RESISTANCE DISTANCES ON NETWORKS

A. Carmona, A.M. Encinas, M. Mitjana

This paper aims to study a family of distances in networks associated with effective resistances. Specifically, we consider the effective resistance distance with respect to a positive parameter and a weight; that is, effective resistance distance associated with an irreducible and symmetric M–matrix. This concept was introduced by the authors in relation with the full extension of Fiedler's characterization of symmetric and diagonal dominant M–matrices as resistive inverses to the case of symmetric M–matrices. The main idea is consider the network embedded in a host network with new edges that reflects the influence of the parameter. The main novelty of these distances is that they take into account not only the influence of shortest and longest weighted paths but also the importance of the vertices. Moreover, we analyse a simple case through some well–known electrical transformation. Finally, we prove that the adjusted forest metric introduced by Chebotarev is nothing else that a distance associated with a Schrödinger operator with constant weight.

2. PRELIMINARIES

Given a finite set V, the set of real valued functions on V is denoted by $\mathcal{C}(V)$. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle\cdot,\cdot\rangle$ and hence, if $u,v\in\mathcal{C}(V)$, then $\langle u,v\rangle = \sum\limits_{x\in V} u(x)\,v(x)$. For any $x\in V$, $\varepsilon_x\in\mathcal{C}(V)$ stands for the Dirac function at x and 1 is the function defined by $1(x)=1$, for any $x\in V$. On the other hand, $\omega\in\mathcal{C}(V)$ is called a *weight* if it satisfies that $\omega(x)>0$ for any $x\in V$ and moreover $\langle\omega,\omega\rangle = n = |V|$. The set of weights on V is denoted by $\Omega(V)$.

The triple $\Gamma=(V,E,c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x,y\}$ has been assigned a *conductance* $c(x,y)>0$. So, the conductance can be considered as a symmetric function $c\colon V\times V\longrightarrow[0,+\infty)$ such that $c(x,x)=0$ for any $x\in V$ and moreover, vertex x is adjacent to vertex y iff $c(x,y)>0$.

Denote by $P_{xy}=\big\{x=x_1\sim x_2\sim\cdots\sim x_k=y\big\}$ a path joining vertices x and y, the *length of the weighted path* P_{xy} is $\ell_c(P_{xy}) = \sum\limits_{i=1}^{k-1}\dfrac{1}{c(x_i,x_{i+1})}$. The *geodesic distance between two vertices x and y* is defined as the less resistive path; that is,

$$d_c(x,y)=\min\Big\{\ell_c(P_{xy}) : P_{xy} \text{ is a path from } x \text{ to } y\Big\}.$$

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The function d_c determines a distance on the network that fulfills the property the triangular inequality is an equality if and only if the central node separates the two others; that is, $d_c(x,y) = d_c(x,z) + d_c(z,y)$ iff every path from x to y passes through z. In general a distance, d, on a network is called *graph geodetic* when $d(x,y) = d(x,z) + d(z,y)$ if and only if every path from x to y passes through z. Therefore, the geodesic distance is graph geodetic.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x,y)\,\Big(u(x) - u(y)\Big), \quad x \in V. \tag{1}$$

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, where $qu \in \mathcal{C}(V)$ is defined as $(qu)(x) = q(x)u(x)$; see for instance [**1, 4**]. It is well–known that any Schrödinger operator is self–adjoint and moreover it is positive semi–definite iff there exist $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that $q = q_\omega + \lambda$; see [**1**]. In addition, $\mathcal{L}_q$ is singular iff $\lambda = 0$, in which case $\langle \mathcal{L}_{q_\omega}(v), v\rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. In any case, λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiple of ω.

If $\mathcal{L}_q$ is positive definite, then it is invertible and its inverse is called *Green operator*. On the other hand, when $\mathcal{L}_q$ is positive semi–definite and singular the operator that assigns to each function $f \in \mathcal{C}(V)$ the unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \frac{1}{n}\langle \omega, f\rangle\omega$ and $\langle u, \omega\rangle = 0$ is called *Green operator*. In any case, the Green operator is denoted by $\mathcal{G}_q$, see [**2**]. Moreover, the function $G_q : V\times V \longrightarrow \mathbb{R}$, defined as $G_q(x,y) = \mathcal{G}_q(\varepsilon_y)(x)$, for any $x, y \in V$, is called *Green function*. Observe that $\mathcal{G}_q(\omega) = \lambda^\dagger\omega$, where $\lambda^\dagger = \lambda^{-1}$ when $\lambda > 0$ and $\lambda^\dagger = 0$ when $\lambda = 0$. Moreover, $\mathcal{G}_q$ is self-adjoint as a consequence of the Fredholm Alternative and $\mathcal{G}_q$ is a symmetric function.

2. EFFECTIVE RESISTANCE DISTANCES

In [**2**], the authors introduced a generalization of the concept of effective resistance with respect to a value $\lambda \geq 0$ and a weight $\omega \in \Omega(V)$. Specifically, from the functional on $\mathcal{C}(V)$ defined as

$$\mathfrak{J}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle \mathcal{L}_q(u), u\rangle \tag{2}$$

where $q = q_\omega + \lambda$, we defined the generalization of the effective resistance.

Definition 2.1. *Given* $x, y \in V$*, the* effective resistance between x and y with respect to λ and ω*, is the value*

$$R_{\lambda,\omega}(x,y) = \max_{u\in\mathcal{C}(V)}\{\mathfrak{J}_{x,y}(u)\}.$$

In the sequel we omit the expression with respect to λ and ω when it does not lead to confusion. When $\lambda = 0$ we usually drop the subindex λ in the above expressions and when ω is constant we also omit the subindex ω. Therefore, R is nothing else than the standard effective resistance of the network.

Since the matrix associated with the Schrödinger operator $\mathcal{L}_q$ is an irreducible, symmetric M–matrix and conversely, each irreducible, symmetric M–matrix appears as associated with a Schrödinger operator, we can assign an effective resistance function to any irreducible, symmetric M–matrix. Notice that λ is the lowest eigenvalue, both of the matrix and the Schrödinger operator, and ω is its corresponding eigenfunction. Therefore, our study includes all the irreducible, symmetric M–matrices, not necessarily diagonally–dominant.

From now on we call ω–*dipole* the function $\tau_{xy} = \omega^{-1}(\varepsilon_x - \varepsilon_y)$. The following result can be found in [**2**] and allows us to express the effective resistances in terms of the solution of a Poisson equation. In particular, these expressions will be useful to prove the main properties of the effective resistances.

Proposition 2.2. *If $u \in \mathcal{C}(V)$ is a solution of the Poisson equation $\mathcal{L}_q(u) = \tau_{xy}$, then*

$$R_{\lambda,\omega}(x,y) = \langle \mathcal{L}_q(u), u \rangle = \frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}.$$

Therefore, $R_{\lambda,\omega}$ is symmetric, non–negative and moreover $R_{\lambda,\omega}(x,y) = 0$ iff $x = y$. In addition,

$$R_{\lambda,\omega}(x,y) = \frac{G_q(x,x)}{\omega^2(x)} + \frac{G_q(y,y)}{\omega^2(y)} - \frac{2G_q(x,y)}{\omega(x)\omega(y)}.$$

Next result, contains the main properties of the effective resistances associated with Schrödinger operators.

Theorem 2.3. *If Γ is a connected network, the effective resistance with respect to a parameter and a weight satisfies the following properties:*

(i) *The effective resistance $R_{\lambda,\omega}$ determines a distance on the network. Moreover, $R_{\lambda,\omega}(x,y) = R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(z,y)$ iff $\lambda = 0$ and z separates x and y.*

(ii) *If $0 \leq \hat{\lambda} \leq \lambda$ and $\hat{q} = q_\omega + \hat{\lambda}$, then $R_{\lambda,\omega} \leq R_{\hat{\lambda},\omega} \leq R_\omega$.*

(iii) *$R_{\lambda,\omega}(x,y) \leq d_{\hat{c}}(x,y)$, where $\hat{c}(x,y) = c(x,y)\omega(x)\omega(y)$, with equality iff $\lambda = 0$ and there exists a unique path from x to y.*

(iv) *$\lim\limits_{\lambda \to +\infty} R_{\lambda,\omega} = 0$ and $\lim\limits_{\lambda \to 0} R_{\lambda,\omega} = R_\omega$.*

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