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Non-Relativistic BMS algebra

Carles Batlle[a], Diego Delmastro[b] and Joaquim Gomis[b]

[a]*Departament de Matemàtiques and IOC,*
Universitat Politècnica de Catalunya
EPSEVG, Av. V. Balaguer 1, E-08808 Vilanova i la Geltrú, Spain
email: carles.batlle@upc.edu
[b]*Departament de Física Quàntica i Astrofísica*
and Institut de Ciències del Cosmos,
Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spain
email: ddelmade7@alumnes.ub.edu, joaquim.gomis@icc.ub.edu

Abstract

We construct two possible candidates for the non-relativistic $\mathfrak{bms}_4$ algebra in four space-time dimensions by contracting the original relativistic $\mathfrak{bms}_4$ algebra. The $\mathfrak{bms}_4$ algebra is infinite-dimensional and it contains the generators of the Poincaré algebra, together with the so-called *super-translations*. Similarly, the proposed $\mathfrak{nrbms}_4$ algebras can be regarded as two infinite-dimensional extensions of the Bargmann algebra. We also study a canonical realisation of one of these algebras in terms of the Fourier modes of a free Schrödinger field, mimicking the canonical realisation of the relativistic $\mathfrak{bms}_4$ algebra using a free Klein-Gordon field.

Contents

1 Introduction

Recently there has been a renewed interest in the Bondi-Metzner-Sachs (BMS) group [1]. This group is the semi-direct product of the Lorentz group with the infinite dimensional group of super-translations [1], which is abelian. The BMS symmetry plays an important role in the understanding of gravitational scattering, soft theorems, memory effects and black holes; see, for example, the recent lectures by Strominger [2] on the subject, where one can also find an extended set of references to the original literature. Moreover, the BMS group could play a crucial role in understanding holography in asymptotically flat space times [3] [4] [5] [6]. The BMS symmetry is an infinite conformal extension of the Carroll symmetry [7], which was introduced in [8] as a limit of the Poincaré symmetry when the velocity of light is scaled down to zero.

The application of holography to condensed matter systems – known as non-relativistic holography – is a subject of current interest. Non-relativistic holography has been used to study phenomena such as cold atoms, quantum critical points and high temperature superconductivity [9] [10] [11]. Moreover, the role of non-relativistic gravities to understand strongly coupled systems in condensed matter has been appreciated; see, for example, the use of Newton-Cartan gravity in the study of the quantum Hall effect [12] [13].

The study of flat-holography for non-relativistic systems is a useful tool because flat space is a good approximation to the real world. Non-relativistic symmetries, in particular the non-relativistic super-translations, are expected to be useful in the study of the infrared behavior of non-relativistic quantum field theories, in the same way that the relativistic super-translations allow us to understand soft theorems [26] as Ward identities.

In order to analyse this point it is natural to consider the non-relativistic limit of holography in asymptotically Minkowski space-times. With this motivation in mind, in this work we will construct two non-relativistic versions of the $\mathfrak{bms}_4$ ($\mathfrak{nrbms}_4$) algebra and an explicit realization of one of them.

In order to construct the non-relativistic analogue of $\mathfrak{bms}_4$ there are several approaches one could take. The simplest one is to consider a suitable İnönü-Wigner contraction of the corresponding relativistic $\mathfrak{bms}_4$ algebra[1], in the same sense that the Galilei algebra can be obtained by contracting the Poincaré one. An alternative to obtain a possible non-relativistic $\mathfrak{bms}_4$ algebra is to mimic the canonical construction of Longhi and Materassi [15], but using a scalar field with Galilean space-time symmetries instead of a Klein-Gordon field. Finally, a third alternative is to follow the same steps that led to the original $\mathfrak{bms}_4$ algebra [1] [16], but in a non-relativistic setting, i.e., one could try to study the set of isometries of asymptotically flat Newtonian space-times [17], characterized by a contravariant degenerate spatial metric $h^{\mu\nu}$ and covariant vector τ_μ; see, for example, [18] and references therein. Here we will only consider the first two possibilities.

The organisation of the paper is as follows. In Section 2 we will construct the two contractions of the $\mathfrak{bms}_4$ that contain the Bargmann algebra [19] as a sub-algebra. In Section 3 we obtain a realisation of one of the $\mathfrak{nrbms}_4$ algebras in terms of a free Schrödinger field. Section 4 is devoted to conclusions and outlook.

2 The algebra $\mathfrak{nrbms}_4$ as a contraction of $\mathfrak{bms}_4$

In this section we will construct two possible $\mathfrak{nrbms}_4$ algebras as the two possible contractions of the standard $\mathfrak{bms}_4$ algebra. The $\mathfrak{bms}_4$ algebra is the semi-direct sum of the Lorentz algebra with the generators of the super-translations, which form an infinite-dimensional abelian sub-algebra [1]. In a similar fashion, the $\mathfrak{nrbms}_4$ algebra will be given by the semi-direct sum of the Bargmann algebra with the generators of spatial super-translations. The extension of the algebra to include super-rotations [20] will not be considered in this paper.

The canonical realisation of the $\mathfrak{bms}_4$ algebra in terms of the Fourier modes of free Klein-

[1] The contraction procedure can be extended to an arbitrary number of space-time dimensions [14].

Gordon field [15] leads to the algebra

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k, & [P_\ell^m, P_{\ell'}^{m'}] &= 0, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k, & [J_i, P_\ell^m] &= i(\mathcal{J}_i)_{\ell\, m'}^{\ell' m}\ P_{\ell'}^{m'}, \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k, & [K_i, P_\ell^m] &= i(\mathcal{K}_i)_{\ell\, m'}^{\ell' m}\ P_{\ell'}^{m'},
\end{aligned}
\tag{2.1}
$$

where J_i, K_i are the generators of Lorentz transformations, and P_ℓ^m are the generators of super-translations. The indices ℓ, m are integers such that $|m| \leq \ell$, and $(\mathcal{J}_i)_{\ell\, m'}^{\ell' m}, (\mathcal{K}_i)_{\ell\, m'}^{\ell' m}$ are the structure constants of the $\mathfrak{bms}_4$ algebra. These matrices furnish a non-unitary, reducible but indecomposable infinite-dimensional representation of the Lorentz group (with the sign of the structure constants negated, corresponding to passive vs. active transformations).

More explicitly, these matrices are given by

$$
\begin{aligned}
[J_1, P_\ell^m] &= \frac{i}{2}\sqrt{(\ell+m)(\ell-m+1)}\ P_\ell^{m-1} - \frac{i}{2}\sqrt{(\ell-m)(\ell+m+1)}\ P_\ell^{m+1}, \\
[J_2, P_\ell^m] &= \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\ P_\ell^{m+1} + \frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\ P_\ell^{m-1}, \\
[J_3, P_\ell^m] &= m\ P_\ell^m, \\
[K_1, P_\ell^m] &= \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ P_{\ell-1}^{m+1} \\
&\quad + \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ P_{\ell-1}^{m-1} \\
&\quad + \frac{1}{2}\frac{(\ell-1)(\ell+3)}{2\ell+3}\sqrt{\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+1)(2\ell+3)}}\ P_{\ell+1}^{m+1} \\
&\quad + \frac{1}{2}\frac{(\ell-1)(\ell+3)}{2\ell+3}\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}}\ P_{\ell+1}^{m-1}, \\
[K_2, P_\ell^m] &= \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ P_{\ell-1}^{m+1} \\
&\quad - \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ P_{\ell-1}^{m-1} \\
&\quad + \frac{i}{2}\frac{(\ell-1)(\ell+3)}{2\ell+3}\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{(2\ell+1)(2\ell+3)}}\ P_{\ell+1}^{m+1} \\
&\quad - \frac{i}{2}\frac{(\ell-1)(\ell+3)}{2\ell+3}\sqrt{\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+1)(2\ell+3)}}\ P_{\ell+1}^{m-1} \\
[K_3, P_\ell^m] &= i\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell+m)}\ P_{\ell-1}^m, \\
&\quad - i\frac{(\ell-1)(\ell+3)}{2\ell+3}\sqrt{\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+1)(2\ell+3)}}\ P_{\ell+1}^m.
\end{aligned}
\tag{2.2}
$$

The form of the $\mathfrak{bms}_4$ algebra as given by Sachs [16] is recovered from the previous expression by multiplying the generators of super-translations by the factor

$$P_\ell^m \to \frac{1}{2}\frac{(\ell+2)!}{(2\ell+1)!!}P_\ell^m \tag{2.3}$$

One should note that Poincaré is a sub-algebra of $\mathfrak{bms}_4$, and no other finite-dimensional sub-algebras exist [15]. The Poincaré sub-algebra is spanned by J_i, K_i, P_0^0, P_1^m, with $i = 1, 2, 3$ and $m = 0, \pm 1$. These operators form a closed sub-algebra of $\mathfrak{bms}_4$ because the matrix coefficients $(\mathcal{J}_i)^{\ell' m}_{\ell\, m'}, (\mathcal{K}_i)^{\ell' m}_{\ell\, m'}$ vanish when $\ell = 1$ and $\ell' \geq 2$.

We now proceed to perform the contraction of this algebra in order to obtain its non-relativistic analogue. Here and in the remainder of this document, we will place a hat over an object to indicate that it corresponds to the contracted, non-relativistic counterpart of the corresponding relativistic object. For example, the generators of non-relativistic super-translations will be denoted by $\hat{P}_\ell^m$, as opposed to the generators of relativistic super-translations, P_ℓ^m.

As the Bargmann algebra contains a central charge, we expect that something similar happens in the case of $\mathfrak{nrbms}$. We therefore consider the direct product of the BMS group with $U(1)$, with generator $\tilde{Z}$, and introduce the following transformation

$$\begin{aligned} H &:= \omega(P_0^0 + \tilde{Z}) & & & P_0^0 &= \frac{1}{2\omega}H + \omega Z \\ Z &:= \frac{1}{\omega}(P_0^0 - \tilde{Z}) & &\overset{\text{inverse}}{\Longleftrightarrow} & \tilde{Z} &= \frac{1}{2\omega}H - \omega Z \\ \hat{K}_i &:= \frac{1}{\omega}K_i & & & K_i &= \omega \hat{K}_i \\ \hat{J}_i &:= J_i & & & J_i &= \hat{J}_i \end{aligned} \tag{2.4}$$

and

$$\hat{P}_\ell^m := \omega^{f(\ell)} P_\ell^m \quad \overset{\text{inverse}}{\Longleftrightarrow} \quad P_\ell^m = \omega^{-f(\ell)}\hat{P}_\ell^m, \quad \ell \geq 1, \tag{2.5}$$

where $f(\ell)$ is an unspecified function of ℓ, and ω is a dimensionless parameter which we shall take $\omega \to \infty$ at the end.

In the limit $\omega \to \infty$, the centrally-extended relativistic algebra (2.2) becomes

$$\begin{aligned} &[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k & &[\hat{P}_\ell^m, H] = 0 & &[\hat{P}_\ell^m, \hat{P}_{\ell'}^{m'}] = 0 \\ &[\hat{J}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}_k & &[\hat{J}_i, H] = 0 & &[\hat{J}_i, \hat{P}_\ell^m] = i(\hat{\mathcal{J}}_i)^{\ell' m}_{\ell\, m'}\ \hat{P}_{\ell'}^{m'} \\ &[\hat{K}_i, \hat{K}_j] = 0 & &[\hat{K}_i, H] = i\hat{P}_i & &[\hat{K}_i, \hat{P}_\ell^m] = i(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'}\ \hat{P}_{\ell'}^{m'} \end{aligned} \tag{2.6}$$

where $(\hat{\mathcal{J}}_i)^{\ell' m}_{\ell\, m'} = (\mathcal{J}_i)^{\ell' m}_{\ell\, m'}$ and

$$(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'} = \lim_{\omega\to\infty} \omega^{f(\ell)-f(\ell')-1}(\mathcal{K}_i)^{\ell' m}_{\ell\, m'} \tag{2.7}$$

In order to have a non-trivial $\omega \to \infty$ limit, we must have

$$f(\ell) - f(\ell') - 1 \equiv 0 \tag{2.8}$$

for some ℓ'. Moreover, the structure constants $(\mathcal{K}_i)^{\ell' m}_{\ell\, m'}$ are non-zero only if $|\ell' - \ell| = 1$. This means that the only non-trivial contractions are the ones that verify

$$f(\ell) = f(0) \pm \ell \tag{2.9}$$

Furthermore, in order to obtain the Bargmann algebra as a sub-algebra, for $\ell = 1$ we should recover the standard contraction $\hat{P}_i = P_i$, so that $f(1) = 0$. With this,

$$f(\ell) = \pm(\ell - 1) \tag{2.10}$$

The conclusion of this discussion is that if we restrict ourselves to scalings of the form $\hat{P}^m_\ell = \omega^{f(\ell)} P^m_\ell$ then there are only two non-trivial contractions of the $\mathfrak{bms}_4$ algebra, corresponding to either sign in $f(\ell) = \pm(\ell - 1)$, which we will call[2] $\mathfrak{nrbms}_4^\pm$. In either case, the structure constants are given by

$$\begin{aligned} (\hat{\mathcal{J}}_i)^{\ell' m}_{\ell\, m'} &= (\mathcal{J}_i)^{\ell' m}_{\ell\, m'} \\ (\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'} &= \begin{cases} (\mathcal{K}_i)^{\ell' m}_{\ell\, m'} & \ell = \ell' \pm 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.11}$$

These two possible algebras contain time and space translations, rotations, Galilean boosts, spatial super-translations, and a central charge, and they both contain a Bargmann sub-algebra (see Fig.1). The matrices $\hat{\mathcal{J}}_i, \hat{\mathcal{K}}_i$ define a non-unitary, reducible but indecomposable, infinite-dimensional realisation of the homogeneous Galilei group, with algebra given by the first column of (2.6).

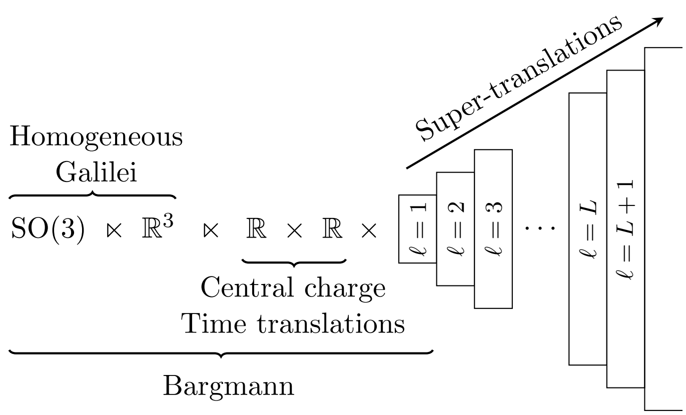

Figure 1: The structure of the $\mathfrak{nrbms}$ algebra. Each box represents an abelian algebra of dimension $2\ell + 1$ corresponding to the super-translations $\{\hat{P}^m_\ell\}$, with $m = -\ell, \cdots, \ell$.

[2] One should note that this '±' has nothing to do with the conventional notation of $\mathrm{BMS}^\pm$ as the group that acts on the asymptotic past and future null infinity in asymptotically flat space-times.

It is important to note that in the case of the $\mathfrak{nrbms}_4^+$ contraction, the boost operators lower the value of ℓ to $\ell-1$. This is in stark contrast with the relativistic case, where we have simultaneous contributions from $\ell-1$ and $\ell+1$. Therefore, unlike in the relativistic case, here the algebra resulting from the + contraction contains an infinite number of finite-dimensional sub-algebras, obtained by considering all the super-translation generators with $1\leq\ell\leq L$ for a given L; the dimension of these sub-algebras is $8+\sum_{\ell=1}^{L}(2\ell+1)=L^2+2L+8$. All these sub-algebras contain a Bargmann sub-algebra, and the associated matrices $\hat{\mathcal{J}}_i$, $\hat{\mathcal{K}}_i$ provide an infinite number of finite-dimensional representations of the homogeneous Galilei group[3], or Euclidean $E(3)$ group, of dimensions $\sum_{\ell=0}^{L}(2\ell+1)=(L+1)^2$, $L\geq 1$.

On the other hand, in the case of the $\mathfrak{nrbms}_4^-$ algebra, the boost operators raise the value of ℓ to $\ell+1$, which means that the only finite-dimensional sub-algebra of $\mathfrak{nrbms}_4^-$ is the Bargmann algebra (see Fig.2).

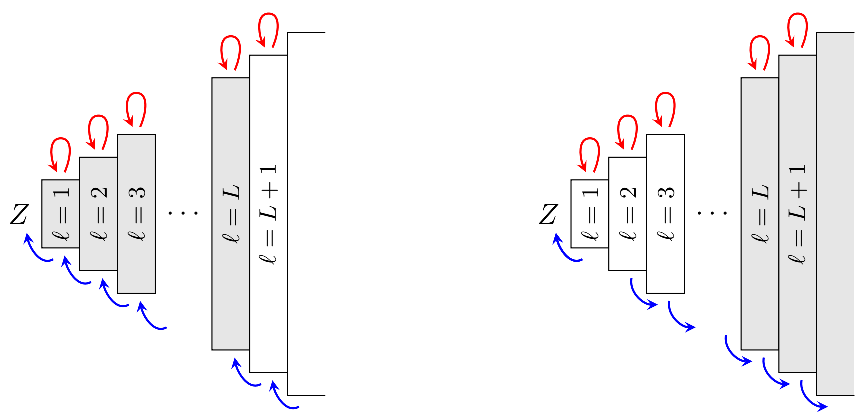

Figure 2: The inhomogeneous part of $\mathfrak{nrbms}^+$ (left) and $\mathfrak{nrbms}^-$ (right) . The upper row arrows represent the action of rotations on the generators of super-translations, while the lower row ones represent that of the Galilean boosts. Rotations do not change the value of ℓ, while boosts take us from ℓ to $\ell\mp 1$. The grey boxes represent the different sub-algebras, obtained from varying L (in the first case, they are all finite-dimensional, while in the second case they are all infinite-dimensional).

In the following section we will construct an explicit realisation of the $\mathfrak{nrbms}_4^+$ algebra corresponding to the plus sign contraction, and we will also discuss the possibility of adding dilatations and expansions in order to obtain a Schrödinger-$\mathfrak{nrbms}$.

3 A canonical realisation of the $\mathfrak{nrbms}_4$ algebra

We now proceed to construct a canonical realisation of the $\mathfrak{nrbms}_4^+$ algebra. Our construction will be analogous to the canonical construction of the $\mathfrak{bms}_4$ algebra, as given by Longhi and

[3] Finite-dimensional indecomposable representations of homegeneous Galilei have been studied in [21].

Materassi [15] (see also [22], [23]). The starting point is a free complex Schrödinger field $\psi(t, \boldsymbol{x})$ in $1+3$ Galilean space-time, with action

$$S[\psi] = \int \mathrm{d}t\, \mathrm{d}\boldsymbol{x} \left(i\psi^*(t,\boldsymbol{x})\dot{\psi}(t,\boldsymbol{x}) + \frac{1}{2\mu}\psi^*(t,\boldsymbol{x})\nabla^2\psi(t,\boldsymbol{x}) \right), \tag{3.1}$$

whose equation of motion is the Schrödinger equation,

$$i\dot{\psi} + \frac{1}{2\mu}\nabla^2\psi = 0 \tag{3.2}$$

The action is invariant under the general Bargmann transformation

$$\begin{aligned} \delta\psi(t,\boldsymbol{x}) = &- i\mu\eta\psi(t,\boldsymbol{x}) + \epsilon\dot{\psi}(t,\boldsymbol{x}) - a_i\partial_i\psi(t,\boldsymbol{x}) - \beta_i(t\partial_i - i\mu x_i)\psi(t,\boldsymbol{x}) \\ &- \omega_i\epsilon_{ijk}(x_j\partial_k - x_k\partial_j)\psi(t,\boldsymbol{x}), \end{aligned} \tag{3.3}$$

where η, ϵ, a_i, β_i and ω_i are, respectively, parameters corresponding to the central charge, time translations, spatial translations, Galilean boosts and spatial rotations. The associated Noether charges are

$$\begin{aligned} Z &= \mu \int \mathrm{d}\boldsymbol{x}\ |\psi|^2 \\ H &= \frac{1}{2\mu}\int \mathrm{d}\boldsymbol{x}\ |\nabla\psi|^2 \\ \hat{P}_i &= -i\int \mathrm{d}\boldsymbol{x}\ \psi^*\partial_i\psi, \\ \hat{K}_i &= t\hat{P}_i - \mu\int \mathrm{d}\boldsymbol{x}\ \psi^* x_i\psi \\ \hat{J}_i &= -i\epsilon_{ijk}\int \mathrm{d}\boldsymbol{x}\ \psi^* x_j\partial_k\psi, \end{aligned} \tag{3.4}$$

and they satisfy the Bargmann algebra,

$$\begin{aligned} \{\hat{J}_i, \hat{J}_j\} &= i\epsilon_{ijk}\hat{J}_k & \{\hat{J}_i, \hat{P}_j\} &= i\epsilon_{ijk}\hat{P}_k \\ \{\hat{J}_i, \hat{K}_j\} &= i\epsilon_{ijk}\hat{K}_k & \{\hat{K}_i, \hat{P}_j\} &= iZ\delta_{ij} \\ \{\hat{K}_i, \hat{K}_j\} &= 0 & \{\hat{K}_i, H\} &= i\hat{P}_i, \end{aligned} \tag{3.5}$$

where $\{\cdot,\cdot\}$ denotes the Dirac bracket (such that the canonical algebra reads $\{\psi(t,\boldsymbol{x}), \psi^*(t,\boldsymbol{y})\} = -i\delta(\boldsymbol{x}-\boldsymbol{y})$).

The field $\psi(t,\boldsymbol{x})$, when on-shell, can be expanded in Fourier modes as

$$\psi(t,\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ \mathrm{e}^{-ik^0t + i\boldsymbol{x}\cdot\boldsymbol{k}} a(\boldsymbol{k}), \tag{3.6}$$

where $k^0 = \frac{1}{2\mu}\boldsymbol{k}^2$. The absence of a^* in the expansion of ψ reflects the fact that no antiparticles would be present in the corresponding second quantised theory.

When expressed in terms of a, a^*, the on-shell Noether charges become

$$
\begin{aligned}
Z &= \mu \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\; a^*(\boldsymbol{k}) a(\boldsymbol{k}) \\
H &= \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\; a^*(\boldsymbol{k}) \frac{\boldsymbol{k}^2}{2\mu} a(\boldsymbol{k}) \\
\hat{P}_i &= \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\; a^*(\boldsymbol{k}) k_i a(\boldsymbol{k}) \\
\hat{K}_i &= t\hat{P}_i - i\mu \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\; a^*(\boldsymbol{k}) \partial_i a(\boldsymbol{k}) \\
\hat{J}_i &= -i\epsilon_{ijk} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\; a^*(\boldsymbol{k}) k_j \partial_k a(\boldsymbol{k}).
\end{aligned}
\tag{3.7}
$$

As far as the homogeneous Galilei algebra is concerned, the time dependent part $t\hat{P}_i$ in $\hat{K}_i$ is irrelevant, and so we will drop it in what follows. Using $i\{a(\boldsymbol{k}), a^*(\boldsymbol{k}')\} = (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}')$, one may easily check that

$$
\begin{aligned}
i\{\hat{K}_i, a(\boldsymbol{k})\} &= \hat{\mathcal{K}}_i a(\boldsymbol{k}) \\
i\{\hat{J}_i, a(\boldsymbol{k})\} &= \hat{\mathcal{J}}_i a(\boldsymbol{k}),
\end{aligned}
\tag{3.8}
$$

where $\hat{\mathcal{K}}_i := i\mu\partial_i$ and $\hat{\mathcal{J}}_i := i\epsilon_{ijk} k_j \partial_k$ are differential operators in $\boldsymbol{k}$ space, which provide a realisation of the homogeneous Galilei algebra,

$$
\begin{aligned}
[\hat{\mathcal{J}}_i, \hat{\mathcal{J}}_j] &= -i\epsilon_{ijk} \hat{\mathcal{J}}_k \\
[\hat{\mathcal{J}}_i, \hat{\mathcal{K}}_j] &= -i\epsilon_{ijk} \hat{\mathcal{K}}_k \\
[\hat{\mathcal{K}}_i, \hat{\mathcal{K}}_j] &= 0,
\end{aligned}
\tag{3.9}
$$

where $[\cdot,\cdot]$ denotes a commutator.

We now consider the quadratic Casimir operator of the homogeneous Galilei group, $\hat{\Delta} := \hat{\mathcal{K}}_i^2$, that is,

$$
\hat{\Delta} = -\mu^2 \partial_i^2 \tag{3.10}
$$

The momenta k_i satisfy an eigenvalue equation with respect to $\hat{\Delta}$,

$$
\hat{\Delta} k_i = 0, \tag{3.11}
$$

i.e., the momenta are zero-modes of $\hat{\Delta}$. As in the relativistic construction [15], this motivates us to consider the most general zero-mode equation

$$
\hat{\Delta}\, \hat{\chi}(\boldsymbol{k}) = 0, \tag{3.12}
$$

whose set of solutions $\{\hat{\chi}\}$ will contain the momenta k_i as a subset. The general set zero-modes will correspond to the complete set of super-translations, in the same sense that k_i corresponds to standard translations (cf. (3.7)). It bears mentioning that the second Casimir, $\hat{\mathcal{J}}_i\hat{\mathcal{K}}_i$, vanishes identically.

The general solution of the equation above is

$$\hat{\chi}_\ell^m(r,\theta,\varphi) = f_\ell(r) Y_\ell^m(\theta,\varphi) \tag{3.13}$$

where r,θ,φ are the spherical coordinates of the dimensionless momentum $\frac{1}{\mu}\boldsymbol{k}$, and Y_ℓ^m are the spherical harmonics, with ℓ, m integers such that $|m| \leq \ell$. Here, f_ℓ is given by the solution of

$$r^2 f_\ell'' + 2r f_\ell' - \ell(\ell+1) f_\ell = 0 \tag{3.14}$$

that is,

$$f_\ell(r) = c_1 r^\ell + c_2 r^{-(\ell+1)} \tag{3.15}$$

We are looking for zero-modes $\hat{\chi}$ that are regular at the origin, like the three momenta k_i, so we drop the second solution. With this,

$$\hat{\chi}_\ell^m(r,\theta,\varphi) = r^\ell Y_\ell^m(\theta,\varphi) \tag{3.16}$$

It is easy to see that the modes corresponding to $\ell = 1$ agree with the spherical components of $\frac{1}{\mu}\boldsymbol{k}$, so that, as expected, the family $\{\hat{\chi}_\ell^m\}$ contains the functions k_i as a special subcase.

As the angular part of the modes $\hat{\chi}_\ell^m$ is given by the spherical harmonics, these functions satisfy

$$\begin{aligned}
\hat{\mathcal{J}}_1\hat{\chi}_\ell^m &= \frac{i}{2}\sqrt{(\ell+m)(\ell-m+1)}\ \hat{\chi}_\ell^{m-1} - \frac{i}{2}\sqrt{(\ell-m)(\ell+m+1)}\ \hat{\chi}_\ell^{m+1} \\
\hat{\mathcal{J}}_2\hat{\chi}_\ell^m &= \frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\ \hat{\chi}_\ell^{m-1} + \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\ \hat{\chi}_\ell^{m+1} \\
\hat{\mathcal{J}}_3\hat{\chi}_\ell^m &= m\,\hat{\chi}_\ell^m
\end{aligned} \tag{3.17}$$

Moreover, the boost differential operators $\hat{\mathcal{K}}_i$ do also have a simple action on the modes $\hat{\chi}_\ell^m$:

$$\begin{aligned}
\hat{\mathcal{K}}_1\hat{\chi}_\ell^m &= \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ \hat{\chi}_{\ell-1}^{m+1} + \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ \hat{\chi}_{\ell-1}^{m-1} \\
\hat{\mathcal{K}}_2\hat{\chi}_\ell^m &= \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ \hat{\chi}_{\ell-1}^{m+1} - \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ \hat{\chi}_{\ell-1}^{m-1} \\
\hat{\mathcal{K}}_3\hat{\chi}_\ell^m &= i\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell+m)}\ \hat{\chi}_{\ell-1}^{m}
\end{aligned} \tag{3.18}$$

With this, we now define the generators of super-translations as

$$\hat{P}_\ell^m := \mu \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ a^*(\boldsymbol{k})\hat{\chi}_\ell^m(\boldsymbol{k}) a(\boldsymbol{k}). \tag{3.19}$$

Unlike the relativistic case [15], here the existence of $\hat{P}_\ell^m$ is not guaranteed by the existence of $\hat{P}_i$, because the non-relativistic modes $\hat{\chi}$ scale as r^ℓ for large r instead of linearly with r. Therefore,

$a(\boldsymbol{k})$ being square-integrable is not enough for the integral defining $\hat{P}_\ell^m$ to converge; we must impose the stronger condition that $|\boldsymbol{k}|^\ell |a(\boldsymbol{k})|^2$ is integrable for all $\ell \in \mathbb{N}$. We also note that, due to the form of $\hat{P}_\ell^m$ in (3.19), the quantum theory would yield an unbroken realisation of the symmetry, i.e., the Fock vacuum would be invariant.

Using (3.17) and (3.18), we see that the functions $\hat{P}_\ell^m$ satisfy the algebra

$$\begin{aligned} \{\hat{J}_1, \hat{P}_\ell^m\} &= \frac{i}{2}\sqrt{(\ell+m)(\ell-m+1)}\ \hat{P}_\ell^{m-1} - \frac{i}{2}\sqrt{(\ell-m)(\ell+m+1)}\ \hat{P}_\ell^{m+1} \\ \{\hat{J}_2, \hat{P}_\ell^m\} &= \frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\ \hat{P}_\ell^{m-1} + \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\ \hat{P}_\ell^{m+1} \\ \{\hat{J}_3, \hat{P}_\ell^m\} &= m\, P_\ell^m \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \{\hat{K}_1, \hat{P}_\ell^m\} &= \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ \hat{P}_{\ell-1}^{m+1} \\ &\quad + \frac{1}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ \hat{P}_{\ell-1}^{m-1} \\ \{\hat{K}_2, \hat{P}_\ell^m\} &= \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell-m-1)}\ \hat{P}_{\ell-1}^{m+1} \\ &\quad - \frac{i}{2}\sqrt{\frac{2\ell+1}{2\ell-1}(\ell+m)(\ell+m-1)}\ \hat{P}_{\ell-1}^{m-1} \\ \{\hat{K}_3, \hat{P}_\ell^m\} &= i\sqrt{\frac{2\ell+1}{2\ell-1}(\ell-m)(\ell+m)}\ \hat{P}_{\ell-1}^{m} \end{aligned} \tag{3.21}$$

which constitutes a realisation of the $\mathfrak{nrbms}_4^+$ algebra as given by (2.6), (2.11). In principle, one may construct a realisation of $\mathfrak{nrbms}_4^-$ by using the second solution of the radial equation, $f_\ell \sim r^{-(\ell+1)}$, but the fact that these functions are singular at the origin implies that the Fourier modes $a(\boldsymbol{k})$ must go to zero faster than any polynomial if we want the integral that defines $\hat{P}_\ell^m$ to converge. We will not consider this possibility any further here.

This constitutes our proposal for the non-relativistic $\mathfrak{bms}$ algebra. It is important to stress that we obtained the same algebra both by means of the abstract group contraction, and the explicit canonical realisation. This agreement is in fact not completely unexpected: one may check that the explicit realisation is nothing but the $m \to \infty$ limit of the canonical realisation of the relativistic $\mathfrak{bms}$ algebra, as given in [15].

Unlike in the relativistic case, the modes $\hat{\chi}(\boldsymbol{k})$ are actually homogeneous polynomials in k_i, of degree ℓ. This means that the symmetries generated by $\hat{P}_\ell^m$ are local when acting on the field ψ, meaning that

$$\delta_\ell^m \psi = \{\hat{P}_\ell^m, \psi\} = \hat{\chi}_\ell^m(-i\partial)\psi \tag{3.22}$$

where $\hat{\chi}_\ell^m$ is a harmonic polynomial of degree ℓ. In other words, $\hat{\chi}_\ell^m(-i\partial)$ is nothing but a polynomial in ∂_i:

$$\hat{\chi}_\ell^m(-i\partial) = \sum_{|\alpha|=\ell} c^\alpha \partial_\alpha \tag{3.23}$$

for a certain set of coefficients c_α, and where α is a multi-index. Differential operators of this form (and generalisations thereof), in the context of symmetries of partial differential equations, have been studied extensively in the literature; see for example [24] [25].

Needless to say, for the operation $\delta_\ell^m \psi = \hat{\chi}_\ell^m(-i\partial)\psi$ to be symmetry of the system, it must be well-defined: ψ must have at least ℓ (spatial) derivatives, and all of them must be square integrable (so that they represent valid wave-functions). In other words, we must have $\psi(t,\cdot) \in W^{\ell,2}(\mathbb{R}^3, \mathbb{C}, \mathrm{d}\boldsymbol{x})$, the Sobolev space of order ℓ. One should note that if $\psi(t,\cdot) \in W^{\ell,2}(\mathbb{R}^3, \mathbb{C}, \mathrm{d}\boldsymbol{x})$ then $|\boldsymbol{k}|^\ell a \in L^2(\mathbb{R}^3, \mathbb{C}, \mathrm{d}\boldsymbol{k})$, and therefore the integral that defines P_ℓ^m, to wit (3.19), converges, as one would expect.

The operators P_ℓ^m are symmetries of the Schrödinger differential operator $\partial_t - \{H, \cdot\}$. Indeed, if $\psi(x)$ is a solution of the Schrödinger equation, then so is $\psi(x) + \delta_\ell^m \psi(x)$ for any ℓ, m. To see this, we note that the general solution of the Schrödinger equation is of the form (3.6)

$$\psi(x) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ \mathrm{e}^{-ik^0 t + i\boldsymbol{k}\cdot\boldsymbol{x}} a(\boldsymbol{k}) \tag{3.24}$$

and therefore

$$\delta_\ell^m \psi(x) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ \mathrm{e}^{-ik^0 t + i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\chi}_\ell^m(\boldsymbol{k}) a(\boldsymbol{k}) \tag{3.25}$$

which is itself of the form (3.24), with $a(\boldsymbol{k}) \to \hat{\chi}_\ell^m(\boldsymbol{k}) a(\boldsymbol{k})$.

Furthermore, the fact that the polynomials $\hat{\chi}_\ell^m(\boldsymbol{k})$ are *homogeneous* and of degree ℓ implies that they satisfy

$$\mathcal{D}\hat{\chi}_\ell^m = \ell \hat{\chi}_\ell^m \tag{3.26}$$

where $\mathcal{D}$ is the homogeneity operator, $\mathcal{D} := k^i \partial_i$. Hence, if we define the dilatation operator as

$$D := 2tH + i \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ a^*(\boldsymbol{k}) \mathcal{D} a(\boldsymbol{k}) \tag{3.27}$$

then the generators of super-translations satisfy

$$\{D, \hat{P}_\ell^m\} = i\ell \hat{P}_\ell^m \tag{3.28}$$

which extends the $\mathfrak{nrbms}_4^+$ algebra to include dilatations (which are also symmetries of the Schrödinger action), giving rise to a Weyl-$\mathfrak{nrbms}$.

Once we check that the algebra admits dilatations, it becomes natural to ask ourselves about the action of the expansion operator (or Schrödinger conformal transformations), given by

$$C := -t^2 H + tD + \frac{\mu}{2} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ a^*(\boldsymbol{k}) \hat{\Delta} a(\boldsymbol{k}). \tag{3.29}$$

Using

$$\begin{aligned}[\hat{\Delta}, \hat{\chi}^m_\ell] &= \hat{\Delta}\hat{\chi}^m_\ell + 2(\partial_i \hat{\chi}^m_\ell)\frac{\partial}{\partial k^i} \\ &= \frac{2}{\mu}(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'}\ \hat{\chi}^{m'}_{\ell'}\frac{\partial}{\partial k^i}\end{aligned} \tag{3.30}$$

we obtain

$$\{C, \hat{P}^m_\ell\} = it\ell\hat{P}^m_\ell + \mu\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ a^*(\boldsymbol{k})\left[(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'}\ \hat{\chi}^{m'}_{\ell'}(\boldsymbol{k})\frac{\partial}{\partial k^i}\right]a(\boldsymbol{k}) \tag{3.31}$$

The *r.h.s.* of (3.31) is not an element of $\mathfrak{nrbms}^+_4$. This means that if we attempt to extend the algebra to include C, the resulting algebra is not closed, which seems to preclude a possible Schrödinger-$\mathfrak{nrbms}$ with only super-translations (that is, without further extending the set of generators).

In any case, the *r.h.s.* (3.31) is the bracket of two conserved quantities, which means that it is conserved as well. Indeed,

$$\left[\frac{\partial}{\partial t} + \{\cdot, H\}\right]\{C, \hat{P}^m_\ell\} = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\ a^*(\boldsymbol{k})\left[k^i(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'}\ \hat{\chi}^{m'}_{\ell'}(\boldsymbol{k}) - i\ell\hat{\chi}^m_\ell(\boldsymbol{k})\right]a(\boldsymbol{k}) \tag{3.32}$$

Using $i\mu\partial_i\hat{\chi}^m_\ell = i(\hat{\mathcal{K}}_i)^{\ell' m}_{\ell\, m'}\ \hat{\chi}^{m'}_{\ell'}$ it follows that the factor in brackets vanishes, which means that $\{C, \hat{P}^m_\ell\}$, despite not being an element of $\mathfrak{nrbms}$, is a conserved operator i.e., it generates a symmetry of the Schrödinger equation. In the particular case $\ell = 1$, this commutator agrees with the generators of boosts, $\{C, \hat{P}_i\} = i\hat{K}_i$. It is tempting to let $\{C, \hat{P}^m_\ell\}$ define a new kind of generator of symmetries, which would generalise the standard generators of boosts; we could dub these objects *super-boosts*. It will be interesting to explore their relation to the relativistic super-rotations.

4 Conclusions and Outlook

We have found the two unique contractions of $\mathfrak{bms}_4 \times \mathfrak{u}(1)$ that contain as a sub-algebra the Bargmann algebra. The $\mathfrak{nrbms}^+_4$ algebra contains an infinite number of finite-dimensional sub-algebras, obtained by considering the first $L \in \mathbb{N}$ super-translations $\hat{P}^m_\ell$. We have also found an infinite number of finite-dimensional indecomposable representations of the homogeneous Galilei algebra given by the matrices $\hat{\mathcal{J}}_i, \hat{\mathcal{K}}_i$. The Bargmann algebra is the only finite dimensional sub-algebra of $\mathfrak{nrbms}^-_4$, the associated indecomposable representation of the homogeneous Galilei group being infinite-dimensional.

We found a canonical realisation of $\mathfrak{nrbms}^+_4$ in terms of the Fourier modes of a free Schrödinger field. We have seen that we can add a dilation generator to $\mathfrak{nrbms}^+_4$, however we cannot extend the algebra with the expansion that is present in the Schrödinger algebra.

One may presume that the conformal symmetry can only be present in the case of an extended NRBMS, that is, in the group that includes super-boosts, as defined by the right-hand side of (3.31), and super-rotations.

The extension of the previous results to other dimensions is under study [14]. In the case of $1+2$ dimensions we expect to obtain an exotic $\mathfrak{nrbms}$ algebra with two central charges.

It will be interesting to study the asymptotic symmetries of asymptotically flat Newtonian space-times and to check whether or not the algebra coincides with the one of the two $\mathfrak{nrbms}$ algebras we have constructed.

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