

Characterizing identifying codes from the spectrum of a graph or digraph *

C. Balbuena¹, C. Dalfo², B. Martínez-Barona¹

¹Departament d'Enginyeria Civil i Ambiental, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain.

²Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain.

e-mails:{m.camino.balbuena, cristina.dalfo, berenice.martinez}@upc.edu

Abstract

A $(1, \leq \ell)$ -identifying code in digraph D is a dominating subset C of vertices of D , such that all distinct subsets of vertices of D with cardinality at most ℓ have distinct closed in-neighbourhoods within C . As far as we know, it is the first time that the relation between identifying codes digraphs and their spectra is studied. The obtained results can also be applied on graphs.

Mathematics Subject Classifications: 05C69, 05C20, 05C50.

Keywords: Graph, digraph, identifying code, adjacency matrix, spectrum, eigenvectors.

1 Introduction

We consider simple digraphs (or directed graphs) without loops or multiple edges. Unless otherwise stated, we follow the textbook by Bang-Jensen and Gutin [4] for terminology

*This research is supported by *MINECO* under project MTM2014-60127-P, and the *Catalan Research Council* under project 2014SGR1147. B. Martínez-Barona has received funding from the scholarship 254379/438356 from CONACYT from Mexico.



C. Dalfo has also received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

and definitions. Besides, regarding spectral graph theory, we use the notation of Godsil and Royle [8].

Let $D = (V, E)$ be a digraph with vertex set V and arc set E . A vertex u is *adjacent* to a vertex v if $(u, v) \in E$. If there exist arcs (u, v) and (v, u) , then we say that both arcs form a *digon*. A digraph is *symmetric* if $(u, v) \in E$ implies $(v, u) \in E$. A digon is often called a *symmetric arc* of D . A digraph D is said to be *oriented* if it has no digon. The girth g of a digraph is the length of a shortest cycle. Hence, an oriented digraph has girth $g \geq 3$. Moreover, observe that every (undirected) graph G with vertex set V and edge set E can be seen as a symmetric digraph denoted by $\overset{\leftrightarrow}{G}$ replacing each edge $uv \in E$ by the digon (u, v) and (v, u) . In other words, a graph can be seen as a symmetric digraph.

The *out-neighborhood* of a vertex u is $N^+(u) = \{v \in V : (u, v) \in E\}$ and the *in-neighborhood* of u is $N^-(u) = \{v \in V : (v, u) \in E\}$. The *closed in-neighbourhood* of u is $N^-[u] = \{u\} \cup N^-(u)$, and the closed out-neighbourhood of u is defined analogously. The *out-degree* of u is $d^+(u) = |N^+(u)|$ and its *in-degree* $d^-(u) = |N^-(u)|$. We denote by $\delta^+ = \delta^+(D)$ the minimum out-degree of the vertices in D , and by $\delta^- = \delta^-(D)$ the minimum in-degree. The minimum degree of D is $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. A digraph D is said to be *d-in-regular* if $|N^-(v)| = d$ for all $v \in V$, and *d-regular* if $|N^+(v)| = |N^-(v)| = d$ for all $v \in V$.

The spectrum of the adjacency matrix A of a graph G or digraph D is denoted by $\text{sp}(A) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where λ_i are the different eigenvalues and the superscripts stand for their (algebraic) multiplicities $m_i = m(\lambda_i)$, whereas $ev(A)$ is the set of the different eigenvalues (without their multiplicities).

Given a vertex subset $U \subset V$, let $N^-[U] = \bigcup_{u \in U} N^-[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subset V$ is a $(1, \leq \ell)$ -*identifying code* in D when, for all distinct subsets $X, Y \subset V$, with $1 \leq |X|, |Y| \leq \ell$, we get

$$N^-[X] \cap C \neq N^-[Y] \cap C. \quad (1)$$

The definition of a $(1, \leq \ell)$ -identifying code for graphs was introduced by Karpovsky, Chakrabarty, and Levitin [9], where the minus sign superscripts in (1) can be omitted. Thus, the definition for digraphs is a natural extension of the concept of $(1, \leq \ell)$ -identifying codes in graphs. A $(1, \leq 1)$ -identifying code is referred to an *identifying code*.

Laihonen [10] proved the following result for graphs.

Theorem 1.1. [10] *Let $k \geq 2$ be an integer.*

1. *If a k -regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$ -identifying code.*
2. *If a k -regular graph has girth $g \geq 5$, then it admits a $(1, \leq k - 1)$ -identifying code.*

Besides, Laihonen and Ranto [11] showed that if G is a connected graph with at least

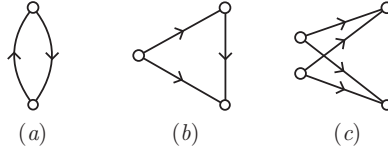


Figure 1: The subdigraphs forbidden by Lemma 1.1: (a) The digon, (b) TT_3 , and (c) $BT_{2,2}$.

three vertices admitting a $(1, \leq \ell)$ -identifying code, then $\ell \leq \delta$, where δ is the minimum degree of G .

Regarding digraphs, the authors proved in [2] that every 1-in-regular digraph has a $(1, \leq 2)$ -identifying code if and only if its girth is at least 5. They also characterized the 2-in-regular digraphs having a $(1, \leq 2)$ -identifying code or a $(1, \leq 3)$ -identifying code. Moreover, they gave some sufficient conditions for a digraph of minimum in-degree $\delta^- \geq 2$ to admit a $(1, \leq \delta^-)$ -identifying code. As a corollary of this result, they proved that a graph of minimum degree $\delta \geq 2$ and girth at least 7 admits a $(1, \leq \delta)$ -identifying code.

Recall that if D admits a $(1, \leq \ell)$ -identifying code, then it admits a $(1, \leq \ell')$ -identifying code for any $\ell' < \ell$.

A digraph with adjacency matrix $A = (a_{uv})$ has eigenvalue λ and eigenvector $\mathbf{x} = (x_u)$ if and only if

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Leftrightarrow \quad \sum_{v \in V} a_{uv}x_v = \sum_{v \in N^+(u)} x_v = \lambda x_u \quad \text{for all } u \in V. \quad (2)$$

This last equation leads to the charge displacement interpretation; for more information about it, see Fiol and Mitjana [6] or Godsil [7]. Moreover, the spectral radius of A is the largest among the absolute values of its eigenvalues.

Recall also that a *transitive tournament* TT_3 is formed by vertices u, v , and w , and arcs (u, v) , (u, w) , and (v, w) . Besides, we called *bipartite tournament* $BT_{2,2}$ to the digraph formed by vertices u, v, w and x , and arcs (u, w) , (u, x) , (v, w) and (v, x) . See both digraphs in Figure 1 (b) and 1 (c), respectively.

Our first lemma is the only non-spectral result of this paper.

Lemma 1.1. *Let D be a d -in-regular digraph on n vertices, without any of the subdigraphs of Figure 1. If D admits a $(1, \leq \ell)$ -identifying code, then $\ell \in \{d, d + 1\}$.*

Proof. Theorem 2 (ii) in [2] states the following: Let D be a twin-free digraph with minimum in-degree $\delta^- \geq 1$. Suppose that D does not contain any subdigraph as those of Figure 1, then D admits a $(1, \leq \delta^-)$ -identifying code. Besides, Corollary 1 in [2] states: Let D be a digraph admitting a $(1, \leq \ell)$ -identifying code. If there exists a vertex u of minimum in-degree $\delta^- \geq 2$ such that $N^+(u) \neq \emptyset$, then $\ell \leq \delta^- + 1$.

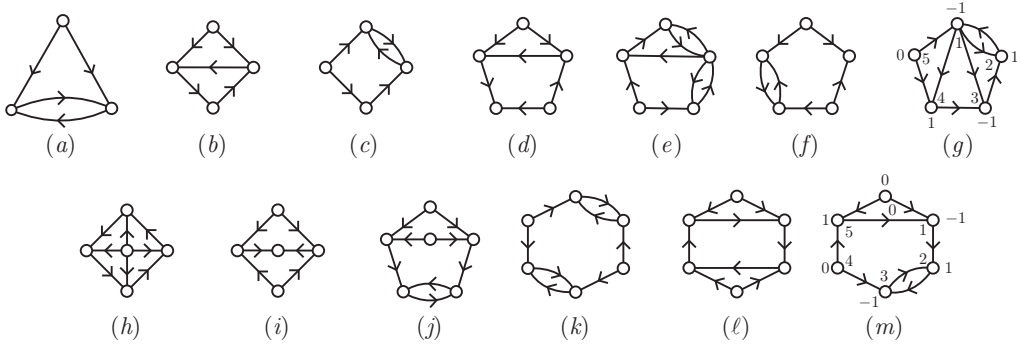


Figure 2: The forbidden subdigraphs characterizing a 2-in-regular digraph admitting a $(1, \leq 1)$ -identifying code (only (a)), or a $(1, \leq 2)$ -identifying code (all of them). The subdigraph (g) and (m) have their vertices numbered in the interior, and the entries of the eigenvector corresponding to eigenvalue -1 in the exterior.

For our d -in-regular digraphs, from the first result, we get that $\ell \geq d$; and, from the second one, $\ell \leq d + 1$. So, $\ell \in \{d, d + 1\}$. \square

2 Main results for digraphs

We begin with a result that gives a sufficient (spectral) condition for a digraph to admit a $(1, \leq 1)$ -identifying code.

Lemma 2.1. *Let D be a digraph with adjacency matrix A and with a set of eigenvalues denoted by $\text{ev}(A)$. If $-1 \notin \text{ev}(A)$, then D admits a $(1, \leq 1)$ -identifying code.*

Proof. We reason by contradiction. The digraph D does not admit a $(1, \leq 1)$ -identifying code if and only if there exists a pair of vertices u and v such that $N^-[u] = N^-[v]$. So, $(A + I)\mathbf{e}_u = (A + I)\mathbf{e}_v$, where \mathbf{e}_u and \mathbf{e}_v are the unitary characteristic vectors corresponding to vertices u and v , respectively, and I is the identity matrix. Then, $(A + I)\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \mathbf{e}_u - \mathbf{e}_v$, whence $A\mathbf{x} = -\mathbf{x}$ and $-1 \in \text{ev}(A)$. \square

Remark 2.1. *The converse is not true since, if $-1 \in \text{ev}(A)$, this does not imply that some of its corresponding eigenvectors are of the form $\mathbf{e}_i - \mathbf{e}_j$. For example, the digraph in Figure 2 (j) has -1 as an eigenvalue, but it does admit a $(1, \leq 1)$ -identifying code.*

In [2] the authors gave the following theorem, which is a combinatorial characterization of a 2-in-regular digraph admitting a $(1, \leq 1)$ -, $(1, \leq 2)$ -, or $(1, \leq 3)$ -identifying code.

Theorem 2.1 ([2]). *Let D be a 2-in-regular digraph. Then,*

- (i) *D admits a $(1, \leq 1)$ -identifying code if and only if it does not contain any subdigraph isomorphic to Figure 2 (a).*

- (ii) D admits a $(1, \leq 2)$ -identifying code if and only if it does not contain any subdigraph isomorphic to one of the digraphs of Figure 2.
- (iii) D admits a $(1, \leq 3)$ -identifying code if and only if it is oriented, TT_3 -free and does not contain any subdigraph isomorphic to one of the digraphs of Figure 3.

Next, we present an algebraic-combinatorial sufficient condition for a 2-in-regular digraph to admit a $(1, \leq 2)$ - or $(1, \leq 3)$ -identifying code, but first we need the following lemma.

Lemma 2.2. *Let D' be a digraph with maximum in-degree Δ^- having an eigenvalue λ with eigenvector $\mathbf{x}' = (x'_u)$, such that $x'_v = 0$ for any vertex $v \in V(D')$ with $d^-(v) < \Delta^-$. Then, any Δ^- -in-regular digraph D containing D' as a subdigraph has also the eigenvalue λ .*

Proof. Let $A' = A'(D')$ be the adjacency matrix of D' . We know that $A'\mathbf{x}' = \lambda\mathbf{x}'$. Let A be the adjacency matrix of D containing D' as a subdigraph, such that $a_{ij} = a'_{ij}$ for all $i, j \in V(D')$. Now let us show that λ also is an eigenvalue of A . To see this, it is enough to check that vector \mathbf{x} , obtained from \mathbf{x}' by adding zeros to the entries of D corresponding to the vertices that are not in D' , is an eigenvector of A with eigenvalue λ . Indeed, from (2), for all $u \in V$ we get

$$\sum_{v \in N^+(u)} x_v = \sum_{\substack{v \in N^+(u) \\ v \in V'}} x_v + \sum_{\substack{v \in N^+(u) \\ v \notin V'}} x_v = \sum_{\substack{v \in N^+(u) \\ v \in V'}} x_v, \quad (3)$$

because by construction of vector \mathbf{x} , the sum when $v \notin V'$ is zero. Then,

- If $u \in V'$, then (3) gives

$$\sum_{v \in N^+(u)} x_v = \lambda x_u.$$

- If $u \notin V'$, then (3) provides

$$\sum_{v \in N^+(u)} x_v = 0 = \lambda x_u,$$

because, again by construction of \mathbf{x} , both $\sum_{\substack{v \in N^+(u) \\ v \in V'}} x_v$ and x_u are zero. □

Theorem 2.2. *Let D be a 2-in-regular digraph with adjacency matrix A .*

- (i) *If $-1 \notin \text{ev}(A)$ and D does not contain any subdigraph isomorphic to (b), (c), (d), (f) and (i) of Figure 2, then D admits a $(1, \leq 2)$ -identifying code.*

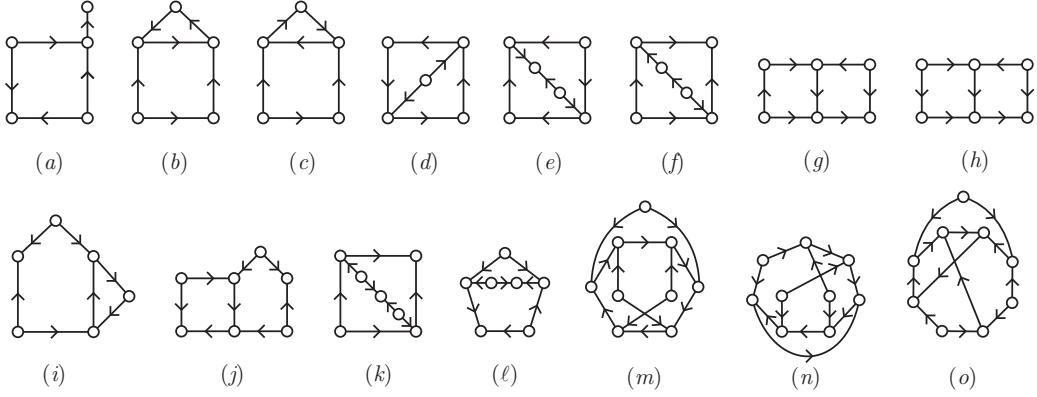


Figure 3: The forbidden subdigraphs characterizing a TT_3 -free, 2-in-regular and oriented digraph admitting a $(1, \leq 3)$ -identifying code.

(ii) If $-1, 0 \notin \text{ev}(A)$ and D does not contain any subdigraph isomorphic to (b)-(l) of Figure 3, then D admits a $(1, \leq 3)$ -identifying code.

Proof. To prove (i), by Theorem 2.1(ii), we know that if a 2-in-regular digraph D does not contain any of the subdigraphs of Figure 2, then D admits a $(1, \leq 2)$ -identifying code. The subdigraphs (a), (e), (g), (h) and (j) – (m) satisfy Lemma 2.2 for $\lambda = -1$. Then, we only need to forbid the subdigraphs (b), (c), (d), (f) and (i) to get the result. Let us see that one of the subdigraphs with $\lambda = -1$ satisfies Lemma 2.2. By way of example, in Figure 2 (g), it is shown a subdigraph with a vertex of in-degree less than 2 having charge 0, so satisfying Lemma 2.2. To prove (ii), reasoning similarly, we get that the subdigraph (a) of Figure 3 satisfies Lemma 2.2 for $\lambda = 0$, and subdigraphs (m)-(o) satisfy Lemma 2.2 for $\lambda = -1$, so we only need to forbid the rest of the subdigraphs in this figure. \square

We provide some necessary notation introduced by Powers [12] and referenced in the book by Cvetković, Rowlinson and Simić [5]. Let $\mathcal{P}(\mathbf{x}) = \{i : x_i > 0\}$ and $\mathcal{N}(\mathbf{x}) = \{i : x_i < 0\}$. Let \mathbf{x} be an eigenvector associated to an eigenvalue λ different from the spectral radius, and let \mathbf{z} be an eigenvector associated to the spectral radius. Hence, \mathbf{x} and \mathbf{z} are orthogonal, yielding that $\mathcal{P}(\mathbf{x})$ and $\mathcal{N}(\mathbf{x})$ are nonempty, because all the entries of \mathbf{z} are positive.

Let us show the meaning of the sign of a real eigenvalue on the sets of in-neighbourhoods of vertices.

Proposition 2.1. *Let $D = (V, E)$ be a digraph with adjacency matrix A having some real eigenvalue, say $\lambda \in \text{ev}(A)$, different from the spectral radius. Let $\mathbf{x} = (x_u)_{u \in V}$ be an eigenvector of A associated to λ such that $X = \mathcal{P}(\mathbf{x})$ and $Y = \mathcal{N}(\mathbf{x})$. Then, depending on the sign of λ , the following holds:*

- (a) If $\lambda < 0$, then $X \cup N^-(X) = Y \cup N^-(Y)$ ($\Leftrightarrow N^-[X] = N^-[Y]$).
- (b) If $\lambda > 0$, then $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) If $\lambda = 0$, then $N^-(X) = N^-(Y)$.

Proof. Let $\Phi_{N^-(i)} = A\mathbf{e}_i$ denote the characteristic vector of the open in-neighbourhood of vertex i , where \mathbf{e}_i is the unitary characteristic vector of i . Let \mathbf{x}_+ be the vector obtained from a vector \mathbf{x} by changing all its negative components to zero. Similarly, \mathbf{x}_- is obtained from \mathbf{x} by changing all its positive components to zero. Then, $\mathbf{x} = \mathbf{x}_+ + \mathbf{x}_-$. Since \mathbf{x} is a λ -eigenvector of A , we get $A\mathbf{x} = \lambda\mathbf{x}$ or $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$. Now we distinguish the possible cases according to the sign of λ :

(a) and (b): If $\lambda < 0$ or $\lambda > 0$, we have

$$\sum_{i \in X} x_i \Phi_{N^-(i)} - \lambda \mathbf{x}_+ + \sum_{j \in Y} x_j \Phi_{N^-(j)} - \lambda \mathbf{x}_- = \mathbf{0}.$$

For the case $\lambda < 0$, $\sum_{i \in X} x_i \Phi_{N^-(i)}$ and $-\lambda \mathbf{x}_+$ are positive, whereas $\sum_{j \in Y} x_j \Phi_{N^-(j)}$ and $-\lambda \mathbf{x}_-$ are negative. Thus, for the global sum to be $\mathbf{0}$, we must have $N^-[X] = N^-[Y]$. For the case $\lambda > 0$, $\sum_{i \in X} x_i \Phi_{N^-(i)}$ and $-\lambda \mathbf{x}_-$ are positive, whereas $-\lambda \mathbf{x}_+$ and $\sum_{j \in Y} x_j \Phi_{N^-(j)}$ are negative. Then, the global sum is $\mathbf{0}$ if $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) Finally, if $\lambda = 0$, the vector equality

$$\sum_{i \in X} x_i \Phi_{N^-(i)} + \sum_{j \in Y} x_j \Phi_{N^-(j)} = \mathbf{0}$$

yields the result. □

The same result holds for graphs by changing $N^-(X)$ and $N^-(Y)$ by $N(X)$ and $N(Y)$, respectively. Moreover, a similar result concerning out-neighbourhoods (instead of in-neighbourhoods) can be obtained by applying Proposition 2.1 to the converse digraph of D or, equivalently, considering the left (instead of right) eigenvectors of D . The next result gives an upper bound for ℓ in a digraph D having a $(1, \leq \ell)$ -identifying code.

Corollary 2.1. *Let D be a digraph admitting a $(1, \leq \ell)$ -identifying code. Let A be its adjacency matrix having at least one negative eigenvalue $-\lambda$ (with $\lambda > 0$) with $\mathbf{x} = (x_1, \dots, x_n)$ any associated eigenvector. Then $\ell < \min_{\mathbf{x}} \max\{|\mathcal{P}(\mathbf{x})|, |\mathcal{N}(\mathbf{x})|\}$.*

Example 2.1. *Consider the digraph of Figure 2 (m). Its spectrum is $\{0^4, 1^1, -1^1\}$. The eigenvector corresponding to the eigenvalue -1 is $(0, -1, 1, -1, 0, 1)$. The positions of the positive entries of this eigenvector give us vertex subset $X = \{2, 5\}$, and the positions of the negatives entries give $Y = \{1, 3\}$. We can check that $N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5\}$. Then, this digraph does not admit a $(1, \leq 2)$ -identifying code.*

To give a general result not depending on some specific eigenvector, but only on the multiplicity of the corresponding eigenvalue, we give the following lemma. More precisely, next result shows that, given a real eigenvalue with geometric multiplicity m , some of its eigenvectors can be chosen with at least $m - 1$ zero entries.

Lemma 2.3. *Let D be a digraph on n vertices with adjacency matrix A , and let λ be a real eigenvalue of A with geometric multiplicity m . For any given index set $I \subset \{1, 2, \dots, n\}$ with $|I| = m - 1$, there exists an eigenvector \mathbf{x} with eigenvalue λ and entries $x_i = 0$ for every $i \in I$.*

Proof. Every eigenvector $\mathbf{x} = (x_1, \dots, x_n)$ associated to the eigenvalue λ is a solution of the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Since the geometric multiplicity of λ is m , it follows that the rank of $A - \lambda I$ is $n - m > 0$. Hence, there is a triangular matrix B having null the last m rows, and diagonal entries $b_i^i = 1$ or $b_i^i = 0$, for $i = 1, \dots, n - m$, such that $B \cdot \mathbf{x} = \mathbf{0}$. Thus, each of the first $n - m$ components of \mathbf{x} can be expressed as a linear combination of the last m ones, that is,

$$x_i = \sum_{j=n-m+1}^n \alpha_j^i x_j, \quad i = 1, \dots, n - m.$$

This implies that

$$\mathbf{x} = \sum_{j=n-m+1}^n (\alpha_j^1 x_j, \dots, \alpha_j^{n-m} x_j, x_{n-m+1}, \dots, x_m),$$

yielding that the set of m vectors

$$\begin{aligned} &(\alpha_{n-m+1}^1, \dots, \alpha_{n-m+1}^{n-m}, 1, 0, \dots, 0), \\ &(\alpha_{n-m+2}^1, \dots, \alpha_{n-m+2}^{n-m}, 0, 1, 0, \dots, 0), \\ &\quad \vdots \\ &(\alpha_n^1, \dots, \alpha_n^{n-m}, 0, \dots, 0, 1). \end{aligned}$$

is a base of eigenvectors associated to λ , and clearly, all of them have at least $m - 1$ zero entries. □

With the following remark, we state that our results can be applied on graphs.

Remark 2.2. *Corollary 2.1 can also be applied to graphs, which always have real eigenvalues.*

Let us see an example on a graph.

Example 2.2. *Consider the Heawood graph (see Figure 4). Its spectrum is $\{3^1, \sqrt{2}^6, -\sqrt{2}^6, -3^1\}$. The eigenvector corresponding to -3 and the 6 eigenvectors of $-\sqrt{2}$ are, respec-*

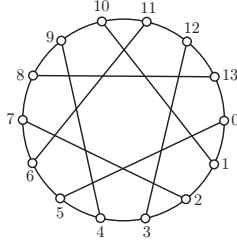


Figure 4: The Heawood graph.

tively,

$$\begin{aligned}
 &(-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1); \\
 &(-1, 0, 1, 0, -1, \sqrt{2}, 0, -\sqrt{2}, 1, 0, 0, 0, 0, 0), \\
 &(-1, 0, 1, -\sqrt{2}, 0, \sqrt{2}, -1, 0, 0, 0, 0, 0, 1, 0), \\
 &(0, -1, \sqrt{2}, -1, 0, 1, -\sqrt{2}, 0, 0, 0, 0, 1, 0, 0), \\
 &(0, -1, \sqrt{2}, 0, -\sqrt{2}, 1, 0, -1, 0, 1, 0, 0, 0, 0), \\
 &(0, -\sqrt{2}, 1, 0, -1, \sqrt{2}, -1, 0, 0, 0, 1, 0, 0, 0), \\
 &(-\sqrt{2}, 0, \sqrt{2}, -1, 0, 1, 0, -1, 0, 0, 0, 0, 0, 1).
 \end{aligned}$$

The last one is one of the eigenvectors whose entries have the maximum number of zeros and with the minimum difference between the number of positive and negative entries. The positions of the positive entries of this eigenvector give us the set of vertices $X = \{2, 5, 13\}$. The positions of the negatives entries give the set $Y = \{0, 3, 7\}$. We can check that $N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 12, 13\}$. Then, the Heawood graph does not admit a $(1, \leq 3)$ -identifying code. By Theorem 1.1 we know that the Heawood graph, which is 3-regular and it has girth 6, admits a $(1, \leq 2)$ -identifying code.

References

- [1] G. Araujo-Pardo, C. Balbuena, L. Montejano, and J. C. Valenzuela, Partial linear spaces and identifying codes, *European J. Combin.* **32** (2011) 344–351.
- [2] C. Balbuena, C. Dalfó, and B. Martínez, Identifying codes in digraphs, submitted, 2017.
- [3] C. Balbuena and J. Salas, On the order of graphs with a given girth pair, *Discrete Math.* **321** (2014) 68–75.
- [4] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2007.

- [5] D. Cvetković, P. Rowlinson, and S. K. Simić, An introduction to the theory of graph spectra, London Mathematical Society Student Texts, 75, Cambridge: Cambridge University Press, 2010.
- [6] M. A. Fiol and M. Mitjana, The spectra of some families of digraphs, *Linear Algebra Appl.* **423** (2007) 109 – 118.
- [7] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York, 1993.
- [8] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
- [9] M. Karpovsky, K. Chakrabarty, and L. Levitin, On a new class of codes for identifying vertices in graphs, *IEEE Trans. Inform. Theory* **44** (1998) 599–611.
- [10] T. Laihonen, On cages admitting identifying codes, *European J. Combin.* **29** (2008) 737–741.
- [11] T. Laihonen and S. Ranto, Codes identifying sets of vertices, *Lect. Notes Comput. Sci.* **2227** (2001) 82–91.
- [12] D. L. Powers, Graph partitioning by eigenvectors, *Linear Algebra Appl.* **101** (1988) 121–133.