

REVIEW ON VISCOUS COSMOLOGY

I. Brevik

Department of Energy and Process Engineering, Norwegian University of
Science and Technology, N-7491 Trondheim, Norway

Ø. Grøn

Oslo and Akershus University College of Applied Sciences, Faculty of
Technology, Art and Design, St. Olavs Plass, N-0130 Oslo, Norway

plus other authors...

February 2, 2017

1 Introduction

The introduction of viscosity coefficients in cosmology theory has in itself a long history, although the physical importance of these phenomenological parameter has traditionally been assumed to be weak or at least subdominant. In connection with the very early universe, the influence from viscosity is assumed to be at largest at the time of neutrino decoupling (end of the lepton era), when the temperature was about 10^{10} K. Misner [1] was probably the first to introduce the viscosity from the standpoint of particle physics; cf. also Zel'dovich and Novikov [2]. If working on a phenomenological level, the viscosity concept was actually introduced much earlier; the first one being Eckart [3].

Now, when considering deviations to the first order from thermal equilibrium in the cosmic fluid one should recognize that there are in principle two different viscosity coefficients, namely the bulk viscosity ζ and the shear viscosity η . In view of the commonly accepted spatial isotropy of the universe, one usually omits the shear viscosity. This is motivated by the WMAP and Planck measurements, and is moreover supported by theoretical calculations showing that in a large class of homogeneous and anisotropic universes, the anisotropy quickly fades away. Eckart's theory, as most other theories, are kept on the first order level. A difficulty in principle with this kind of theory is that one becomes confronted with a non-causal behavior. In order to prevent this, one has to go to the second order approximation away

from thermal equilibrium. The interest in viscosity theories in cosmology has increased in recent years, for various reasons, perhaps especially from a fundamental viewpoint. It is well known among hydrodynamicists that the ideal (nonviscous) theory is after all only an approximation to the real world.

For reviews on both causal and non-causal theories, the reader may consult Grøn [4] (surveying the literature up to 1990), and later treatises by Maartens [5, 6], and Brevik and Grøn [7]. Some other research articles, most of them new, discussing viscous cosmology from various perspectives can be found in

Refs. [8, 9, 10, 62, 87, 13, 30, 15, 16, 17, 18, 19, 20, 38, 22].

1.1 Basic formalism

We begin by an outline of the general relativistic theory, setting, as usual, k_B and c equal to one. The formalism below is taken from Ref. [9]. We adopt the Minkowski metric in the form $(-+++)$, let Latin indices go from 1 to 3 and Greek indices from 0 to 3. If $U^\mu = (U^0, U^i)$ denotes the four-velocity of the cosmic fluid we have thus $U^0 = 1, U^i = 0$ in a local comoving frame.

With $g_{\mu\nu}$ being a general metric tensor we introduce the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \quad (1.1)$$

and the rotation tensor

$$\omega_{\mu\nu} = h_\mu^\alpha h_\nu^\beta U_{(\alpha;\beta)} = \frac{1}{2}(U_{\mu;\alpha} h_\nu^\alpha - U_{\nu;\alpha} h_\mu^\alpha). \quad (1.2)$$

The expansion tensor is

$$\theta_{\mu\nu} = h_\mu^\alpha h_\nu^\beta U_{(\alpha;\beta)} = \frac{1}{2}(U_{\mu;\alpha} h_\nu^\alpha + U_{\nu;\alpha} h_\mu^\alpha), \quad (1.3)$$

and has the trace $\theta \equiv \theta^\mu{}_\mu = U^\mu{}_{;\mu}$. The third tensor that we shall introduce is the shear tensor,

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\theta, \quad (1.4)$$

which satisfies $\sigma^\mu{}_\mu = 0$.

It is often useful to make use of the three tensors above in the following decomposition of the covariant derivative of the fluid velocity,

$$U_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}h_{\mu\nu}\theta - A_\mu U_\nu, \quad (1.5)$$

where A_μ means the four-acceleration, $A_\mu = \dot{U}_\mu = U^\nu U_{\mu;\nu}$.

The above formalism was quite general. Let us now specialize to the case of Friedmann-Robertson-Walker (FRW) geometry, which is of main interest in cosmology. The line element is then

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right], \quad (1.6)$$

where $a(t)$ is the scale factor and $k = 1, 0, -1$ the spatial curvature parameter. In this case the coordinates x^μ are numerated as (t, r, θ, φ) . In these coordinates the covariant derivatives of the velocity become quite simple,

$$U_{\mu;\nu} = Hh_{\mu\nu}, \quad (1.7)$$

with $H = \dot{a}/a$ being the Hubble parameter. The rotation tensor, the shear tensor, and the four-acceleration all vanish,

$$\omega_{\mu\nu} = \sigma_{\mu\nu} = 0, \quad A_\mu = 0, \quad (1.8)$$

and the relation between scalar expansion and Hubble parameter is simply

$$\theta = 3H. \quad (1.9)$$

Consider now the fluid's energy-momentum tensor $T_{\mu\nu}$, when viscosity as well as heat conduction is accounted for. If κ is the thermal conductivity taken in its nonrelativistic meaning, we have for the spacelike heat flux density four-vector the expression

$$Q^\mu = -\kappa h^{\mu\nu}(T_{,\nu} + TA_\nu), \quad (1.10)$$

with T the temperature. The last term in this expression is of relativistic origin. If one omits it, one obtains in a local rest inertial frame (a 'hat' on subscripts) the usual expression $Q_{\hat{i}} = -\kappa T_{,\hat{i}}$ for the heat flux density through a surface orthogonal to the unit vector $\mathbf{e}_{\hat{i}}$.

We can now write the energy-momentum tensor as

$$T_{\mu\nu} = \rho U_\mu U_\nu + (p - 3H\zeta)h_{\mu\nu} - 2\eta\sigma_{\mu\nu} + Q_\mu U_\nu + Q_\nu U_\mu, \quad (1.11)$$

assuming the FRW metric.

Turn now to thermodynamics, especially the production of entropy. The simplest way of getting the relativistic formulas is to generalize the known formalism from nonrelativistic thermodynamics. Let σ be the nondimensional entropy per particle. As 'particle' we will for definiteness mean a baryon. The nonrelativistic entropy density becomes thus $nk_B\sigma$, where n is the baryon number density. We may make use of the relationship [23]

$$\frac{dS}{dt} = \frac{2\eta}{T}(\theta_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\mathbf{u})^2 + \frac{\zeta}{T}(\nabla\cdot\mathbf{u})^2 + \frac{\kappa}{T^2}(\nabla T)^2, \quad (1.12)$$

where \mathbf{u} denotes the nonrelativistic velocity. We can now generalize to a relativistic formalism simply by making the effective substitutions

$$\theta_{ik} \rightarrow \theta_{\mu\nu}, \quad \delta_{ik} \rightarrow h_{\mu\nu}, \quad \nabla\cdot\mathbf{u} \rightarrow 3H, \quad -\kappa T_{,k} \rightarrow Q_\mu, \quad (1.13)$$

whereby we obtain the desired equation

$$S^\mu{}_{;\mu} = \frac{2\eta}{T}\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{9\zeta}{T}H^2 + \frac{1}{\kappa T^2}Q_\mu Q^\mu, \quad (1.14)$$

in which S^μ means the entropy current four-vector

$$S^\mu = nk_B\sigma U^\mu + \frac{1}{T}Q^\mu. \quad (1.15)$$

More careful derivations of these results can be found, for instance, in Refs. [8] or [24].

1.2 Brief review of some characteristic properties of the viscous fluid

We now turn to a presentation of some central properties of the viscous cosmic fluid, basing the present subsection essentially on Ref. [7]. We assume a homogeneous and isotropic universe with geodesic fluid flow, take the metric in the form (1.6) with $k = 0$, so that $a^\mu{}_{;\mu} = \omega = \sigma = 0$, and $\theta = 3H$.

Assume first that the fluid is nonviscous. According to the standard model the total energy density and pressure are

$$\rho_{\text{tot}} = \rho + \rho_\Lambda, \quad p_{\text{tot}} = p + p_\Lambda = -\rho_\Lambda, \quad (1.16)$$

where ρ (the density of cold dust matter) is zero, $\rho_\Lambda = \Lambda/8\pi G$ is the Lorentz invariant vacuum energy density (LIVE), and $p_\Lambda = -\Lambda/8\pi G$ is the

vacuum pressure corresponding to a positive tensile stress. With the critical energy density ρ_c , the matter density parameter Ω_M , and the Einstein gravitational constant κ defined as

$$\rho_c = \frac{3H^2}{8\pi G}, \quad \Omega_M = \frac{\rho}{\rho_c}, \quad \kappa = 8\pi G, \quad (1.17)$$

we obtain for the scale factor in Eq. (1.6) [25]

$$a(t) = K_s^{1/3} \sinh^{2/3} \left(\frac{t}{t_\Lambda} \right), \quad t_\Lambda = \frac{2}{3H_0\sqrt{\Omega_{\Lambda 0}}}, \quad K_s = \frac{1 - \Omega_{\Lambda 0}}{\Omega_{\Lambda 0}}. \quad (1.18)$$

Subscript zero refers to the present time $t = t_0$. At is usual, we take $a(t_0) = 1$. The present age of the universe is

$$t_0 = t_\Lambda \operatorname{arctanh} \sqrt{\Omega_{\Lambda 0}}, \quad (1.19)$$

giving $t_\Lambda = 11.4 \times 10^9$ years if we insert $t_0 = 13.7 \times 10^9$ years and $\Omega_{\Lambda 0} = 0.7$.

In terms of these quantities we get for the Hubble parameter

$$H = \frac{2}{3t_\Lambda} \coth \left(\frac{t}{t_\Lambda} \right), \quad (1.20)$$

whereas the deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \quad (1.21)$$

becomes

$$q = \frac{1}{2} \left[1 - 3 \tanh^2 \left(\frac{t}{t_\Lambda} \right) \right]. \quad (1.22)$$

It is of interest to determine the time $t = t_1$ when deceleration turns into acceleration. The condition for this is $q(t_1) = 0$, and leads to

$$t_1 = t_\Lambda \operatorname{arctanh} \frac{1}{\sqrt{3}}, \quad (1.23)$$

with corresponding redshift

$$z = \frac{1}{a(t_1)} - 1 = \left(\frac{2\Omega_{\Lambda 0}}{1 - \Omega_{\Lambda 0}} \right)^{1/3} - 1. \quad (1.24)$$

It gives $t_1 = 7.4 \times 10^9$ years and $z(t_1) = 0.67$.

Let t_e be the time of emission of a signal that arrives at the time t_0 . Taking time in units of Gyr we then obtain, when inserting $t_0 = 13.7$ and $\Omega_{\Lambda 0} = 0.7$, the useful formula

$$t_e = 11.3 \operatorname{arctanh}[1.53(1+z)^{-1.5}]. \quad (1.25)$$

Now introduce the viscosity coefficients in the cosmic fluid, still assuming spatial isotropy and the metric (1.6) with $k = 0$. The isotropy implies that only the bulk viscosity ζ contributes. The energy conservation equation $T^{0\nu}{}_{;\nu} = 0$ leads to

$$\dot{\rho} + 3H(\rho + p) = 9\zeta H^2, \quad (1.26)$$

while the Raychaudhuri equation (originally expressing the time derivative of the scalar expansion) can in our case be written in the form

$$\dot{H} + H^2 = \frac{\kappa}{6}(9\zeta H - \rho - 3p) + \frac{1}{3}\Lambda. \quad (1.27)$$

Now assume the equation of state to be

$$p = w\rho, \quad (1.28)$$

where in its simplest version w is a constant. Friedmann's equations take the form

$$3H^2 = \kappa\rho + \Lambda, \quad (1.29)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(1+3w)\rho + \frac{3\kappa\zeta}{2}H + \frac{\Lambda}{3}. \quad (1.30)$$

In view of the relation $\ddot{a}/a = \dot{H} + H^2$ we derive from the above equations

$$\dot{H} = -\frac{3}{2}(1+w)H^2 + \frac{3\kappa\zeta}{2}H + \frac{1}{2}(1+w)\Lambda. \quad (1.31)$$

The following non-dimensional parameters are used,

$$\Omega_M = \frac{\rho}{\rho_c}, \quad \Omega_\zeta = \frac{\kappa\zeta}{H}, \quad \Omega_\Lambda = \frac{\Lambda}{\kappa\rho_c}, \quad (1.32)$$

where the critical mass density follows from $3H^2 = \kappa\rho_c$. With an extra subscript zero referring to present time, we can express the current deceleration parameter as

$$q_0 = \frac{1}{2}(1 + 3w) - \frac{3}{2}[\Omega_{\zeta 0} + (1 + w)\Omega_{\Lambda 0}]. \quad (1.33)$$

If the cosmic fluid is cold, as is often assumed, we get

$$\Omega_{\zeta 0} = \frac{1}{3}(1 - 2q_0) - \Omega_{\Lambda 0}. \quad (1.34)$$

In principle, this equation enables one to estimate the viscosity parameter $\Omega_{\zeta 0}$ if one has at hand accurate measured values of q_0 and $\Omega_{\Lambda 0}$. Several of the early attempts failed in this respect, however, because of large uncertainties. We mention, though, an interesting study of Mathews et al. [26] in which the production of viscosity was associated with the decay of dark matter particles into relativistic particles in a recent epoch with redshift $z < 1$.

Before reviewing further models we will consider the most simple model in some detail. It was proposed by Padmanabhan and Chitre already in 1987 [27], and is based upon a dust model for matter, vanishing cosmological constant, and constant viscosity coefficient $\zeta = \zeta_0$. The from Eq. (1.31)

$$\dot{H} = -\frac{3}{2}H^2 + \frac{3}{2}\Omega_{\zeta 0}H_0H, \quad (1.35)$$

which upon integration two times gives $H = H(t)$ from which we derive

$$a = \left[\frac{1 - \Omega_{\zeta 0}}{\Omega_{\zeta 0}} \right] \left(e^{\frac{3}{2}\Omega_{\zeta 0}H_0t} - 1 \right)^{2/3}. \quad (1.36)$$

This means that the age of the universe when expressed in terms of the present Hubble parameter H_0 becomes

$$t_0 = \frac{4}{3\Omega_{\zeta 0}H_0} \operatorname{arctanh} \frac{\Omega_{\zeta 0}}{2 - \Omega_{\zeta 0}}. \quad (1.37)$$

Thus it is seen that for early times $\Omega_{\zeta 0}H_0t \ll 1$ the viscosity can be neglected, as we obtain $a \approx [(3/2)H_0t]^{2/3}$ corresponding to the evolution of a dust universe. At late times $\Omega_{\zeta 0}H_0t \gg 1$ the expansion becomes exponential

with $H = \kappa\zeta_0$, $a \propto \exp(\kappa\zeta_0)$, $\rho = 3\kappa\zeta_0^2$, and so the universe enters into a late inflationary era with accelerated expansion. A drawback of this model is however that the time when the bulk viscosity becomes dominant is predicted to be unrealistically large.

Let us now consider briefly the following model, which has attracted a good deal of attention [28, 29, 30],

$$\zeta = \zeta_0 + \zeta_1 \frac{\dot{a}}{a} + \zeta_2 \frac{\ddot{a}}{a}. \quad (1.38)$$

It is based on the physical idea that the dynamic state of the fluid influences its viscosity. We then obtain

$$a\dot{H} = -bH^2 + cH + d, \quad (1.39)$$

where

$$a = 1 - \frac{3\kappa\zeta_2}{2}, \quad b = \frac{3}{2}[1+w-\kappa(\zeta_1+\zeta_2)], \quad c = \frac{3\kappa\zeta_0}{2}, \quad d = \frac{1}{2}(1+w)\Lambda. \quad (1.40)$$

Integrating this equation with $a(0) = 0$, $a(t_0) = 1$ and assuming $\kappa(\zeta_1 + \zeta_2) < 1$ and $w \geq 0$ so that $b > 0$ and $4bd + c^2 > 0$ we obtain

$$H(t) = \frac{c}{2b} + \frac{a}{b} \hat{H} \coth(\hat{H}t), \quad \hat{H}^2 = \frac{bd}{a^2} + \frac{c^2}{4a^2}, \quad (1.41)$$

The age of the universe becomes in this model

$$t_0 = \frac{1}{\hat{H}} \operatorname{arctanh} \frac{2a\hat{H}}{2bH_0 - c} \quad (1.42)$$

Thus the viscosity increases the age of the universe. Assuming that $\kappa\zeta_0 \ll H_0$ the increase of the age because of viscosity is roughly

$$t_0 - t_{00} \approx \Omega_{\zeta_0}^2 t_0. \quad (1.43)$$

Let us return to the solution (1.41) and apply it to the case when the universe contains no matter, only dark energy, with $w = -1$ and moreover with linear viscosity ($\zeta_1 = \zeta_2 = 0$) so that $b = 0$. Cataldo et al. [31] found in this case, by integrating

$$\dot{H} = \frac{3\kappa\zeta_0}{2} H, \quad (1.44)$$

Integration with $a(t_0) = 1$ gives

$$H(t) = H_0 \exp \left[\frac{3\Omega_{\zeta_0} H_0}{2} (t - t_0) \right], \quad (1.45)$$

$$a(t) = \exp \left\{ \frac{2}{3\Omega_{\zeta_0}} \left[e^{\frac{3\Omega_{\zeta_0} H_0}{2} (t-t_0)} - 1 \right] \right\}. \quad (1.46)$$

Hence a universe dominated by viscous dark energy with constant viscosity coefficient expands exponentially faster than a corresponding universe model without viscosity.

One may now ask: how does the introduction of a bulk viscosity conform with the observed *acceleration* of the universe? there have been several papers dealing with this issue; cf., for instance, Refs. [32, 33, 34]. In the model of Avelino and Nucamendi [34] $\zeta_1 = \zeta_2 = 0, w = 0, \Omega_M = 1, \Omega_\Lambda = 0$, and the scale factor can be written as

$$a(t) = \left(\frac{1 - \Omega_{\zeta_0}}{\Omega_{\zeta_0}} \right)^{2/3} \left(e^{\frac{3}{2}\Omega_{\zeta_0} H_0 t} - 1 \right)^{2/3}. \quad (1.47)$$

This form of the solution satisfies the boundary conditions $a(0) = 0, a(t_0) = 1$.

The age of the universe in this model becomes

$$t_0 = \frac{4}{3\Omega_{\zeta_0} H_0} \operatorname{arctanh} \frac{\Omega_{\zeta_0}}{2 - \Omega_{\zeta_0}} = -\frac{2}{3\Omega_{\zeta_0} H_0} \ln(1 - \Omega_{\zeta_0}). \quad (1.48)$$

This universe model was actually considered earlier, by Brevik and Gorbunova [73] and by Grøn [36], and is also similar to the model of Padmanabhan and Chitre considered above [27]. The Hubble parameter is

$$H(t) = \frac{\Omega_{\zeta_0} H_0}{1 - e^{-(3/2)\Omega_{\zeta_0} H_0 t}}. \quad (1.49)$$

This approaches a de Sitter model for $t \gg 1/\Omega_{\zeta_0} H_0$, with a constant Hubble parameter equal to $\Omega_{\zeta_0} H_0$. The deceleration parameter is

$$q = \frac{3}{2 \exp[(3/2)\Omega_{\zeta_0} H_0 t]} - 1. \quad (1.50)$$

The value of it at present is

$$q(t_0) = (1/2)(1 - 3\Omega_{\zeta_0}). \quad (1.51)$$

The expansion thus starts from a Big Bang with an infinitely large velocity, but decelerates to a finite value. When $t = t_1$ given by $q(t_1) = 0$ there is a transition to an accelerated eternal expansion. The transition happens at

$$t_1 = \frac{2 \ln(3/2)}{3\Omega_{\zeta 0} H_0}, \quad (1.52)$$

at which time the scale factor is

$$a(t_1) = \left(\frac{1 - \Omega_{\zeta 0}}{2\Omega_{\zeta 0}} \right)^{2/3}, \quad (1.53)$$

and the redshift is

$$z_1 = \left(\frac{2\Omega_{\zeta 0}}{1 - \Omega_{\zeta 0}} \right)^{2/3} - 1. \quad (1.54)$$

We see that the bulk viscosity must have been sufficiently large, $\Omega_{\zeta 0} > 1/3$, for this transition to have happened in the past, $a(t_1) < 1$.

For this universe model, with spatial curvature $k = 0$, the matter density is equal to the critical density,

$$\rho = \frac{3H^2}{\kappa} = \frac{3\Omega_{\zeta 0}^2 H_0^2}{\kappa (1 - e^{-(3/2)\Omega_{\zeta 0} H_0 t})^2}. \quad (1.55)$$

Hence, the matter density approaches a constant value, $\rho \rightarrow (3/\kappa)\Omega_{\zeta 0}^2 H_0^2$.

In the mentioned study of Avelino and Nucamendi [34], supernova data were used to estimate the value of $\Omega_{\zeta 0}$ giving the best fit for a universe model containing dust with constant viscosity coefficient. The result was that $\Omega_{\zeta 0} = 0.64$ had to be several orders of magnitude greater than estimates based upon kinetic gas theory [9]. However, as an unorthodox idea we may mention here the probability for producing greater viscosity via dark matter particles into relativistic products; cf. Singh [37].

The comparison between magnitude of bulk viscosity and astronomical observations were also done in a recent paper by Normann and Brevik [38], basing the analysis on various experimentally-based sources [39, 40]. Various options for the bulk viscosity were analyzed: (i) $\zeta = \text{constant}$, (ii) $\zeta \propto \sqrt{\rho}$, and (iii) $\zeta \propto \rho$. The differences between the predictions of the options were found to be small. As a simple estimate based upon this analysis, we suggest that

$$\zeta_0 \sim 10^6 \text{ Pa s} \quad (1.56)$$

can serve as a reasonable mean estimate for the present viscosity.

The behavior of a viscous universe in its final stages has been discussed by Brevik and Gorbunova [73] and by Cataldo, Cruz and Lepe [31]. Consider first a universe without viscosity and dark energy, containing only a non-viscous fluid with $p = w\rho$. In this case Eq. (1.39) reduces to

$$\dot{H} = -bH^2, \quad b = \frac{3}{2}(1+w). \quad (1.57)$$

For this universe model there is a Big Rip at the instant

$$t_{R0} = t_0 + \frac{2}{3(1+w)H_0}. \quad (1.58)$$

In Ref. [31] a fluid was considered having $w < -1$ and constant viscosity coefficient ξ_0 , implying $b < 0$ and $d = 0$. In this case Eq. (1.39) reduces to

$$\dot{H} = -\frac{3}{2}(1+w)H^2 + \frac{3}{2}\Omega_{\zeta 0}H_0H. \quad (1.59)$$

The Hubble parameter, scale factor and density for this universe model are

$$H = \frac{H_0}{\frac{1+w}{\Omega_{\zeta 0}} + \left(1 - \frac{1+w}{\Omega_{\zeta 0}}\right) e^{-\frac{3}{2}\Omega_{\zeta 0}(t-t_0)}}, \quad (1.60)$$

$$a = \left[1 - \frac{1+w}{\Omega_{\zeta 0}} + \frac{1+w}{\Omega_{\zeta 0}} e^{\frac{3}{2}\Omega_{\zeta 0}H_0(t-t_0)}\right]^{\frac{2}{3(1+w)}}, \quad (1.61)$$

and

$$\rho = \frac{\rho_0}{\left[\frac{1+w}{\Omega_{\zeta 0}} + \left(1 - \frac{1+w}{\Omega_{\zeta 0}}\right) e^{-\frac{3}{2}\Omega_{\zeta 0}(t-t_0)}\right]^2}. \quad (1.62)$$

In this case there is a Big Rip singularity at

$$t_R = t_0 + \frac{2}{3\Omega_{\zeta 0}H_0} \ln \left(1 - \frac{\Omega_{\zeta 0}}{1+w}\right). \quad (1.63)$$

Similar universe models with variable gravitational and cosmological 'constants' have been investigated by Singh et al. [41, 42].

We have in this brief review considered isotropic spaces only. A good deal of work has been done on viscous fluids in spatially anisotropic spaces, belonging to the Bianchi type-I class. The interested reader might consult, for instance, the discussion in Ref. [7].

2 Cold and warm inflation

Usually, one is concerned with cold inflationary models, for which dissipation coming from decay of inflaton energy to radiation is omitted from consideration. This contrasts the characteristic feature of so-called warm inflation: dissipation is included as an important factor, and inflaton energy becomes dissipated into heat. This means in turn that the inflationary period lasts longer than it does in the cold case. Readers introduced in introductions to this theme may consult, for instance, Refs. [43, 44, 45, 46]. In this section we essentially follow Ref. [47]. The warm inflation scenario implies that no reheating at the end of the inflationary era is needed, and the transition to the radiation era becomes a smooth one.

2.1 Cold inflation

To begin with, let us consider cold inflation, when there is a scalar field ϕ , called the inflaton field, present.

The first Friedmann equation is

$$H^2 = \frac{\kappa}{3}\rho = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right), \quad (2.1)$$

where here ρ is the energy density of the inflaton field, and $V = V(\phi)$ is the corresponding potential. The continuity equation is

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (2.2)$$

Thus

$$\dot{\rho} = -\sqrt{3\kappa\rho}(\rho + p). \quad (2.3)$$

The equation generating the dark-energy repulsive gravity during inflation is

$$\ddot{\phi} + 3H\dot{\phi} = -V', \quad (2.4)$$

where $V' = dV/d\phi$. Equation (??) shows that a constant inflaton field requires a flat scalar potential, $V' = 0$. It follows that the inflaton field is either constant, or increases with time if the potential is flat.

From the second Friedmann equation it follows that the acceleration is given by

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p). \quad (2.5)$$

Often, one describes the inflaton field as a perfect fluid with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V, \quad (2.6a)$$

$$p = \frac{1}{2}\dot{\phi}^2 - V. \quad (2.6b)$$

Hence, the fluid satisfies

$$p = w\rho, \quad (2.7a)$$

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \quad (2.7b)$$

The fluid lies intermediate between an invariant vacuum energy with $w = -1$ for a constant inflaton field, and a stiff (Zel'dovich) fluid with $w = 1$ and $V = 0$. From the above we have

$$\dot{\phi}^2 = \frac{1+w}{1-w}2V. \quad (2.8)$$

The acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}(\dot{\phi}^2 - V). \quad (2.9)$$

Then we obtain

$$\dot{H} = -\frac{\kappa}{2}\dot{\phi}^2, \quad \dot{\phi} = -\frac{2}{\kappa}\dot{H}' \quad (2.10)$$

where $H' = dH/d\phi = \dot{H}/\dot{\phi}$ since $\dot{H} < 0$ and $\dot{\phi} < 0$. The inflaton field rolls down the potential. It follows that

$$\kappa^2 V = 3\kappa H^2 - 2H'^2. \quad (2.11)$$

For a constant inflaton field, $H = \text{constant}$ and there is a de Sitter universe with exponential expansion. It also follows that for a variable scalar field, H decreases with time.

During the main part of the inflationary epoch (except in the transitional stages), the scalar field changes slowly so that $\ddot{\phi} \ll H\dot{\phi}$. For a moderately sized potential V the condition $\dot{\phi} \ll V$ may also hold, implying that $w \simeq -1$ meaning that the inflaton field behaves as a Lorentz invariant vacuum energy (LIVE) with approximately constant energy density.

From the equations above one may derive

$$\dot{H} = -\frac{3}{2}(1+w)H^2. \quad (2.12)$$

Integration of this equation for constant $w \neq -1$ gives

$$a = a_1 \left(\frac{t}{t_1} \right)^{\frac{2}{3(1+w)}}. \quad (2.13)$$

Thus a power law expansion corresponds to a constant parameter $w \neq -1$ during the inflationary epoch, while an exponential expansion corresponds to $w = -1$. Insertion into Eq. (2.3) gives

$$\dot{\rho} = -\sqrt{3\kappa\rho}(1+w)\rho, \quad (2.14)$$

which upon integration with $\rho(0) = \rho_0$ (assuming $w \neq -1$) gives

$$\rho(t) = \frac{\rho_0}{\left[1 + \frac{1}{2}(1+w)\sqrt{3\kappa\rho_0}t\right]^2}. \quad (2.15)$$

When $\sqrt{\rho_0}t \gg M_p$ ($M_p = 4.3 \times 10^{-9}\text{kg}$ the Planck mass), the energy density of an inflaton field with $w = \text{constant}$ is thus approximately inversely proportional to t^2 .

2.2 Slow roll parameters

In inflationary theory, the so-called slow roll parameters have become in common use. One set of such parameters is defined via derivatives of the

potential with respect to the inflaton field. These 'potential' slow roll parameters, conventionally called ε , η , ξ , are defined as

$$\varepsilon = \frac{1}{2\kappa} \left(\frac{V'}{V} \right)^2, \quad (2.16a)$$

$$\eta = \frac{1}{\kappa} \frac{V''}{V}, \quad (2.16b)$$

$$\xi = \frac{1}{\kappa^2} \frac{V'V'''}{V^2}. \quad (2.16c)$$

The magnitudes of the slow roll parameters are small during the slow roll period, meaning that the graph of $V(\phi)$ is very flat.

In the *slow roll* approximation the condition $\ddot{\phi} \ll H\dot{\phi}$ is satisfied. Then Eq. (2.4) reduces to

$$V' \simeq -3H\dot{\phi}, \quad (2.17)$$

which implies

$$\dot{\phi}^2 \simeq \frac{V'^2}{9H^2} \simeq \frac{V'^2}{3\kappa V} = \frac{2}{3}\varepsilon V \ll V. \quad (2.18)$$

One may also define the slow roll parameters in a somewhat different form, by taking the derivatives of the Hubble parameter with respect to the inflaton field. Calling these parameters $\varepsilon_H, \eta_H, \xi_H$, we then have

$$\varepsilon_H = \frac{2}{\kappa} \left(\frac{H'}{H} \right)^2, \quad (2.19a)$$

$$\eta_H = \frac{2}{\kappa} \frac{H''}{H}, \quad (2.19b)$$

$$\xi_H = \frac{4}{\kappa^2} \frac{H'H'''}{H^2}. \quad (2.19c)$$

This means that the inflaton potential satisfies

$$\kappa V = 3 - \varepsilon_H H^2, \quad (2.20)$$

meaning that in the slow roll epoch where $|\varepsilon_H| \ll 1$,

$$\kappa V \simeq 3H^2 \quad (2.21)$$

will be a good approximation.

As the slow roll era differentiation with respect to time and with respect to inflaton field are related by

$$\frac{d}{dt} = -\frac{2}{\kappa} H' \frac{d}{d\phi}, \quad (2.22)$$

one may derive alternative expressions for ε_H and η_H which are convenient in some cases. We shall not go into further detail on this point, however, but mention the relationship

$$\varepsilon = \varepsilon_H \left(\frac{3 - \eta_H}{3 - \varepsilon_H} \right)^2, \quad (2.23)$$

which is exact and does not depend upon the slow roll approximation. Often $\varepsilon_H \simeq \varepsilon$ will be a good approximation. Differentiating Eq. (2.17) and making use of the expressions for ε_H and η_H one may derive

$$V'' = -3H^2(\eta_H + \varepsilon_H). \quad (2.24)$$

In turn, this leads to the slow roll relation

$$\eta = \eta_H + \varepsilon_H. \quad (2.25)$$

We now define the quantity N , the number of e-folds in the slow roll era, as the logarithm of the ratio between the final value a_f of the scale factor during inflation and the initial value $a(N) = a$,

$$N = \ln(a_f/a). \quad (2.26)$$

Note that $N = 0$ at the end of inflation. Thus N is equal to the number of e-folds until inflation ends; it increases as we go backward in time. The definition above is most usual, although there exist also other definitions.

Using the property

$$\frac{d}{dN} = -\frac{1}{H} \frac{d}{dt}, \quad (2.27)$$

we can derive various relationships between the slow roll parameters. We give here three of them,

$$\varepsilon \simeq \frac{\kappa}{2} \left(\frac{d\phi}{dN} \right)^2, \quad (2.28)$$

as well as

$$\frac{d \ln \varepsilon}{dN} = 2(\eta - 2\varepsilon), \quad \frac{d\eta}{dN} = \xi - 2\varepsilon\eta. \quad (2.29)$$

2.3 Power spectra

We will briefly review the formalism that is used to describe the temperature fluctuations in the CMB. Power spectra of scalar and tensor fluctuations are written as [48]

$$P_S = A_S(k_*) \left(\frac{k}{k_*} \right)^{n_S - 1 + (1/2)\alpha_S \ln(k/k_*)}, \quad P_T = A_T(k_*) \left(\frac{k}{k_*} \right)^{n_T + (1/2)\alpha_T \ln(k/k_*)} \quad (2.30)$$

$$A_S = \frac{V}{24\pi^2 \varepsilon M_p^4} = \left(\frac{H^2}{2\pi \dot{\phi}} \right)^2, \quad A_T = \frac{2V}{3\pi^2 \varepsilon M_p^4} = \varepsilon \left(\frac{2H^2}{\pi \dot{\phi}} \right)^2. \quad (2.31)$$

Here k is the wave number of the perturbation, and k_* is a reference scale usually chosen as the wave number at horizon crossing (the pivot scale). Often one writes $k = \dot{a} = aH$, with a the scale factor. The quantities A_S and A_T are amplitudes at the pivot scale, while n_S and n_T are called the *spectral indices* of scalar and tensor fluctuations. Moreover $-\delta_{n_s} = n_S - 1$ and n_T are called the tilts of the power spectrum, since they describe deviations from the scale invariant spectrum where $\delta_{n_s} = n_t = 0$. The factors α_S and α_T are called *running spectral indices* and are defined by

$$\alpha_S = \frac{dn_S}{d \ln k}, \quad \alpha_T = \frac{dn_T}{d \ln k}. \quad (2.32)$$

If $n_s = 1$ the spectrum of the scalar fluctuations is said to be *scale invariant*. An invariant mass-density power spectrum ($n_S = 1$) is called a *Harrison-Zel'dovich spectrum*. It turns out that the mass distribution is approximately scale invariant. Analysis of the observations from the Planck satellite give the result $n_S = 0.968(6) \pm 0.006$ [49, 50, 51, 52]. Further, the observations give $\alpha_S = -0.003 \pm 0.007$. The tilt of the curvature fluctuations is $\delta_{n_s} = 0.032$. An analysis of several relevant experiments gives the restriction $n_T < 0.36$ [53].

The *tensor-to-scalar ratio* r is defined by

$$r = \frac{P_T(k_*)}{P_S(k_*)} = \frac{A_T}{A_S}. \quad (2.33)$$

This is a measure of the inflationary energy scale. From the above equations we see that

$$r = 16\varepsilon. \quad (2.34)$$

2.4 How the spectral indices are related to the slow roll parameters

From the above equations we derive

$$\delta_{ns} = - \left[\frac{d \ln P_S(k)}{d \ln k} \right]_{k=aH}, \quad n_T = - \left[\frac{d \ln P_T(k)}{d \ln k} \right]_{k=aH}, \quad (2.35)$$

where quantities are evaluated at the horizon crossing ($k = k_*$), and $k = aH$. Making use of the relationships

$$\frac{d}{d \ln k} = \frac{d}{d \ln N} \times \frac{dN}{d \ln k}, \quad \frac{d}{d \ln k} \approx -\frac{d}{dN}, \quad (2.36)$$

we can derive the useful equations

$$\delta_{ns} = 2(3\varepsilon - \eta), \quad n_T \approx -2\varepsilon. \quad (2.37)$$

The actual value of n_T is not well known. The combined BICEP2/Planck and LIGO data give $n_T = -0.76^{+1.37}_{-0.52}$ [54], while the BICEP/Planck data alone constraint the tensor tilt to be $n_T = 0.66^{+1.83}_{-1.44}$.

A consistency relation between r and n_T follows from

$$n_T = -\frac{r}{8}, \quad (2.38)$$

which can be derived from the equations above.

The preferred BICEP2/Planck value of $r = 0.05$ then gives $n_T = -0.006$.

2.5 Intermediate inflation

Intermediate inflation models, introduced by Barrow in 1990 [55], have later been considered by other investigators [56, 57]. These models make use of a scalar field. We will in the following give a brief account of simple cases of these models.

We take the time dependent scale factor in the form

$$a(t) = \frac{a_p}{\exp A} \exp [A(M_p t)^\alpha], \quad 0 < \alpha < 1. \quad (2.39)$$

Here A is a positive non-dimensional constant, while a_p refers to the Planck time ($M_p t = 1$). The reason why these models are called intermediate, is that the expansion is faster than power law expansion and slower than an exponential expansion one (the latter corresponding to $\alpha = 1$). We calculate

$$H = AM_p \alpha (M_p t)^{\alpha-1}, \quad \dot{H} = AM_p^2 \alpha (\alpha - 1) (M_p t)^{\alpha-2}. \quad (2.40)$$

As $\dot{H} < 0$ for $\alpha < 1$, the Hubble parameter decreases with time. Inserting these equations into Eqs. (2.1) and (2.5) we obtain

$$\rho = 3A^2(M_p)^4 \alpha^2 (M_p t)^{2\alpha-1}, \quad p = A(M_p)^4 \alpha (M_p t)^{\alpha-2} [2(1-\alpha) - 3\alpha A(M_p t)^\alpha]. \quad (2.41)$$

As $\rho + p = \dot{\phi}^2$ we obtain by integration, using the initial condition $\phi(0) = 0$,

$$\phi(t) = 2M_p \sqrt{2A \frac{1-\alpha}{\alpha}} (M_p t)^{\frac{\alpha}{2}}. \quad (2.42)$$

As $V = \frac{1}{2}(\rho - p)$ we get.

$$V(t) = M_p^4 A \alpha (M_p t)^{\alpha-2} [3A \alpha (M_p)^\alpha - 2(1-\alpha)] \quad (2.43)$$

The potential can also be expressed as a function of the inflaton field,

$$V(\phi) = M_p^4 A \alpha \left[\frac{\alpha}{2A(1-\alpha)} \right]^{\frac{\alpha-2}{\alpha}} \left(\frac{\phi}{2M_p} \right)^{\frac{2(\alpha-2)}{\alpha}} \left[\frac{3\alpha^2}{2(1-\alpha)} \left(\frac{\phi}{2M_p} \right)^2 - 2(1-\alpha) \right]. \quad (2.44)$$

For these models the spectral parameters are conveniently expressed in terms of the Hubble slow roll parameters (2.19),

$$\varepsilon_H = \frac{1-\alpha}{A\alpha} (M_p t)^{-\alpha}, \quad \eta_H = \frac{2-\alpha}{2(1-\alpha)} \varepsilon_H. \quad (2.45)$$

We will not go into much detail on this point, but mention that the number of e-folds becomes

$$N = A [(M_p t_f)^\alpha - (M_p t_i)^\alpha], \quad (2.46)$$

t_i and t_f being the initial and final instants of the inflationary era.

For the spectral parameters δ_{ns} , n_T and r one can derive the expressions

$$\delta_{ns} \equiv 1 - n_s = \frac{2 - 3\alpha}{N\alpha + 1 - \alpha}, \quad n_T = \frac{2(\alpha - 1)}{N\alpha + 1 - \alpha}, \quad r = \frac{16(1 - \alpha)}{N\alpha + 1 - \alpha}. \quad (2.47)$$

Moreover, the relationship between r and δ_{ns} becomes

$$r = \frac{16(1 - \alpha)}{2 - 3\alpha} \delta_{ns}. \quad (2.48)$$

2.6 Warm inflation

Characteristic for the warm inflationary models is that the inflaton field energy ρ_ϕ is taken to depend on the temperature T . Similarly, also the electromagnetic radiation density ρ_r depends on T . For the Friedmann equation we have

$$H^2 = \frac{\kappa}{3}(\rho_\phi + \rho_r), \quad (2.49)$$

and the continuity equations for the two fluid components are

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\Gamma\dot{\phi}^2, \quad \dot{\rho}_r + 4H\rho_r = \Gamma\dot{\phi}^2. \quad (2.50)$$

Here Γ , in general time dependent, is a dissipation coefficient describing the transfer of dark energy into radiation.

In warm inflation the dark energy is the dominating component, $\rho_\phi \gg \rho_r$, and H , ϕ and Γ vary slowly such that $\ddot{\phi} \ll H\dot{\phi}$, $\dot{\rho}_r \ll 4H\rho_r$ and $\dot{\rho}_r \ll \Gamma\dot{\phi}^2$. In the slow roll epoch, all radiation is produced by dark energy dissipation. Then,

$$3H^2 = \kappa\rho_\phi = \kappa V, \quad (3H + \Gamma)\dot{\phi} = -V'. \quad (2.51)$$

Defining the so-called dissipative ratio by

$$Q = \frac{\Gamma}{3H}, \quad (2.52)$$

we see that in the warm inflation era the second of Eqs. (2.50) yields

$$\rho_r = \frac{3}{4}Q\dot{\phi}^2. \quad (2.53)$$

During warm inflation $T > H$ (in geometric units), and it turns out that the tensor-to-scalar ratio becomes modified in comparison to the cold inflation case [58]

$$r = \frac{H/T}{(1+Q)^{5/2}}r. \quad (2.54)$$

Thus this ratio is suppressed by a factor $(T/H)(1+Q)^{5/2}$ compared to the cold inflationary case.

The slow roll parameters in the present models are calculated at the beginning $t = t_i$ of the slow roll epoch. From the definition equation (2.16) we get

$$\varepsilon = -(1+Q)\frac{\dot{H}}{H^2}. \quad (2.55)$$

Manipulation of the above equations then yields for the parameter η

$$\eta = \frac{Q}{1+Q}\frac{1}{\kappa}\frac{\Gamma V'}{\Gamma V} - \frac{1+Q}{H}\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{H}}{H^2}. \quad (2.56)$$

For convenience we introduce the quantity β ,

$$\beta = \frac{1}{\kappa}\frac{\Gamma V'}{\Gamma V}. \quad (2.57)$$

Then, this quantity appears in the expression for the relative rate of change of the radiation energy density,

$$\frac{\dot{\rho}_r}{H\rho_r} = -\frac{1}{1+Q}\left(2\eta - \beta - \varepsilon + 2\frac{\beta - \varepsilon}{1+Q}\right). \quad (2.58)$$

With n_S the scalar spectral index we define δ_{ns} by $\delta_{ns} = 1 - n_S$. With ω a second quantity introduced for convenience, defined as

$$\omega = \frac{T}{H}\frac{2\sqrt{3}\pi Q}{\sqrt{3+4\pi Q}}, \quad (2.59)$$

one can then obtain after some manipulations [59]

$$\delta_{ns} = \frac{1}{1+Q} \left[4\varepsilon - 2 \left(\eta - \beta + \frac{\beta - \varepsilon}{1+Q} \right) + \frac{\omega}{1+\omega} \left(\frac{2\eta + \beta - 7\varepsilon}{4} + \frac{6 + (3 + 4\pi)Q}{(1+Q)(3 + 4\pi Q)} (\beta - \varepsilon) \right) \right]. \quad (2.60)$$

The cold inflationary case corresponds to the limit $Q \rightarrow 0$ and $T \ll H$. Then $\omega \rightarrow 0$, and

$$\delta_{ns} \rightarrow 2(3\varepsilon - \eta). \quad (2.61)$$

When the warm inflation is strong, $Q \gg 1$, $\omega \gg 1$,

$$\delta_{ns} = \frac{2}{2Q} \left[\frac{3}{2}(\varepsilon + \beta) - \eta \right], \quad (2.62)$$

whereas when it is weak, $Q \ll 1$,

$$\delta_{ns} = 2(3\varepsilon - \eta) - \frac{\omega/4}{1+\omega} (15\varepsilon - 2\eta - 9\beta). \quad (2.63)$$

Visinelli found the following expression for tensor-to-scalar ratio in warm inflation [59],

$$r = \frac{16\varepsilon}{(1+Q)^2(1+\omega)}. \quad (2.64)$$

In the limit of cold inflation,

$$r \rightarrow 16\varepsilon. \quad (2.65)$$

and in the limit of strong dissipative warm inflation,

$$r \rightarrow \frac{16}{Q^2\omega} \varepsilon \ll \varepsilon. \quad (2.66)$$

Thus the warm inflation models with $Q \gg 1$ and $\omega \gg 1$ yield a very small tensor-to-scalar ratio.

2.7 Warm and viscous intermediate inflation

As the last theme in this section we will discuss briefly how the cold and nonviscous inflationary models considered above can be generalized to the warm and viscous case. These latter kind of models are most likely more physical than the earlier more idealized ones, since they take into account

the presence of massive particles produced from the decaying inflaton field. Moreover, an important advantage of these models is that they give rise to a much smaller tensor-to-scalar ratio than the cold models. This conforms with the Planck data. The presence of massive particles provides a natural reason why the cosmic fluid can be associated with a bulk viscosity.

We now abstain from using the simple equation of state $p = (1/3)\rho$ holding for radiation, and assume instead the more general form $p = w\rho$ with w constant. Alternatively it is sometimes convenient to use the form $p = (\gamma - 1)\rho$ with $\gamma = 1 + w$. The effective pressure becomes $p_{\text{eff}} = p + p_\zeta$, where

$$p_\zeta = -3H\zeta \tag{2.67}$$

is the viscous pressure and ζ the bulk viscosity.

The second of Eqs. (2.50) now becomes generalized to [60]

$$\dot{\rho} + 3H(\rho + p - 3\zeta H) = \Gamma\dot{\phi}^2. \tag{2.68}$$

The usual condition about quasi-stationarity implies $\dot{\rho} \ll 3H(\gamma\rho - 3\zeta H)$ and $\dot{\rho} \ll \Gamma\dot{\phi}^2$. From the definition Eqs. (2.19) we obtain

$$\varepsilon_H = \frac{\kappa}{2}(1 + Q)\frac{\dot{\phi}^2}{H^2}, \tag{2.69}$$

and so the slow-roll condition $\varepsilon_H \ll 1$ leads to the inequality

$$\rho_\phi \gg (3/2)(\gamma\rho - 3\zeta H). \tag{2.70}$$

We will henceforth follow the formalism as presented by Setare and Kamali [60] for the strong dissipative case, $Q \gg 1$; cf. also the related treatment of Sharif and Saleem [61]. The scale factor and the Hubble parameters are found above in Eqs. (2.39) and (2.40).

We will base the analysis on the basic assumptions

$$\Gamma(\phi) = \kappa V(\phi)/M_p, \quad \zeta = \zeta_1 \rho. \tag{2.71}$$

Here the proportionality of ζ to ρ is a frequently used assumption. From Eqs. (2.51) and (2.52) we then have $Q = H/M_p$. Of main interest is the strong dissipative case $Q \gg 1$. Manipulations of the equations give the following expression for the inflaton field as a function of time,

$$\phi(t) = 2M_p \sqrt{2M_p(1-\alpha)t}. \quad (2.72)$$

This equation, predicting $\phi(t)$ to increase with time, is seen to be different from the corresponding Eq. (2.42) for cold intermediate inflation.

Taking into account the expression (2.40) for H we can express the potential as a function of time,

$$V(t) = 3A^2\alpha^2 M_p^4 (M_p t)^{2(\alpha-1)}, \quad (2.73)$$

which can alternatively be represented as a function of ϕ ,

$$V(\phi) = 3A^2\alpha^2 M_p^4 \left(\frac{\phi}{2M_p \sqrt{2(1-\alpha)}} \right)^{4(\alpha-1)}. \quad (2.74)$$

As

$$\rho = \frac{V\dot{\phi}^2}{3H(\gamma - 3\zeta_1 H)}, \quad (2.75)$$

we see that it is necessary for the constant ζ_1 in Eq. (2.71) to satisfy the condition $\zeta_1 < \gamma/3H$ in order to make ρ positive. The density varies with time as

$$\rho(t) = \frac{2A\alpha(1-\alpha)M_p^4(M_p t)^{\alpha-2}}{\gamma - 3\zeta_1 A\alpha M_p (M_p t)^{2(\alpha-1)}}, \quad (2.76)$$

while when considered as a function of time,

$$\rho(t) = \frac{2A\alpha(1-\alpha)M_p^4 \left[\phi/2M_p \sqrt{2(1-\alpha)} \right]^{2(\alpha-2)}}{\gamma - 3\zeta_1 A\alpha M_p \left[\phi/2M_p \sqrt{2(1-\alpha)} \right]^{2(\alpha-1)}}. \quad (2.77)$$

The Hubble slow roll parameters ε_H and η_H become in the strong dissipative epoch when $Q \gg 1$,

$$\varepsilon_H = \frac{1}{2Q} \left(\frac{V'}{V} \right)^2, \quad \eta_H = \frac{1}{Q} \left[\frac{V''}{V} - \frac{1}{2} \left(\frac{V'}{V} \right)^2 \right], \quad (2.78)$$

giving in turn for the spectral parameter δ_{ns}

$$\delta_{ns} = \frac{3\alpha - 2}{1 - \alpha} \varepsilon_H = \frac{3\alpha - 2}{\alpha A} \left(\frac{\phi}{2M_p \sqrt{2(1-\alpha)}} \right)^{-2\alpha}. \quad (2.79)$$

The Harrison-Zel'dovich spectrum (independent of scale) corresponds to $\alpha = 2/3$.

The number of e-folds becomes in this case

$$N = \frac{1}{\sqrt{3}M_p} \int_{\phi_f}^{\phi} \frac{V^{3/2}}{V'} d\phi = A \frac{1-\alpha}{\alpha} - A \left(\frac{\phi}{2M_p \sqrt{2(1-\alpha)}} \right)^{2\alpha}. \quad (2.80)$$

Here ϕ_f is the inflaton field at the end of the slow roll epoch, characterized by $\varepsilon_H(\phi_f) = 1$, corresponding to $\varepsilon(\phi_f) = Q$.

A similar analysis can be made for the case where Γ and ζ are assumed constants [60, 61]. We will however not go into further detail here.

3 Special topics

We will in the following consider various topics from viscous cosmological theory, mostly topics in which the present authors have taken part. We will first focus on rather recent topics, but also draw into consideration topics that were dealt with some years ago.

3.1 Estimate for the present bulk viscosity. Further remarks on the future universe

Considerable attention has been given in the past to calculating the behavior of the cosmic fluid in the far future. There may appear various kinds of singularities: the Big Rip [62, 63], the Little Rip [64, 15, 65], the Pseudo-Rip [66], the Quasi-Rip [67], as well as other kinds of soft singularities (so-called type IV finite time singularities [68]).

Naturally, what value we can estimate for the (effective) bulk viscosity at present time, will here be an important ingredient of such a description. Recent observations from the Planck satellite have given us a better ground for estimate the value $\zeta = \zeta_0$ of the bulk viscosity at the present time $t = t_0$. As discussed already in Section 1, referring to the paper [38] as well as to several other theoretical and experimental papers, the estimate

$$\zeta_0 \sim 10^6 \text{ Pa s} \quad (3.1)$$

was suggested as a reasonable (logarithmic) mean value. The uncertainty is however quite large; there have appeared proposals ranging from about 10^4 Pa s to about 10^7 Pa s, depending on analysis of different sources.

We extract in the following some information from a recent paper [69] in which two different cosmological models were analyzed: (1) a one-component dark energy model where the bulk viscosity ζ was associated with the cosmic fluid as a whole, and (2) a two-component model where ζ was associated with a dark matter component ρ_m only, the latter component assumed nonviscous. We limit ourselves here to the one-component model only.

We assume here the simple equation of state

$$p = w\rho, \quad w = \text{constant}, \quad (3.2)$$

and assume a spatially flat FRW space (the scalar expansion is thus $\theta = 3H$). The governing equations are

$$3H^2 = \kappa\rho, \quad 2\dot{H} + 3H^2 = -\kappa[p - 3H\zeta(\rho)], \quad (3.3)$$

with $\kappa = 8\pi G$ as before, and the energy conservation equation is

$$\dot{\rho} + 3H(\rho + p) = 9H^2\zeta(\rho). \quad (3.4)$$

It is now straightforward to derive a governing equation for ρ . Solving it, and introducing for convenience the small quantity α via $w = -1 + \alpha$, we find ($t_0 = 0$)

$$t = \frac{1}{\sqrt{3\kappa}} \int_{\rho_0}^{\rho} \frac{d\rho}{\rho^{3/2}[-\alpha + \sqrt{3\kappa}\zeta(\rho)/\sqrt{\rho}]}, \quad (3.5)$$

where the integration goes into the future.

For the bulk viscosity we will assume the form

$$\zeta = \zeta_0 \left(\frac{H}{H_0} \right)^{2\lambda} = \zeta_0 \left(\frac{\rho}{\rho_0} \right)^\lambda, \quad (3.6)$$

with λ a constant. This form for ζ has often been adopted in the literature.

We will consider two options for the value of λ ; both of them are physically reasonable.

(i) $\lambda = 1$ ($\zeta \propto \sqrt{\rho}$). From Eq. (3.5) we then obtain

$$t = \frac{2}{3H_0 X_0} \left(1 - \frac{1}{\sqrt{\Omega}} \right), \quad (3.7)$$

where we have for convenience introduced the nondimensional quantities

$$X_0 = \frac{2B}{3H_0} - \alpha, \quad B = \frac{3}{2}\kappa\zeta_0, \quad \Omega = \frac{\rho}{\rho_0}. \quad (3.8)$$

The point worth attention here is that even if the fluid is initially in the quintessence region $\alpha > 0$ at $t = 0$ it will, if $X_0 > 0$, inevitably be driven into a Big Rip singularity ($\rho = \infty$) after a finite time [73, 17, 38]

$$t_s = \frac{2}{3H_0 X_0}, \quad (\zeta \propto \sqrt{\rho}). \quad (3.9)$$

If on the other hand the combination of equation-of-state parameter α and viscosity ζ_0 is such that $X_0 < 0$, then the cosmic fluid becomes gradually diluted as $\rho \propto 1/t^2$ in the far future.

(ii) $\lambda = 0$ ($\zeta = \text{constant} = \zeta_0$). We then get the solution

$$t = \frac{1}{B} \ln \left[\frac{X_0}{-\alpha + 2B/(\theta_0 \sqrt{\Omega})} \right], \quad (\zeta = \zeta_0). \quad (3.10)$$

which means for the energy density

$$\Omega = \frac{\rho}{\rho_0} = \left[\frac{\alpha + X_0}{\alpha + X_0 e^{-Bt}} \right]^2. \quad (3.11)$$

Hence in the far future $\rho \rightarrow \text{const}$, which implies $H \rightarrow \text{const}$, thus a de Sitter solution. Let us denote the limiting value of the density by ρ_{dS} . Then

$$\rho_{\text{dS}} = \rho_0 \left(1 + \frac{X_0}{\alpha} \right)^2 = \frac{3\kappa\zeta_0^2}{\alpha^2}. \quad (3.12)$$

This expression tells us that both α and X_0 are of importance for the future fate of the cosmic fluid:

1. If $\alpha > 0$ and $X_0 > 0$, then $\rho_{\text{dS}} > \rho_0$.
2. If $\alpha > 0$ and $X_0 < 0$, then $\rho_{\text{dS}} < \rho_0$.
3. If $\alpha < 0$ then $X_0 > 0$, and $\rho_{\text{dS}} < \rho_0$.

This case may be defined as a pseudo-rip in accordance with the definition given by Frampton et al. [66], since the limiting value for the density reached after an infinite span of time is finite.

It is appropriate here to give some numbers from the 2015 Planck observations. From Ref. [52] Table 5, we have $w = -1.019_{-0.080}^{+0.075}$. Thus, $\alpha = 1 + w$ will be lying within two limits,

$$\alpha_{\min} = -0.099, \quad \alpha_{\max} = +0.056. \quad (3.13)$$

As mentioned above, we took $\zeta_0 = 10^6$ Pa s to be a reasonable mean value for the present viscosity. We can then evaluate the quantity $B = (3/2)\kappa\zeta_0 \sim 1 \text{ km s}^{-1}\text{Mpc}^{-1}$ in astronomical units. With $H_0 = 67.74 \text{ km s}^{-1}\text{Mpc}^{-1}$ this leads to $2B/(3H_0) = 0.00984$ as an estimate. Then, according to Eq. (3.8) we have

$$X_0(B = 1, \alpha_{\max}) = -0.0462, \quad X_0(B = 1, \alpha_{\min}) = +0.109. \quad (3.14)$$

In this way we recover the cases 2 and 3 above: the future de Sitter energy density will become lower than ρ_0 .

3.2 Is the bulk viscosity large enough to permit the phantom divide crossing?

This subsection is a continuation of the previous one, and is motivated by the following question: is the value of ζ_0 as inferred from the analysis of recent experiments actually large enough to permit the crossing of the phantom divide, i.e. the transition from the quintessence region to the phantom region? To analyze this question we have to consider more carefully the uncertainties in the data found from different sources. We will here present some material from the recent paper [17], discussing this point.

Assume that the bulk viscosity varies with density as $\zeta \propto \sqrt{\rho}$. The condition for phantom divide crossing, as noted above, is that the quantity X_0 defined in Eq. (3.8) has to be positive. In the analysis of Wang and Meng [39]. various assumptions for the bulk viscosity in the early universe were considered and the corresponding theoretical curves for $H = H(z)$ were compared with a number of observations. The comparison gets somewhat complicated, but for our purpose it is sufficient to note that the preferred value of the magnitude $|B|$ of $B = (3/2)\kappa\zeta_0$ was given as

$$|B| \sim 50 \frac{\text{km}}{\text{s Mpc}} = 1.6 \times 10^{-18} \text{s}^{-1} \quad (3.15)$$

(cf. also the discussion in Ref. [38]). As in dimensional units $12\pi G/c^2 = 2.79 \times 10^{-26} \text{ m/kg}$, this gives the estimate

$$\zeta_0 \sim 5 \times 10^7 \text{ Pa s.} \quad (3.16)$$

This value is probably high. We may in this context compare with the formula for the bulk viscosity in a photon fluid [8],

$$\zeta = 4a_{\text{rad}} T^4 \tau_f \left[\frac{1}{3} - \left(\frac{\partial p}{\partial \rho} \right)_n \right]^2, \quad (3.17)$$

where $a_{\text{rad}} = \pi^2 k_B^4 / 15 \hbar^3 c^3$ is the radiation constant and τ_f the mean free time. If we estimate $\tau_f = 1/H_0$ (the inverse Hubble radius), we obtain $\zeta \sim 10^4 \text{ Pa s}$, which is considerably lower. In all, it seems that one has to allow for a quite wide span in the value of the present bulk viscosity. At least all suggestions in the literature should be encompassed if we write

$$10^4 \text{ Pa s} < \zeta_0 < 10^7 \text{ Pa s.} \quad (3.18)$$

Now, rewrite the condition for phantom divide crossing as

$$\zeta_0 > \frac{H_0}{\kappa} \alpha = (1.18 \times 10^8) \alpha, \quad (3.19)$$

where we have inserted $H_0 = 67.80 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.20 \times 10^{-18} \text{ s}^{-1}$.

As noted above, from the observed data we derive the maximum value of α to be $\alpha_{\text{max}} = 0.056$. This yields

$$\zeta_0 > \frac{H_0}{\kappa} \alpha_{\text{max}} = 6.6 \times 10^6 \text{ Pa s.} \quad (3.20)$$

Comparison between the expressions (3.18) and (3.20) tells us that a phantom divide crossing is actually possible, on the basis of available data, even if $\alpha = \alpha_{\text{max}}$.

3.3 Inclusion of isotropic turbulence

From a hydrodynamicist's point of view the inclusion of turbulence in the theory of the cosmic fluid seem most natural, at least in the final stage of the universe's evolution when the fluid motion may well turn out to be quite vigorous.

The local Reynolds number must then be expected to be very high. On a local scale this brings the *shear* viscosity concept into consideration, as it has to furnish the transport of eddies over the wave number spectrum until the local Reynolds number becomes of order unity, marking the transfer of kinetic energy into heat. Because of the assumed isotropy in the fluid, we must expect that the kind of turbulence is isotropic when looked upon on a large scale. According to standard theory of isotropic turbulence in hydrodynamics we then expect to find a Loitziankii distribution for low wave numbers (energy density varying as k^4), whereas for higher k we expect an inertial subrange in which the energy distribution is

$$E(k) = \alpha \epsilon^{2/3} k^{-5/3}, \quad (3.21)$$

in which α denotes the Kolmogorov constant and ϵ is the mean energy dissipation per unit mass and unit time. When k reaches the inverse Kolmogorov length η_K ,

$$k \rightarrow k_L = \frac{1}{\eta_L} = \left(\frac{\epsilon}{\nu^3} \right)^{1/4} \quad (3.22)$$

(ν the kinematic viscosity), the dissipative region is reached.

We will in the following consider a dark fluid developing into the future from the present time $t = 0$ when turbulence is accounted for. We will do this in two different ways: either assuming a two-fluid model with one and one turbulent constituent, or assuming simply a one-component fluid.

We follow the earlier papers [7, 16, 70, 71]. Assume first a two-component model, where the effective energy is written as a sum of two parts,

$$\rho_{\text{eff}} = \rho + \rho_{\text{turb}}, \quad (3.23)$$

ρ denoting the conventional energy density. Taking ρ_{turb} to be proportional to the scalar expansion $\theta = 3H$, calling the proportionality factor τ , we get

$$\rho_{\text{eff}} = \rho(1 + 3\tau H). \quad (3.24)$$

The effective pressure p_{eff} is split similarly,

$$p_{\text{eff}} = p + p_{\text{turb}}. \quad (3.25)$$

Now assume homogeneous equations of state for each component,

$$p = w\rho, \quad p_{\text{turb}} = w_{\text{turb}}\rho_{\text{turb}}, \quad (3.26)$$

The Friedmann equations are now written (recall that $\kappa = 8\pi G$)

$$H^2 = \frac{1}{3}\kappa\rho(1 + 3\tau H). \quad (3.27)$$

$$\frac{2\ddot{a}}{a} + H^2 = -\kappa\rho(w + 3\tau H w_{\text{turb}}), \quad (3.28)$$

leading to the following governing equation for H ,

$$(1 + 3\tau H)\dot{H} + \frac{3}{2}\gamma H^2 + \frac{9}{2}\tau\gamma_{\text{turb}}H^3 = 0. \quad (3.29)$$

We have here made use of the obvious notation

$$\gamma = 1 + w, \quad \gamma_{\text{turb}} = 1 + w_{\text{turb}}. \quad (3.30)$$

The input parameters in this model are $\{w, w_{\text{turb}}, \tau\}$, all of them assumed constant.

Before considering the specific options about w and w_{turb} , let us make a remark on the energy balance equation, when the energy dissipation is assumed to be

$$\epsilon = \epsilon_0(1 + 3\tau H), \quad (3.31)$$

Then the energy balance may be written as

$$\dot{\rho} + 3H(\rho + p) = -\rho\epsilon_0(1 + 3\tau H). \quad (3.32)$$

3.3.1 The case $w_{\text{turb}} = w < -1$

This assumption means that we equalize the ordinary and turbulent components as far as the EoS is concerned. From Eq. (3.29) we get

$$H = \frac{H_0}{Z}, \quad Z = 1 + \frac{3}{2}\gamma H_0 t. \quad (3.33)$$

This implies a Big Rip singularity after a finite time

$$t_s = \frac{2}{3|\gamma|H_0}, \quad (3.34)$$

and we obtain correspondingly

$$a = a_0 Z^{2/3\gamma}, \quad \rho = \frac{3H_0^2}{\kappa} \frac{1}{Z} \frac{1}{Z + 3\tau H_0}. \quad (3.35)$$

Near t_s we find, using that $Z = 1 - t/t_s$,

$$H \sim \frac{1}{t_s - t}, \quad a \sim \frac{1}{(t_s - t)^{2/3|\gamma|}}, \quad (3.36)$$

$$\rho \sim \frac{1}{t_s - t}, \quad \frac{\rho_{\text{turb}}}{\rho} \sim \frac{1}{t_s - t}. \quad (3.37)$$

which shows the same kind of behavior for H and a as in conventional cosmology but shows also that the singularity in ρ has become more weak. The physical reason for this is obviously the presence of the factor τ .

It is of interest to see how these solutions compare with our assumed form (3.32) for the energy equation. The left hand side of Eq. (3.32) can be calculated, and we obtain in the limit $t \rightarrow t_s$ (details omitted here) the following expression for the present energy dissipation

$$\epsilon_0 = \frac{1}{2} \frac{|\gamma|}{\tau}. \quad (3.38)$$

This result could hardly have been seen without calculation; it shows that the specific dissipation ϵ_0 is closely related to the EoS parameter γ and the parameter τ .

3.3.2 The case $w < -1$, $w_{\text{turb}} > -1$

The turbulent component is accordingly not only a passive component in the fluid. The present assumption encompasses also the region $-1 < w_{\text{turb}} < 0$, in which the turbulent pressure will be negative as before. But if $w_{\text{turb}} > 0$, the turbulent pressure becomes positive as in ordinary hydrodynamics.

The governing equation (3.29) can be solved with respect to t ,

$$t = \frac{2}{3|\gamma|} \left(\frac{1}{H_0} - \frac{1}{H} \right) - \frac{2\tau}{|\gamma|} \left(1 + \frac{\gamma_{\text{turb}}}{|\gamma|} \right) \ln \left[\frac{|\gamma| - 3\tau\gamma_{\text{turb}}H}{|\gamma| - 3\tau\gamma_{\text{turb}}H_0} \frac{H_0}{H} \right], \quad (3.39)$$

showing that the kind of singularity encountered in this case is of the *Little Rip* type.

As $t \rightarrow \infty$, H approaches the finite value

$$H_{\text{crit}} = \frac{1}{3\tau} \frac{|\gamma|}{\gamma_{\text{turb}}}. \quad (3.40)$$

Physically, the role of γ_{turb} is here to soften the evolution toward the future singularity.

3.3.3 One-component model

Instead of assuming the fluid to consist of two components as above, we can instead introduce a one-component model in which the fluid starts out from $t = 0$ as an ordinary viscous non-turbulent fluid and then after some time, here called $t = t_*$, enters a turbulent state of motion. This picture, of course, corresponds more closely to ordinary hydrodynamics.

Now follow the development of such a fluid, assuming as before that $w < -1$, so that the fluid would develop toward a future singularity. After the sudden transition to turbulent motion at t_* , $w \rightarrow w_{\text{turb}}$ and correspondingly $p_{\text{turb}} = w_{\text{turb}} \rho_{\text{turb}}$. As before, we assume $w_{\text{turb}} > -1$, and for simplicity we assume here that ζ is a constant.

We can now easily solve the Friedmann equations, requiring the density of the fluid to be continuous at $t = t_*$. It is convenient to introduce the "viscosity time"

$$t_c = \left(\frac{3}{2} \kappa \zeta \right)^{-1}. \quad (3.41)$$

We then get, for $0 < t < t_*$, [73]

$$H = \frac{H_0 e^{t/t_c}}{1 - \frac{3}{2}|\gamma|H_0 t_c (e^{t/t_c} - 1)}, \quad (3.42)$$

$$a = \frac{a_0}{\left[1 - \frac{3}{2}|\gamma|H_0 t_c (e^{t/t_c} - 1)\right]^{2/3|\gamma|}}, \quad (3.43)$$

$$\rho = \frac{\rho_0 e^{2t/t_c}}{\left[1 - \frac{3}{2}|\gamma|H_0 t_c (e^{t/t_c} - 1)\right]^2}, \quad (3.44)$$

whereas for $t > t_*$,

$$H = \frac{H_*}{1 + \frac{3}{2}\gamma_{\text{turb}}H_*(t - t_*)}, \quad (3.45)$$

$$a = \frac{a_*}{\left[1 + \frac{3}{2}\gamma_{\text{turb}}H_*(t - t_*)\right]^{2/3\gamma_{\text{turb}}}}, \quad (3.46)$$

$$\rho = \frac{\rho_*}{\left[1 + \frac{3}{2}\gamma_{\text{turb}}H_*(t - t_*)\right]^2}. \quad (3.47)$$

Thus the density ρ at first increases with time, and then decreases again until it goes to zero as t^{-2} when $t \rightarrow \infty$. Note that in the turbulent region $p_* = w_{\text{turb}} \rho_*$ will even be greater than zero, when $w_{\text{turb}} > 0$.

We ought to mention finally that the presence of turbulence, or of viscosity in general, may alternatively be dealt with in terms of a more general equation of state, admitting inhomogeneity terms.

3.4 Viscous cosmology and the Cardy-Verlinde formula

The apparent deep connection between general relativity, conformal field theory (CFT), and thermodynamics has aroused considerable interest for several years. In the following we will consider one specific aspect of this problem complex, namely to what extent the Cardy-Verlinde entropy formula remains valid if we allow for bulk viscosity in the cosmic fluid. For simplicity we will assume a one-component fluid model, and we assume the bulk viscosity ζ to be constant. For more details, the reader may consult Refs. [72, 73], and also the related Ref. [74].

We start with the Cardy entropy formula for a (1+1) dimensional CFT,

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}, \quad (3.48)$$

where c is here the central charge and L_0 the lowest Virasoro generator [75]. Compare this with the first Friedmann equation for a closed universe ($k = +1$) when $\Lambda = 0$,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{1}{a^2}. \quad (3.49)$$

As pointed out by Verlinde [76], formal agreement is achieved if we choose

$$L_0 \rightarrow \frac{1}{3}Ea, \quad c \rightarrow \frac{3}{\pi} \frac{V}{Ga}, \quad S \rightarrow \frac{HV}{2G}, \quad (3.50)$$

where V is the volume.

One noteworthy fact is evident already at this stage: the correspondence holds also if the fluid possess viscosity. There is no explicit appearance of viscosity in the first Friedmann equation.

Moreover, the equation of state for the fluid is so far not involved.

To highlight the physical importance of the formal substitutions (3.50), let us consider the thermodynamic entropy of the fluid. As is known, there exist several definitions, the

Bekenstein entropy, the Bekenstein-Hawking entropy, and the Hubble entropy. We will consider only the last quantity here, called S_H . Its order of magnitude can easily be found by observing

that the holographic entropy $A/4G$ of a black hole with the same size as the universe may be written, as $A \sim H^{-2}$, in the form

$$S_H \sim \frac{H^{-2}}{4G} \sim \frac{HV}{4G}. \quad (3.51)$$

Here we have also used that $V \sim H^{-3}$. More careful arguments lead to the replacement of the factor 4 in the denominator with 2. Thus equality is achieved with the last of

Eqs. (3.50), indicating that the formal substitutions above have a physical basis.

Consider now the Casimir energy E_C , defined in this context to be the violation of the Euler identity,

$$E_C = 3(E + pV - TS). \quad (3.52)$$

We may now make use of scaling arguments for the extensive part E_E and the Casimir part E_C that make up the total energy E . These arguments give (details omitted here)

$E(S, V) = E_C(S, V) + \frac{1}{2}E_C(S, V)$. An essential point is the property of conformal invariance is that the products $E_E a$ and $E_C a$ are volume independent and dependent only on S .

We get

$$E_E = \frac{\alpha}{4\pi a} S^{4/3}, \quad E_C = \frac{\beta}{2\pi a} S^{2/3}, \quad (3.53)$$

α, β being constants. Their product is known in CFT, $\sqrt{\alpha\beta} = 3$ for $n = 3$ spatial dimensions. From the formulas above we get

$$S = \frac{2\pi a}{3} \sqrt{E_C(2E - E_C)}. \quad (3.54)$$

This is the Cardy-Verlinde formula. With the substitutions $Ea \rightarrow L_0$ and $E_C \rightarrow c/12$ it is seen that Eqs. (3.53) and (3.48) are in agreement, apart from a numerical pre-factor. This is caused by our assumption about $n = 3$ spatial dimensions instead of the $n = 1$ assumption in the Cardy formula.

The above arguments were made for a radiation dominant, conformally invariant, universe. Do the same arguments apply for a viscous universe also? The subtle point here is the earlier pure entropy dependence of the product Ea , which is now lost. To analyze this question we may consider the following equation, holding for a $k = 1, \Lambda = 0$ universe with EoS $p = \rho/3$,

$$\frac{d}{dt}(\rho a^4) = 9\zeta H^2 a^4. \quad (3.55)$$

This is essentially an equation for the rate of change of the quantity Ea . Compare this with the entropy production formula

$$n\dot{\sigma} = \frac{9H^2}{T}\zeta, \quad (3.56)$$

where n is the particle number density and σ the entropy per particle. Both time derivatives in Eqs. (3.57) and (3.55) are seen to be proportional to ζ . If ζ is small, we can insert for the scale factor the usual solution for the non-viscous case, $a(t) = \sqrt{(8\pi G/3)\rho_{\text{in}} a_{\text{in}}^4} \sin \eta$, with η the conformal time

(“in” means the initial instant). As the densities $\zeta^{-1}\rho a^4$ and $\zeta^{-1}n\sigma$ can then be regarded as functions of t (recall that $\zeta = \text{constant}$), we conclude that ρa^4 can be regarded as a function of $n\sigma$. This implies in turn that Ea can be regarded as a function of S . This property, originally based upon CFT, can thus be carried over also to the viscous case, assumed that the viscosity is small.

The following conceptual point ought however to be observed. The specific entropy σ in Eq. (3.57) is a conventional thermodynamic quantity, whereas the identification $S \rightarrow HV/(2G)$ in Eq. (3.50) is based on the holographic principle. The latter entropy is identified with the Hubble entropy S_H , so we can put $n\sigma_H = H/(2G)$, with σ_H the specific Hubble entropy. The quantity σ_H is holographic-based, whereas the quantity σ is not.

Finally it is to be noticed that the same kind of arguments carry through also in the more general case where the EoS has the form

$$p = (\gamma - 1)\rho, \quad (3.57)$$

with γ a constant. For the nonviscous case this was worked out by Youm [77], with the result

$$S = \left[\frac{2\pi a^{3(\gamma-1)}}{\sqrt{\alpha\beta}} \sqrt{E_C(2E - E_C)} \right]^{\frac{3}{3\gamma-1}}. \quad (3.58)$$

Also in this case, the generalization to a weak viscous case can be made [72, 73]. When $\gamma = 4/3$, the radiation dominant result is recovered.

4 The accelerating universe with generalized fluids

In cosmology, it is well-known that for the Friedmann-Lemaître-Robertson-Walker geometry, when the equation of state modeling the matter content was a linear equation with an equation of state parameter greater than -1 , a singularity named Big Bang appears at early times, where the energy density of the universe diverges. Moreover, dealing with nonlinear equations of state one can see that other kind of singularities such as Sudden singularity [78, 79, 80] or Big Freeze [81, 82, 83] appear.

In fact, the future singularities are classified as follows [81] (see also [84] for a more detailed classification):

- Type I (Big Rip): $t \rightarrow t_s, a \rightarrow \infty, \rho \rightarrow \infty$ and $|p| \rightarrow \infty$.
- Type II (Sudden): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \rho_s$ and $|p| \rightarrow \infty$.
- Type III (Big Freeze): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \infty$ and $|p| \rightarrow \infty$.
- Type IV (Generalized Sudden): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow 0, |p| \rightarrow 0$ and derivatives of H diverge.

Analogously as the future ones, one can define the past singularities:

- Type I (Big Bang): $t \rightarrow t_s, a \rightarrow 0, \rho \rightarrow \infty$ and $|P| \rightarrow \infty$.
- Type II (Past Sudden): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \rho_s$ and $|P| \rightarrow \infty$.
- Type III (Big Hottest): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow \infty$ and $|P| \rightarrow \infty$.
- Type IV (Generalized past Sudden): $t \rightarrow t_s, a \rightarrow a_s, \rho \rightarrow 0, |P| \rightarrow 0$ and derivatives of H diverge.

For the simple case of a linear equation of state $p = w\rho$. It is well-known that for non-phantom fluid ($w > -1$) one obtains a Big Bang singularity and for a phantom fluid the singularity is a future Type I (Big Rip).

To obtain the other type of singularities one has to consider phantom fluids modeled by non-linear equations of state of the form $p = -\rho - f(\rho)$, where f is a positive function. The simplest model is obtained taking $f(\rho) = A\rho^\alpha$ with $A > 0$. In this case from the conservation equation $\dot{\rho} = -3H(\rho + p)$ and the Friedmann equation $H^2 = \frac{\kappa\rho}{3}$ one obtains the dynamical equation

$$\dot{\rho} = \sqrt{3\kappa A}\rho^{\alpha+\frac{1}{2}}, \quad (4.59)$$

which solution is

$$\rho = \begin{cases} \left(\frac{\sqrt{3\kappa A}}{2}(t - t_0)(1 - 2\alpha) + \rho_0^{\frac{1}{2}-\alpha} \right)^{\frac{2}{1-2\alpha}} & \text{when } \alpha \neq \frac{1}{2} \\ \rho_0 e^{\sqrt{3\kappa A}(t-t_0)} & \text{when } \alpha = \frac{1}{2}. \end{cases} \quad (4.60)$$

To obtain the evolution of the scale factor we will integrate the conservation equation, obtaining

$$a = a_0 \exp \left(\frac{1}{3} \int_{\rho_0}^{\rho} \frac{\bar{\rho}}{f(\bar{\rho})} \right), \quad (4.61)$$

which leads, in our case to

$$a = \begin{cases} a_0 \exp\left(\frac{1}{3A(1-\alpha)}(\rho^{1-\alpha} - \rho_0^{1-\alpha})\right) & \text{when } \alpha \neq 1 \\ a_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3A}} & \text{when } \alpha = 1. \end{cases} \quad (4.62)$$

Once we have calculated these quantities, we have the following different situations (see also [81]):

1. When $\alpha < 0$ we have a past singularity Type II, because for the energy density vanishes for $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} < t_0$, meaning that the pressure diverges at $t = t_s$.
2. When $\alpha = 0$ there are not singularities. The dynamics is defined from $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \rho_0^{\frac{1}{2}}$ (the energy density is zero) to ∞ .
3. When $0 < \alpha < \frac{1}{2}$ there are two different cases:
 - (a) $\frac{1}{1-2\alpha}$ is not a natural number. One has a past Type IV singularity at $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha}$ because higher derivatives of H diverge at $t = t_s$.
 - (b) $\frac{1}{1-2\alpha}$ is a natural number. In that case there are not singularities and the dynamics is defined from $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha}$ to ∞ .
4. When $\alpha = \frac{1}{2}$ there are not singularities in cosmic time.
5. When $\frac{1}{2} < \alpha < 1$, one has future singularities Type I, because in this case ρ , p and a diverges at $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} > t_0$.
6. When $\alpha = 1$ the equation of state is linear, so we obtain a Big Rip singularity.
7. When $\alpha > 1$, the energy density and the pressure diverge but the scale factor remains finite at $t = t_s$, meaning that we have a future Type III singularity.

The remarkable case appears when $0 < \alpha < \frac{1}{2}$ and $\frac{1}{1-2\alpha}$ is a natural number. In that case, from the Friedmann equation $H^2 = \frac{\kappa\rho}{3}$ and the solution (4.60) one obtains

$$H = \sqrt{\frac{\kappa}{3}} \left(\frac{\sqrt{3\kappa A}}{2} (t - t_0)(1 - 2\alpha) + \rho_0^{\frac{1}{2}-\alpha} \right)^n, \quad (4.63)$$

with $n = \frac{1}{1-2\alpha}$. As we have already seen, this solution modeling a universe in the expanding phase driven by a phantom fluid is defined from $t_s = t_0 - \frac{2}{\sqrt{3\kappa A}} \frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha}$ (where $H = 0$) to ∞ . However, the solution (4.63) could be extended analytically back in time. There are two different situations: When n is odd, this extended solution shows a universe driven by a phantom field that goes to the contracting to expanding phase, bouncing at time t_s . On the contrary, when n is even, the universe only moves in the expanding phase, before t_s it is driven by a non-phantom field, and after t_s the universe enters in a phantom era. We will explain this phenomenon in more detail in next subsection.

4.1 Inhomogeneous equation of state of the universe: phantom era and singularities

Motivated by the introduction of bulk viscous terms in an ideal fluid (see equation (1.26)), one can include, in the equation of state, a term depending of the Hubble parameter, obtaining

$$p = -\rho - f(\rho) + G(H). \quad (4.64)$$

Then, the conservation equation becomes $\dot{\rho} = 3H(f(\rho) - G(H))$, that using the Friedmann equation, in the expanding phase has the form

$$\dot{\rho} = 3H \left(f(\rho) - G \left(\sqrt{\frac{\kappa\rho}{3}} \right) \right) \equiv 3HF(\rho), \quad (4.65)$$

what shows that, this formalism is equivalent to consider a fluid with an effective equation of state equal to

$$p = -\rho - F(\rho) = -\rho - f(\rho) + G \left(\sqrt{\frac{\kappa\rho}{3}} \right). \quad (4.66)$$

It is clear that in general, the equation of state (4.64), does not lead to a universe crossing the phantom barrier. A simple way, to obtain transitions to the non-phantom to the phantom regime, is to consider implicit inhomogeneous equations of state of the form $F(\rho, p, H) = 0$, for example [85]

$$(\rho + p)^2 - C_0 \rho^2 \left(1 - \frac{H_0}{H}\right) = 0, \quad (4.67)$$

being C_0 and H_0 some positive constants.

Taking the square of the equation $\dot{H} = -\frac{\kappa}{2}(\rho + p)$, and inserting on it (4.67) one obtains the the bi-valued dynamical equation

$$\dot{H}^2 = \frac{9}{4} C_0 H^4 \left(1 - \frac{H_0}{H}\right). \quad (4.68)$$

From this equation, since there are two square roots and the effective equation of state parameter is given by $w_{eff} \equiv -1 - \frac{2\dot{H}}{3H^2}$, one can see that there are two different dynamics: the one which corresponds to the branch with $H < 0$ depicting a universe in a non-phantom regime and the one corresponding to the branch $H > 0$ depicting a universe in the phantom era.

In fact, the equation (4.68) can be integrated as

$$H(t) = \frac{16}{9C_0^2 H_0 (t - t_-)(t_+ - t)}, \quad (4.69)$$

where we have introduced the notation $t_{\pm} = \pm \frac{4}{3C_0 H_0}$.

It is easy to check that, $H(t)$ is only defined for time between t_- and t_+ , because at t_{\pm} H diverges (Big Bang at t_- and Big Rip at t_+). Moreover, it is a decreasing function for $t \in (t_-, 0)$ and increasing for $t \in (0, t_+)$, meaning that at $t = 0$ the universe crosses the phantom barrier (it passes from the non-phantom to the phantom era).

Another interesting example, is given by the equation of state

$$(\rho + p)^2 + \frac{16H_1}{\kappa^2 t_0^2} (H_0 - H) \ln \left(\frac{H_0 - H}{H_1} \right) = 0, \quad (4.70)$$

where t_0, H_0, H_1 are some parameters satisfying $H_0 > H_1 > 0$. The corresponding bi-valued dynamical equation is

$$\dot{H}^2 = -\frac{4H_1}{t_0^2} (H_0 - H) \ln \left(\frac{H_0 - H}{H_1} \right), \quad (4.71)$$

which has two fixed points H_0 and $H_0 - H_1$. As we have already explained, when $\dot{H} < 0$ (resp. $\dot{H} < 0$) the universe is in a non-phantom (resp. phantom) era. Then, when the universe is in the branch with $\dot{H} < 0$ it moves from H_0 to $H_0 - H_1$, when it reaches $H = H_0$ and enters in the other branch ($\dot{H} > 0$) going from $H_0 - H_1$ to H_0 . In fact, in [85] the authors found the following solution,

$$H(t) = H_0 - H_1 \exp\left(-\frac{t^2}{t_0^2}\right), \quad (4.72)$$

which satisfy all the properties described above.

A final remark is in order: One can assume more general equations of state of the form $F(\rho, p, H, \dot{H}, \ddot{H}, \dots) = 0$ containing higher order derivatives of the Hubble parameter. In this case, using the Friedmann equations

$$H^2 = \frac{\kappa\rho}{3}, \quad \dot{H} = -\frac{\kappa}{2}(\rho + p), \quad (4.73)$$

the equation of state becomes the dynamical equation

$$F\left(\frac{3H^2}{\kappa}, -\frac{2\dot{H}}{\kappa} - \frac{3H^2}{\kappa}, \dot{H}, \ddot{H}, \dots\right) = 0 \iff G(H, \dot{H}, \ddot{H}, \dots) = 0. \quad (4.74)$$

A non-trivial example, is the following equation of state [85]:

$$p = w\rho - G_0 - \frac{2}{\kappa}\dot{H} + G_1\dot{H}^2, \quad (4.75)$$

where G_0 and G_1 are some constant. Then, the dynamical equation becomes

$$-\frac{3H^2(1+w)}{\kappa} = -G_0 + G_1\dot{H}^2. \quad (4.76)$$

We look for periodic solutions of the form $H(t) = H_0 \cos(\Omega t)$ depicting and oscillatory universe. Inserting this expression in (4.76), we obtain the algebraic system:

$$G_0 = G_1\Omega^2 H_0^2, \quad G_0 = \frac{3H_0^2(1+w)}{\kappa}, \quad (4.77)$$

which solution is given by

$$H_0 = \sqrt{\frac{\kappa G_0}{3(1+w)}}, \quad \Omega = \sqrt{\frac{3(1+w)}{\kappa G_1}}, \quad (4.78)$$

provided by $G_0(1+w) > 0$ and $G_1(1+w) > 0$.

On the other hand, when $G_1(1+w) < 0$, one can look for solutions of the form $H(t) = H_0 \cosh(\Omega t)$, obtaining

$$H_0 = \sqrt{\frac{\kappa G_0}{3(1+w)}}, \quad \Omega = \sqrt{-\frac{3(1+w)}{\kappa G_1}}. \quad (4.79)$$

4.2 Unification of inflation with dark energy in viscous cosmology

The simplest way to unify early inflationary epoch with the current cosmic acceleration is using scalar fields. Starting with the action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa} R - \frac{1}{2} \omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\}, \quad (4.80)$$

where ω and V are some functions of the scalar field ϕ , when one deals with the flat FLRW geometry one obtains the following dynamical equation

$$\omega(\phi) \ddot{\phi} + \frac{1}{2} \omega'(\phi) \dot{\phi}^2 + 3H\omega(\phi) \dot{\phi} + V'(\phi) = 0. \quad (4.81)$$

The relevant fact, is that given a function $f(\phi)$, the equation (4.81) has always the solution $\phi = t$ and $H = f(t)$, provided that (see for details [87])

$$\omega(\phi) = -\frac{2}{\kappa} f'(\phi), \quad \text{and} \quad V(\phi) = \frac{1}{\kappa} (3f^2(\phi) + f'(\phi)). \quad (4.82)$$

An interesting example is obtained when one considers the function

$$f(\phi) = H_0 \left(\frac{\phi_s}{\phi} + \frac{\phi_s}{\phi_s - \phi} \right), \quad (4.83)$$

where H_0 and ϕ_s are the two positive parameters of the model. In this case one has

$$\begin{aligned} \omega(\phi) &= \frac{2H_0\phi_s^2(\phi_s - 2\phi)}{\kappa\phi^2(\phi_s - \phi)^2}, \\ V(\phi) &= \frac{H_0\phi_s^2}{\kappa\phi^2(\phi_s - \phi)^2} (3H_0\phi_s^2 - \phi_s + 2\phi), \end{aligned} \quad (4.84)$$

whose dynamics is given by

$$H = \frac{H_0 t_s^2}{t(t_s - t)}, \quad a = a_0 \left(\frac{t}{t_s - t} \right)^{H_0 t_s}, \quad (4.85)$$

where we have introduced the notation $t_s = \phi_s$.

Since, H diverges at $t = 0$ and $t = t_s$ the dynamics is defined in $(0, t_s)$. In fact at $t = 0$ one has $a = 0$, what means that we have a Big Bang singularity and at $t = t_s$ the scale factor diverges, meaning that we have a Big Rip singularity.

On the other hand the derivative of the Hubble parameter is

$$\dot{H} = \frac{H_0 t_s^2}{t^2 (t_s - t)^2} (2t - t_s), \quad (4.86)$$

that is, $w_{eff} > -1$ when $0 < t < t_s/2$, and the universe is in the phantom phase ($w_{eff} > -1$) for $t_s/2 < t < t_s$. Then, we conclude that this model could depict the current cosmic acceleration. To see what happens at early times we note that near $t = 0$ one can make the approximation $a = a_0 \left(\frac{t}{t_s} \right)^{H_0 t_s}$, and thus, its second derivative at early times is approximately

$$\ddot{a} = a \frac{H_0 t_s (H_0 t_s - 1)}{t^2}, \quad (4.87)$$

meaning that if one chooses $H_0 t_s > 1$ the universe will have an early period of acceleration.

Another example is to consider

$$f(\phi) = H_0 \sin(\nu\phi), \quad (4.88)$$

with H_0 and ν positive parameters. A simple calculation leads to

$$\begin{aligned} \omega(\phi) &= -\frac{2H_0\nu}{\kappa} \cos(\nu\phi) \\ V(\phi) &= \frac{2}{\kappa} (H_0\nu \cos(\nu\phi) + H_0^2 \sin^2(\nu\phi)). \end{aligned} \quad (4.89)$$

In this case one obtains a non-singular oscillating universe whose dynamics is given by

$$H = H_0 \sin(\nu t), \quad a = a_0 \exp\left(-\frac{H_0}{\nu} \cos(\nu t)\right). \quad (4.90)$$

This solution depicts a universe that bounces at time $t = \frac{n\pi}{\nu}$ where n is an integer, and since $\dot{H} = H_0\nu \cos(\nu t)$ one can easily check that the universe is in a phantom phase when $\frac{\pi}{\nu}(-\frac{1}{2} + 2n) < t < \frac{\pi}{\nu}(\frac{1}{2} + 2n)$ and in a non-phantom one when $\frac{\pi}{\nu}(\frac{1}{2} + 2n) < t < \frac{\pi}{\nu}(\frac{3}{2} + 2n)$.

To introduce viscosity we consider the equation (1.31) with $\Lambda = 0$ and $w = 1$. Based in the equivalence between bulk viscous and open cosmology, where isentropic particle production is allowed [88], we choose the following viscosity coefficient [89]

$$\zeta(H) = \frac{1}{\kappa} \left(-\xi_0 + 2H + \frac{\xi_0^2}{8H} \right), \quad (4.91)$$

where $\xi_0 > 0$ is a constant.

Then, the corresponding dynamical equation is

$$\dot{H} = -\frac{3}{2}H\xi_0 + \frac{3}{16}\xi_0^2, \quad (4.92)$$

which only has $H = \frac{\xi_0}{8}$ as a fixed point.

If one consider the dynamics in the domain $\frac{\xi_0}{8} \leq H \leq \infty$, it is easy to check that the effective equation of state parameter is greater than -1 , what means that the Hubble parameter goes from ∞ to $\frac{\xi_0}{8}$. Moreover, since

$$w_{eff} = -1 + \frac{\xi_0}{H} - \frac{\xi_0^2}{8H^2}, \quad (4.93)$$

$w_{eff} \cong -1$ at early ($H \gg \xi_0$) and late ($H \cong \frac{\xi_0}{8}$) times, what means that the model unifies inflation with the current cosmic acceleration. On the other hand, w_{eff} is positive when $\xi_0 \frac{\sqrt{2}-1}{2\sqrt{2}} < H < \frac{\sqrt{2}+1}{2\sqrt{2}}$, having the maximum value $w_{eff} = 1$ at $H = \frac{\xi_0}{4}$, then in the model the universe starts from an inflationary epoch, evolving through a kination ($w_{eff} = 1$), radiation ($w_{eff} = 1/3$) and a matter-domination ($w_{eff} = 0$) epoch, and finally finishing in a de Sitter phase.

In fact, the solution of the equation (4.92) is

$$H = \frac{\xi_0}{8} \left(e^{-\frac{3}{2}\xi_0 t} + 1 \right), \quad (4.94)$$

and the scalar field that leads to this dynamics, if one chooses $\omega(\phi) \equiv 1$, is the following Higgs-style potential (see for details [89])

$$V(\phi) = \frac{27\xi_0^2\kappa}{256} \left(\phi^2 - \frac{2}{3\kappa} \right)^2. \quad (4.95)$$

We want to stress that this models, leads to theoretical values of the spectral index, its running and the ratio of tensor to scalar perturbations that match at 2σ Confidence Level, with the observational data provided by PLANCK+WP 2013 [90] (see for a detailed discussion [89, 91]).

To end the section, we consider a quintessential-inflation potential [92] which unifies inflation with the late time acceleration

$$V(\phi) = \begin{cases} \frac{9}{2} \left(H_E^2 - \frac{\Lambda}{3} \right) \left(\phi^2 - \frac{2}{3\kappa} \right) & \text{for } \phi \leq \phi_E \\ \frac{\Lambda}{\kappa} & \text{for } \phi \geq \phi_E, \end{cases} \quad (4.96)$$

where $\phi_E \equiv -\sqrt{\frac{2}{3\kappa}} \frac{H_E}{\sqrt{H_E^2 - \frac{\Lambda}{3}}}$, being $H_E > 0$ the parameter of the model.

This model leads to the following dynamics

$$\dot{H} = \begin{cases} -3H_E^2 + \Lambda & \text{for } H \geq H_E \\ -3H^2 + \Lambda & \text{for } H \leq H_E, \end{cases} \quad (4.97)$$

whose solution has the following expression

$$H(t) = \begin{cases} (-3H_E^2 + \Lambda)t + 1 & t \leq 0 \\ \sqrt{\frac{\Lambda}{3}} \frac{3H_E + \sqrt{3\Lambda} \tanh(\sqrt{3\Lambda}t)}{3H_E \tanh(\sqrt{3\Lambda}t) + \sqrt{3\Lambda}} & t \geq 0, \end{cases} \quad (4.98)$$

and the corresponding scale factor is

$$a(t) = \begin{cases} a_E e^{[(-3H_E^2 + \Lambda)\frac{t^2}{2} + t]} & t \leq 0 \\ a_E \left(\frac{3H_E}{\sqrt{3\Lambda}} \sinh(\sqrt{3\Lambda}t) + \cosh(\sqrt{3\Lambda}t) \right)^{\frac{1}{3}} & t \geq 0. \end{cases} \quad (4.99)$$

Note that, this dynamics also comes from a universe with a cosmological constant Λ and filled by a fluid with the simple linear equation of state

$$P = \begin{cases} -\rho + 2\rho_E & \rho \geq \rho_E \\ \rho & \rho \leq \rho_E, \end{cases} \quad (4.100)$$

where $\rho_E = \frac{3H_E^2 - \Lambda}{\kappa}$, or equivalently from a viscous fluid. Effectively choosing in $w = 2$ and the following viscosity coefficient

$$\zeta = \begin{cases} \frac{2}{\kappa} \left(H - \frac{H_E^2}{H} \right) & H \geq H_E \\ 0 & H \leq H_E, \end{cases} \quad (4.101)$$

and inserting it in (1.31) one obtains the dynamics (4.97).

On the other hand, for this model, the effective equation of state parameter is given by

$$w_{eff} = \begin{cases} -1 + \frac{2}{3H^2} (3H_E^2 - \Lambda) & H \geq H_E \\ 1 - \frac{2\Lambda}{3H^2} & H \leq H_E, \end{cases} \quad (4.102)$$

which shows that for $H \gg H_E$ one has $w_{eff}(H) \cong -1$ (early quasi de Sitter period). When $H \cong H_E$, the equation of state parameter satisfies $w_{eff}(H) \cong 1$ (kination or deflationary period), and finally, for $H \cong \sqrt{\frac{\Lambda}{3}}$ one also has $w_{eff}(H) \cong -1$ (late quasi de Sitter period).

References

- [1] C. W. Misner, *Astrophys. J.* **151**, 431 (1968).
- [2] Ya. B. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics*, Vol. II (The University of Chicago Press, Chicago, 1983).
- [3] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [4] Ø. Grøn, *Astrophys. Space Sci.* **173**, 191 (1990).
- [5] R. Maartens, *Class. Quant. Grav.* **12**, 1455 (1995).
- [6] R. Maartens, Causal thermodynamics in relativity. In *Proceedings of Hanno Rund Workshop on Relativity and Thermodynamics*, South Africa, June 1996; arXiv:astro-ph/9609119.
- [7] I. Brevik and Ø. Grøn, Relativistic Viscous universe models. In *Recent Advances in Cosmology* (edited by A. Travena and B. Soren, Nova Scient. Publ., New York, 2013), p. 99; arXiv:1409.8561.
- [8] S. Weinberg, *Astrophys. J.* **68**, 175 (1971).
- [9] I. Brevik and L. T. Heen, *Astrophys. Space Sci.* **219**, 99 (1994).
- [10] W. Zimdahl, *Mon. Not. R. Astronom. Soc.* **280**, 1239 (1996).
- [11] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003).
- [12] S. Nojiri and S. D. Odintsov, *Phys. Rev. D* **72**, 023003 (2005).
- [13] V. F. Cardone, C. Tortora, A. Troisi, and S. Capozziello, *Phys. Rev. D* **73**, 043508 (2006).
- [14] N. Mostafapoor and Ø. Grøn, *Astrophys. Space Sci.* **333**, 357 (2011).
- [15] I. Brevik, E. Elizalde, S. Nojiri and S.D. Odintsov, *Phys. Rev. D* **84**, 103508 (2011).
- [16] I. Brevik, R. Myrzakulov, S. Nojiri and S.D. Odintsov, *Phys. Rev. D* **86**, 063007 (2012).

- [17] I. Brevik, *Entropy* **17**, 6318 (2015).
- [18] I. Brevik, V. V. Obukhov and A. V. Timoshkin, *Astrophys. Space. Sci.* **355**, 399 (2015).
- [19] A. Sasidharan and T. K. Mathew, *JHEP* **06**, 138 (2016).
- [20] K. Bamba and S. D. Odintsov, *Eur. Phys. J. C* **76**, 18 (2016).
- [21] B. D. Normann and I. Brevik, *Entropy* **18**, 215 (2016).
- [22] C. E. Laciana, arXiv:1611.02611 [gr-qc].
- [23] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, Oxford, 1987).
- [24] A. H. Taub, *Annual Rev. Fluid Mechanics* **10**, 301 (1978).
- [25] Ø. Grøn, *Eur. J. Phys.* **23**, 135 (2002).
- [26] G. J. Mathews, L. Q. Lan and C. Kolda, *Phys. Rev. D* **78**, 043525 (2008).
- [27] T. Padmanabhan and S. M. Chitre, *Phys. Lett.* **120**, 433 (1987).
- [28] J. Ren and X. Meng, *Phys. Lett. B* **633**, 1 (2006).
- [29] M. Hu and X. Meng, *Phys. Lett. B* **635**, 186 (2006).
- [30] M. Mostafapoor and Ø. Grøn, *Astrophys. Space Sci.* **333**, 357 (2011).
- [31] M. Cataldo, N. Cruz and S. Lepe, *Phys. Lett. B* **619**, 5 (2005).
- [32] G. M. Kremer and F. P. Devecchi, *Phys. Rev. D* **67**, 047301 (2003).
- [33] J. C. Fabris, S. V. B. Goncalves and R. de Sá Ribeiro, *Gen. Relativ. Gravit.* **38**, 495 (2006).
- [34] A. Avelino and U. Nucamendi, *JCAP*, **006** (2009).
- [35] I. Brevik and O. Gorbunova, *Gen. Relativ. Gravit.* **37**, 2039 (2005).

- [36] Ø. Grøn, Statefinder analysis of universe models with a viscous cosmic fluid and with a fluid having a non-linear equation of state. In *The Casimir Effect and Cosmology*, Tomsk State Pedagogical University Press 2009 ISBN 978-5-89428-311-1, p. 75.
- [37] C. P. Singh, *Pramana* **71**, 30 (2008).
- [38] B. D. Normann and I. Brevik, *Entropy* **18**, 215 (2016).
- [39] J. Wang and X. Meng, *Mod. Phys. Lett. A* **29**, 390 (2014).
- [40] Y. Chen et al., *J. Cosmolog. Astropart. Phys.* **2**, (2015).
- [41] C. P. Singh, A. Beesham and W. S. Mbokazi, *Gen. Relativ. Gravit.* **30**, 573 (1998).
- [42] C. P. Singh, S. Kumar and A. Pradhan, *Class. Quant. Grav.* **24**, 455 (2007).
- [43] A. Berera, I. G. Moss and R. O. Ramos, *Rep. Prog. Phys.* **72**, 026901 (2009).
- [44] S. del Campo, in *Aspects of Today's Cosmology*. Edited by A. Alfonso-Faus, InTech (2011).
- [45] S. Bartrum, A. Berera and J. G. Rosa, *JCAP* **06**, 025 (2013); arXiv:1303.508 [astro-ph.CO].
- [46] G. Panotopoulos and N. Videla, arXiv:1510.0698 [gr-qc].
- [47] Ø. Grøn, *Universe* **2**, 20 (2016).
- [48] W. H. Kinney, Cosmology, inflation, and the physics of nothing, in *Techniques and Concepts of High-Energy Physics XII*, NATO Science Series **123**, 189 (2003); arXiv:0301448 [astro-ph].
- [49] P. Ade et al., Planck 2013 results. XVI. Cosmological parameters. *Astronom. Astrophys.* **571**, A16, arXiv:1303.5076 [astro-ph.CO].
- [50] P. Ade et al., Planck 2013 results. XXII. Constraints on inflation. *Astron. Astrophys.* **571**, A22 (2014), arXiv:1303.5082 [astro-ph.CO].

- [51] P. Ade et al., A joint analysis of BICEP2/Keck Array and Planck Data. *Stron. Astrophys.* **594**, A13, arXiv:1502.00612 [astro-ph.CO].
- [52] P. Ade et al., Planck 2015 results.XX. Cosmological parameters. *Astron. Astrophys.* **594**, A13 (2016), arXiv:1502.01589 [astro-ph.CO].
- [53] P. D. Lasky et al., *Phys. Rev. X* **6**, 011035 (2016).
- [54] Q. Huang and S. Wang, arXiv:1502.02541 [astro-ph.CO].
- [55] J. D. Barrow, *Phys. Lett. B* **235**, 40 (1990).
- [56] A. Mohammadi, Z. Ossoulian, T. Golanbari and K. Saaidi, *Astrophys. Space Sci.* **359**, 7 (2015).
- [57] K. Rezazadeh, K. Karami and P. Karimi, *JCAP* **09** 053 (2015), arXiv:1411.7302v4 [gr-qc].
- [58] M. Bastero-Gil and A. Berera, *Int. J. Mod. Phys. A* **24**, 2207 (2009).
- [59] L. Vasinelli, *JCAP* **1607**, 054 (2016), arXiv:1605.06449.
- [60] M. R. Setare and V. Kamali, *Gen. Rel. Gravitat.* **46**, 1698 (2014).
- [61] M. Sharif and R. Saleem, *Astroparticle Physics* **62**, 241 (2015).
- [62] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003).
- [63] S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **562**, 147 (2003).
- [64] H. Frampton, K. J. Ludwick and R. J. Scherrer, *Phys. Rev. D* **84**, 063003 (2011).
- [65] P. H. Frampton, K. J. Ludwick, S. Nojiri, S. D. Odintsov and R. J. Scherrer, *Phys. Lett. B* **708**, 204 (2012).
- [66] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, *Phys. Rev. D* **85**, 083001 (2012).
- [67] H. Wei H., L. F. Wang and X. J. Guo, *Phys.Rev. D* **86**, 083003 (2012).
- [68] I. Brevik and A. V. Timoshkin, *Int. J. Geom. Methods Mod. Phys.*, in press; arXiv:1612.06689 [gr-qc].

- [69] B. D. Normann and I. Brevik, Mod. Phys. Lett. A, in press; arXiv:1612.01794 [gr-qc].
- [70] I. Brevik, A. V. Timoshkin, Y. Rabochaya and S. Zerbini, Astrophys. Space Sci. **347**, 203 (2013).
- [71] I. Brevik, O. Gorbunova, S. Nojiri and S. D. Odintsov, Eur. Phys. J. C **71**, 1629 (2011).
- [72] I. Brevik, Int. J. Mod. Phys. A **18**, 2145 (2003).
- [73] I. Brevik, Viscous cosmology, entropy, and the Cardy-Verlinde formula. In *Horizons in World Physics*, Vol. **246**, pp. 165-183, Nova Science Publishers, 2005; gr-qc/0404095.
- [74] I. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, Phys. Rev. D **70**, 043520 (2004).
- [75] J. L. Cardy, Nucl. Phys. B **270**, 186 (1986); B **275**, 200 (1986).
- [76] E. Verlinde, hep-th/0008140.
- [77] D. Youm, Phys. Lett. B **531**, 276 (2002).
- [78] J. D. Barrow, Class. Quant. Grav., **21**, L79 (2004).
- [79] S. Nojiri and S. D. Odintsov, Phys. Lett. B **595**, 1 (2004).
- [80] J. D. Barrow, Class. Quant. Grav. **21**, 5619 (2004).
- [81] S. Nojiri, S. D. Odintsov, and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005).
- [82] S. Nojiri, and S. D. Odintsov, Phys. Lett. B **686**, 44 (2010).
- [83] A. V. Astashenok, S. Nojiri, Sergei D. Odintsov, R. J. Scherrer, Phys.Lett. B **713**, 145 (2012).
- [84] L. Fernández-Jambrina, *Phys. Rev. D* **90**, 064014 (2014).
- [85] S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005).

- [86] K. Bamba, S. Nojiri, S. D. Odintsov and D. Sáez-Gómez, Phys. Rev. D **90**, 124061 (2014).
- [87] S. Nojiri and S. D. Odintsov, Gen.Rel.Grav. **38**, 1285 (2006).
- [88] I. Prigogine, J. Geheniau, E. Gunzig, and P. Nardone, Proc. Natl. Acad. Sci **85**, 7428 (1998).
- [89] J. Haro and S. Pan, arXiv:1512.03033 [gr-qc] (2015).
- [90] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 571, A22 (2014).
- [91] J. de Haro, J. Amorós and S. Pan, Phys. Rev. D **93**, 084018 (2016).
- [92] J. de Haro, J. Amorós and S. Pan, Phys. Rev. D. **94**, 064060 (2016).