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TIME DECAY IN DUAL-PHASE-LAG THERMOELASTICITY: CRITICAL CASE

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ABSTRACT. This note is devoted to the study of the time decay of the one-dimensional dual-phase-lag thermoelasticity. In this theory two delay parameters τ_q and τ_θ are proposed. It is known that the system is exponentially stable if $\tau_q < 2\tau_\theta$ [22]. We here make two new contributions to this problem. First, we prove the polynomial stability in the case that $\tau_q = 2\tau_\theta$ as well the optimality of this decay rate. Second, we prove that the exponential stability remains true even if the inequality only holds in a proper sub-interval of the spatial domain, when τ_θ is spatially dependent.

1. Introduction. The Fourier constitutive law for the heat flux vector proposes the well-known linear parabolic equation for the heat conduction. This equation has received criticism from a physical point of view because it implies that the thermal disturbances at some point will be felt instantly anywhere for every distance. Different theories describing the heat conduction have been established to save this drawback. The most famous one is the Maxwell-Cattaneo law that proposes an hyperbolic damped equation for the heat conduction. Two hyperbolic theories of thermoelasticity have been proposed for this heat conduction. They are the theory of Lord and Shulman [16] and the theory of Green and Lindsay [6], which are under intensive study in the recent years. In the 1990's Green and Naghdi also proposed three different thermoelastic theories [7, 8, 9] based on the axioms of the continuum mechanics. The main difference in these theories was the kind of heat conduction they proposed. More details of the above theories can be found in the books and review articles [10, 11, 13, 23].

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In 1995 Tzou [24] suggested an alternative to modify of the classical Fourier law. He introduced delay parameters in the constitutive equations. To be precise, the equation he proposed is:

$$q(x, t + \tau_q) = -k\nabla\theta(x, t + \tau_\theta), \quad k > 0. \quad (1)$$

Here θ is the temperature, q is the heat flux vector and τ_θ and τ_q are two delay parameters. This equation assumes that the temperature gradient across a material volume at position x and at time $t + \tau_\theta$ results in the heat flux to flow at a different instant $t + \tau_q$. However the proposition of this theory only builds on an intuitive point of view, and there is no axiomatic mathematical foundation for it. In fact, when we adjoin this equations with the classical energy equation

$$\dot{\theta} + \operatorname{div} q = 0, \quad (2)$$

it has been proved the existence of a sequence of solutions $T_n(x, t) = \exp(\omega_n t)\Psi_n(x)$, $n \geq 1$, such that the real part of ω_n becomes positively unbounded [4]. This says that the mathematical problem is ill posed in the sense of Hadamard. Therefore, we see an explosive behavior of the solutions which is not a suitable property for a heat conduction theory. Nevertheless this theory has caught researchers' attention when equation (1) is replaced by its Taylor approximations to the delay parameters. For instance, by taking the first order approximation of the flux vector and zeroth order approximation of the temperature gradient we recover the Maxwell-Cattaneo proposition; and by taking first order approximation to both, we recover the theory proposed by Morro, Payne and Straughan [18] which is considered in the book by Flavini and Rionero [5]. Several other order of approximations have proposed some new and stimulating heat conduction equations to be studied from a mathematical viewpoint. In this paper we will adopt the following approximation proposed by Tzou,

$$\dot{\theta} + \tau_q \ddot{\theta} + \frac{\tau_q^2}{2} \ddot{\theta} = k\Delta\theta + k\tau_\theta \Delta\dot{\theta}. \quad (3)$$

To clarify the stability of this equation is the aim of many research works [1, 19, 21, 15] (among others). It is known that the solutions of this equation decay exponentially when $\tau_q < 2\tau_\theta$ and polynomially when $\tau_q = 2\tau_\theta$. Moreover, they are unstable when $\tau_q > 2\tau_\theta$. Thus, $\tau_q = 2\tau_\theta$ is called the critical case of stability when the phase-lag τ_q and τ_θ are constants.

In this paper we concentrate our attention to the thermoelastic theory based on the heat conduction equation (3). The system was proposed by Chandrasekharaiah [3] and studied in [22]. There Quintanilla and Racke considered the one-dimensional problem and obtained by means of the energy arguments the exponential decay of solutions in case that $\tau_q < 2\tau_\theta$. Instability of solutions was also proved when $\tau_q > 2\tau_\theta$ [20] by means of the spectral analysis. We would like to continue the study of this problem and concentrate our attention to two cases. First, for constant phase-lag, we want to clarify whether $\tau_q = 2\tau_\theta$ is still the critical case. The second problem corresponds to the case that the phase-lag τ_θ depends on the material point. A new case arises between $\tau_q < 2\tau_\theta(x)$ and $\tau_q = 2\tau_\theta(x)$ on the entire spatial domain, i.e., $\tau_q < 2\tau_\theta(x)$ only on a proper sub-domain.

To prove our main theorems we will use the semigroup arguments. For the first case we apply a theorem obtained by Borichev and Tomilov [2] which characterize the polynomial decay of solutions by the resolvent operator behavior on the imaginary axis. This kind of treatment has been applied to a large class of problems in the literature. For the second case we adopt the semigroup arguments similar to

the one in [12]. It is worth noting that special multipliers are introduced to handle both elastic equation and heat equations. We refer the readers to the comment after (89).

This paper is organized as follows. Section 2 is devoted to the statement of the problem and review of several results we will use later. In Section 3 we consider the first case, and prove the polynomial decay of the solutions and its optimality. In the last Section we consider the second case, and obtain the exponential decay of the solutions. This confirms that the behavior of solutions for the critical case of the thermoelastic system (8)-(9) coincide with the known one for the dual-phase-lag heat equation (3).

2. Preliminary. The basic system which determined the thermoelastic vibrations for the dual-phase lag thermoelasticity is determined by the system

$$u_{tt} = a_1 u_{xx} - m\theta_x, \quad (4)$$

$$\theta_{ttt} = -\frac{2}{\tau_q} \theta_{tt} - \frac{2}{\tau_q^2} \theta_t - \frac{2m\theta_0}{\tau_q^2} \tilde{u}_{tx} + \frac{2k}{\tau_q^2} (\tau_\theta(x)\theta_{tx})_x + \frac{2k}{\tau_q^2} \theta_{xx}, \quad (5)$$

where we have used the following notation $\tilde{f} = f + \tau_q \dot{f} + \frac{\tau_q^2}{2} \ddot{f}$. Our system becomes

$$\tilde{u}_{tt} = a_1 \tilde{u}_{xx} - \frac{\tau_q^2 m}{2} \theta_{ttx} - \tau_q m \theta_{tx} - m\theta_x, \quad (6)$$

$$\theta_{ttt} = -\frac{2}{\tau_q} \theta_{tt} - \frac{2}{\tau_q^2} \theta_t - \frac{2m\theta_0}{\tau_q^2} \tilde{u}_{tx} + \frac{2k}{\tau_q^2} (\tau_\theta(x)\theta_{tx})_x + \frac{2k}{\tau_q^2} \theta_{xx}, \quad (7)$$

Notice that if this system can be solved, then the first system can be solved too, because $\tilde{u} = u + \tau_q \dot{u} + \frac{\tau_q^2}{2} \ddot{u}$ and this is an ordinary differential equation. In fact, the asymptotic behavior will agree for the variables u and \tilde{u} . Hence, we can consider the system

$$u_{tt} = a_1 u_{xx} - \frac{\tau_q^2 m}{2} \theta_{ttx} - \tau_q m \theta_{tx} - m\theta_x, \quad (8)$$

$$\theta_{ttt} = -\frac{2}{\tau_q} \theta_{tt} - \frac{2}{\tau_q^2} \theta_t - \frac{2m\theta_0}{\tau_q^2} u_{tx} + \frac{2k}{\tau_q^2} (\tau_\theta(x)\theta_{tx} + \theta_x)_x, \quad (9)$$

$$u(0, t) = \theta(0, t) = u(L, t) = \theta(L, t) = 0, \quad (10)$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), \theta_t(x, 0) = \theta_1(x), \theta_{tt}(x, 0) = \theta_2(x),$$

where we have deleted the tilde to simplify the notation. This system will be the target of our investigation in this paper.

Define

$$a(x) = \tau_\theta(x) - \frac{\tau_q}{2}. \quad (11)$$

We say that $a(x)$ is locally positive if

$$a(x) = \begin{cases} > 0, & x \in (x_1, x_2) \\ = 0, & \text{otherwise} \end{cases} \quad (12)$$

for some $x_1, x_2 \in [0, L]$.

For the simplicity of notation, we use prime to denote the derivative about x hereafter. In order to obtain a properly defined energy for this system, we multiply

equation (8) and (9) by $\theta_0 u_t$ and $\frac{\tau_q^2}{2}(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta)$, respectively. Then we integrate for x to get

$$\frac{d}{dt} \frac{\theta_0}{2} \int_0^L (|u_t|^2 + a_1|u'|^2) dx = -m\theta_0 \int_0^L (\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta)' u_t dx \quad (13)$$

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_0^L |\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta|^2 dx &= m\theta_0 \int_0^L u_t (\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta)' dx \\ &- k \int_0^L (\tau_\theta(x)\theta_t' + \theta') (\frac{\tau_q^2}{2}\theta_{tt}' + \tau_q\theta_t' + \theta') dx. \end{aligned} \quad (14)$$

Note that

$$\begin{aligned} &k \int_0^L \theta' (\frac{\tau_q^2}{2}\theta_{tt}' + \tau_q\theta_t' + \theta') dx \\ &= k \int_0^L \left(|\theta'|^2 - \frac{\tau_q^2}{2} |\theta_t'|^2 \right) dx + \frac{d}{dt} \int_0^L \left(\frac{k\tau_q}{2} |\theta'|^2 + \frac{k\tau_q^2}{2} \theta' \theta_t' \right) dx, \end{aligned} \quad (15)$$

and

$$\begin{aligned} &k \int_0^L \tau_\theta(x) \theta_t' (\frac{\tau_q^2}{2}\theta_{tt}' + \tau_q\theta_t' + \theta') dx \\ &= k \int_0^L \tau_q \tau_\theta(x) |\theta_t'|^2 dx + \frac{d}{dt} \int_0^L \left(\frac{k\tau_\theta(x)}{2} |\theta'|^2 + \frac{k\tau_q^2 \tau_\theta(x)}{4} |\theta_t'|^2 \right) dx. \end{aligned} \quad (16)$$

Then the sum of (13) and (14) yields

$$\begin{aligned} \frac{1}{2} \frac{dE}{dt} &= -k \int_0^L |\theta'|^2 dx - k\tau_q \int_0^L \left(\tau_\theta(x) - \frac{\tau_q}{2} \right) |\theta_t'|^2 dx \\ &= -k \int_0^L |\theta'|^2 dx - k\tau_q \int_0^L a(x) |\theta_t'|^2 dx, \end{aligned} \quad (17)$$

where

$$\begin{aligned} E(t) &= \int_0^L \left(\theta_0 |u_t|^2 + a_1 \theta_0 |u'|^2 + k(\tau_\theta(x) + \tau_q) |\theta'|^2 + \frac{k\tau_q^2 \tau_\theta(x)}{2} |\theta_t'|^2 + k\tau_q^2 \theta' \theta_t' \right. \\ &\quad \left. + |\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta|^2 \right) dx \\ &= \int_0^L \left(\theta_0 |u_t|^2 + a_1 \theta_0 |u'|^2 + \frac{k\tau_q}{2} |\theta'|^2 + k\tau_q |\theta'| + \frac{\tau_q}{2} \theta_t'^2 + |\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta|^2 \right. \\ &\quad \left. + ka(x) |\theta'|^2 + \frac{k\tau_q^2}{2} a(x) |\theta_t'|^2 \right) dx. \end{aligned} \quad (18)$$

Let $v = u_t$, $\phi = \theta_t$, $\psi = \theta_{tt}$, and $z = (u, v, \theta, \phi, \psi)^T$. Define the state space

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product which induces the energy norm in (18). Then, we can rewrite (8)-(9) as a first order evolution equation in \mathcal{H}

$$\frac{dz}{dt} = \mathcal{A}z, \quad z(0) = z_0, \quad (19)$$

where

$$\mathcal{A}z = \begin{pmatrix} v \\ -a_1 u'' - \frac{\tau_q^2 m}{2} \psi' - \tau_q m \phi' - m \theta' \\ \phi \\ \psi \\ -\frac{2}{\tau_q} \psi - \frac{2}{\tau_q^2} \phi - \frac{2m\theta_0}{\tau_q^2} v' + \frac{2k}{\tau_q^2} (\tau_\theta(x)\phi')' + \frac{2k}{\tau_q^2} \theta'' \end{pmatrix} \quad (20)$$

with domain

$$D(\mathcal{A}) = \{z \in \mathcal{H} \mid u, \theta, \phi \in H^2(\Omega)\}. \quad (21)$$

It is obvious that $D(\mathcal{A})$ is dense in \mathcal{H} . From (17), we also have the dissipativeness of \mathcal{A} . Moreover, $0 \in \rho(\mathcal{A})$ which will be proved in next section where we show the imaginary axis is in $\rho(\mathcal{A})$. By Theorem 1.2.4 in [17], we have

Theorem 2.1. *Let operator \mathcal{A} defined above is the infinitesimal generator of a C_0 -semigroup $e^{\mathcal{A}t}$ of contractions in the Hilbert space \mathcal{H} .*

Our main tools are the following two theorems.

Theorem 2.2. [12] *Let $S(t) = e^{\mathcal{A}t}$ be a C_0 -semigroup of contractions in a Hilbert space \mathcal{H} . Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (22)$$

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < \infty. \quad (23)$$

Theorem 2.3. [2] *Let $S(t) = e^{\mathcal{A}t}$ be a C_0 -semigroup of contractions in a Hilbert space \mathcal{H} . Then, $S(t)$ is polynomially stable of order $\frac{1}{k}$ if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (24)$$

$$\sup_{|\beta| > 1} \frac{1}{\beta^k} \|(i\beta - \mathcal{A})^{-1}\| \leq \infty \quad \text{for some } k > 0. \quad (25)$$

Here, polynomial stability of order $\frac{1}{k}$ means that there exists a positive constant $C > 0$ such that

$$\|e^{t\mathcal{A}} z_0\| \leq C \left(\frac{1}{t}\right)^{\frac{1}{k}} \|z_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \quad (26)$$

for all $z_0 \in D(\mathcal{A})$.

Throughout this paper, we use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the standard inner product and norm in L^2 .

3. Polynomial Stability for $a(x) \equiv 0$. In this section, we consider the case

$$a(x) \equiv 0, \quad x \in [0, L], \quad (27)$$

i.e., τ_θ and τ_q are positive constants, and $\tau_\theta = \frac{\tau_q}{2}$.

Theorem 3.1. *The semigroup $e^{t\mathcal{A}}$ is polynomially stable of order $\gamma = \frac{1}{2}$ when $2\tau_\theta = \tau_q$, i.e., for all $z_0 \in D(\mathcal{A})$, there is a constant $C > 0$ such that the solution z of (19) satisfies*

$$\|z\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}} \|z_0\|_{D(\mathcal{A})}. \quad (28)$$

Proof. Assume that (25) is false. Then by the uniform boundedness theorem, there exist a sequence $\beta_n \rightarrow \infty$ and a unit norm vector sequence $z_n = (u_n, v_n, \theta_n, \phi_n, \psi_n)^T \in D(\mathcal{A})$ such that

$$\beta_n^\gamma \|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} \rightarrow 0, \quad (29)$$

which implies that

$$\beta^\gamma (i\beta u - v) = o(1), \quad \text{in } H_0^1(\Omega), \quad (30)$$

$$\beta^\gamma \left(i\beta v - a_1 u'' + \frac{\tau_q^2 m}{2} \psi' + \tau_q m \phi' + m \theta' \right) = o(1), \quad \text{in } L^2(\Omega), \quad (31)$$

$$\beta^\gamma (i\beta \theta - \phi) = o(1), \quad \text{in } H_0^1(\Omega), \quad (32)$$

$$\beta^\gamma (i\beta \phi - \psi) = o(1), \quad \text{in } H_0^1(\Omega), \quad (33)$$

$$\beta^\gamma \left(i\beta \psi + \frac{2}{\tau_q} \psi + \frac{2}{\tau_q} \phi + \frac{2m\theta_0}{\tau_q^2} v' - \frac{2\tau_\theta k}{\tau_q^2} \phi'' - \frac{2k}{\tau_q^2} \theta'' \right) = o(1), \quad \text{in } L^2(\Omega). \quad (34)$$

For the simplicity of notation, we omit the subscript n hereafter.

Let $\gamma = 2$. From dissipation, we have

$$\beta \|\theta'\| = o(1). \quad (35)$$

Combining (35) and (32), we also have

$$\|\phi'\| = o(1), \quad (36)$$

which further implies that, due to (33),

$$\frac{1}{\beta} \|\psi'\| = o(1). \quad (37)$$

Take the inner product of $\frac{\psi}{\beta^2}$ and $\frac{\phi}{\beta^2}$ with (33) and (34) in $L^2(\Omega)$, respectively. We have

$$i\beta \langle \phi, \psi \rangle - \|\psi\|^2 = o(1) \quad (38)$$

$$i\beta \langle \psi, \phi \rangle + \frac{2}{\tau_q} \langle \psi, \phi \rangle + \frac{2}{\tau_q^2} \|\phi\|^2 + \frac{2m\theta_0}{\tau_q^2} \langle v', \phi \rangle - \langle \frac{2k}{\tau_q^2} (\tau_\theta \phi'' + \theta''), \phi \rangle = o(1). \quad (39)$$

Integrating by parts shows that last three terms on the left-hand side of (39) converges to zero due to (35) and (36). It is clear the second and third term on the left-hand side of (39) also converges to zero. Therefore, (38),(39) now lead to

$$\|\psi\| = o(1). \quad (40)$$

Dividing (34) by β^3 and using (30), it can be simplified into

$$i \frac{2m\theta_0}{\tau_q^2} u' - \frac{2k}{\beta \tau_q^2} (\tau_\theta \phi'' + \theta'') = o(1). \quad (41)$$

The inner product of (41) with u' in $L^2(\Omega)$ leads to

$$i \frac{2m\theta_0}{\tau_q^2} \|u'\|^2 + \langle \frac{2k}{\tau_q^2} (\tau_\theta \phi' + \theta'), \frac{1}{\beta} u'' \rangle - \frac{2k}{\beta \tau_q^2} (\tau_\theta \phi'(x) + \theta'(x)) \bar{u}'(x) \Big|_0^L = o(1). \quad (42)$$

It follows from (31) and (35)-(37) that $\frac{1}{\beta} \|u''\|$ is bounded. Hence, the second term on the left-hand side of (42) converges to zero. On the other hand, the boundary

terms can be estimated by the Gagliardo-Nirenberg inequality as follows.

$$\begin{aligned} & \left| \frac{1}{\beta} (\tau_\theta \phi'(\xi) + \theta'(\xi)) \bar{u}'(\xi) \right| \\ & \leq C \left(\|\tau_\theta \phi' + \theta'\|^{1/2} \left\| \frac{1}{\beta} (\tau_\theta \phi'' + \theta'') \right\|^{1/2} + \frac{\|\tau_\theta \phi' + \theta'\|}{\beta^{1/2}} \right) \left(\|u'\|^{1/2} \left\| \frac{1}{\beta} u'' \right\|^{1/2} + \frac{\|u'\|}{\beta^{1/2}} \right) \\ & = o(1), \end{aligned} \quad (43)$$

for $\xi = 0, L$. Here, we have also employed the fact that $\left\| \frac{1}{\beta} (\tau_\theta \phi'' + \theta'') \right\|$ is bounded which is from (41). Therefore,

$$\|u'\| = o(1). \quad (44)$$

Finally, simplify (31) by diving β^3 . We obtain

$$iv - \frac{a_1}{\beta} u'' = o(1), \quad (45)$$

where we have used (36)-(37). Then, the inner product of (45) with v in $L^2(\Omega)$ yields

$$i\|v\|^2 + a_1 \langle u', \frac{1}{\beta} v' \rangle = o(1) \quad (46)$$

which further leads to, in reference to (44) and (30),

$$\|v\| = o(1). \quad (47)$$

This completes the proof of condition (25).

To verify condition (24), we assume $i\omega \in \sigma(\mathcal{A})$. Then, there exist a sequence $\beta \rightarrow \omega$ and a unit norm vector sequence $z = (u, v, \theta, \phi, \psi)^T \in D(\mathcal{A})$ such that

$$i\beta u - v = o(1), \quad \text{in } H_0^1(\Omega), \quad (48)$$

$$i\beta v - a_1 u'' + \frac{\tau_q^2 m}{2} \psi' + \tau_q m \phi' + m \theta' = o(1), \quad \text{in } L^2(\Omega), \quad (49)$$

$$i\beta \theta - \phi = o(1), \quad \text{in } H_0^1(\Omega), \quad (50)$$

$$i\beta \phi - \psi = o(1), \quad \text{in } H_0^1(\Omega), \quad (51)$$

$$i\beta \psi + \frac{2}{\tau_q} \psi + \frac{2}{\tau_q^2} \phi + \frac{2m\theta_0}{\tau_q^2} v' - \frac{2\tau_\theta k}{\tau_q^2} \phi'' - \frac{2k}{\tau_q^2} \theta'' = o(1), \quad \text{in } L^2(\Omega). \quad (52)$$

It follows from the dissipation of \mathcal{A} that

$$\|\theta'\| = o(1), \quad (53)$$

which further implies

$$\|\phi'\|, \|\psi'\| = o(1) \quad (54)$$

since β is bounded in (50) and (51). By the Poincaré inequality, we also have

$$\|\psi\| = o(1). \quad (55)$$

Replace v' in (52) by $i\beta u'$. Repeating the same argument between (41) and (47), we obtain

$$\|u'\|, \|v\| = o(1). \quad (56)$$

Thus, condition (24) holds. \square

Theorem 3.2. *The polynomial decay rate found in Theorem 3.1 is optimal.*

Proof. Notice that when $m = 0$ in system (8)-(9), the system decoupled into a dissipative phase-lag heat equation and a conservative wave equation. It has been proved in [15] that the energy of that heat equation, which coincides the heat energy defined in this paper, decays at a rate of $\frac{1}{\sqrt{t}}$, same as the one we derived in Theorem 3.1. If we can show $\frac{1}{\sqrt{t}}$ is the optimal energy decay rate for the heat equation, then the energy decay rate in Theorem 3.1 can not be faster.

Denote by \mathcal{A}_1 the restriction of \mathcal{A} to the last three components where $m = 0$, and by \mathcal{H}_1 the last three spaces in \mathcal{H} . Consider the sequence

$$F_n = (0, 0, \sin p_n x)^T$$

where $p_n = \frac{n\pi}{L}$. This sequence is bounded in \mathcal{H}_1 . Indeed,

$$\|F_n\|_{\mathcal{H}_1} = \frac{\tau_q^2 L}{2}, \quad \text{for all } n.$$

We will find solution in the form $z_n = (\theta_n, \phi_n, \psi_n)^T$ to the equation

$$(i\beta_n I - \mathcal{A}_1)z_n = F_n. \quad (57)$$

Eliminating ϕ_n, ψ_n in (57), we obtain

$$\left(-i\frac{\tau^2}{2}\beta_n^3 - \tau\beta_n^2 + i\beta_n\right)\theta_n - k\left(1 + i\beta_n\frac{\tau}{2}\right)\theta_n'' = \sin p_n x, \quad (58)$$

where τ is used to denote τ_q and $2\tau_\theta$. Due to the boundary conditions, the functions

$$\theta_n(x) = A_n \sin p_n x \quad (59)$$

solves the equation (58) if and only if A_n satisfies

$$\left[i\left(-\frac{\tau^2}{2}\beta_n^3 + \left(\frac{k\tau}{2}p_n^2 + 1\right)\beta_n\right) + (kp_n^2 - \tau\beta_n^2)\right]A_n = 1. \quad (60)$$

We choose β_n such that

$$\beta_n^2 = \frac{2}{\tau^2} \left(\frac{k\tau}{2}p_n^2 + 1\right). \quad (61)$$

Then,

$$kp_n^2 - \tau\beta_n^2 = -\frac{2}{\tau}.$$

This gives $A_n = -\frac{\tau}{2}$. Now, the vector function

$$z_n = (A_n \sin p_n x, iA_n \beta_n \sin p_n x, -\beta_n^2 A_n \sin p_n x)^T$$

is the solution to the resolvent equation (57).

Therefore,

$$\|z_n\|_{\mathcal{H}_1} \geq C\|\psi\| = O(\beta_n^2) = O(n^2), \quad (62)$$

which implies that for any $\epsilon > 0$, we can find $\beta_n \rightarrow \infty$ such that

$$\frac{1}{\beta_n^{2-\epsilon}} \|(i\beta_n I - \mathcal{A}_1)^{-1} F_n\|_{\mathcal{H}_1} = \frac{1}{\beta_n^{2-\epsilon}} \|z_n\|_{\mathcal{H}_1} = O(n^\epsilon) \rightarrow \infty. \quad (63)$$

This shows that $\frac{1}{\sqrt{t}}$ is the optimal decay rate. \square

Remark 1. It is worthwhile to point out that by coupling the dual-phase-lag heat equation with a conservative wave equation, the optimal polynomial decay rate remains the same. The heat dissipation is passed to the conservative equation most efficiently.

4. Exponential Stability for Locally Positive $a(x)$. In this section, we consider the case of locally positive $a(x)$. Assume that there is a positive constants C such that

$$|a'(x)| \leq Ca^{\frac{1}{2}}(x) \quad x \in (x_1, x_2). \quad (64)$$

Remark 2. Condition (64) is satisfied if $a(x) \geq a_0 > 0$ and differentiable on (x_1, x_2) . But if $a(x) = 0$ at the interface, it imposes a restriction on the slope of $a(x)$ near the interface. In this case, condition (64) can be satisfied, for example, if $a''(x_i)$, $i = 1, 2$ are finite. We believe that condition (64) is only needed because of the multiplier method we will use in the proof of next theorem.

Theorem 4.1. *The semigroup e^{tA} is exponentially stable when $a(x)$ satisfies condition (64), i.e., for all $z_0 \in \mathcal{H}$, there exist constant $M, \omega > 0$ such that the solution z of (19) satisfies*

$$\|z\|_{\mathcal{H}} \leq Me^{-\omega t} \|z_0\|_{\mathcal{H}}. \quad (65)$$

Proof. Since (22) has been proved in last section, we only need to verify condition (23). Assume that it is false. Then by the uniform boundedness theorem, there exists a sequence $\beta \rightarrow \infty$ and a unit norm vector sequence $z = (u, v, \theta, \phi, \psi)^T \in D(\mathcal{A})$ such that

$$i\beta u - v = f_1 = o(1), \quad \text{in } H_0^1(\Omega), \quad (66)$$

$$i\beta v - a_1 u'' + \frac{\tau_q^2 m}{2} \psi' + \tau_q m \phi' + m \theta' = f_2 = o(1), \quad \text{in } L^2(\Omega), \quad (67)$$

$$i\beta \theta - \phi = f_3 = o(1), \quad \text{in } H_0^1(\Omega), \quad (68)$$

$$i\beta \phi - \psi = f_4 = o(1), \quad \text{in } H_0^1(\Omega), \quad (69)$$

$$i\beta \psi + \frac{2}{\tau_q} \psi + \frac{2}{\tau_q^2} \phi + \frac{2m\theta_0}{\tau_q^2} v' - \frac{2k}{\tau_q^2} (\tau_\theta(x) \phi' + \theta')' = f_5 = o(1), \quad \text{in } L^2(\Omega). \quad (70)$$

From the dissipation,

$$\|\theta'\| = o(1), \quad \|a^{\frac{1}{2}}(x) \phi'\| = o(1). \quad (71)$$

We are going to derive $\|z\|_{\mathcal{H}} = o(1)$ from (66)-(71), which is a contradiction. The rest of the proof is divided into two parts. First, we show $\|z\|_{\mathcal{H}}$ is of $o(1)$ locally. More precisely, $\|a^{\frac{1}{2}}(x) z\|_{\mathcal{H}} = o(1)$. Then, we extend the local result to global. This strategy is adopted from [14].

Step 1

The inner product of (70) with $\frac{1}{\beta} a(x) \psi$ in $L^2(\Omega)$ yields

$$i \|a^{\frac{1}{2}}(x) \psi\|^2 - \frac{2m\theta_0}{\tau_q^2} \langle v, \frac{(a(x)\psi)'}{\beta} \rangle + \frac{2k}{\tau_q^2} \langle \tau_\theta(x) \phi' + \theta', \frac{(a(x)\psi)'}{\beta} \rangle = o(1). \quad (72)$$

Here, we have removed two inner product terms since $\|\phi'\|, \|\psi\|$ are bounded, and $\|\frac{(a(x)\psi)'}{\beta}\| = o(1)$ due to (69) and (71). This equation can be further simplified into

$$\|a^{\frac{1}{2}}(x) \psi\| = o(1) \quad (73)$$

because $\|v\|, \|\phi'\|$ are bounded, and $\|\theta'\| = o(1)$.

Next, dividing (70) by β and using (66) to obtain

$$i\psi + i \frac{2m\theta_0}{\tau_q^2} u' - \frac{2k}{\beta \tau_q^2} (\tau_\theta(x) \phi' + \theta')' = o(1), \quad \text{in } L^2(\Omega). \quad (74)$$

The inner product of (74) with $a(x)u'$ in $L^2(\Omega)$ leads to

$$\begin{aligned} & i\langle \psi, a(x)u' \rangle + i\frac{2m\theta_0}{\tau_q^2} \|a^{\frac{1}{2}}(x)u'\|^2 + \frac{2k}{\tau_q^2} \langle \tau_\theta(x)\phi' + \theta', \frac{1}{\beta}(a(x)u')' \rangle \\ & - \frac{2k}{\beta\tau_q^2} (\tau_\theta(x)\phi'(x) + \theta'(x))a(x)\bar{u}'(x)\Big|_0^L = o(1). \end{aligned} \quad (75)$$

The first term on the left-hand side of (75) is of $o(1)$ because $\|u'\|$ is bounded and (73). The third term can be estimated as follows.

$$\langle \tau_\theta(x)\phi' + \theta', \frac{1}{\beta}(a(x)u')' \rangle = \langle \tau_\theta(x)\phi' + \theta', \frac{1}{\beta}a'(x)u' \rangle + \langle a(x)(\tau_\theta(x)\phi' + \theta'), \frac{1}{\beta}u'' \rangle = o(1) \quad (76)$$

since $\|\frac{1}{\beta}u''\|$ is bounded which can be seen from (67). Here, we also used (71). The boundary terms in (75) vanish if $a(0) = a(L) = 0$. Otherwise, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned} & \left| \frac{1}{\beta} a(\xi)(\tau_\theta(\xi)\phi'(\xi) + \theta'(\xi))\bar{u}'(\xi) \right| \\ & \leq C \left(\|a(x)(\tau_\theta\phi' + \theta')\|^{\frac{1}{2}} \left\| \frac{1}{\beta} [a(x)(\tau_\theta\phi' + \theta')]' \right\|^{\frac{1}{2}} + \frac{\|a(x)(\tau_\theta\phi' + \theta')\|}{\beta^{\frac{1}{2}}} \right) \times \\ & \quad \times \left(\|u'\|^{\frac{1}{2}} \left\| \frac{1}{\beta} u'' \right\|^{\frac{1}{2}} + \frac{\|u'\|}{\beta^{\frac{1}{2}}} \right) = o(1), \end{aligned} \quad (77)$$

for $\xi = 0, L$. Therefore, we obtain

$$\|a^{\frac{1}{2}}(x)u'\| = o(1). \quad (78)$$

Finally, we take the inner product of (67) with $\frac{1}{\beta}a(x)v$ in $L^2(\Omega)$, i.e.,

$$i\|a^{\frac{1}{2}}(x)v\|^2 + a_1\langle u', \frac{(a(x)v)'}{\beta} \rangle + m\langle \frac{\tau_q^2}{2}\psi' + \tau_q\phi' + \theta', \frac{a(x)v}{\beta} \rangle = o(1). \quad (79)$$

It follows from the local results (71),(78), and the equations (66),(69), that the above two inner product terms converge to zero. We arrive at

$$\|a^{\frac{1}{2}}(x)v\| = o(1). \quad (80)$$

Step 2

In this step, we will establish the following identity

$$\begin{aligned} & \frac{4\theta_0}{\tau_q^4} \int_0^L (q(x)\tau_\theta(x))' |\beta u|^2 dx + \frac{4a_1\theta_0}{\tau_q^4} \int_0^L (q(x)\tau_\theta(x))' |u'|^2 dx + \int_0^L (q(x)\tau_\theta(x))' |\beta\phi|^2 dx \\ & + \frac{2k}{\tau_q^2} \int_0^L q'(x) |\tau_\theta(x)\phi' + \theta'|^2 dx - \frac{4a_1\theta_0}{\tau_q^4} q(L)\tau_\theta(L) |u'(L)|^2 - \frac{2k}{\tau_q^2} q(L) |\tau_\theta(L)\phi'(L) + \theta'(L)|^2 \\ & = o(1). \end{aligned} \quad (81)$$

First, we substitute (66) into (67) to get

$$-\beta^2 u - a_1 u'' + \frac{\tau_q^2 m}{2} \psi' + \tau_q m \phi' + m \theta' = f_2 + \beta f_1. \quad (82)$$

For any nonnegative function $q_1(x) \in C^1(\Omega)$ with $q_1(0) = 0$, we take the real part of the inner product of (82) with $2q_1(x)u'$ in $L^2(\Omega)$. Then,

$$\langle q_1'(x)\beta u, \beta u \rangle + a_1 \langle q_1'(x)u', u' \rangle - q_1(L)a_1|u'(L)|^2 + \operatorname{Re} \langle \frac{\tau_q^2 m}{2} \psi' + \tau_q m \phi', 2q_1(x)u' \rangle = o(1), \quad (83)$$

since we have $\|\theta'\| = o(1)$ in (71), $\|f_1'\| = o(1)$, and

$$\langle f_2 + \beta f_1, q_1(x)u' \rangle = \langle f_2, q_1(x)u' \rangle - \langle (q_1(x)f_1)'\prime, \beta u \rangle = o(1). \quad (84)$$

Next, Substitute (69) into (70) and use the fact that $\|\phi\| = o(1)$ to get

$$-\beta^2 \phi + \frac{2}{\tau_q} \psi + \frac{2m\theta_0}{\tau_q^2} v' - \frac{2k}{\tau_q^2} (\tau_\theta(x)\phi')' - \frac{2k}{\tau_q^2} \theta'' = f_5 + \beta f_4 + o(1), \quad \text{in } L^2(\Omega). \quad (85)$$

For any function $q(x) \in C^1[0, L]$ with $q(0) = 0$, taking the real part of the inner product of (85) with $2q(x)(\tau_\theta(x)\phi' + \theta')$ in $L^2(\Omega)$ yields

$$\begin{aligned} & \langle (q(x)\tau_\theta(x))'\beta\phi, \beta\phi \rangle + \frac{2k}{\tau_q^2} \langle q'(x)(\tau_\theta(x)\phi' + \theta'), (\tau_\theta(x)\phi' + \theta') \rangle \\ & - q(L) \frac{2k}{\tau_q^2} |\tau_\theta(L)\phi'(L) + \theta'(L)|^2 - \operatorname{Re} \langle \beta^2 \phi, 2q(x)\theta' \rangle + \operatorname{Re} \langle \frac{2}{\tau_q} \psi, 2q(x)\tau_\theta(x)\phi' \rangle \\ & + \operatorname{Re} \langle \frac{2m\theta_0}{\tau_q^2} v', 2q(x)(\tau_\theta(x)\phi' + \theta') \rangle = o(1) \end{aligned} \quad (86)$$

since

$$\begin{aligned} & \langle f_5 + \beta f_4, q(x)(\tau_\theta(x)\phi' + \theta') \rangle = \langle f_5, q(x)(\tau_\theta(x)\phi' + \theta') \rangle - \langle (q(x)\tau_\theta(x)f_4)'\prime, \beta\phi \rangle \\ & + \langle (q(x)f_4)'\prime, \beta\theta \rangle = o(1). \end{aligned} \quad (87)$$

Note that two terms in (86) can be simplified and removed as follows.

$$\begin{aligned} & - \operatorname{Re} \langle \beta^2 \phi, 2q(x)\theta' \rangle + \operatorname{Re} \langle \frac{2}{\tau_q} \psi, 2q(x)\tau_\theta(x)\phi' \rangle \\ & = - \operatorname{Re} \langle i\beta\phi, 2q(x)i\beta\theta' \rangle + \operatorname{Re} \langle \psi, q(x) \frac{4\tau_\theta(x)}{\tau_q} \phi' \rangle \\ & = - \operatorname{Re} \langle \psi, 2q(x)\phi' \rangle + \operatorname{Re} \langle \psi, q(x) \frac{4\tau_\theta(x)}{\tau_q} \phi' \rangle + o(1) \\ & = \frac{4}{\tau_q} \operatorname{Re} \langle \psi, q(x)(\tau_\theta(x) - \frac{\tau_q}{2})\phi' \rangle + o(1) \\ & = \frac{4}{\tau_q} \operatorname{Re} \langle \psi, q(x)a(x)\phi' \rangle + o(1) = o(1) \end{aligned} \quad (88)$$

due to the local dissipation of ϕ' in (71).

Furthermore, the last term on the left-hand side of (83) and (86) are related. If we chose

$$q_1(x) = \frac{4\theta_0}{\tau_q^4} q(x)\tau_\theta(x),$$

then

$$\begin{aligned}
& \operatorname{Re}\langle \frac{\tau_q^2 m}{2} \psi', 2q_1(x)u' \rangle + \operatorname{Re}\langle \frac{2m\theta_0}{\tau_q^2} v', 2q(x)\tau_\theta(x)\phi' \rangle \\
&= \operatorname{Re}\langle \frac{\tau_q^2 m}{2} (i\beta\phi'), 2q_1(x)u' \rangle + \operatorname{Re}\langle \phi', \frac{4m\theta_0}{\tau_q^2} q(x)\tau_\theta(x)v' \rangle + o(1) \\
&= \operatorname{Re}\langle \phi', -\tau_q^2 m q_1(x)v' \rangle + \operatorname{Re}\langle \phi', \frac{4m\theta_0}{\tau_q^2} q(x)\tau_\theta(x)v' \rangle + o(1) \\
&= \operatorname{Re}\langle \phi', [\frac{4m\theta_0}{\tau_q^2} q(x)\tau_\theta(x) - \tau_q^2 m q_1(x)]v' \rangle + o(1) \\
&= o(1).
\end{aligned} \tag{89}$$

Here, we remark that the inner product of ψ' with u' , and v' with ϕ' are not small. However, by using related multipliers $q_1(x)$ and $q(x)$, the sum of these two terms is small as shown in (89). On the other hand, we also have

$$\begin{aligned}
& \operatorname{Re}\langle \tau_q m \phi', 2q_1(x)u' \rangle + \operatorname{Re}\langle \frac{2m\theta_0}{\tau_q^2} v', 2q(x)\theta' \rangle \\
&= \operatorname{Re}\langle u', 2q_1(x)\tau_q m \phi' \rangle + \operatorname{Re}\langle \frac{2m\theta_0}{\tau_q^2} (i\beta u'), 2q(x)\theta' \rangle + o(1) \\
&= \operatorname{Re}\langle u', 2q_1(x)\tau_q m \phi' \rangle + \operatorname{Re}\langle u', -\frac{4m\theta_0}{\tau_q^2} q(x)(i\beta\theta') \rangle + o(1) \\
&= \operatorname{Re}\langle u', \frac{8m\theta_0}{\tau_q^3} q(x)\tau_\theta(x)\phi' \rangle + \operatorname{Re}\langle u', -\frac{4m\theta_0}{\tau_q^2} q(x)\phi' \rangle + o(1) \\
&= \operatorname{Re}\langle u', \frac{8m\theta_0}{\tau_q^3} a(x)q(x)\phi' \rangle + o(1) \\
&= o(1).
\end{aligned} \tag{90}$$

We now add (83) and (86). Taking into account of (88)-(89), this gives us (81).

Step 3

Pick $q(x) = \int_0^x a^{\frac{1}{2}}(\xi) d\xi$ in (81). Then, $q'(x) = a^{\frac{1}{2}}(x)$, and

$$\begin{aligned}
|(q(x)\tau_\theta(x))'| &= |a^{\frac{1}{2}}(x)\tau_\theta(x) + q(x)\tau_\theta'(x)| \\
&= |a^{\frac{1}{2}}(x)\tau_\theta(x) + q(x)a'(x)| \\
&\leq C a^{\frac{1}{2}}(x)
\end{aligned} \tag{91}$$

due to the condition (64). Therefore, by the local dissipation derived in **Step 1**, (81) become

$$-\frac{4a_1\theta_0}{\tau_q^4} \bar{a}\tau_\theta(L)|u'(L)|^2 - \frac{2k}{\tau_q^2} \bar{a}|\tau_\theta(L)\phi'(L) + \theta'(L)|^2 = o(1), \tag{92}$$

where $\bar{a} = \int_0^L a^{\frac{1}{2}}(x)dx > 0$. We now can simplify (81) into

$$\begin{aligned} & \frac{4\theta_0}{\tau_q^4} \int_0^L (q(x)\tau_\theta(x))' |\beta u|^2 dx + \frac{4a_1\theta_0}{\tau_q^4} \int_0^L (q(x)\tau_\theta(x))' |u'|^2 dx + \int_0^L (q(x)\tau_\theta(x))' |\beta\phi|^2 dx \\ & + \frac{2k}{\tau_q^2} \int_0^L q'(x) |\tau_\theta(x)\phi' + \theta'|^2 dx = o(1). \end{aligned} \quad (93)$$

Pick $q(x) = \frac{x}{\tau_\theta(x)}$ in (93). Then,

$$\begin{aligned} & \frac{4\theta_0}{\tau_q^4} \int_0^L |\beta u|^2 dx + \frac{4a_1\theta_0}{\tau_q^4} \int_0^L |u'|^2 dx + \int_0^L |\beta\phi|^2 dx + \frac{2k}{\tau_q^2} \int_0^L \tau_\theta(x) |\phi'|^2 dx \\ & = o(1) + \frac{2k}{\tau_q^2} \int_0^L x\tau'_\theta(x) |\phi'|^2 dx. \end{aligned} \quad (94)$$

Since

$$\begin{aligned} & \left| \int_0^L x\tau'_\theta(x) |\phi'|^2 dx \right| = \left| \int_0^L xa'(x) |\phi'|^2 dx \right| \\ & \leq C \int_0^L a^{\frac{1}{2}}(x) |\phi'|^2 dx = o(1), \end{aligned}$$

it follows from (94) that

$$\|\beta u\|, \|u'\|, \|\beta\phi\|, \|\phi'\| = o(1), \quad (95)$$

which further implies that

$$\|v\|, \|\psi\| = o(1) \quad (96)$$

due to (66) and (69). We have reached the promised contradiction. \square

Remark 3. The conclusion in Theorem 3.1, 3.2 and Theorem 4.1 also hold for other boundary conditions, such as the Dirichlet-Neumann boundary conditions

$$u(0, t) = u(L, t) = \theta_x(0, t) = \theta_x(L, t) = 0,$$

or the Neumann-Dirichlet boundary conditions

$$u_x(0, t) = u_x(L, t) = \theta(0, t) = \theta(L, t) = 0.$$

The proof above can be repeated with very minor changes, such as the definition of the state space, the boundary terms from integration by parts. In fact, we don't need to use the Gagliardo-Nirenberg inequality to estimate boundary terms since they vanish in the above two cases.

Remark 4. The most general case corresponds to that both delay parameters depend on the material point. However, the analysis seems to be very cumbersome. We are unable to find a proper dissipative energy even just for the dual phase-lag heat equation (3). It is our plan to address this problem in future studies.

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