\beta}\overline{\varepsilon} - \delta\overline{\gamma}\beta\varepsilon = 0, \\ F_3 &= (\overline{\varepsilon} - \varepsilon)\alpha\beta + \gamma(\delta\varepsilon - \overline{\delta}\overline{\varepsilon}) = 0. \end{aligned}$$

In Figure 2, we show our numerical evidence of the fact that the sets  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are non-empty.



**Fig. 2** Regions  $\mathcal{N}_1$  and  $\mathcal{N}_2$  in the  $(c, d)$ -plane, with their boundaries  $F_1 = F_2 = F_3 = 0$ .

Let  $\mu$  be the normalized mass mapping  $\mu : \mathcal{N}_1 \cup \mathcal{N}_2 \subseteq (\mathbb{R}^2)^+ \rightarrow (\mathbb{R}^5)^+ \setminus (0, 0, 0, 0, 0)$  given by

$$\mu(c, d) = (m_1(c, d), m_2(c, d), m_3(c, d), m_4 = 1, m_5 = 1),$$

where  $m_i(c, d)$ ,  $i = 1, 2, 3$  are defined in (3).

The next proposition states that each point  $(c, d) \in \mathcal{N}_1 \cup \mathcal{N}_2$  determines a unique mass vector  $\mu$ .

**Proposition 1** *If  $\mathbf{q}$  is a symmetric configuration of the planar five-body problem, where the axis of symmetry contains the bodies  $m_1$ ,  $m_2$  and  $m_3$ , ordered from left to the right and  $r_{12} = r_{23}$ , then the corresponding positive normalized mass vector  $(m_1, m_2, m_3, m_4 = 1, m_5 = 1)$  is unique, as long as the collinear configuration of masses  $m_4$ ,  $m_2$  and  $m_5$  is excluded.*

*Proof* Equations (2) can be reduced to a non-homogeneous linear system  $Bm = b$  where

$$B = \begin{pmatrix} 0 & (R_{12} - R_{24})\Delta_{142} & (R_{13} - R_{34})\Delta_{143} \\ (R_{12} - R_{14})\Delta_{241} & 0 & (R_{23} - R_{34})\Delta_{243} \\ (R_{13} - R_{14})\Delta_{341} & (R_{23} - R_{24})\Delta_{342} & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} (-R_{14} + R_{45})\Delta_{145} \\ (-R_{24} + R_{45})\Delta_{245} \\ (-R_{34} + R_{45})\Delta_{345} \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

Given the mutual distances, the existence and uniqueness of  $m_1$ ,  $m_2$  and  $m_3$  (positive or not) depends on the non-vanishing determinant of the matrix  $B$ ,

$$\det(B) = [(R_{13} - R_{14})(R_{12} - R_{24})(R_{23} - R_{34}) - (R_{12} - R_{14})(R_{23} - R_{24})(R_{13} - R_{34})] \Delta_{341} \Delta_{142} \Delta_{243}. \quad (4)$$

When  $r_{12} = r_{23}$  expression (4) factors nicely into

$$\det(B) = (R_{12} - R_{24})(R_{34} - R_{14})(R_{12} - R_{13}) \Delta_{341} \Delta_{142} \Delta_{243}.$$

Since the three signed areas  $\Delta_{341}$ ,  $\Delta_{142}$  and  $\Delta_{243}$  are different from zero, and since  $r_{12} = 1 \neq 2 = r_{13}$  and  $r_{14} \neq r_{34}$  ( $c > 0$ ), the positive normalized mass vector  $(m_1, m_2, m_3, m_4 = 1, m_5 = 1)$  is unique as long as  $r_{12} \neq r_{24}$ .

When  $r_{12} = r_{24}$ , the configuration lies on a circle with  $m_2$  at its center. From Corollary 3 in (Alvarez-Ramírez et. al. 2013), it follows that the only possible central configuration with the four bodies sharing a common circle is a square with equal masses at its vertices. The square corresponds to  $c = 0$  and  $d = 1$ , where line  $m_4 m_5$  passing through  $m_2$ . Hence, the proposition follows.

We remark that  $c = 0$  and  $d \neq 1$  corresponds to 1+rhombus configuration, which was also studied by Gidea and Llibre (2010). However, the authors stated in (a) of Theorem 1 that the masses are uniquely determined for each configuration belonging to the family. Their assertion is not true as was stated by Roberts on page 144 in (Roberts 1999), where it is proved that fixing the size of the rhombus yields a one-parameter family of relative equilibria for which the masses  $m_1$  and  $m_2$  change linearly with respect to each other.

### 3 Stacked central configurations: Euler plus two

Assume that the masses  $m_1$ ,  $m_2$  and  $m_3$  are in an Euler central configuration of the three-body problem. In our case, this forces  $m_1 = m_3$ . Then, equations (3) become

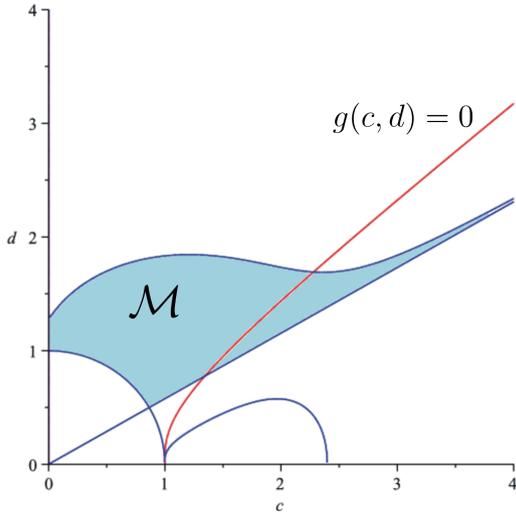
$$m_1 = \frac{(\bar{\varepsilon} - \varepsilon)\alpha}{\bar{\gamma} - \gamma}, \quad (5a)$$

$$m_2 = \frac{-2m_1\bar{\beta} - 2\delta\varepsilon}{\gamma + \delta - \alpha}, \quad (5b)$$

$$0 = \frac{(1+c)d}{((1+c)^2 + d^2)^{3/2}} - \frac{(1-c)d}{((c-1)^2 + d^2)^{3/2}} - \frac{2cd}{(c^2 + d^2)^{3/2}}. \quad (5c)$$

We remark that equation (5c) is the same that was considered by Gidea and Llibre (2010) for the case (iii) in the proof of part (a) of Theorem 1.

We define the function  $g = g(c, d)$  equal to the right hand side of equation (5c). Hence a family of stacked central configurations will be given by points along the curve  $g(c, d) = 0$ , excluding the curves  $g(c, 0) = 0$  and  $g(0, d) = 0$ ,



**Fig. 3** The curve  $g(c, d) = 0$  and the region  $\mathcal{M}$ .

with  $(c, d) \in \mathcal{M}_1 \cap \mathcal{M}_2 = \mathcal{M}$ . In Figure 3 we provide numerical evidence that curve  $g(c, d) = 0$  has a nonempty intersection with  $\mathcal{M}$ . The rest of the paper is devoted to proving analytically their existence.

From (5a),  $m_1 = 0$  if and only if  $(\bar{\varepsilon} - \varepsilon)\alpha = 0$ ; but  $\bar{\varepsilon} - \varepsilon \neq 0$ , so  $\alpha = 0$ . This implies that  $d = \frac{c}{\sqrt{3}}$ . Moreover,  $(\bar{\varepsilon} - \varepsilon) < 0$  and  $\bar{\gamma} - \gamma < 0$ ; thus, when  $\alpha > 0$  we have that  $m_1 > 0$ . It follows that  $\mathcal{M}_1 = \{(c, d) \mid d > \frac{c}{\sqrt{3}}\}$ .

To study the intersection between the curves  $m_1 = 0$  and  $g(c, d) = 0$ , we define  $\chi(c) = \frac{\sqrt{3}}{c} g\left(c, \frac{c}{\sqrt{3}}\right)$ . It is easy to see that  $\chi(c)$  is well defined in the interval  $[1, 7]$ . Moreover,  $\chi(1) < 0$  and  $\chi(7) > 0$ . Thus, the curves  $m_1 = 0$  and  $g(c, d) = 0$  intersect at least at one point  $(c_0, \frac{c_0}{\sqrt{3}})$  with  $1 < c_0 < 7$ . Also, due to the change of sign of  $\chi(c)$ , the curve  $g(c, d) = 0$  intersects the region  $\mathcal{M}_1$ .

Finally, we have to check that  $m_2 > 0$  at least, at one of the points belonging to the intersection of  $m_1 = 0$  and  $g(c, d) = 0$ . When  $m_1 = 0$ , equation (5b) becomes

$$m_2 = \frac{-2\delta\varepsilon}{\gamma + \delta} = -\frac{16\sqrt{3}(1+c)c^3 \left( \frac{2c}{\sqrt{3}} - \sqrt{1+2c+\frac{4}{3}c^2} \right)}{2c(8c^3 - 3^{3/2})\sqrt{1+2c+\frac{4}{3}c^2}}.$$

A straightforward computation shows that  $m_2 > 0$  for  $c > 1$ . Thus, we have proven the following theorem.

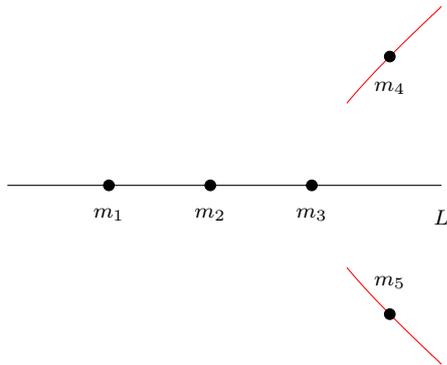
**Theorem 1** *Consider the following configuration of the five-body problem: Three collinear masses  $m_1, m_2, m_3$ , ordered from left to right, with  $m_1 =$*

$m_3$  in an Euler central configuration of the three-body problem, and the two remaining masses,  $m_4, m_5$  placed symmetrically with respect to the collinear central configuration. Then there exist central configurations of the five-body problem with  $m_4 = m_5 = 1$ , such that the line through  $m_4$  and  $m_5$  crosses the collinear configuration to the right of  $m_3$ , or by symmetry, to the left of  $m_1$ .

Numerically, we observe that Theorem 1 is valid for any fixed value of  $c \in (c_i, c_f)$  with  $c_i = 1.34528\dots$ ,  $c_f = 2.27866\dots$  and that the intersection between  $g(c, d) = 0$  and  $\mathcal{M}$  is empty when  $0 < c < 1$ . Thus, there are no central configurations when the line  $m_4m_5$  crosses  $L$  between  $m_2$  and  $m_3$ .

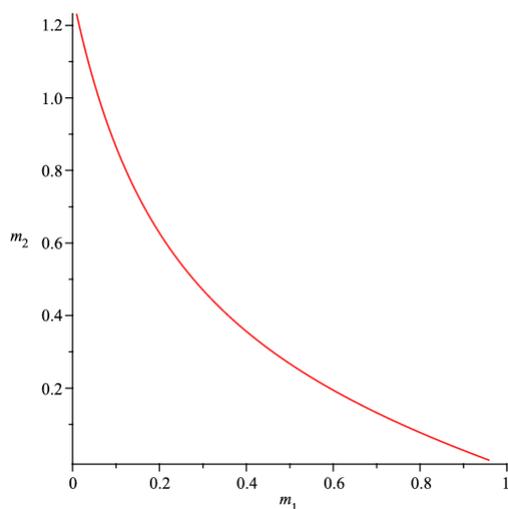
In Figure 4 a stacked central configuration is shown when  $m_1 = m_2 = m_3 = 0.378378411156148\dots$  and  $m_4 = m_5 = 1$ . The curve containing  $m_4$  represents all their admissible positions, that is, the family of stacked central configurations as the values of  $m_1$  and  $m_2$  varies. When the value of  $m_1$  tends to zero, the value of  $m_2$  tends to  $1.28240390152325\dots$ , and the configuration approaches a Lagrange configuration of the three-body problem. On the other hand, when the value of  $m_2$  tends to zero, the value of  $m_1$  tends to  $0.961839715898175\dots$ , and the configuration approaches a four-body central configuration, see (Leandro 2003).

Figure 5 provides numerical evidence that the reverse result stated in Proposition 1 is also true, that is, given a vector of masses  $(m_1, m_2, m_3, m_4, m_5)$  with  $m_1 = m_3$  and  $0 < m_1 < 0.96183\dots$ ,  $m_2 = m_2(m_1)$ , and  $m_4 = m_5 = 1$ , there exists a unique stacked central configuration with  $c > 1$ , ‘‘Euler plus two’’.



**Fig. 4** A stacked central configuration: ‘‘Euler plus two’’, when  $m_1 = m_2 = m_3 = 0.378378411156148\dots$ , and  $m_4 = m_5 = 1$ . In that case,  $c = 1.76905011042108651\dots$  and  $d = 1.21909474808793510\dots$

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**Fig. 5** Graph of  $m_2$  versus  $m_1$  along the family of stacked central configurations.  
 $\lim_{m_1 \rightarrow 0} m_2 = 1.28240\dots$  and  $\lim_{m_2 \rightarrow 0} m_1 = 0.96183\dots$

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