MASTER'S THESIS

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Title: Random Tug-of-War games and the infinity Laplacian

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Supervised by Dr. Fernando Charro Caballero.

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Abstract

In this work we introduce and analyze a new random Tug-of-War game in which one of the players has the power to decide at each turn whether to play a round of classical random Tug-of-War, or let the other player choose the new game position in exchange of a fixed payoff.

We prove that this game has a value and identify the partial differential equation to which it is related, namely

$$\min\left\{|\nabla u(x)| - 1, -\Delta_\infty^N u(x)\right\} = 0.$$

This equation is known to have a key role in Jensen's celebrated proof of uniqueness of infinity harmonic functions and can also be obtained as a limit of p-Laplace equations. However, this is the first time that such equation is found to have a relation with game theory. Moreover, our analysis relies on comparison and viscosity tools, in contrast to probabilistic arguments which are more common in the literature.

The work also includes a review of the infinity Laplacian and its connection to the classical random Tug-of-War game, as well as an introduction to the theory of viscosity solutions. Furthermore, some explicit examples of the new game are considered.

Keywords: Infinity Laplacian, Tug-of-War, Viscosity solutions, Comparison principle

AMS 2010 Subject Classification: 35J60, 35J70, 91A05, 35D40, 35B51

Contents

Introduction

Random Tug-of-War games were introduced in [23] in connection with partial differential equations (see also the survey [25]). Informally, random Tug-of-War games play for the normalized infinity Laplacian

$$\Delta_\infty^N u = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j}, \tag{1}$$

the role that the Brownian motion plays for the Laplacian. Observe that (1) is the pure second derivative of u in the direction of the gradient.

In the classical random Tug-of-War game, two players are in contest and the total earnings of one player are the losses of the other (it is a two-person, zero-sum game). In the simplest setup, the game is played by turns on a domain Ω; at each turn a fair coin is tossed and the winner of the toss is allowed to decide the next game position within a radius ϵ of the current one. If the game position reaches $\partial\Omega$, the game stops and Player I earns a terminal payoff given by a function $F : \partial\Omega \to \mathbb{R}$ (Player II earnings are given by $-F$). The terminal payoff F is known to both players beforehand.

In [23] it was first proved that the random Tug-of-War game has a value, that is, a function $u_\epsilon(x)$ which represents the expected outcome of the game just described when starting at a point x and both players play optimally. Moreover, the game value satisfies the following Dynamic Programming Principle (DPP)

$$u_\epsilon(x) = \frac{1}{2}\left(\sup_{y\in \overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in \overline{B}_\epsilon(x)} u_\epsilon(y) \right) \quad \text{for all} \ \ x \in \Omega.$$

The key observation in [23] is that the DPP can be seen as a "discretization" of the normalized infinity Laplacian. In other words, the limit $u = \lim_{\epsilon\to 0} u_\epsilon$, known as the continuous value of the game, is a viscosity solution of the Dirichlet problem for the normalized infinity Laplacian, that is,

$$\begin{cases} -\Delta_\infty^N u(x) = 0, & x \in \Omega; \\ u(x) = F(x), & x \in \partial\Omega. \end{cases}$$

This work is divided in six chapters, which we briefly summarize now. They are mostly self-contained and our main contributions are in Chapters 4, 5 and 6 as discussed below. Chapter 1 presents some results on viscosity solutions which are necessary for all the analysis in this work and related literature. We also introduce the concept of jets, needed for the comparison results considered in Chapter 6.

Chapter 2 is a brief review of the infinity Laplacian, from its definition and notion of infinity harmonic function to the description of the normalized infinity Laplacian, which is the one that appears in connection to game theory. Due to the degenerate elliptic nature of these operators, we include a short appendix describing ellipticity of general second-order partial differential operators.

Game theory, and more precisely random Tug-of-War games, is the topic of Chapter 3. It includes first a description of random Tug-of-War games in a very general configuration. Then, the classical random Tug-of-War game is commented in detail, that is, not just the instructions of the game but the player's strategies and the concepts of value of the game and Dynamic Programming Principle (DPP) which we have briefly discussed above. At the end of the chapter we point out the connection between random Tug-of-War games and the normalized infinity Laplacian, which occurs via the DPP.

In Chapter 4 we introduce a new game, the *Totalitarian Tug-of-War*, which is a modified version of the classical random Tug-of-War game reviewed in Chapter 3. Now, at every turn Player I has the power to decide whether to play a round of classical random Tug-of-War, or let Player II choose the new game position in exchange of a fixed payoff of value ϵ. The fact that Player I somehow imposes the type of game that is played, is the reason why we refer to this game as Totalitarian Tug-of-War.

After a description of the new game, where we point out the differences and similarities with respect the classical one, we prove that the Totalitarian Tug-of-War has a value when played either on graphs, or in bounded domains $\Omega \subset \mathbb{R}^n$. We prove that the game has a value by means of a comparison principle for the particular DPP associated to each game. In the graph case the DPP is

$$u_i = \max\left\{ \min_{j\in\{i'\}} u_j + \epsilon,\ \frac{1}{2}\left(\max_{j\in\{i'\}} u_j + \min_{j\in\{i'\}} u_j \right) \right\},$$

where u_i stands for the expected value of the game associated to the game position x_i, and $\{i'\}$ denotes the finite set of indices associated to the nodal neighbors $x_{i'}$ of x_i in the graph. In the case of a bounded domain $\Omega \subset \mathbb{R}^n$ the DPP is

$$u_\epsilon(x) = \max\left\{ \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \epsilon,\ \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) \right) \right\}, \tag{2}$$

for all $x \in \Omega$. These comparison results are new and, in our opinion, of independent interest. We also want to remark that these proofs do not use any tools from probability, which is a novel approach.

Chapter 5 contains several explicit examples of Totalitarian Tug-of-War game on graphs. More precisely, we consider a game on a graph segment with two running nodes and its generalization to n running nodes, and the equivalent version for a graph with three terminal nodes on a Y configuration. The first example, which is the simplest game configuration, describes in detail how the players decide their strategy and how we end up with a value of the game (recall that we have proved in the previous chapter the existence and uniqueness of the game value).

Finally, in Chapter 6 we show that there exists a limit $u = \lim_{\epsilon\to 0} u_\epsilon$, known as the continuous value of the Totalitarian Tug-of-War, and that it can be characterized as the unique solution of a PDE problem. This is, next to the existence of a value for the game, our main contribution in this work. It is summarized in the following theorem.

Theorem. *Let u_ϵ be the solution of the Dirichlet problem*

$$\begin{cases} \mathcal{G}[u_\epsilon](x) = 0, & x \in \Omega; \\ u_\epsilon(x) = F(x), & x \in \Gamma_\epsilon, \end{cases}$$

where $\Gamma_\epsilon = \{x \in \mathbb{R}^n \setminus \Omega : \operatorname{dist}(x, \partial\Omega) \le \epsilon\}$, $F : \Gamma_\epsilon \to \mathbb{R}$ is a bounded function and

$$\mathcal{G}[u_\epsilon](x) = \min\left\{ u_\epsilon(x) - \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) - \epsilon,\ u_\epsilon(x) - \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) \right) \right\}.$$

Then, $u = \lim_{\epsilon\to 0} u_\epsilon$ uniformly in $\overline{\Omega}$ is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} \min\left\{ |\nabla u(x)| - 1, -\Delta_\infty^N u(x) \right\} = 0, & x \in \Omega; \\ u(x) = F(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

The proof of this result has two parts: the uniform convergence of u_ϵ to u, solution of (3), and the uniqueness of this solution. The key point in our proof of convergence is to observe that the DPP can be seen as a "numerical scheme" which meets the requirements for convergence stated in the seminal article [3]. The uniqueness of solution u follows via a comparison principle which was already known in the literature (see [7, 16]). We include the proof of the comparison principle for the sake of completeness.

We conclude pointing out that the equation in (3) is known to have a key role in Jensen's celebrated proof of uniqueness (see [15]) of infinity harmonic functions (i.e., viscosity solutions to the equation $-\Delta_\infty u = 0$) and can also be obtained as a limit of p-Laplace equations. However, this is the first time that such equation is found to have a relation with game theory. Moreover, our analysis relies on comparison and viscosity tools, in contrast to probabilistic arguments which are more common in the literature.

Furthermore, with small modifications of the Totalitarian Tug-of-War game, more precisely, considering payoffs of value $\lambda\epsilon$ instead of ϵ, and considering a Totalitarian Tug-of-War which favors Player II instead of Player I, the results in this work can be extended to treat both Jensen's equations

$$\min\left\{|\nabla u(x)| - \lambda, -\Delta_\infty^N u(x)\right\} = 0 \qquad \text{and} \qquad \max\left\{\lambda - |\nabla u(x)|, -\Delta_\infty^N u(x)\right\} = 0. \tag{4}$$

Even more interestingly, the methods described here could be applied to treat inhomogeneous versions of (4) with a general right-hand side, and these equations would not be a limit of p-Laplace equations.

Finally, we would like to mention that the results in this work and the extensions described above will become a research article in the upcoming months.

Chapter 1

Some results on viscosity solutions

The theory of viscosity solutions applies to certain second order partial differential equations of the form $G(x, r, p, X) = 0$, where $G : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$, for $\mathcal{S}^n$ the set of real symmetric $n \times n$ matrices. Some of the primary virtues of this theory are that it allows merely continuous functions to be solutions of fully nonlinear equations of second order, that it provides very general existence and uniqueness theorems and that it yields precise formulations of general boundary conditions (see [7] for a complete analysis of viscosity solutions of second order partial differential equations). We remark that all our analysis relies on partial differential equations and viscosity arguments, in contrast to probabilistic arguments that are more usual in the literature.

Moreover, we require G to satisfy the following degenerate ellipticity condition (we say that G is *degenerate elliptic*),

$$G(x, r, p, X) \leq G(x, r, p, Y) \quad \text{whenever} \quad Y \leq X, \tag{1.1}$$

where $\mathcal{S}^n$ is equipped with the usual order in the space of matrices. We remark that the sign convention we use in (1.1) corresponds to $G = -\Delta$. Another monotonicity condition that is often (but not always) required is the following,

$$G(x, r, p, X) \leq G(x, s, p, X) \quad \text{whenever } r \leq s. \tag{1.2}$$

Conditions (1.1) and (1.2) can be formulated together as

$$G(x, r, p, X) \leq G(x, s, p, Y) \quad \text{whenever } r \leq s \ \text{ and } \ Y \leq X. \tag{1.3}$$

When (1.3) holds, G is said to be *proper* in the terminology of [7], where more details on the notations and definitions below can be found.

1.1 Definitions and equivalences

The type of functions defined next is the one required for the viscosity solutions. Moreover, from now on we will consider $\Omega \subset \mathbb{R}^n$ open.

Definition 1.1 (Semicontinuous function)**.** *A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is* upper semicontinuous *if the set $\{x \in \Omega : u(x) < \lambda\}$ is open for every $\lambda \in \mathbb{R}$, or equivalently, if*

$$\limsup_{y \to x} u(y) \leq u(x) \quad \text{for all } x \in \Omega.$$

Similarly, a function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ *is* lower semicontinuous *if the set* $\{x \in \Omega : u(x) > \lambda\}$ *is open for every* $\lambda \in \mathbb{R}$*, or equivalently, if*

$$\liminf_{y \to x} u(y) \geq u(x) \quad \text{for all } x \in \Omega.$$

Alternatively, a function u *is lower semicontinuous if* $-u$ *is upper semicontinuous and vice versa.*

Remark 1.2. *The continuity of a function is equivalent to being both upper and lower semicontinuous.*

Definition 1.3 (Viscosity solution)**.** *A* viscosity subsolution *of the equation* $G = 0$ *(equivalently a viscosity solution of* $G \leq 0$*) in* Ω *is an upper semicontinuous function* $u : \Omega \to \mathbb{R}$ *such that*

$$G\left(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \leq 0, \tag{1.4}$$

whenever $\hat{x} \in \Omega$ *and* $\varphi \in \mathcal{C}^2(\Omega)$ *are such that* $u(\hat{x}) = \varphi(\hat{x})$ *and* $u(x) \leq \varphi(x)$*, for all* x *in a neighborhood of* $\hat{x}$ *(in other words,* φ *touches* u *at* $\hat{x}$ *from above in a neighborhood of* $\hat{x}$*, or equivalently,* $u - \varphi$ *has a local maximum at* $\hat{x}$*).*

Similarly, a viscosity supersolution *of* $G = 0$ *(equivalently a viscosity solution of* $G \geq 0$*) in* Ω *is a lower semicontinuous function* $u : \Omega \to \mathbb{R}$ *such that*

$$G\left(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \geq 0,$$

whenever $\hat{x} \in \Omega$ *and* $\varphi \in \mathcal{C}^2(\Omega)$ *are such that* $u(\hat{x}) = \varphi(\hat{x})$ *and* $u(x) \geq \varphi(x)$*, for all* x *in a neighborhood of* $\hat{x}$ *(in other words,* φ *touches* u *at* $\hat{x}$ *from below in a neighborhood of* $\hat{x}$*, or equivalently,* $u - \varphi$ *has a local minimum at* $\hat{x}$*).*

Finally, a function $u : \Omega \to \mathbb{R}$ *is a* viscosity solution *of* $G = 0$ *in* Ω *if it is both a viscosity subsolution and viscosity supersolution of* $G = 0$ *in* Ω*.*

According to Definition 1.3, we require that the so-called test functions φ touch the viscosity subsolutions from above and viscosity supersolutions from below. This motivates to ask for upper semicontinuity in the case of viscosity subsolutions and lower semicontinuity for viscosity supersolutions.

Remark 1.4. *In some situations it is necessary that the test function* φ *touches* u *at* $\hat{x}$ *strictly from above in a neighborhood of* $\hat{x}$ *(or equivalently,* $u - \varphi$ *has a strict local maximum at* $\hat{x}$*) for the case of viscosity subsolutions, and analogously that* φ *touches* u *at* $\hat{x}$ *strictly from below in a neighborhood of* $\hat{x}$ *(or equivalently,* $u - \varphi$ *has a strict local minimum at* $\hat{x}$*) for the case of viscosity supersolutions.*

Note that this is an equivalent definition of viscosity sub- and supersolutions. On the one hand, it is clear that the new definition implies the original one since it is more restrictive (inequalities in Definition 1.3 are replaced by strict inequalities). On the other hand, the new definition can be derived from Definition 1.3 as follows. For the case of viscosity subsolutions, it is enough to consider $\tilde{\varphi}(x) = \varphi(x) + |x - \hat{x}|^4$ *as test function. Due to this particular choice of* $\tilde{\varphi} \in \mathcal{C}^2(\Omega)$*,* $\tilde{\varphi}$ *touches* u *at* $\hat{x} \in \Omega$ *strictly from above in a neighborhood of* $\hat{x}$ *(or equivalently,* $u - \tilde{\varphi}$ *has a strict local maximum at* $\hat{x}$*) and*

$$G\left(\hat{x}, \tilde{\varphi}(\hat{x}), \nabla\tilde{\varphi}(\hat{x}), D^2\tilde{\varphi}(\hat{x})\right) = G\left(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\right) \leq 0$$

is satisfied, in view of (1.4). The case of viscosity supersolutions follows analogously taking $\tilde{\varphi}(x) = \varphi(x) - |x - \hat{x}|^4$*. Just to mention that, abusing of notation, the new test functions are also denoted as* $\varphi(x)$*.*

With the following result we show that the notion of viscosity solution is consistent with the notion of classical solution, under the assumption of enough regularity on the solution.

Proposition 1.5 (Consistency of the definition)**.** *Let $u \in \mathcal{C}^2(\Omega)$. Then u is a classical solution of $G = 0$ in Ω, that is,*

$$G(x, u(x), \nabla u(x), D^2u(x)) = 0 \quad \text{for all} \quad x \in \Omega, \tag{1.5}$$

if and only if u is a viscosity solution to $G = 0$ in Ω.

Proof. Consider first that u is a classical solution of $G = 0$ in Ω. Let $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$, then by calculus $\nabla u(\hat{x}) = \nabla\varphi(\hat{x})$ and $D^2u(\hat{x}) \leq D^2\varphi(\hat{x})$. By degenerate ellipticity of G, i.e., assuming that (1.1) holds, it follows that

$$G(\hat{x}, \varphi(\hat{x}), \nabla\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq G(\hat{x}, u(\hat{x}), \nabla u(\hat{x}), D^2u(\hat{x})) = 0, \tag{1.6}$$

which shows that u is a viscosity subsolution to $G = 0$ in Ω, according to Definition 1.3. By a similar argument, one can prove that u is also a viscosity supersolution to $G = 0$ in Ω and therefore a viscosity solution.

Conversely, if $u \in \mathcal{C}^2(\Omega)$ is a viscosity solution of $G = 0$ in Ω, u itself works as test function φ in Definition 1.3. It then follows that (1.5) holds and therefore u is a solution in the classical sense to $G = 0$ in Ω. □

1.2 The concept of jets

Assume, for instance, that $\varphi \in \mathcal{C}^2(\Omega)$ touches u at $\hat{x} \in \Omega$ from above in a neighborhood of $\hat{x}$. In other words, $\hat{x}$ is a local maximum of $u - \varphi$, so that

$$u(\hat{x}) - \varphi(\hat{x}) \geq u(x) - \varphi(x)$$

for all x in a neighborhood of $\hat{x}$. Rearranging the terms as $u(x) \leq u(\hat{x}) - \varphi(\hat{x}) + \varphi(x)$ and doing a Taylor expansion of φ around $\hat{x}$ at x, we get

$$u(x) \leq u(\hat{x}) + \langle p, (x - \hat{x})\rangle + \frac{1}{2}\langle X\,(x - \hat{x}), (x - \hat{x})\rangle + o\left(|x - \hat{x}|^2\right) \quad \text{as} \quad x \to \hat{x}, \tag{1.7}$$

where $p = \nabla\varphi(\hat{x})$ and $X = D^2\varphi(\hat{x})$. Moreover, if (1.7) holds for some $(p, X) \in \mathbb{R}^n \times \mathcal{S}^n$ and u is twice differentiable at $\hat{x}$, then $p = \nabla u(\hat{x}) = \nabla\varphi(\hat{x})$ and $D^2u(\hat{x}) \leq X = D^2\varphi(\hat{x})$ since φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$. Thus, if u is a classical solution of $G \leq 0$, it follows from (1.1) that $G(\hat{x}, u(\hat{x}), p, X) \leq 0$ whenever (1.7) holds. Note that the same discussion can be analogously reproduced for $G \geq 0$, where $\varphi \in \mathcal{C}^2(\Omega)$ touches u at $\hat{x}$ from below in a neighborhood of $\hat{x}$.

In case u is not regular enough, the following concepts appear necessary for the definition of non-differentiable solutions u to the equation $G = 0$.

Definition 1.6 (Subjet/Superjet)**.** *Given a function $u : \Omega \to \mathbb{R}$, the (second-order)* superjet *$J^{2,+}_\Omega u(\hat{x})$ at the point $\hat{x} \in \Omega$ is defined to be the set of points*

$$J^{2,+}_\Omega u(\hat{x}) = \Big\{(p, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x) \leq u(\hat{x}) + \langle p, (x - \hat{x})\rangle + \frac{1}{2}\langle X\,(x - \hat{x}), (x - \hat{x})\rangle + o\left(|x - \hat{x}|^2\right), \ \text{as } x \to \hat{x}\Big\}. \tag{1.8}$$

Similarly, the (second-order) subjet *$J^{2,-}_\Omega u(\hat{x})$ at the point $\hat{x} \in \Omega$ is defined as*

$$J^{2,-}_\Omega u(\hat{x}) = \Big\{(p, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x) \geq u(\hat{x}) + \langle p, (x - \hat{x})\rangle + \frac{1}{2}\langle X\,(x - \hat{x}), (x - \hat{x})\rangle + o\left(|x - \hat{x}|^2\right), \ \text{as } x \to \hat{x}\Big\}. \tag{1.9}$$

If u is twice differentiable at $\hat{x} \in \Omega$, then $p = \nabla u(\hat{x})$ and $X = D^2 u(\hat{x})$ belong to both sets of pairs (p, X) in (1.8) and (1.9). Note also that the subjets can be equivalently defined in terms of superjets as $J_\Omega^{2,-} u(\hat{x}) = -J_\Omega^{2,+}(-u)(\hat{x})$. Furthermore, we can consider their closure, which follows now.

Definition 1.7 (Closure of subjet/superjet)**.** *The closures of $J_\Omega^{2,+}u(\hat{x})$ and $J_\Omega^{2,-}u(\hat{x})$ at the point $\hat{x} \in \Omega$, are respectively defined as the sets*

$$\overline{J}_\Omega^{2,+} u(\hat{x}) = \Big\{ (p, X) \in \mathbb{R}^n \times \mathcal{S}^n : \exists\, (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n \ \text{ such that } \\ (p_n, X_n) \in J_\Omega^{2,+} u(x_n) \ \text{ and } \ (x_n, u(x_n), p_n, X_n) \to (\hat{x}, u(\hat{x}), p, X)\,, \ \text{as } n \to \infty \Big\}$$

and

$$\overline{J}_\Omega^{2,-} u(\hat{x}) = \Big\{ (p, X) \in \mathbb{R}^n \times \mathcal{S}^n : \exists\, (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n \ \text{ such that } \\ (p_n, X_n) \in J_\Omega^{2,-} u(x_n) \ \text{ and } \ (x_n, u(x_n), p_n, X_n) \to (\hat{x}, u(\hat{x}), p, X)\,, \ \text{as } n \to \infty \Big\}.$$

Just to mention that since $J_\Omega^{2,+} u(\hat{x})$ is the same for all sets Ω for which $\hat{x}$ is an interior point, it is usual to write all these sets as $J^{2,+}u(\hat{x})$. The same convention applies to subjets, $J^{2,-}u(\hat{x})$, and their respective closure, $\overline{J}^{2,+}u(\hat{x})$ and $\overline{J}^{2,-}u(\hat{x})$.

The following lemma gives an equivalent definition to Definition 1.3 for viscosity solutions in terms of jets.

Lemma 1.8. *A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of $G = 0$ in Ω if and only if u is upper semicontinuous and*

$$G(\hat{x}, u(\hat{x}), p, X) \leq 0 \quad \text{for all } \hat{x} \in \Omega \ \text{ and all } \ (p, X) \in J_\Omega^{2,+} u(\hat{x}). \tag{1.10}$$

Similarly, a function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of $G = 0$ in Ω if and only if u is lower semicontinuous and

$$G(\hat{x}, u(\hat{x}), p, X) \geq 0 \quad \text{for all } \hat{x} \in \Omega \ \text{ and all } \ (p, X) \in J_\Omega^{2,-} u(\hat{x}).$$

Proof. Assume first that u is a viscosity subsolution of $G = 0$ in the sense of Definition 1.3 and let $(p, X) \in J_\Omega^{2,+} u(\hat{x})$. We want to show (1.10). Define

$$\varphi(x) = u(\hat{x}) + \langle p, (x - \hat{x}) \rangle + \frac{1}{2} \langle X\,(x - \hat{x}), (x - \hat{x}) \rangle\,,$$

so that $\varphi \in \mathcal{C}^2(\Omega)$ satisfies $u(\hat{x}) = \varphi(\hat{x})$ and, according to (1.8), $u(x) \leq \varphi(x)$ for all x in some neighborhood of $\hat{x}$. Then, $G(\hat{x}, \varphi(\hat{x}), p, X) \leq 0$ by Definition 1.3, so we end up with

$$G\left(\hat{x}, u(\hat{x}), p, X\right) = G\left(\hat{x}, \varphi(\hat{x}), p, X\right) \leq 0$$

for all $\hat{x} \in \Omega$ and all $(p, X) \in J_\Omega^{2,+} u(\hat{x})$.

Consider now the reverse implication. We assume that u satisfies (1.10) and seek to prove that given $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$, (1.4) holds. In fact, from the Taylor expansion up to second order terms of φ around $\hat{x}$ at x, we get

$$u(x) \leq \varphi(x) = \varphi(\hat{x}) + \langle \nabla \varphi(\hat{x}), (x - \hat{x}) \rangle + \frac{1}{2} \left\langle D^2 \varphi(\hat{x})\,(x - \hat{x}), (x - \hat{x}) \right\rangle + o\left(|x - \hat{x}|^2\right) \quad \text{as } x \to \hat{x}$$

and therefore $\left(\nabla \varphi(\hat{x}), D^2 \varphi(\hat{x})\right) \in J_\Omega^{2,+} u(\hat{x})$, according to (1.8). This implies that

$$G(\hat{x}, \varphi(\hat{x}), \nabla \varphi(\hat{x}), D^2 \varphi(\hat{x})) = G(\hat{x}, u(\hat{x}), \nabla \varphi(\hat{x}), D^2 \varphi(\hat{x})) \leq 0,$$

whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$.

The proof for the case of viscosity supersolution follows analogously. $\square$

We would like to stress that according to Lemma 1.8, it follows that for every pair $(p, X) \in J^{2,+}_{\Omega} u(\hat{x})$ there exists $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$ and $(p, X) = \big(\nabla\varphi(\hat{x}), D^2\varphi(\hat{x})\big)$. The analogous result holds for every pair $(p, X) \in J^{2,-}_{\Omega} u(\hat{x})$.

Chapter 2

A brief review of the infinity Laplacian

Informally, the infinity Laplacian $\Delta_\infty u$ is the second derivative of u in the direction of its gradient, properly normalized (see Section 2.2). More precisely, we have the following definition.

Definition 2.1 (Infinity Laplacian). *For a smooth function $u = u(x_1, x_2, \ldots, x_n) : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, the* infinity Laplacian *(also known as the* ∞-Laplace operator*) is the fully nonlinear second-order degenerate partial differential operator denoted by Δ_∞ and given by*

$$\Delta_\infty u := \left\langle D^2 u\, \nabla u, \nabla u \right\rangle = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}. \tag{2.1}$$

The operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a given Lipschitz boundary function (see [25, Section 3] for instance). Moreover, solutions to the ∞-*Laplace equation*, $-\Delta_\infty u = 0$, are used in several applications, such as optimal transportation and image processing (see, e.g., [10, 13]). The eigenvalue problem for the infinity Laplacian has been exhaustively studied too (see [16, Section 6]) and is formulated as

$$\min \left\{ |\nabla u(x)| - \frac{1}{d} u(x), -\Delta_\infty^N u(x) \right\} = 0 \quad \text{for} \quad x \in \Omega,$$

where $d = \|\text{dist}(x, \partial\Omega)\|_{\infty,\Omega}$. Note also that the ∞-Laplace equation can be given a probabilistic interpretation in terms of game theory (see, e.g., [23, 25]), which is precisely the approach that we will follow.

Now, in order to motivate the elliptic nature of the ∞-Laplace operator, we present a short review on elliptic operators (some extra comments are included in Appendix A). A deep look into second-order elliptic equations can be found in [9, Chapter 6].

Thus, ellipticity for a second-order partial differential operator G (see Definition A.2, according to [9, Section 6.1.1]) means that for each point $x \in \Omega$, the symmetric coefficient matrix $A(x) = \left(a_{i,j}(x)\right)_{i,j}$ is positive definite, with smallest eigenvalue greater than or equal to λ.

For the particular case $a_{i,j} \equiv \delta_{i,j}$ and $b_i \equiv c \equiv 0$, the operator G is $-\Delta$, that is, minus the Laplace operator, which is clearly elliptic. The infinity Laplacian corresponds to the choice $a_{i,j} \equiv \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$ and $b_i \equiv c \equiv 0$ according to (2.1), so that (see [28, Section 1.5])

$$\sum_{i,j=1}^{n} a_{i,j}(x)\, \xi_i\, \xi_j = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, \xi_i\, \xi_j = \left(\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\, \xi_i \right)^2 \geq 0.$$

Therefore, the matrix A associated to $-\Delta_\infty$ is not positive definite but positive semidefinite. The reason is that $-\Delta_\infty$ is not an elliptic but degenerate elliptic operator, as it is described now.

According to (1.1), $-\Delta_\infty u = -\langle X\,p, p\rangle$ is degenerate elliptic if

$$-\langle X\,p, p\rangle \leq -\langle Y\,p, p\rangle \quad \text{whenever} \quad Y \leq X, \tag{2.2}$$

for $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}^n$. This is the case, since $Y \leq X$ means that $X - Y$ is a positive semidefinite matrix, i.e., $\langle (X-Y)\,\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$, which by bilinearity of the scalar product can be equivalently written as $-\langle X\,\xi, \xi\rangle \leq -\langle Y\,\xi, \xi\rangle$. Therefore, condition (2.2) holds.

2.1 Notion of infinity harmonic function

The framework of viscosity solutions (see Chapter 1) turns out to be the natural one for the definition and study of the ∞-Laplace operator. The reason is not only the (possible) lack of regularity of the function u, but the fact that $-\Delta_\infty$ is not in divergence form (see (A.1)), so we cannot integrate by parts to define a notion of weak solution.

The reason why one does not restrict the study only to $\mathcal{C}^2$ solutions is that in the Dirichlet problem for the equation $-\Delta_\infty u = 0$, one can prescribe smooth boundary values that no $\mathcal{C}^2$ solution can attain. This is formalized in the following theorem, which was proved for the two-dimensional case by Aronsson in [1]. The case for dimension $n \geq 3$ is included in [29].

Theorem 2.2. *Suppose that $u \in \mathcal{C}^2(\Omega)$ and that $-\Delta_\infty u = 0$ in Ω. Then, either $\nabla u \neq 0$ in Ω or u reduces to a constant.*

As a remedy to the previous discussion there arise the following notion of solution, which is suitable when u is not regular enough. It is just Definition 1.3 for the particular case of the ∞-Laplace equation (these and other related results can be found in [27, Section 4] and [28, Section 2.1]).

Definition 2.3 (Viscosity solution of the ∞-Laplace equation)**.** *A* viscosity subsolution of $-\Delta_\infty u = 0$ *(equivalently a viscosity solution of $-\Delta_\infty u \leq 0$) in Ω is an upper semicontinuous function $u : \Omega \to \mathbb{R}$ such that*

$$-\Delta_\infty \varphi(\hat{x}) \leq 0,$$

whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$.

A viscosity supersolution of $-\Delta_\infty u = 0$ *(equivalently a viscosity solution of $-\Delta_\infty u \geq 0$) in Ω is a lower semicontinuous function $u : \Omega \to \mathbb{R}$ such that*

$$-\Delta_\infty \varphi(\hat{x}) \geq 0,$$

whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from below in a neighborhood of $\hat{x}$.

A function $u : \Omega \to \mathbb{R}$ is a viscosity solution of $-\Delta_\infty u = 0$ *in Ω if it is both a viscosity subsolution and viscosity supersolution of $-\Delta_\infty u = 0$ in Ω.*

Definition 2.4 (Infinity harmonic function)**.** *An* infinity harmonic function *in Ω is a viscosity solution of $-\Delta_\infty u = 0$ in Ω.*

We remark that the previous definitions are also satisfied when the solution is of class $\mathcal{C}^2$, as it is shown in the following proposition.

Proposition 2.5 (Consistency of the definition)**.** *If $u \in \mathcal{C}^2(\Omega)$, then u is infinity harmonic in Ω if and only if $-\Delta_\infty u = 0$ in Ω in the classical pointwise sense.*

Proof. Suppose that u is infinity harmonic in Ω. Since $u \in \mathcal{C}^2(\Omega)$, the function itself works as a test function in Definition 2.3, so that $-\Delta_\infty u = 0$ in Ω in the pointwise sense.

Reciprocally, assume that $u \in \mathcal{C}^2(\Omega)$ satisfies $-\Delta_\infty u = 0$ in Ω in the pointwise sense. In particular, $-\Delta_\infty u \leq 0$ in Ω in the pointwise sense. Then, whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$, it follows that

$$\nabla\varphi(\hat{x}) = \nabla u(\hat{x}) \qquad \text{and} \qquad D^2\varphi(\hat{x}) \geq D^2 u(\hat{x}).$$

Using this, we have that

$$\begin{aligned} -\Delta_\infty\varphi(\hat{x}) = -\left\langle D^2\varphi(\hat{x})\,\nabla\varphi(\hat{x}), \nabla\varphi(\hat{x})\right\rangle &= -\left\langle D^2\varphi(\hat{x})\,\nabla u(\hat{x}), \nabla u(\hat{x})\right\rangle \\ &\leq -\left\langle D^2 u(\hat{x})\,\nabla u(\hat{x}), \nabla u(\hat{x})\right\rangle = -\Delta_\infty u(\hat{x}) \leq 0. \end{aligned}$$

Similarly, since $-\Delta_\infty u = 0$ holds in Ω in the pointwise sense for $u \in \mathcal{C}^2(\Omega)$, in particular $-\Delta_\infty u \geq 0$ in Ω in the pointwise sense. Then, whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from below in a neighborhood of $\hat{x}$, it follows that

$$\nabla\varphi(\hat{x}) = \nabla u(\hat{x}) \qquad \text{and} \qquad D^2\varphi(\hat{x}) \leq D^2 u(\hat{x}).$$

Thus, one gets that

$$\begin{aligned} -\Delta_\infty\varphi(\hat{x}) = -\left\langle D^2\varphi(\hat{x})\,\nabla\varphi(\hat{x}), \nabla\varphi(\hat{x})\right\rangle &= -\left\langle D^2\varphi(\hat{x})\,\nabla u(\hat{x}), \nabla u(\hat{x})\right\rangle \\ &\geq -\left\langle D^2 u(\hat{x})\,\nabla u(\hat{x}), \nabla u(\hat{x})\right\rangle = -\Delta_\infty u(\hat{x}) \geq 0. \end{aligned}$$

Hence, in view of Definition 2.3, u is infinity harmonic in Ω. □

By Proposition 2.5, a classical solution of $-\Delta_\infty u = 0$ is a viscosity solution. However, the converse might not be true, as it is the case of the function

$$u(x, y) = x^{4/3} - y^{4/3},$$

which is not an infinity harmonic function in the classical sense but in the viscosity sense. In fact, this particular u has regularity $\mathcal{C}^{1,1/3}$, but its second derivatives do not exist on the lines $x = 0$ and $y = 0$ (see [6] for more details).

All this leads us to focus now on the regularity of the infinity Laplacian, which turns out to be a very tough question (see comments on [8, Section 3], [27, Section 6], [28, Section 1.5 and 1.8]). According to [12], infinity harmonic functions are differentiable everywhere, while it remains an open problem to prove the $\mathcal{C}^1$ or $\mathcal{C}^{1,\alpha}$ regularity in general dimensions, they are known to hold only in two dimensions after the breakthroughs of [11] and [26].

Related to the previous discussion, the following theorem is a regularity result for infinity harmonic functions, the proof of which can be found in [27, Theorem 7].

Theorem 2.6. *If $u : \Omega \to \mathbb{R}$ is an infinity harmonic function in Ω, then it is locally Lipschitz continuous in Ω.*

Remark 2.7. *By Rademacher's theorem (see below), infinity harmonic functions are differentiable almost everywhere in its domain.*

Theorem 2.8 (Rademacher)**.** *A Lipschitz continuous function u is differentiable almost everywhere in its domain, i.e., the expansion*

$$u(y) = u(x) + \langle\nabla u(x), (y - x)\rangle + o\left(|y - x|\right) \quad \text{as} \quad y \to x$$

holds at almost every point x of the domain.

2.1.1 On the derivation of the ∞-Laplace operator

As the nomenclature "infinity Laplacian" suggests, the ∞-Laplace equation, $-\Delta_\infty u = 0$, is proved to be the limit as $p \to \infty$ in the *p-Laplace equation*, $-\Delta_p u = 0$. More details on this fact appear in Section 2.1.2, however a detailed and rigorous proof can be found in [5].

In order to get some intuition on this derivation of the infinity Laplacian, consider the expansion of the p-Laplacian operator under the assumption that u is regular enough,

$$\begin{aligned}\Delta_p u &= \nabla \cdot \left(|\nabla u|^{p-2}\,\nabla u\right)\\ &= |\nabla u|^{p-2}\,\Delta u + (p-2)\,|\nabla u|^{p-4}\sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\\ &= (p-2)\,|\nabla u|^{p-4}\left(\frac{|\nabla u|^2}{p-2}\,\Delta u + \sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right).\end{aligned}$$

By assuming that u is *p-harmonic*, i.e., $-\Delta_p u = 0$, and that $|\nabla u| \neq 0$ (so that we can divide by $(p-2)\,|\nabla u|^{p-4}$), it follows from the previous expansion that

$$\frac{|\nabla u|^2}{p-2}\,\Delta u + \sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = 0.$$

Assuming uniform convergence and taking the limit $p \to \infty$ we obtain

$$\Delta_\infty u = \sum_{i,j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} = 0.$$

We remark that this is not a rigorous derivation (see [5]), which has to be done in the framework of viscosity solutions for the case when functions are not regular enough.

2.1.2 Variational approach

In Section 2.1.1 it is shown heuristically that the infinity Laplacian is limit as $p \to \infty$ of the p-Laplacian. Our aim now is to analyze this fact in more detail in the variational framework.

Note first that since we are only looking at very large values of p (recall that we want to take $p \to \infty$), one can assume $p > n$, for n the dimension of $\mathbb{R}^n$. Under this consideration and provided $\partial\Omega$ is $\mathcal{C}^1$, for $\Omega \subset \mathbb{R}^n$ a bounded domain, the Sobolev space $W^{1,p}(\Omega)$ contains only continuous functions and the boundary values are taken in the classical sense, due to Morrey's inequality (this and other results on Sobolev spaces and functional analysis can be found in [9]).

Let us start then with the statement of the following variational result on the p-Laplace equation (see [18, Theorem 2.4] for a proof).

Theorem 2.9. *Given $n < p < \infty$, a bounded domain Ω and $F \in \mathcal{C}(\overline{\Omega}) \cap W^{1,p}(\Omega)$, there exists a unique solution to the minimizing problem*

$$u_p(x) = \min_v \left\{ \int_\Omega |\nabla v(x)|^p\,dx \;:\; v \in \mathcal{C}(\overline{\Omega}) \cap W^{1,p}(\Omega) \text{ and } v|_{\partial\Omega} = F \right\}. \tag{2.3}$$

The minimizer is also a weak solution of $-\Delta_p u = -\nabla \cdot \left(|\nabla u|^{p-2}\,\nabla u\right) = 0$ in Ω, i.e.,

$$\int_\Omega \left\langle |\nabla u|^{p-2}\,\nabla u, \nabla\eta \right\rangle dx = 0 \quad \text{for all} \quad \eta \in \mathcal{C}_c^\infty(\Omega). \tag{2.4}$$

Conversely, a weak solution to (2.4) in $\mathcal{C}(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is always a minimizer u_p (among functions with its own boundary values).

Consider now $u_\infty = \lim_{p\to\infty} u_p$ uniformly in $\overline{\Omega}$, for u_p a weak solution to the p-Laplace equation. According to this, the following definition holds when u_∞ solves the ∞-Laplace equation in the viscosity sense, which is the limit equation of the p-Laplace equation as $p \to \infty$, in view of Section 2.1.1.

Definition 2.10 (Variational solution)**.** *A* variational solution *of the ∞-Laplace equation is the uniform limit of weak solutions to the p-Laplace equation via a suitable subsequence $(p_k)_k$ for $p_k \to \infty$, i.e.,*

$$u_\infty = \lim_{p_k\to\infty} u_{p_k}.$$

Furthermore, the limit function u_∞ is Lipschitz continuous in Ω and since

$$\lim_{p\to\infty} \left(\int_D |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} = \operatorname{ess\,sup}_{x\in D} |\nabla u(x)| = \|\nabla u\|_{L^\infty(D)}$$

for $D \subset \Omega$ a subdomain, one might guess in view of (2.3) that it minimizes somehow $\|\nabla u\|_{L^\infty(D)}$. This is so according to the following theorem, which asserts the existence of variational solution under certain conditions. Note that the space $W^{1,\infty}(\Omega)$ appearing in the statement of the theorem, consists of all Lipschitz continuous functions defined in Ω, provided $\partial\Omega$ is $\mathcal{C}^1$ (see [9, Section 5.8.2.b]).

Theorem 2.11 (Theorem 3.2 in [18])**.** *Given $F \in \mathcal{C}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, there exists a function $u_\infty \in \mathcal{C}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with boundary values $u_\infty = F$ on $\partial\Omega$ having the following minimizing property in each subdomain $D \subset \Omega$: if $v \in \mathcal{C}(\overline{D}) \cap W^{1,\infty}(D)$ and $v = u_\infty$ on $\partial\Omega$, then*

$$\|\nabla u_\infty\|_{L^\infty(D)} \leq \|\nabla v\|_{L^\infty(D)}.$$

This u_∞ can be obtained as the uniform limit in $\overline{\Omega}$ of u_{p_k}, where u_{p_k} denotes the weak solution of the p_k-Laplace equation such that $u_{p_k} = F$ on $\partial\Omega$.

Remark 2.12. *Weak solutions u_p of (2.4) are viscosity solutions of $-\Delta_p u = 0$ in Ω, by [18, Theorem 2.4], and variational solutions u_∞ of the ∞-Laplace equation are viscosity solutions to $-\Delta_\infty u = 0$ in Ω, according to [18, Theorem 4.6]. Because of this, every variational solution u_∞ is an infinity harmonic function. The importance of being viscosity solution is due to the fact that the framework of viscosity solutions is the natural one for the type of equations we are dealing with, as we mentioned before.*

The inhomogeneous equation $-\Delta_\infty u = f$

In some problems the limit equation of which the limit function u_∞ is solution, is not the expected one, as it is the case of the so-called ∞-*Poisson equation*,

$$-\Delta_\infty u(x) = f(x). \tag{2.5}$$

It has to be observed that, in general (2.5) is not the limit as $p \to \infty$ of the corresponding p-*Poisson equation* $-\Delta_p u = f$. For instance, let u_p be the weak solution to the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = 1, & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.6}$$

which is also a viscosity solution, as it occurred with the weak solutions of the p-Laplace equation (see Remark 2.12 and [18, Theorem 2.4]).

The results in [5, 17] show that $u_\infty = \lim_{p\to\infty} u_p$ is the distance function, i.e., $u_\infty = \operatorname{dist}(x, \partial\Omega)$ for $x \in \overline{\Omega}$. In fact, according to [16, Lemma 6.10], the distance function u_∞ is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} \min\{|\nabla u(x)| - 1, -\Delta_\infty u(x)\} = 0, & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

which turns out to be the limit problem of (2.6) as $p \to \infty$ (all the details can be found in [5]). Thus, it is not true that $-\Delta_\infty u(x) = 1$ is the limit equation as $p \to \infty$ of (2.6), as one might expect, and therefore there is no variational solution of the ∞-Poisson equation $-\Delta_\infty u(x) = 1$.

Remark 2.13. *We want to stress that (2.7) has a probabilistic interpretation in terms of certain random Tug-of-War games, which is a new result and our main contribution in this work.*

More results on the Dirichlet problem for the ∞-Poisson equation with boundary values not necessarily null, are considered in [18, Chapter 10]. Some of these results are the existence of solution to the Dirichlet problem, which is proved via the celebrated Perron's method, and a comparison principle for viscosity solutions of the ∞-Poisson equation.

2.1.3 An asymptotic mean value formula

It was discovered in [19] that p-harmonic functions satisfy the following nonlinear mean value formula

$$u(x) = \left(\frac{p-2}{p+n}\right) \frac{\max_{\overline{B}_\epsilon(x)} u + \min_{\overline{B}_\epsilon(x)} u}{2} + \left(\frac{2+n}{p+n}\right) \fint_{B_\epsilon(x)} u(y)\, dy + o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{2.8}$$

Moreover, it has a game-theoretic interpretation, where it is essential that the coefficients add up to 1, so that they play the role of conditional probabilities.

Note that the second term of (2.8) is linear, while the first one is nonlinear (the greater the p, the more weight in the formula and stronger nonlinearity). In particular, the case $p = 2$ reduces to

$$u(x) = \fint_{B_\epsilon(x)} u(y)\, dy \quad \text{for all} \quad x \in \Omega, \tag{2.9}$$

which is the well-known mean value property that characterizes harmonic functions. The following theorem (see [14, Theorem 2.7] for a proof and more details on this topic) formalizes this fact.

Theorem 2.14. *A function u is harmonic in Ω if and only if $u \in \mathcal{C}(\Omega)$ and for every ball compactly embedded in Ω, it satisfies the mean value property (2.9).*

On the other hand, when $p \to \infty$ one gets the asymptotic formula, fundamental in image processing and for some numerical algorithms (see, e.g., [22]),

$$u(x) = \frac{1}{2}\left(\max_{\overline{B}_\epsilon(x)} u + \min_{\overline{B}_\epsilon(x)} u\right) + o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{2.10}$$

It provides an alternative definition to infinity harmonic functions through the following result, a proof of which can be found in [18, Theorem 5.3] and also appears later as part of the proof of Theorem 6.1.

Theorem 2.15. *A function u is a viscosity solution to the ∞-Laplace equation in Ω, i.e., is infinity harmonic in Ω, if and only if $u \in \mathcal{C}(\Omega)$ and the mean value formula (2.10) holds in Ω in the viscosity sense.*

2.2 The normalized infinity Laplacian

The normalized version of the infinity Laplacian (see (2.1)) is the one that appears in the framework of game theory and is defined as (according to [25, Section 3.4])

$$\Delta_\infty^N u(x) := \begin{cases} \left\langle D^2u(x)\, \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\nabla u(x)}{|\nabla u(x)|} \right\rangle = |\nabla u(x)|^{-2}\, \Delta_\infty u(x), & \text{if } \nabla u(x) \neq 0; \\ \lim_{y \to x} \frac{2(u(y)-u(x))}{|y-x|^2}, & \text{otherwise.} \end{cases} \tag{2.11}$$

Note that when $\nabla u(x) \neq 0$, the normalized ∞-Laplacian coincides with the Laplace operator, Δu, in the segment case. Moreover, the homogeneous equation $-\Delta_\infty^N u = 0$ is equivalent to the ∞-Laplace equation.

About the expression of $\Delta_\infty^N u(x)$ in (2.11) when $\nabla u(x) = 0$, it follows from the Taylor expansion of $u(y)$ at x up to second order terms in the case when $u \in \mathcal{C}^2$, that is,

$$\begin{aligned} u(y) &= u(x) + \langle \nabla u(x), (y-x)\rangle + \frac{1}{2}\left\langle D^2u(x)\,(y-x), (y-x)\right\rangle + o\left(|y-x|^2\right) \\ &= u(x) + \frac{1}{2}\left\langle D^2u(x)\,(y-x), (y-x)\right\rangle + o\left(|y-x|^2\right) \quad \text{as } y \to x. \end{aligned}$$

Rearranging terms in the last expression and dividing by $|y-x|^2$, we get

$$\frac{2(u(y)-u(x))}{|y-x|^2} = \left\langle D^2u(x)\,\frac{y-x}{|y-x|}, \frac{y-x}{|y-x|}\right\rangle + o(1) \quad \text{as } y \to x.$$

Consider now the second derivative of u in the direction v, i.e.,

$$D_v^2u(x) = \frac{d^2}{d\xi^2}\bigg|_{\xi=0} u(x+\xi v) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(x)\, v_i\, v_j.$$

In view of (2.11), at the points where $\nabla u(x) \neq 0$ we can take $v = \frac{\nabla u(x)}{|\nabla u(x)|}$, so that $\Delta_\infty^N u$ is the second derivative of u in the direction of its gradient. On the other hand, at the points where $\nabla u(x) = 0$ it is said that $D^2u(x)$ is the same in every direction (no direction is preferred), that is, the limit of $\frac{u(y)-u(x)}{|y-x|^2}$ exists as $y \to x$.

Chapter 3

The classical random Tug-of-War and its connection to the infinity Laplacian

3.1 Description of the game

A random Tug-of-War game is a two-person zero-sum random-turn game, that is, two players are in a contest which involves some randomness for the turn choice, and the total earnings of one player are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his or her expected outcome, while the other, say Player II, is trying to minimize Player I's outcome (or in other words, since the game is zero-sum, maximize his or her own outcome).

The configuration of such games can be described in terms of some parameters. These are a set X of states of the game, a non-empty set $Y \subset X$ of terminal states, a terminal payoff function $F : Y \to \mathbb{R}$ and a running payoff function $f : X \backslash Y \to \mathbb{R}$. Moreover, consider E_I and E_{II} the graphs with vertex set X, which describe the possible move options for Players I and II, respectively, at any game state. In the case of the random Tug-of-War, $E := E_I = E_{II}$, i.e., players have identical move options and E is undirected, i.e., all moves are reversible.

In the classical random Tug-of-War, the game starts with a token placed at position $x^0 \in X \backslash Y$. A fair coin is tossed and the player who wins the toss chooses a new position for the token among all positions adjacent to the current one (x^0 in the first turn), according to the graph E. Moreover, both players are assumed to play optimally, that is, they choose the position which is most favorable to increase the final payoff they receive.

This procedure is repeated at each turn until the token reaches for the first time any $x^k \in Y$. Then the game ends and Player I receives the total payoff $F(x^k) + \sum_{i=0}^{k-1} f(x^i)$ from Player II. This is so since $F(x^k)$ is the payoff associated to the terminal position of the game, x^k, and $f(x^i)$, for $i = 0, \dots, k-1$, refers to the payoff associated at each running node (game position) where the token has been placed along the game. In other words, the running payoff $f(x^i)$ represents the reward for Player I (the cost for Player II) at each intermediate state x^i of the game.

Remark 3.1. *The previous description follows [23], where all results are presented in general length spaces. Now the classical random Tug-of-War game, which has just been described in a general metric space, is formulated in a bounded domain $\Omega \subset \mathbb{R}^n$, as it is done in [25].*

Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\epsilon > 0$ fixed, so that

$$\Gamma_\epsilon = \{x \in \mathbb{R}^n \backslash \Omega : \mathrm{dist}(x, \partial\Omega) \leq \epsilon\} \tag{3.1}$$

denotes a compact boundary strip of width ϵ, and $F : \Gamma_\epsilon \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ two continuous functions. At an initial time, a token is placed at a point $x^0 \in \Omega$. Then, a fair coin is tossed and the winner of the toss is allowed to move the token to any $x^1 \in \overline{B}_\epsilon(x^0)$, and another turn starts. The two players proceed in this way at each turn until the token reaches some $x^k \in \Gamma_\epsilon$. This implies the end of the game and a payoff of value $F(x^k) + \sum_{i=0}^{k-1} f(x^i)$, which Player I receives from Player II. Alternatively, we can say that Player II receives a payoff $-F(x^k) - \sum_{i=0}^{k-1} f(x^i)$ from Player I.

Note that the reason for using the boundary strip Γ_ϵ instead of simply using the boundary $\partial\Omega$ of our domain, is the fact that, for $x \in \Omega$, the ball $\overline{B}_\epsilon(x)$ is not necessarily contained in Ω but in $\Omega \cup \Gamma_\epsilon$. However, it is also possible to define the game using $\overline{B}_\epsilon(x) \cap \overline{\Omega}$ instead of Γ_ϵ. In this case, the game token (in the game position x^i, for $i = 0, \ldots, k-1$) can be moved to any $x^{i+1} \in \overline{B}_\epsilon(x^i) \cap \overline{\Omega}$, and the game ends when the token reaches some $x^k \in \partial\Omega$.

3.2 Players' strategies

A *pure strategy* $S_\alpha = \{S_\alpha^k\}_k$ for player α is a sequence of mappings from histories H_k to actions a^k. The *history* up to stage k is the sequence of game positions and actions up to the kth turn of the game, written as $H^k = (x^0, a^0, x^1, a^1, \ldots, a^{k-1}, x^k)$, where x^i for $i = 0, \ldots, k$ stands for the game position at the ith turn and a^i means the action carried out by the player who moved in this turn from position x^i.

Roughly speaking, at every turn the strategy indicates the player's next move, provided such player is given the choice, as a function of the current game position and past history. In other words, the strategy of a player says what action to take at each running node of the game, but it depends on the evolution of the game that these actions are accomplished or not. More information about the general notion of strategy can be found in [25, Section 3.1].

For some games, the action of a player at a given node is independent of both the stage of the game where the decision is made and the history up to that stage. In particular, for these games (the classical random Tug-of-War and the game presented in Chapter 4 are examples), strategies will be independent of time and initial position.

3.3 Value of the game

For a game where Players I and II adopt strategies S_I and S_{II} respectively and start the game at position x_i (this is not the same as x^i, which was the position at the ith turn), the expected payoff Player I receives from Player II is defined as

$$V_{S_I,S_{II}}^{x_i}(I) := \begin{cases} \mathbb{E}_{S_I,S_{II}}^{x_i}[F_\tau], & \text{if the game ends almost surely;} \\ -\infty, & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\mathbb{E}_{S_I,S_{II}}^{x_i}[F_\tau]$ is the expected payoff that Player I receives from Player II when they follow strategies S_I and S_{II} respectively and the game starts at the node x_i. F_τ is the payoff associated to the terminal node x_τ. On the contrary, there might exist games for which the players can choose strategies so that the game does not end almost surely. In order to penalize these strategies, the expected payoff that each player receives from the other is the worst possible, that is, $-\infty$ in the case of Player I and $+\infty$ for Player II.

Analogously, the expected payoff that Player II has to pay Player I is defined as

$$V^{x_i}_{S_I,S_{II}}(II) := \begin{cases} \mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau], & \text{if the game ends almost surely;} \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

Note that $V^{x_i}_{S_I,S_{II}}(I), V^{x_i}_{S_I,S_{II}}(II)$ defined above represent the expected payoff that players receive from each other when a game, played using a pair of strategies S_I, S_{II}, ends almost surely. These can be used to assign a value to the game as follows. Consider first the least payoff which Player I expects to receive from Player II when Player I's strategy is subordinated to Player II's strategy. It is called the *value of the game for Player I* and defined as, when the game starts at the node x_i,

$$u^I(x_i) := \sup_{S_I} \inf_{S_{II}} V^{x_i}_{S_I,S_{II}}(I). \tag{3.4}$$

In the same way, the largest payoff which Player I expects to receive from Player II occurs when Player II's strategy is subordinated to Player I's strategy. This is called the *value of the game for Player II* and defined as, for a game starting at the x_i position,

$$u^{II}(x_i) := \inf_{S_{II}} \sup_{S_I} V^{x_i}_{S_I,S_{II}}(II). \tag{3.5}$$

Provided $u^I(x_i) = u^{II}(x_i)$ for all $x_i \in \overline{\Omega}$, we say that the game has a value. This means that the pair of strategies S_I, S_{II} involved in both $u^I(x_i)$ and $u^{II}(x_i)$ leads to the same value $u(x_i) := u^I(x_i) = u^{II}(x_i)$, the so called *value of the game* associated to x_i. Note also that this pair of strategies S_I, S_{II} needs not be unique (see Section 5.1 for an example in a variant of the classical random Tug-of-War game) for a given x_i. In other words, different pair of strategies S_I, S_{II} associated to $u^I(x_i)$ might lead to the same value of the game for Player I, and the same occurs with the value of the game for Player II. However, among all equivalent pair of strategies S_I, S_{II} (if any), just one of them is optimal.

Related to this, we point out that when the game starting at x_i has a value, the pair of strategies S_I, S_{II} associated to $u^I(x_i)$ and $u^{II}(x_i)$ respectively, are the same. It then follows that the game of value $u(x_i)$ is played according to this particular pair of strategies S_I, S_{II}. We also remark that the existence of a value for the classical random Tug-of-War game (and therefore the existence of the same and unique optimal pair of strategies for both players) was proved in [23] using probabilistic arguments.

The value of the game associated to the classical random Tug-of-War in $\Omega \subset \mathbb{R}^n$ for $\epsilon > 0$ fixed (this game is sometimes known as the classical random ϵ-Tug-of-War), is denoted by u_ϵ and it is not continuous in general (some examples can be found in [25, Section 3 and 4]). Related to this, we will refer to u as the *continuous value of the game*, which is the uniform limit of the game value u_ϵ in $\overline{\Omega}$, as $\epsilon \to 0$.

3.4 Dynamic Programming Principle (DPP)

The function $u_\epsilon(x)$, the value of the game of the classical random ϵ-Tug-of-War, is the unique solution (see [23, 25]) of the Dirichlet problem

$$\begin{cases} u_\epsilon(x) = \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y)\right) + f(x), & x \in \Omega; \\ u_\epsilon(x) = F(x), & x \in \Gamma_\epsilon, \end{cases}$$

where the expression

$$u_\epsilon(x) = \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y)\right) + f(x) \quad \text{for all } \ x \in \Omega$$

is known as the *Dynamic Programming Principle* (abbreviated as DPP). This name is motivated by the fact that the players choose their optimal global strategy by choosing the optimal strategy at each turn. In other words, the solution to the global problem is divided recursively into subproblems, which are solved optimally. This has to do with *Bellman's Principle of Optimality*, which is stated below according to [2, Section 1.4 and 1.7] and [4, Section 3.3a]. In fact, this optimality principle is the reason why the strategies for some games are independent of time and initial position, as commented before.

Theorem 3.2 (Bellman's Principle of Optimality)**.** *An optimal policy has the property that whatever are the initial state and initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. In more simple words, components of a globally optimal solution are themselves globally optimal.*

For the case $f(x) \equiv 0$, it can be proved by similar arguments to those in [25, Theorem 4.11], that the uniform limit $u = \lim_{\epsilon\to 0} u_\epsilon$ in $\overline{\Omega}$ is the unique viscosity solution to

$$\begin{cases} -\Delta_\infty^N u(x) = 0, & x \in \Omega; \\ u(x) = F(x), & x \in \partial\Omega. \end{cases} \tag{3.6}$$

Theorem 6.1 states the analogous result for a new random Tug-of-War game presented in Chapter 4. In the case of a more general $f(x)$, similar results are included and proved via probabilistic arguments in [23].

According to this, the Dirichlet problem (3.6) can be given a probabilistic interpretation in terms of game theory. This is similar to what occurs with the Dirichlet problem associated to the Laplace equation, a probabilistic approach of which is provided by the Brownian motion. The main difference between both problems is that all directions are equally probable for the Brownian motion, while the classical random Tug-of-War favors only two directions, as it follows from its DPP.

Chapter 4

Introducing a new game: the Totalitarian Tug-of-War

4.1 Description of the game

In this chapter we introduce a new game, which we call *Totalitarian Tug-of-War*. This is a variant of a classical random Tug-of-War game in which one of the players is given extra options. More precisely, the game is played by turns starting with a token placed at a running node x^0. Then, at turn k, Player I has the power to decide whether to play a round of classical random Tug-of-War (that is, they toss a coin and the winner decides the new game position x^{k+1} among the neighbors of x^k), or let Player II choose the new game token position x^{k+1} among the neighbors of x^k in exchange of a fixed payoff of value ϵ. The fact that Player I somehow imposes at each turn the type of game that is played, is the reason why we refer to this game as *Totalitarian Tug-of-War*.

The game ends the first time the token reaches a terminal state x_τ, and the payoff Player I receives from Player II is $F_\tau + k_\tau\,\epsilon$ (the game is a two-person, zero-sum game). F_τ corresponds to the terminal payoff at x_τ and $k_\tau \in \{0, 1, \ldots, \tau\}$ is a positive integer that represents the number of times Player I has let Player II decide the next move in exchange of an ϵ-payoff throughout the game.

Just as in the classical random Tug-of-War game, Player I wants to maximize the final payoff that receives from Player II, who in turn wants to minimize it. In order to attain this objective, the players follow strategies according to which they take a particular action in each turn. Note that all terminal payoffs and the value of ϵ are known to both players beforehand.

For a game where Players I and II adopt strategies S_I and S_{II} respectively and start the game at position x_i, the expected payoff that Player I receives from Player II is defined as

$$V^{x_i}_{S_I,S_{II}}(I) := \begin{cases} \mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau + k_\tau\,\epsilon], & \text{if the game ends almost surely;} \\ -\infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau + k_\tau\,\epsilon]$ represents the expected payoff that Player I receives from Player II when they follow strategies S_I and S_{II} respectively and the game starts at the node x_i. Analogously, the expected payoff that Player II has to pay Player I is defined as

$$V^{x_i}_{S_I,S_{II}}(II) := \begin{cases} \mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau + k_\tau\,\epsilon], & \text{if the game ends almost surely;} \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that $V^{x_i}_{S_I,S_{II}}(I), V^{x_i}_{S_I,S_{II}}(II)$ defined in (4.1) and (4.2) respectively, can be used to assign a value to the game in the same way as in the classical random Tug-of-War (see Section 3.3). Namely, the *value of the game for Player I* when the game starts at the node x_i, is defined as

$$u^I(x_i) := \sup_{S_I} \inf_{S_{II}} V^{x_i}_{S_I,S_{II}}(I), \tag{4.3}$$

and the *value of the game for Player II* when the game starts at the node x_i, as

$$u^{II}(x_i) := \inf_{S_{II}} \sup_{S_I} V^{x_i}_{S_I,S_{II}}(II). \tag{4.4}$$

Similarly to the classical random Tug-of-War game, we say that the Totalitarian Tug-of-War game has a value at x_i provided $u^I(x_i) = u^{II}(x_i)$, for x_i the starting point of the game. We say that the game has a value when $u^I = u^{II}$.

In Sections 4.2 and 4.3 we prove that the Totalitarian Tug-of-War has a value when played on graphs and in bounded domains $\Omega \subset \mathbb{R}^n$ respectively. A key point in the analysis of this result is that both u^I and u^{II}, as well as the value of the game solve a Dynamic Programming Principle. In the case of a graph, the aforementioned DPP is, for all interior nodes x_i,

$$u_i = \max\left\{\min_{j\in\{i'\}} u_j + \epsilon,\ \frac{1}{2}\left(\max_{j\in\{i'\}} u_j + \min_{j\in\{i'\}} u_j\right)\right\}, \tag{4.5}$$

where $u_i = u(x_i)$ and $\{i'\}$ denotes the finite set of indices associated to the nodal neighbors $x_{i'}$ of x_i in the graph. When the game is played in $\Omega \subset \mathbb{R}^n$, the associated DPP is

$$u_\epsilon(x) = \max\left\{\inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \epsilon,\ \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y)\right)\right\} \quad \text{for all} \quad x \in \Omega. \tag{4.6}$$

Equations (4.5) and (4.6) are obtained considering the two possible choices for Player I and then applying conditional probabilities for the coin toss.

4.2 The Totalitarian Tug-of-War on graphs has a value

We present now a novel mathematical analysis of the Totalitarian Tug-of-War game played on graphs. We refer to this game as the discrete Totalitarian Tug-of-War. For notational simplicity, we will consider here the case where the graph is a segment; however, the general case follows with the same ideas.

Let us point out that the mathematical analysis of a random Tug-of-War game is based on its Dynamic Programming Principle (DPP), which has been already presented in Section 3.4 for the classical random Tug-of-War game. On the other hand, for the case of the discrete Totalitarian Tug-of-War game, we will denote by $\mathcal{N}$ (nodes) the entire finite set of game position indices and $\mathcal{I}$ (interior nodes) the set of indices for the running nodes of the game.

According to all this, the DPP associated to the discrete Totalitarian Tug-of-War in a discrete graph segment, corresponds to

$$u_i = \max\left\{\min\{u_{i-1},\, u_{i+1}\} + \epsilon,\ \frac{1}{2}(u_{i-1} + u_{i+1})\right\} \quad \text{for all} \quad i \in \mathcal{I},$$

where u_i stands for the expected value of the game associated to the game position x_i (compare with (4.5)). Note that this DPP can be equivalently written in several different ways as follows. Namely, for all $i \in \mathcal{I}$,

$$u_i = \max\left\{\min\{u_{i-1},\, u_{i+1}\} + \epsilon,\ \frac{1}{2}(u_{i-1} + u_{i+1})\right\}$$

$$
\begin{aligned}
&\iff u_i = -\min\left\{-\min\{u_{i-1},\, u_{i+1}\} - \epsilon, -\frac{1}{2}(u_{i-1}+u_{i+1})\right\}\\
&\iff u_i = -\min\left\{\max\{-u_{i-1}, -u_{i+1}\} - \epsilon, -\frac{1}{2}(u_{i-1}+u_{i+1})\right\}\\
&\iff u_i + \min\left\{\max\{-u_{i-1}, -u_{i+1}\} - \epsilon, -\frac{1}{2}(u_{i-1}+u_{i+1})\right\} = 0\\
&\iff \min\left\{u_i + \max\{-u_{i-1}, -u_{i+1}\} - \epsilon,\, u_i - \frac{1}{2}(u_{i-1}+u_{i+1})\right\} = 0. \qquad (4.7)
\end{aligned}
$$

For convenience, and in order to simplify the notation, we will refer to it as

$$
\mathcal{G}^i[u] = \min\left\{u_i + \max\{-u_{i-1}, -u_{i+1}\} - \epsilon,\, u_i - \frac{1}{2}(u_{i-1}+u_{i+1})\right\} \qquad (4.8)
$$

for all $i \in \mathcal{I}$ and $u = (u_0, \ldots, u_{n+1})$, so that (4.7) reads $\mathcal{G}^i[u] = 0$. The combination of the DPP with the expected value of the game at the terminal nodes, given by the bounded values F_i for all $i \in \mathcal{N}\backslash\mathcal{I}$, gives us the following Dirichlet problem associated to the discrete Totalitarian Tug-of-War:

$$
\begin{cases} \mathcal{G}^i[u] = 0, & i \in \mathcal{I};\\ u_i = F_i, & i \in \mathcal{N}\backslash\mathcal{I}. \end{cases} \qquad (4.9)
$$

Related to this is the following proposition, which is derived considering the two possible choices for Player I and then applying conditional probabilities for the coin toss.

Proposition 4.1. *The value functions of the game for Players I and II, u_i^I and u_i^{II} respectively, are both solution to the discrete Dirichlet problem (4.9).*

We intend to provide in an upcoming work another proof of Proposition 4.1 in the line of [20], where an alternative and more precise proof of [23, Lemma 1.1] is given.

The importance of Proposition 4.1 lies on the fact that it allows us to apply PDE methods to study u_i^I and u_i^{II}. In particular, a key novelty that we introduce in our analysis is a discrete comparison principle (see Theorem 4.2 below). This result allows us to prove that (4.9) has a unique solution and therefore conclude that the game has a value without appealing to probabilistic arguments.

4.2.1 A discrete comparison principle on graphs

Now follows the statement and proof of a discrete comparison principle, Theorem 4.2, which is one of the key results of this section as commented before.

Theorem 4.2. *Let u and v be respectively a subsolution and supersolution of (4.7), i.e., $\mathcal{G}^i[u] \le 0 \le \mathcal{G}^i[v]$ for all $i \in \mathcal{I}$, where $\mathcal{G}^i$ is defined in (4.8). Assume also that v is bounded from above and $u_i \le v_i$ for all $i \in \mathcal{N}\backslash\mathcal{I}$. Then $u_i \le v_i$ for all $i \in \mathcal{N}$.*

An important consequence of Theorem 4.2 is the following bound.

Proposition 4.3. *Let u_i be a solution to the Dirichlet problem (4.9) and F_i bounded for all $i \in \mathcal{N}\backslash\mathcal{I}$. Then, for all $i \in \mathcal{N}$ and $K = \max_{i\in\mathcal{N}\backslash\mathcal{I}} |F_i|$,*

$$
\min\{(n+1) - i,\, i\} - K \le u_i \le \min\{(n+1) - i,\, i\} + K.
$$

Proof. Consider, for F_i the same in (4.9), the Dirichlet problem

$$\begin{cases} \mathcal{G}^i[v] = 0, & i \in \mathcal{I}; \\ v_i = K, & i \in \mathcal{N} \backslash \mathcal{I}, \end{cases}$$

a solution of which is $v_i = \min\{(n+1) - i,\, i\} + K$. Then, since it is bounded for all $i \in \mathcal{N}$, by Theorem 4.2 it follows that $u_i \leq v_i$ for all $i \in \mathcal{N}$ and therefore the upper bound for u_i is proved.

About the lower bound of u_i, consider now the Dirichlet problem

$$\begin{cases} \mathcal{G}^i[v] = 0, & i \in \mathcal{I}; \\ v_i = -K, & i \in \mathcal{N} \backslash \mathcal{I}, \end{cases}$$

a solution of which is $v_i = \min\{(n+1) - i,\, i\} - K$. On the other hand, since u_i is a solution to (4.9) by hypothesis, it is in particular a supersolution and bounded from above due to the first part of the proof. Then, by Theorem 4.2 it follows that $v_i \leq u_i$ for all $i \in \mathcal{N}$ and the proof is finished. □

The following lemma, which is just the segment version of Lemma 4.13, is necessary for the proof of Theorem 4.2.

Lemma 4.4. *Let v be a supersolution to $\mathcal{G}^i[v] = 0$ and bounded from above for all $i \in \mathcal{N}$. Then, for every $\gamma > 0$ there exists a supersolution $\tilde{v}$ to the equation $\mathcal{G}^i[\tilde{v}] = \mu$ for some constant $\mu = \mu(\gamma, v) > 0$. Moreover, $\tilde{v}_i - v_i \leq \gamma$ for all $i \in \mathcal{I}$ and $\tilde{v}_i - v_i \geq -\gamma$ for all $i \in \mathcal{N} \backslash \mathcal{I}$.*

Proof. We look for $\tilde{v}$ of the form $\tilde{v}_i = g(v_i)$ for all $i \in \mathcal{I}$, where

$$g(\alpha) = (1 + \varepsilon)\, \alpha - \frac{\varepsilon}{4C}\, \alpha^2$$

for $\varepsilon > 0$ and $C = \max_{i \in \mathcal{I}} |v_i|$ (recall that v is bounded from above by hypothesis). The constant γ in the statement of the lemma will be chosen later as a function of ε.

We assume, without loss of generality, that $\max\{v_i - v_{i-1},\, v_i - v_{i+1}\} = v_i - v_{i-1}$. Note also that $\max\{v_i - v_{i-1},\, v_i - v_{i+1}\} - \epsilon \geq 0$, in view of (4.7). According to all this,

$$\begin{aligned}
\max\{\tilde{v}_i - \tilde{v}_{i-1},\, \tilde{v}_i - \tilde{v}_{i+1}\} - \epsilon &= \max\{g(v_i) - g(v_{i-1}),\, g(v_i) - g(v_{i+1})\} - \epsilon \\
&= \max\Big\{(1+\varepsilon)\,(v_i - v_{i-1}) - \frac{\varepsilon}{4C}\,(v_i^2 - v_{i-1}^2), \\
&\qquad\qquad (1+\varepsilon)\,(v_i - v_{i+1}) - \frac{\varepsilon}{4C}\,(v_i^2 - v_{i+1}^2)\Big\} - \epsilon \\
&= \max\Big\{\Big((1+\varepsilon) - \frac{\varepsilon}{4C}\,(v_i + v_{i-1})\Big)\,(v_i - v_{i-1}), \\
&\qquad\qquad \Big((1+\varepsilon) - \frac{\varepsilon}{4C}\,(v_i + v_{i+1})\Big)\,(v_i - v_{i+1})\Big\} - \epsilon \\
&\geq \Big((1+\varepsilon) - \frac{\varepsilon}{4C}\,2C\Big)\,(v_i - v_{i-1}) - \epsilon \\
&= \Big(1 + \frac{\varepsilon}{2}\Big)\,(v_i - v_{i-1}) - \epsilon \\
&\geq \Big(1 + \frac{\varepsilon}{2}\Big)\,\epsilon - \epsilon \\
&= \frac{\varepsilon}{2}\,\epsilon.
\end{aligned}$$

On the other hand,

$$(\tilde{v}_i - \tilde{v}_{i-1}) + (\tilde{v}_i - \tilde{v}_{i+1}) = (g(v_i) - g(v_{i-1})) + (g(v_i) - g(v_{i+1}))$$

$$
\begin{aligned}
&= (1+\varepsilon)\,\left((v_i - v_{i-1}) + (v_i - v_{i+1})\right) - \frac{\varepsilon}{4C}\,\left(v_i^2 - v_{i-1}^2\right) - \frac{\varepsilon}{4C}\,\left(v_i^2 - v_{i+1}^2\right)\\
&= (1+\varepsilon)\,\left(2v_i - v_{i-1} - v_{i+1}\right) + \frac{\varepsilon}{4C}\,\left(-2v_i^2 + v_{i-1}^2 + v_{i+1}^2\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
-2v_i^2 + v_{i-1}^2 + v_{i+1}^2 &= -2v_i^2 + v_{i-1}^2 + v_{i+1}^2 \pm \left(2v_i^2 + 2v_i\,v_{i-1} + 2v_i\,v_{i+1}\right)\\
&= -2v_i^2 + (v_{i-1} - v_i)^2 + (v_{i+1} - v_i)^2 + 2v_i\,v_{i-1} + 2v_i\,v_{i+1} - 2v_i^2\\
&= -4v_i^2 + (v_{i-1} - v_i)^2 + (v_{i+1} - v_i)^2 + 2v_i\,(v_{i-1} + v_{i+1})\\
&= -2v_i\,(2v_i - v_{i-1} - v_{i+1}) + (v_{i-1} - v_i)^2 + (v_{i+1} - v_i)^2.
\end{aligned}
$$

From these, it follows that

$$
\begin{aligned}
(\tilde v_i - \tilde v_{i-1}) + (\tilde v_i - \tilde v_{i+1}) &= (1+\varepsilon)\,\left(2v_i - v_{i-1} - v_{i+1}\right) + \frac{\varepsilon}{4C}\,\left(-2v_i^2 + v_{i-1}^2 + v_{i+1}^2\right)\\
&= (1+\varepsilon)\,\left(2v_i - v_{i-1} - v_{i+1}\right)\\
&\qquad + \frac{\varepsilon}{4C}\,\left(-2v_i\,(2v_i - v_{i-1} - v_{i+1}) + (v_{i-1} - v_i)^2 + (v_{i+1} - v_i)^2\right)\\
&= \left(1 + \varepsilon - \frac{\varepsilon\, v_i}{2C}\right)\,\left(2v_i - v_{i-1} - v_{i+1}\right) + \frac{\varepsilon}{4C}\,\left((v_{i-1} - v_i)^2 + (v_{i+1} - v_i)^2\right)\\
&\geq \left(1 + \varepsilon - \frac{\varepsilon\, v_i}{2C}\right)\,\left(2v_i - v_{i-1} - v_{i+1}\right) + \frac{\varepsilon}{4C}\,\left(\max\{v_i - v_{i-1},\, v_i - v_{i+1}\}\right)^2\\
&\geq \frac{\varepsilon}{4C}\,\epsilon^2,
\end{aligned}
$$

where in the last inequality we have used that, since $\mathcal{G}^i[v] \geq 0$ by hypothesis for all $i \in \mathcal{I}$, $2v_i - v_{i-1} - v_{i+1} \geq 0$ and $\max\{v_i - v_{i-1},\, v_i - v_{i+1}\} \geq \epsilon$.

Then, according to the previous results, we get that

$$
\begin{aligned}
\min\left\{\tilde v_i + \max\left\{-\tilde v_{i-1}, -\tilde v_{i+1}\right\} - \epsilon,\ \tilde v_i - \frac{1}{2}\left(\tilde v_{i-1} + \tilde v_{i+1}\right)\right\} &\geq \min\left\{\frac{\varepsilon}{2}\,\epsilon,\ \frac{\varepsilon}{8C}\,\epsilon^2\right\}\\
&= \frac{\varepsilon}{2}\,\epsilon\,\min\left\{1,\ \frac{\epsilon}{4C}\right\} = \mu
\end{aligned}
$$

for all $i \in \mathcal{I}$, where $\mu = \mu(\varepsilon, v) > 0$.

About the second part of this lemma, since $g(\alpha) - \alpha \leq \frac{3}{4}\,\varepsilon\, C$ for $\alpha \leq C$ and because $\tilde v_i = g(v_i)$, it follows that $\tilde v_i - v_i \leq \frac{3}{4}\,\varepsilon\, C$ for all $i \in \mathcal{I}$. Similarly, since $g(\alpha) - \alpha \geq -\varepsilon\, D\left(1 + \frac{D}{4C}\right)$ for $\alpha \geq -D = -\left|\min_{i\in\mathcal{N}\setminus\mathcal{I}} v_i\right|$, it follows that $\tilde v_i - v_i \geq -\varepsilon\, D\left(1 + \frac{D}{4C}\right)$ for all $i \in \mathcal{N}\setminus\mathcal{I}$. The results hold for every

$$
\gamma = \varepsilon \max\left\{\frac{3}{4}\,C,\ D\left(1 + \frac{D}{4C}\right)\right\} > 0,
$$

provided ε is small enough. □

We proceed now with the proof of the discrete comparison principle, Theorem 4.2, which is inspired by the proof of Theorem 4.10 in the continuous domain $\Omega \subset \mathbb{R}^n$.

Proof of Theorem 4.2. Arguing by contradiction, we suppose that $\max_{i\in\mathcal{N}}(u_i - v_i) > 0$. Since $u_i \leq v_i$ for all $i \in \mathcal{N}\setminus\mathcal{I}$, it follows that there is an index $j \in \mathcal{I}$ such that $u_j - v_j = \max_{i\in\mathcal{N}}(u_i - v_i) > 0$. On the other hand, by Lemma 4.4, for every $\gamma > 0$ there exists $\tilde v$ such that $\tilde v_i - v_i \leq \gamma$ for all $i \in \mathcal{I}$. As a result, $u_j - v_j > \gamma \geq \tilde v_j - v_j$ for γ small enough and therefore $u_j > \tilde v_j$.

This implies that there is an index $k \in \mathcal{N}$ such that $u_k - \tilde v_k = \max_{i\in\mathcal{N}}(u_k - \tilde v_k) > 0$. In fact, $k \in \mathcal{I}$ since by Lemma 4.4, we can assume $\tilde v_i - v_i \geq -\gamma$ for all $i \in \mathcal{N}\setminus\mathcal{I}$ and therefore

$u_k - \tilde{v}_k > \gamma \geq v_i - \tilde{v}_i \geq u_i - \tilde{v}_i$ for all $i \in \mathcal{N} \backslash \mathcal{I}$. For the sake of simplicity let us assume this index k to be j.

It follows by definition that

$$u_j - \tilde{v}_j \geq u_i - \tilde{v}_i \quad \text{for all} \quad i \in \mathcal{N}.$$

In particular $u_j - u_{j-1} \geq \tilde{v}_j - \tilde{v}_{j-1}$ and $u_j - u_{j+1} \geq \tilde{v}_j - \tilde{v}_{j+1}$. According to this and writing (4.7) as

$$\min \{\max \{u_i - u_{i-1},\, u_i - u_{i+1}\} - \epsilon,\, (u_i - u_{i-1}) + (u_i - u_{i+1})\} = 0,$$

we have that $\mathcal{G}^j[\tilde{v}] \leq \mathcal{G}^j[u]$, in terms of (4.8). Combining this with the fact that, by Lemma 4.4, $\mathcal{G}^i[u] < \mathcal{G}^i[\tilde{v}]$ for all $i \in \mathcal{I}$, it follows $\mathcal{G}^i[u] < \mathcal{G}^j[\tilde{v}] \leq \mathcal{G}^j[u]$ for all $i \in \mathcal{I}$. Setting $i = j$ yields a contradiction. □

A result that follows directly from Theorem 4.2 is the following maximum principle.

Corollary 4.5. *Under the same assumptions of Theorem 4.2 and provided u_i is bounded from above for all $i \in \mathcal{N} \backslash \mathcal{I}$, it holds*

$$\max_{i \in \mathcal{N}} (u_i - v_i) = \max_{i \in \mathcal{N} \backslash \mathcal{I}} (u_i - v_i).$$

Proof. Let K be a constant such that $u_i - K \leq v_i$ for all $i \in \mathcal{N} \backslash \mathcal{I}$. Since $\mathcal{G}^i[u - K] = \mathcal{G}^i[u] \leq 0$ for all $i \in \mathcal{I}$, $u_i - K$ is a subsolution of equation (4.7). Then, by Theorem 4.2 we have that, for all $i \in \mathcal{N}$, $u_i - K \leq v_i$, i.e., $u_i - v_i \leq K$. The result follows by taking $K = \max_{i \in \mathcal{N} \backslash \mathcal{I}} (u_i - v_i)$, which is finite since u_i and $-v_i$ are both bounded from above for all $i \in \mathcal{N} \backslash \mathcal{I}$. □

4.2.2 Existence and uniqueness of the value of the game

As it occurs with the classical random Tug-of-War game, the discrete Totalitarian Tug-of-War has a value associated to the position x_i for $i \in \mathcal{N}$, denoted as u_i, provided that $u_i := u_i^I = u_i^{II}$, for u_i^I, u_i^{II} the respective expected value of the game for Players I and II at x_i. Therefore, in order to prove that the discrete Totalitarian Tug-of-War game has a value, i.e., $u_i^I = u_i^{II}$ for all $i \in \mathcal{N}$, it will be enough to show that both $u_i^I \leq u_i^{II}$ and $u_i^{II} \leq u_i^I$ hold for all $i \in \mathcal{N}$.

Remark 4.6. *If the discrete Totalitarian Tug-of-War ends almost surely, then the value of the game for Player I, (4.3), and II, (4.4), is finite. This is so in view of (4.1) and (4.2), and the fact that for every game that ends almost surely, the number of coin tosses is finite and so it is $\mathbb{E}_{S_I, S_{II}}^{x_i}[F_\tau + k_\tau\, \epsilon]$. Hence, u_i^I and u_i^{II} are both bounded in all the game nodes. Note also that, since we are assuming that F_i is bounded for all $i \in \mathcal{N} \backslash \mathcal{I}$, it is derived that if the discrete Totalitarian Tug-of-War has a finite value, the game ends almost surely. This is clear since, according to (4.1) and (4.2), the only possibility for the game to have an infinite value is that it does not end.*

At this point, the following result provides one of the two inequalities necessary to guarantee the existence (and uniqueness) of a value for the discrete Totalitarian Tug-of-War.

Proposition 4.7. *It holds that $u_i^I \leq u_i^{II}$ for all $i \in \mathcal{N}$.*

Proof. Note first that, by definition,

$$V_{S_I, S_{II}}^{x_i}(I) \leq V_{S_I, S_{II}}^{x_i}(II) \quad \text{for all} \quad i \in \mathcal{N}.$$

According to this, it holds

$$\inf_{S_{II}} V_{S_I, S_{II}}^{x_i}(I) \leq \sup_{S_I} V_{S_I, S_{II}}^{x_i}(II) \quad \text{for all} \quad i \in \mathcal{N}.$$

Then, it follows

$$\sup_{S_I}\inf_{S_{II}} V^{x_i}_{S_I,S_{II}}(I) \leq \sup_{S_I} V^{x_i}_{S_I,S_{II}}(II) \quad \text{for all} \quad i \in \mathcal{N},$$

and since the left-hand side of the inequality holds for all S_{II}, one finally gets

$$u_i^I = \sup_{S_I}\inf_{S_{II}} V^{x_i}_{S_I,S_{II}}(I) \leq \inf_{S_{II}}\sup_{S_I} V^{x_i}_{S_I,S_{II}}(II) = u_i^{II} \quad \text{for all} \quad i \in \mathcal{N}.$$

□

It remains to prove that $u_i^{II} \leq u_i^I$ holds for all $i \in \mathcal{N}$ (see Theorem 4.8). Then, we will be able to conclude the existence and uniqueness of the game value for the discrete Totalitarian Tug-of-War.

Theorem 4.8. *The discrete Totalitarian Tug-of-War game has a value, which is unique.*

Proof of Theorem 4.8. Let u_i^I, u_i^{II} be the respective value of the Totalitarian Tug-of-War game for Players I and II, for $i \in \mathcal{N}$. By Proposition 4.1, both u_i^I and u_i^{II} are solutions to the discrete Dirichlet problem (4.9) and hence, according to Proposition 4.3 they are bounded for all $i \in \mathcal{N}$ (note that the boundedness follows alternatively by Remark 4.6). In particular, they are respectively a supersolution and subsolution of (4.7) for all $i \in \mathcal{N}$ and $u_i^{II} \leq u_i^I$ for all $i \in \mathcal{N}\backslash\mathcal{I}$. Thus, we can apply Theorem 4.2 for $u_i = u_i^{II}$ and $v_i = u_i^I$, so that $u_i^{II} \leq u_i^I$ for all $i \in \mathcal{N}$. On the other hand, by Proposition 4.7, $u_i^I \leq u_i^{II}$ for all $i \in \mathcal{N}$. It then follows that $u_i^{II} = u_i^I$ for all $i \in \mathcal{N}$ so that the discrete Totalitarian Tug-of-War game has a unique value. □

Just to mention that a more exhaustive analysis on discretized degenerate elliptic equations and its use in numerical analysis can be found in [21], for instance.

4.3 The ϵ-Totalitarian Tug-of-War in $\mathbb{R}^n$ has a value

A similar analysis to the one in the previous section is considered now for the Totalitarian Tug-of-War game played on a bounded domain $\Omega \subset \mathbb{R}^n$, instead of a finite set of points (discrete case). For $\epsilon > 0$ fixed, we denote the compact boundary strip of width ϵ by

$$\Gamma_\epsilon = \{x \in \mathbb{R}^n\backslash\Omega : \operatorname{dist}(x, \partial\Omega) \leq \epsilon\}.$$

Note that this is the same boundary strip (3.1) presented in Section 3.1 for the classical random Tug-of-War game.

More precisely, we are considering a Totalitarian Tug-of-War game that is played in the way described in Section 4.1, but now the players can move the game token (currently placed at $y \in \Omega$) to any position $x \in \overline{B}_\epsilon(y)$, where $\epsilon > 0$ is fixed at the beginning of the game. This is why we refer to this game as the ϵ-Totalitarian Tug-of-War.

The Dynamic Programming Principle associated to the ϵ-Totalitarian Tug-of-War game is

$$u_\epsilon(x) = \max\left\{\inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \epsilon,\, \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y)\right)\right\} \quad \text{for all} \quad x \in \Omega$$

and can be equivalently written, following the same argument that we did for the derivation of (4.7), as

$$\min\left\{u_\epsilon(x) - \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) - \epsilon,\, u_\epsilon(x) - \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y)\right)\right\} = 0 \qquad (4.10)$$

for all $x \in \Omega$. For our convenience and in order to simplify the notation, we will denote

$$\mathcal{G}[u_\epsilon](x) = \min\left\{ u_\epsilon(x) - \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) - \epsilon,\, u_\epsilon(x) - \frac{1}{2}\left(\sup_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) + \inf_{y\in\overline{B}_\epsilon(x)} u_\epsilon(y) \right) \right\}, \tag{4.11}$$

so that (4.10) reads $\mathcal{G}[u_\epsilon](x) = 0$.

The Dirichlet problem that results from the combination of the DPP (4.10) and the expected value of the game on its domain boundary Γ_ϵ, for $F : \Gamma_\epsilon \to \mathbb{R}$ a bounded function, corresponds to

$$\begin{cases} \mathcal{G}[u_\epsilon](x) = 0, & x \in \Omega; \\ u_\epsilon(x) = F(x), & x \in \Gamma_\epsilon. \end{cases} \tag{4.12}$$

Related to this is the following proposition, that is derived considering the two possible choices for Player I and then applying conditional probabilities for the coin toss.

Proposition 4.9. *The value functions for Players I and II of the ϵ-Totalitarian Tug-of-War game, $u_\epsilon^I(x)$ and $u_\epsilon^{II}(x)$ respectively, are both solution to the Dirichlet problem (4.12).*

We intend to provide in an upcoming work a proof of Proposition 4.9 in the line of [20], where an alternative and more precise proof of [23, Lemma 1.1] is given.

The importance of Proposition 4.9 lies on the fact that it allows us to apply partial differential equation methods to study $u_\epsilon^I(x)$ and $u_\epsilon^{II}(x)$. In particular, a key novelty that we introduce in our analysis is a discrete comparison principle (see Theorem 4.10 below) which allows us to prove that (4.12) has a unique solution and therefore conclude that the game has a value without appealing to probabilistic arguments.

4.3.1 A comparison principle for the DPP

Now follows the statement and proof of a discrete comparison principle, Theorem 4.10, which is one of the key results of this section as commented before.

Theorem 4.10. *Let the functions $u_\epsilon, v_\epsilon : \Omega \cup \Gamma_\epsilon \to \mathbb{R}$ be respectively a subsolution and supersolution of (4.10), i.e., $\mathcal{G}[u_\epsilon] \le 0 \le \mathcal{G}[v_\epsilon]$ in Ω, where $\mathcal{G}$ is defined in (4.11). Assume also that u_ϵ is bounded from above in $\Omega \cup \Gamma_\epsilon$, v_ϵ is bounded from above and below in $\Omega \cup \Gamma_\epsilon$ and $u_\epsilon \le v_\epsilon$ in Γ_ϵ. Then $u_\epsilon \le v_\epsilon$ in $\Omega \cup \Gamma_\epsilon$.*

A result which follows from Theorem 4.10 is the following proposition.

Proposition 4.11. *Let u_ϵ be a solution to the Dirichlet problem (4.12) and $F : \partial\Omega \to \mathbb{R}$ bounded. Then, for all $x \in \Omega_\epsilon := \Omega \cup \Gamma_\epsilon$ and $K = \|F\|_{\infty,\Gamma_\epsilon}$,*

$$\operatorname{dist}(x, \partial\Omega_\epsilon) - K - \epsilon \le u_\epsilon(x) \le \operatorname{dist}(x, \partial\Omega_\epsilon) + K. \tag{4.13}$$

Proof. Consider, for $F(x)$ the same in (4.12), the Dirichlet problem

$$\begin{cases} \mathcal{G}[v_\epsilon](x) = 0, & x \in \Omega; \\ v_\epsilon(x) = K, & x \in \Gamma_\epsilon, \end{cases} \tag{4.14}$$

a supersolution of which is $v_\epsilon(x) = \operatorname{dist}(x, \partial\Omega_\epsilon) + K$. To see this, let

$$\mathcal{M}_\epsilon = \{x \in \Omega : \|\operatorname{dist}(\cdot, \partial\Omega_\epsilon)\|_{\infty,\Omega_\epsilon} \le \operatorname{dist}(x, \partial\Omega_\epsilon) + \epsilon\} \tag{4.15}$$

and observe that

i) when $x \in \mathcal{M}_\epsilon$, on the one hand

$$\begin{aligned} v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \inf_{\overline{B}_\epsilon(x)} (\operatorname{dist}(\cdot, \partial\Omega_\epsilon) + K) \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \inf_{\overline{B}_\epsilon(x)} \operatorname{dist}(\cdot, \partial\Omega_\epsilon) - K \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - (\operatorname{dist}(x, \partial\Omega_\epsilon) - \epsilon) - K \\ &= \epsilon, \end{aligned}$$

and on the other hand,

$$\begin{aligned} v_\epsilon(x) - \sup_{\overline{B}_\epsilon(x)} v_\epsilon &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \sup_{\overline{B}_\epsilon(x)} (\operatorname{dist}(\cdot, \partial\Omega_\epsilon) + K) \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \sup_{\overline{B}_\epsilon(x)} \operatorname{dist}(\cdot, \partial\Omega_\epsilon) - K \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \|\operatorname{dist}(\cdot, \partial\Omega_\epsilon)\|_{\infty, \overline{B}_\epsilon(x)} - K \\ &\geq (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - (\operatorname{dist}(x, \partial\Omega_\epsilon) + \epsilon) - K \\ &= -\epsilon, \end{aligned}$$

where the inequality follows from (4.15). By adding up the two previous expressions, one gets

$$\left(v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon \right) + \left(v_\epsilon(x) - \sup_{\overline{B}_\epsilon(x)} v_\epsilon \right) \geq \epsilon - \epsilon = 0.$$

Thus, $v_\epsilon(x) = \operatorname{dist}(x, \partial\Omega_\epsilon) + K$ is a supersolution to $\mathcal{G}[v_\epsilon] = 0$ in $\mathcal{M}_\epsilon$.

ii) When $x \in \Omega \backslash \mathcal{M}_\epsilon$, on the one hand

$$v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon = \epsilon,$$

as before, and on the other hand

$$\begin{aligned} v_\epsilon(x) - \sup_{\overline{B}_\epsilon(x)} v_\epsilon &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \sup_{\overline{B}_\epsilon(x)} (\operatorname{dist}(\cdot, \partial\Omega_\epsilon) + K) \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - \sup_{\overline{B}_\epsilon(x)} \operatorname{dist}(\cdot, \partial\Omega_\epsilon) - K \\ &= (\operatorname{dist}(x, \partial\Omega_\epsilon) + K) - (\operatorname{dist}(x, \partial\Omega_\epsilon) + \epsilon) - K \\ &= -\epsilon. \end{aligned}$$

Adding up the two last expressions, one gets

$$\left(v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon \right) + \left(v_\epsilon(x) - \sup_{\overline{B}_\epsilon(x)} v_\epsilon \right) = \epsilon - \epsilon = 0,$$

so that it follows that $v_\epsilon(x) = \operatorname{dist}(x, \partial\Omega_\epsilon) + K$ is a solution to $\mathcal{G}[v_\epsilon] = 0$ in $\Omega \backslash \mathcal{M}_\epsilon$.

iii) Finally, when $x \in \Gamma_\epsilon$, it is clear that $\operatorname{dist}(x, \partial\Omega_\epsilon) + K \geq K$ since $\operatorname{dist}(x, \partial\Omega_\epsilon) \geq 0$. That is, $v_\epsilon \geq K$ in Γ_ϵ.

Thus, we conclude that $v_\epsilon(x) = \operatorname{dist}(x, \partial\Omega_\epsilon) + K$ is a supersolution to the Dirichlet problem (4.14) and then also to (4.12). Moreover, since $v_\epsilon(x)$ is bounded in Ω_ϵ, by Theorem 4.10 it follows that $u_\epsilon \leq v_\epsilon$ in Ω_ϵ and therefore the upper bound for u_ϵ is proved.

About the lower bound of u_ϵ, consider now the Dirichlet problem

$$\begin{cases} \mathcal{G}[v_\epsilon](x) = 0, & x \in \Omega; \\ v_\epsilon(x) = -K, & x \in \Gamma_\epsilon, \end{cases} \tag{4.16}$$

a subsolution of which is $v_\epsilon(x) = \operatorname{dist}(x, \partial\Omega_\epsilon) - K - \epsilon$. In order to prove this, note first that the same computations which prove that v_ϵ is a supersolution of (4.14) apply here. That is, for all $x \in \Omega$ it holds

$$v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \epsilon = 0 \qquad \text{and} \qquad \left(v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \left(v_\epsilon(x) - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right) \geq 0. \tag{4.17}$$

Therefore, $v_\epsilon = \operatorname{dist}(x, \partial\Omega_\epsilon) - K - \epsilon$ is a solution to $\mathcal{G}[v_\epsilon] = 0$ in Ω, since $\mathcal{G}[v_\epsilon]$ is the minimum of the two quantities in (4.17). When $x \in \Gamma_\epsilon$, we have that $\operatorname{dist}(x, \partial\Omega_\epsilon) - K - \epsilon \leq -K$ since $\operatorname{dist}(x, \partial\Omega_\epsilon) \leq \epsilon$, so that $v_\epsilon \leq -K$ in Γ_ϵ.

Hence, $v_\epsilon = \operatorname{dist}(x, \partial\Omega_\epsilon) - K - \epsilon$ is a subsolution to the Dirichlet problem (4.16) and consequently to (4.12). On the other hand, since u_ϵ is a solution to (4.12) by hypothesis, it is in particular a supersolution and bounded from above due to the first part of the proof. Then, by Theorem 4.10 it follows that $v_\epsilon \leq u_\epsilon$ in Ω_ϵ and the proof is finished. $\square$

As a consequence, we get the following uniform bound.

Remark 4.12. *For a given value of $\epsilon > 0$ fixed, it follows from Proposition 4.11 that, for all $x \in \Omega \cup \Gamma_\epsilon$, u_ϵ is uniformly bounded with respect to ϵ, provided ϵ is small enough. That is, taking $\epsilon = 1$ for instance and in view of (4.13), it holds for all $x \in \Omega \cup \Gamma_\epsilon$ that*

$$\operatorname{dist}(x, \Omega_1) - \|F\|_{\infty,\Gamma_1} - 1 \leq u_\epsilon(x) \leq \operatorname{dist}(x, \Omega_1) + \|F\|_{\infty,\Gamma_1}.$$

The following lemma, which is a simpler version (a discrete case) of Lemma 6.10, is necessary for the proof of Theorem 6.3.

Lemma 4.13. *Let $v_\epsilon : \Omega \cup \Gamma_\epsilon \to \mathbb{R}$ be a supersolution to $\mathcal{G}[v_\epsilon] = 0$, bounded from above in $\Omega \cup \Gamma_\epsilon$. Then, for every $\gamma > 0$ there exists a function $\tilde{v}_\epsilon : \Omega \cup \Gamma_\epsilon \to \mathbb{R}$ which is a supersolution to the equation $\mathcal{G}[\tilde{v}_\epsilon] = \mu$ for some constant $\mu = \mu(\gamma, v_\epsilon) > 0$. Moreover, $\tilde{v}_\epsilon - v_\epsilon \leq \gamma$ in Ω and $\tilde{v}_\epsilon - v_\epsilon \geq -\gamma$ in Γ_ϵ.*

Proof. We look for $\tilde{v}_\epsilon$ of the form $\tilde{v}_\epsilon = g(v_\epsilon)$, where

$$g(\alpha) = (1+\varepsilon)\,\alpha - \frac{\varepsilon}{4C}\,\alpha^2$$

for $\varepsilon > 0$ and C a positive constant such that $\sup_{\Omega\cup\Gamma_\epsilon} v_\epsilon \leq C$ (v_ϵ is bounded from above by hypothesis). The constant γ in the statement of the lemma will be chosen later as a function of ε.

According to this,

$$\begin{aligned}
\tilde{v}_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} \tilde{v}_\epsilon - \epsilon &= g(v_\epsilon) - g\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \epsilon \\
&= (1+\varepsilon)\,v_\epsilon - \frac{\varepsilon}{4C}\,v_\epsilon^2 - (1+\varepsilon)\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \frac{\varepsilon}{4C}\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 - \epsilon \\
&= (1+\varepsilon)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \frac{\varepsilon}{4C}\left(v_\epsilon^2 - \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2\right) - \epsilon \\
&= (1+\varepsilon)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \frac{\varepsilon}{4C}\left(v_\epsilon + \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \epsilon \\
&= \left((1+\varepsilon) - \frac{\varepsilon}{4C}\left(v_\epsilon + \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)\right)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \epsilon
\end{aligned}$$

$$
\begin{aligned}
&\geq \left((1+\varepsilon) - \frac{\varepsilon}{4C}\,2C\right)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \epsilon\\
&= \left(1+\frac{\varepsilon}{2}\right)\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) - \epsilon\\
&\geq \left(1+\frac{\varepsilon}{2}\right)\epsilon - \epsilon\\
&= \frac{\varepsilon}{2}\,\epsilon,
\end{aligned}
$$

where we used the fact that $v_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \epsilon \geq 0$, since $\mathcal{G}[v_\epsilon] \geq 0$ in Ω.

On the other hand,

$$
\begin{aligned}
\left(\tilde v_\epsilon - \inf_{\overline{B}_\epsilon(x)} \tilde v_\epsilon\right) + \left(\tilde v_\epsilon - \sup_{\overline{B}_\epsilon(x)} \tilde v_\epsilon\right) &= \left(g(v_\epsilon) - g\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)\right) + \left(g(v_\epsilon) - g\left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)\right)\\
&= (1+\varepsilon)\left(\left(v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \left(v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)\right)\\
&\qquad - \frac{\varepsilon}{4C}\left(v_\epsilon^2 - \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2\right) - \frac{\varepsilon}{4C}\left(v_\epsilon^2 - \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2\right)\\
&= (1+\varepsilon)\left(2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \frac{\varepsilon}{4C}\left(-2v_\epsilon^2 + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
-2v_\epsilon^2 + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 &= -2v_\epsilon^2 + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2\\
&\qquad \pm \left(2v_\epsilon^2 + 2v_\epsilon \inf_{\overline{B}_\epsilon(x)} v_\epsilon + 2v_\epsilon \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)\\
&= -2v_\epsilon^2 + \left(\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 - 2v_\epsilon \inf_{\overline{B}_\epsilon(x)} v_\epsilon + v_\epsilon^2\right) + \left(\left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)^2 - 2v_\epsilon \sup_{\overline{B}_\epsilon(x)} v_\epsilon + v_\epsilon^2\right)\\
&\qquad + 2v_\epsilon \inf_{\overline{B}_\epsilon(x)} v_\epsilon + 2v_\epsilon \sup_{\overline{B}_\epsilon(x)} v_\epsilon - 2v_\epsilon^2\\
&= -4v_\epsilon^2 + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2 + 2v_\epsilon\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon + \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)\\
&= -2v_\epsilon\left(2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2.
\end{aligned}
$$

From these, it follows that

$$
\begin{aligned}
\left(\tilde v_\epsilon - \inf_{\overline{B}_\epsilon(x)} \tilde v_\epsilon\right) + \left(\tilde v_\epsilon - \sup_{\overline{B}_\epsilon(x)} \tilde v_\epsilon\right) &= (1+\varepsilon)\left(2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right)\\
&\qquad + \frac{\varepsilon}{4C}\left(-2v_\epsilon\left(2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2\right)\\
&= \left(1+\varepsilon - \frac{\varepsilon\, v_\epsilon}{2C}\right)\left(2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon\right) + \frac{\varepsilon}{4C}\left(\left(\inf_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2 + \left(\sup_{\overline{B}_\epsilon(x)} v_\epsilon - v_\epsilon\right)^2\right)
\end{aligned}
$$

$$\geq \frac{\varepsilon}{4C}\,\epsilon^2,$$

where in the last inequality we have used that $2v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \sup_{\overline{B}_\epsilon(x)} v_\epsilon \geq 0$ and $v_\epsilon - \inf_{\overline{B}_\epsilon(x)} v_\epsilon - \epsilon \geq 0$, since $\mathcal{G}[v_\epsilon] \geq 0$ in Ω by hypothesis.

Then, according to the previous computations we get that

$$\begin{aligned}\min\left\{\left(\tilde{v}_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} \tilde{v}_\epsilon\right) - \epsilon,\, \tilde{v}_\epsilon(x) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x)} \tilde{v}_\epsilon + \inf_{\overline{B}_\epsilon(x)} \tilde{v}_\epsilon\right)\right\} &\geq \min\left\{\frac{\varepsilon}{2}\,\epsilon, \frac{\varepsilon}{8C}\,\epsilon^2\right\} \\ &= \frac{\varepsilon}{2}\,\epsilon\,\min\left\{1, \frac{\epsilon}{4C}\right\} = \mu\end{aligned}$$

for all $x \in \Omega$, where $\mu = \mu(\varepsilon, v_\epsilon) > 0$.

About the second part of the lemma, since $g(\alpha) - \alpha \leq \frac{3}{4}\,\varepsilon C$ for $\alpha \leq C$ and because $\tilde{v}_\epsilon = g(v_\epsilon)$, it follows that $\tilde{v}_\epsilon - v_\epsilon \leq \frac{3}{4}\,\varepsilon C$ in Ω. Similarly, since $g(\alpha) - \alpha \geq -\varepsilon\, D\left(1 + \frac{D}{4C}\right)$ for $\alpha \geq -D = -\left|\min_{\Gamma_\epsilon} v_\epsilon\right|$, it follows that $\tilde{v}_\epsilon - v_\epsilon \geq -\varepsilon\, D\left(1 + \frac{D}{4C}\right)$ in Γ_ϵ. The results hold for every

$$\gamma = \varepsilon \max\left\{\frac{3}{4}\,C,\, D\left(1 + \frac{D}{4C}\right)\right\} > 0,$$

provided ε is small enough. □

We proceed now with the proof of the discrete comparison principle, Theorem 4.10, which is inspired by the proof of Theorem 6.3 in the continuous case.

Proof of Theorem 4.10. Arguing by contradiction, we suppose that $\sup_{\Omega\cup\Gamma_\epsilon}(u_\epsilon - v_\epsilon) > 0$. Since $\Omega \cup \Gamma_\epsilon$ is compact and both u_ϵ and $-v_\epsilon$ are bounded from above by hypothesis, the supremum is attained (this is just the extension to semicontinuous functions of the extreme value theorem). And because $u_\epsilon \leq v_\epsilon$ in Γ_ϵ, it follows that there is a point $\hat{x} \in \Omega$ such that $(u_\epsilon - v_\epsilon)(\hat{x}) = \max_{\Omega\cup\Gamma_\epsilon}(u_\epsilon - v_\epsilon) > 0$. On the other hand, by Lemma 4.13, for every $\gamma > 0$ there exists a function $\tilde{v}_\epsilon$ such that $\tilde{v}_\epsilon - v_\epsilon \leq \gamma$ in Ω. As a result, $u_\epsilon(\hat{x}) - v_\epsilon(\hat{x}) > \gamma \geq \tilde{v}_\epsilon(\hat{x}) - v_\epsilon(\hat{x})$ for γ small enough and therefore $u_\epsilon(\hat{x}) > \tilde{v}_\epsilon(\hat{x})$.

This implies that there is a point $\tilde{x} \in \Omega \cup \Gamma_\epsilon$ such that $u_\epsilon(\tilde{x}) - \tilde{v}_\epsilon(\tilde{x}) = \sup_{\Omega\cup\Gamma_\epsilon}(u_\epsilon - \tilde{v}_\epsilon) > 0$. Note that the supremum is attained since $-\tilde{v}_\epsilon$ is bounded from above by construction. In fact, $\tilde{x} \in \Omega$ since by Lemma 4.13, we can assume $\tilde{v}_\epsilon - v_\epsilon \geq -\gamma$ in Γ_ϵ and therefore $u_\epsilon(\tilde{x}) - \tilde{v}_\epsilon(\tilde{x}) > \gamma \geq v_\epsilon(x) - \tilde{v}_\epsilon(x) \geq u_\epsilon(x) - \tilde{v}_\epsilon(x)$ for all $x \in \Gamma_\epsilon$. For the sake of simplicity let us assume this point $\tilde{x}$ to be $\hat{x}$.

It follows by definition that

$$u_\epsilon(\hat{x}) - \tilde{v}_\epsilon(\hat{x}) \geq u_\epsilon(x) - \tilde{v}_\epsilon(x) \quad \text{for all} \;\; x \in \Omega,$$

which is equivalent to say $u_\epsilon(\hat{x}) - u_\epsilon(x) \geq \tilde{v}_\epsilon(\hat{x}) - \tilde{v}_\epsilon(x)$ for all $x \in \Omega$. According to this and writing (4.10) as

$$\min\left\{\sup_{y\in\overline{B}_\epsilon(\hat{x})}(u_\epsilon(\hat{x}) - u_\epsilon(y)) - \epsilon,\; \sup_{y\in\overline{B}_\epsilon(\hat{x})}(u_\epsilon(\hat{x}) - u_\epsilon(y)) + \inf_{y\in\overline{B}_\epsilon(\hat{x})}(u_\epsilon(\hat{x}) - u_\epsilon(y))\right\} = 0, \tag{4.18}$$

we have that $\mathcal{G}[\tilde{v}_\epsilon](\hat{x}) \leq \mathcal{G}[u_\epsilon](\hat{x})$, where $\mathcal{G}$ is given by (4.11). The combination of this inequality with the fact that by Lemma 4.13 one gets $\mathcal{G}[u_\epsilon] < \mathcal{G}[\tilde{v}_\epsilon]$ in Ω, yields a contradiction. □

Remark 4.14. *Expression (4.18) displays the DPP (4.10) as a discrete elliptic, and in particular monotone, scheme in the language of [21] and [3] respectively.* Discrete ellipticity *in our case* *amounts to showing (see [21, Definition 2]) that, for a given $x \in \Omega$, (4.18) is a non-decreasing fucntion of $u_\epsilon(x)$ and the differences*

$$u_\epsilon(x) - u_\epsilon(y) \quad \text{whenever} \quad y \neq x,$$

which is satisfied by (4.18). On the other hand, monotonicity requires (4.18) to be non-increasing as a function of $u_\epsilon(y)$, $y \neq x$, for $u_\epsilon(x)$ fixed (see [3, Section 2]), which is also satisfied by (4.18). Let us recall that discrete ellipticity implies monotonicity, as pointed out in [21].

A result that follows directly from Theorem 4.10 is the following maximum principle.

Corollary 4.15. *Under the same assumptions of Theorem 4.10, it holds*

$$\sup_{x \in \Omega \cup \Gamma_\epsilon} (u_\epsilon(x) - v_\epsilon(x)) = \sup_{x \in \Gamma_\epsilon} (u_\epsilon(x) - v_\epsilon(x)).$$

Proof. Let K be a constant such that $u_\epsilon - K \leq v_\epsilon$ on Γ_ϵ. Since $\mathcal{G}[u_\epsilon - K] = \mathcal{G}[u_\epsilon] \leq 0$ in Ω, $u_\epsilon - K$ is a subsolution of equation (4.10) and bounded from above in $\Omega \cup \Gamma_\epsilon$, as it is u_ϵ. By Theorem 4.10 we have that in $\Omega \cup \Gamma_\epsilon$, $u_\epsilon - K \leq v_\epsilon$, i.e., $u_\epsilon - v_\epsilon \leq K$. Hence, the result follows by taking $K = \sup_{\Gamma_\epsilon}(u_\epsilon - v_\epsilon)$, which is finite since Γ_ϵ is compact and u_ϵ and $-v_\epsilon$ are bounded from above in $\Omega \cup \Gamma_\epsilon$ by hypothesis (this is just the extension to semicontinuous functions of the extreme value theorem). □

4.3.2 Existence and uniqueness of the value of the game

In a similar way to classical random Tug-of-War (see Chapter 3), we say that an ϵ-Totalitarian Tug-of-War game (see Chapter 4) has a value associated to the point $x \in \Omega \cup \Gamma_\epsilon$, denoted as $u_\epsilon(x)$, provided that $u_\epsilon(x) := u_\epsilon^I(x) = u_\epsilon^{II}(x)$, for $u_\epsilon^I(x), u_\epsilon^{II}(x)$ the respective expected values of the game for Players I and II at x. Therefore, in order to prove that an ϵ-Tug-of-War game has a value, i.e., $u_\epsilon^I(x) = u_\epsilon^{II}(x)$ for all $x \in \Omega \cup \Gamma_\epsilon$, it will be enough to show that both $u_\epsilon^I \leq u_\epsilon^{II}$ and $u_\epsilon^{II} \leq u_\epsilon^I$ hold in $\Omega \cup \Gamma_\epsilon$.

Remark 4.16. *If the ϵ-Totalitarian Tug-of-War ends almost surely, then the value of the game for Player I, (4.3), and II, (4.4), is finite. This is so in view of (4.1) and (4.2), and the fact that for every game that ends almost surely, the number of coin tosses is finite and so it is $\mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau + k_\tau\,\epsilon]$. Hence, u_ϵ^I and u_ϵ^{II} are both bounded in all the game domain. Note also that, since we are assuming that $F(x)$ is bounded for all $x \in \Gamma_\epsilon$, it is derived that if the ϵ-Totalitarian Tug-of-War has a finite value, the game ends almost surely. This is clear since, according to (4.1) and (4.2), the only possibility for the game to have an infinite value is that it does not end.*

At this point, the following result provides us with one of the two inequalities necessary to guarantee the existence (and uniqueness) of a value for the ϵ-Totalitarian Tug-of-War.

Proposition 4.17. *It holds that $u_\epsilon^I(x) \leq u_\epsilon^{II}(x)$ for all $x \in \Omega \cup \Gamma_\epsilon$.*

Proof. Note first that, by definition,

$$V^x_{S_I,S_{II}}(I) \leq V^x_{S_I,S_{II}}(II) \quad \text{for all} \quad x \in \Omega \cup \Gamma_\epsilon.$$

According to this, it holds

$$\inf_{S_{II}} V^x_{S_I,S_{II}}(I) \leq \sup_{S_I} V^x_{S_I,S_{II}}(II) \quad \text{for all} \quad x \in \Omega \cup \Gamma_\epsilon.$$

Then, it follows

$$\sup_{S_I} \inf_{S_{II}} V^x_{S_I,S_{II}}(I) \leq \sup_{S_I} V^x_{S_I,S_{II}}(II) \quad \text{for all} \quad x \in \Omega \cup \Gamma_\epsilon,$$

and since the left-hand side holds for all S_{II}, one finally gets

$$u_\epsilon^I(x) = \sup_{S_I}\inf_{S_{II}} V^x_{S_I,S_{II}}(I) \leq \inf_{S_{II}}\sup_{S_I} V^x_{S_I,S_{II}}(II) = u_\epsilon^{II}(x) \quad \text{for all} \quad x \in \Omega\cup\Gamma_\epsilon.$$

□

It remains to prove that $u_\epsilon^{II} \leq u_\epsilon^I$ holds in $\Omega\cup\Gamma_\epsilon$ for the ϵ-Totalitarian Tug-of-War (see Theorem 4.18). Then, we will be able to conclude the existence and uniqueness of the game value for the ϵ-Totalitarian Tug-of-War.

Theorem 4.18. *The ϵ-Totalitarian Tug-of-War game has a value, which is unique.*

Proof of Theorem 4.18. Let $u_\epsilon^I(x), u_\epsilon^{II}(x)$ be the respective values of the ϵ-Totalitarian Tug-of-War for Players I and II at $x \in \Omega\cup\Gamma_\epsilon$. By Proposition 4.9 both u_ϵ^I and u_ϵ^{II} are solutions to the Dirichlet problem (4.12) and hence, according to Proposition 4.11 they are bounded in $\Omega\cup\Gamma_\epsilon$ (note that the boundedness follows alternatively by Remark 4.16). In particular, they are respectively a supersolution and subsolution of (4.10) in Ω and $u_\epsilon^{II} \leq u_\epsilon^I$ in Γ_ϵ. Thus, we can apply Theorem 4.10 for $u_\epsilon = u_\epsilon^{II}$ and $v_\epsilon = u_\epsilon^I$, so that $u_\epsilon^{II} \leq u_\epsilon^I$ in $\Omega\cup\Gamma_\epsilon$. On the other hand, by Proposition 4.17, $u_\epsilon^I \leq u_\epsilon^{II}$ in $\Omega\cup\Gamma_\epsilon$. It then follows that $u_\epsilon^{II} = u_\epsilon^I$ in $\Omega\cup\Gamma_\epsilon$ so that the ϵ-Totalitarian Tug-of-War game has a unique value. □

Chapter 5

Examples of Totalitarian Tug-of-War on graphs

This chapter is devoted to four examples of the Totalitarian Tug-of-War, introduced in Chapter 4, played on graphs. More precisely, we describe in detail the game for a graph in a segment with two running and terminal nodes. Adding an extra terminal node to this graph, one gets the configuration (Y-shape) of the graph for a different example. Finally, the generalization to n running nodes of the two previous cases are the remaining examples.

5.1 Game on a segment with two running nodes

5.1.1 Configuration of the game

Consider a random Tug-of-War game, the positions of which are represented in Figure 5.1, played in the following way. If the game token is at a running node, either x_1 or x_2, Player I can either play a round of classical random Tug-of-War game (see Section 3.1) with Player II in that game position or let Player II move the game token to any adjacent game position. In the second case, Player I will receive in return a payment of value ϵ from Player II. The game proceeds according to these instructions until a terminal node is reached, namely, x_0 or x_3.

When the game token reaches a terminal node the game ends and Player I receives from Player II a final payoff. This value depends on each particular terminal node x_i, so we will denote it as F_i. The final payoff is independent of the ϵ-payments that took place throughout the game and as it occurs with the particular value of ϵ, all F_i are fixed and known to both players beforehand. Just to mention that due to the symmetry of the game positions configuration (see Figure 5.1) we can assume without loss of generality that $F_0 \leq F_3$.

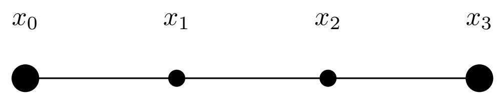

Figure 5.1: Game positions of the game on a segment with two running nodes.

	$S_I^*(x_1)$	$S_I^{x_2}(x_1)$
$S_{II}^{x_0}(x_1)$	A	B

	$S_I^*(x_2)$	$S_I^{x_3}(x_2)$	$S_I^{x_1}(x_2)$
$S_{II}^{x_1}(x_2)$	1	2	3
$S_{II}^{x_3}(x_2)$	4	5	6

Table 5.1: Table of all possible pairs of strategies for the game on a segment with two running nodes.

5.1.2 Game strategies

According to the description of the game, both players can choose between two possible actions at each running node. However, Player II should always move the game token from x_1 to the terminal node x_0, whenever possible, since doing otherwise is against his/her interest (this behavior is reasonable since we are assuming that both players are playing optimally). Taking this into account, Table 5.1 shows all possible strategies of the game, labeling with a letter A or B and a number from 1 to 6 all possible combinations of actions of both players for all running nodes (a letter when the game token is in the running node x_1 and a number when the game token is in the running node x_2).

These actions are indicated as $S_\alpha^{x_p}(x_i)$, where $\alpha \in \{I, II\}$ indicates the player who is taking that action, x_p $(p = 0, 1, 2, 3)$ is the node that player α wants to be the new game token position, $*$ means that Player I lets Player II choose the new game token position, and x_i $(i = 1, 2)$ stands for the current position of the game token.

5.1.3 Value of the game

Once the action of both players in all running nodes is established, we are able to compute the expected value of the game associated to this particular pair of strategies for all game positions. We denote the expected value of the game at node x_i as u_i (recall that the expected value of the game at the terminal nodes x_0 and x_3 is respectively $u_0 = F_0$ and $u_3 = F_3$).

The computation of the expected value of the game u_i at the running nodes x_i $(i = 1, 2)$ is

$$u_i = u_q + \epsilon \quad \text{for } S_I^*(x_i) \text{ and } S_{II}^{x_q}(x_i) \tag{5.1}$$

and

$$u_i = \frac{1}{2}(u_p + u_q) \quad \text{for } S_I^{x_p}(x_i) \text{ and } S_{II}^{x_q}(x_i). \tag{5.2}$$

Proceeding in the way detailed in (5.1) and (5.2) with all pairs of strategies that derive from Table 5.1, we end up with the following expected values of the game u_1 and u_2, for each given pair of strategies:

$$A1: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + 2\epsilon \end{cases} \qquad A2: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = \frac{1}{2}(F_0 + F_3 + \epsilon) \end{cases} \qquad A3: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + \epsilon \end{cases}$$

$$A4: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_3 + \epsilon \end{cases} \qquad A5: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_3 \end{cases} \qquad A6: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = \frac{1}{2}(F_0 + F_3 + \epsilon) \end{cases}$$

$$B1: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + 2\epsilon \end{cases} \qquad B2: \begin{cases} u_1 = \frac{1}{3}(2F_0 + F_3) \\ u_2 = \frac{1}{3}(F_0 + 2F_3) \end{cases} \qquad B3: \begin{cases} u_1 = F_0 \\ u_2 = F_0 \end{cases} \tag{5.3}$$

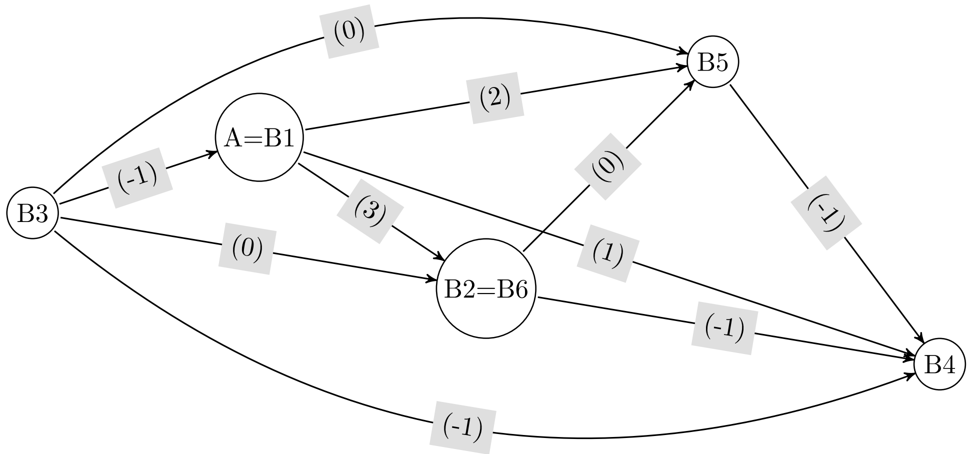

Figure 5.2: Directed graph where the nodes are the pairs of strategies described in Table 5.1 and computed in (5.3), and edges point towards the largest expected value of the game at the x_1-node. The letter A stands for all possible combinations Ax, $x = 1, \ldots, 6$.

$$B4 : \begin{cases} u_1 = \frac{1}{2}(F_0 + F_3 + \epsilon) \\ u_2 = F_3 + \epsilon \end{cases} \qquad B5 : \begin{cases} u_1 = \frac{1}{2}(F_0 + F_3) \\ u_2 = F_3 \end{cases} \qquad B6 : \begin{cases} u_1 = \frac{1}{3}(2F_0 + F_3) \\ u_2 = \frac{1}{3}(F_0 + 2F_3) \end{cases}$$

Through the comparison of the values u_i among all pairs of strategies, we are able to construct, for each fixed i, a directed graph where the nodes are the pairs of strategies and the edges relate each pair of strategies with different associated value u_i (pairs of strategies with the same associated value u_i are represented by the same node), see Figure 5.2 and Figure 5.3. Each edge points towards the node with largest value u_i between the two nodes that the edge relates.

Note that, when trying to establish an order among all pairs of strategies for the value u_i, some conditions appear. In our particular example, Figure 5.2 and Figure 5.3 show the directed graph comparing all possible pairs of strategies associated to the values u_1 and u_2, respectively. On each edge it is indicated the condition which must hold so that the edge points towards the pair of strategies with largest expected value of the game, u_1 in Figure 5.2 and u_2 in Figure 5.3. We summarize all the conditions appearing at some edge, in either one or both graphs, as follows:

$$\begin{cases} (-1): & \epsilon > 0, \\ (0): & F_0 < F_3, \\ (1): & F_0 + \epsilon < F_3, \\ (1.5): & F_0 + 1.5\epsilon < F_3, \\ (2): & F_0 + 2\epsilon < F_3, \\ (3): & F_0 + 3\epsilon < F_3. \end{cases} \tag{5.4}$$

In case some condition in (5.4) does not hold, this means that the reverse inequality is satisfied and therefore the edges associated to such condition in Figures 5.2 and 5.3 have to be reversed.

Now we are ready to compute the value of the game for each player at each running node. According to (4.1) and under the assumption that the game ends almost surely (see Remark 4.6 for an extended comment), the value of the game for Player I in our case reads as

$$u_i^I = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_i} [F_\tau + k_\tau\, \epsilon], \tag{5.5}$$

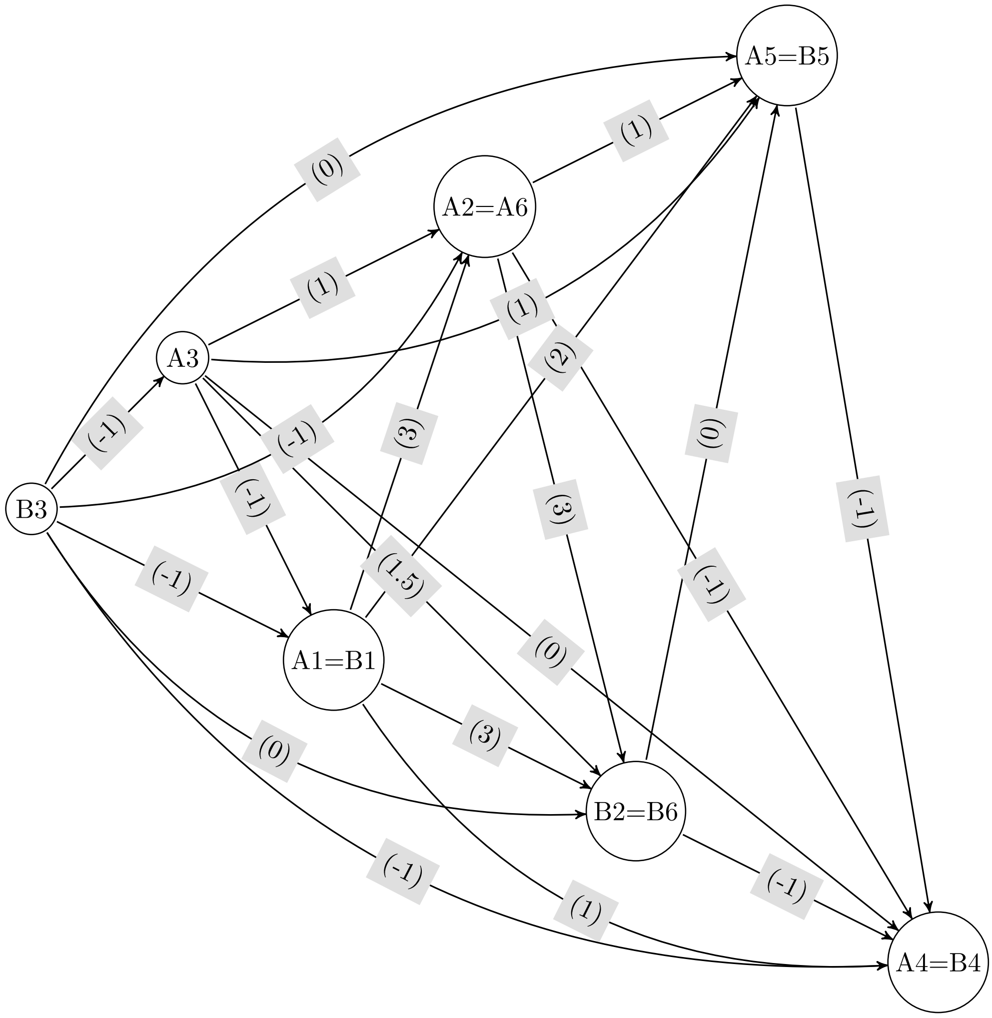

Figure 5.3: Directed graph where the nodes are the pairs of strategies described in Table 5.1 and computed in (5.3), and edges point towards the largest expected value of the game at the x_2-node.

Similarly, the value of the game for Player II is written here as

$$u_i^{II} = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{x_i}_{S_I,S_{II}}[F_\tau + k_\tau\,\epsilon]. \tag{5.6}$$

As we already explained in detail in Section 4.1, the game starting at the node x_i will have a value provided $u_i^I = u_i^{II}$, which holds according to Theorem 4.8. Here, we are going to compute this value explicitly in the case of the current game.

Let us assume, for instance, that all conditions in (5.4) hold, so that the orientations of the edges in the directed graphs displayed in Figures 5.2 and 5.3 are correct. We are going to show that the pairs of strategies B2 and B6 in Table 5.1, which have associated the same value of the game u_1 in x_1, and u_2 in x_2, are the only pair of strategies which fulfill the expressions (5.5) and (5.6), that is, u_i^I and u_i^{II} for $i = 1, 2$.

In order to conclude that B2 or B6 are the pairs of strategies S_I, S_{II} that the players should follow, one can reason in the following way. According to the expression (5.5) for u_i^I, Player II chooses his/her strategy first, taking into account that Player I will choose, afterwards, the best among all compatible strategy. At node x_1, Player II has just one possible action according to Table 5.1, move to x_0 (denoted $S_{II}^{x_0}(x_1)$ in the strategies notation, commented above). At game position x_2, Player II will decide to move to x_1 ($S_{II}^{x_1}(x_2)$ in the strategies notation).

Then, the compatible strategies available for Player I to choose yield either A or B and 1 or 2 or 3. Note that the expected value of the game associated to these choices is smaller than the one associated to A or B and 4 or 5 or 6, but the latter are not compatible with Player II's choice, according to Figures 5.2 and 5.3. Among the compatible strategies, $S_I^{x_2}(x_1)$ and $S_I^{x_3}(x_2)$ is the strategy that maximizes u_i^I for $i = 1, 2$ and therefore the strategy to be chosen by Player I. Hence, we conclude that B2 is the pair of strategies associated to u_i^I for $i = 1, 2$.

Remark 5.1. *The pairs of strategies associated to u_i^I for $i = 1, 2$ is B2, but B6 brings associated the same expected value u_i^I, so that formally it is also a valid choice. However, looking at the behavior of the players if they follow B6, one realizes that this pair of strategies is a classical random Tug-of-War game where players interchange their roles. That is, with this pair of strategies, both players make choices that are opposite to their original goals (while in x_1 Player I wants to move towards x_3 and Player II wants to move towards x_0, in x_2 Player I wants to move towards x_0 and Player II wants to move towards x_3).*

This is an unstable equilibrium, since both players would be able to get a higher expected payoff by changing their own strategy (this situation is similar to the Prisoner's Dilemma*, fully analyzed in [24]). This argument leads us to say that B2 is the only optimal strategy associated to u_i^I for $i = 1, 2$.*

Remark 5.2. *Although u_i^I was defined taking into account the starting running node x_i of the game, it turns out that the pair of strategies associated to the value of the game is independent of the starting game position. The key point of this fact is that the inequalities in (5.4) do not change during the game, as we describe now.*

Suppose that, at the start, Player I lets Player II move the game token, so that Player I receives a payment of amount ϵ. Then, the second turn of the game can be interpreted as a new game which starts in the current position of the game token and the terminal nodes x_0 and x_3 now have associated an expected value of $F_0 + \epsilon$ and $F_3 + \epsilon$, respectively. The "update" of F_0 and F_3 by the respective new terminal payoffs $F_0 + \epsilon$ and $F_3 + \epsilon$ does not modify any of the inequalities in (5.4).

Consider then the value u_i^I of a game starting at the running node x_i, with terminal payoffs F_0 and F_3 and following the optimal pair of strategies S_I, S_{II}, such that after n turns the game token is placed at the running node x_j. From the previous argument, it is derived that this u_i^I is the same for a game which follows the same pair of strategies S_I, S_{II} but starts at the running node x_j and has terminal payoffs $F_0 + k_n\,\epsilon$ and $F_3 + k_n\,\epsilon$, for some $k_n \in \mathbb{N}$ that takes into account the update in the payoffs due to the n turns already played.

Due to this equivalence, if the strategies associated to u_j^I were different from the ones in u_i^I, this would mean that the strategies associated to u_i^I were not optimal, which contradicts the choice of strategies for u_i^I. This argument is known as Bellman's Principle of Optimality *(see Theorem 3.2).*

We proceed analogously in the computation of u_i^{II} for $i = 1, 2$, and obtain that B2 is the associated pair of strategies whatever is the starting node of the game. Then, since u_i^I and u_i^{II} have both the same associated pair of strategies for each i (the pair of strategies B2 in this case), we get that $u_i^I = u_i^{II}$ for $i = 1, 2$ and therefore the game has a value, already computed in (5.3).

Remark 5.3. *The solution corresponding to B2, i.e., $u_1 = \frac{1}{3}(2F_0 + F_3)$ and $u_2 = \frac{1}{3}(F_0 + 2F_3)$ according to (5.3), is the only one that satisfies the DPP (4.7) associated to the game. This is in agreement with the uniqueness result in Theorem 4.8.*

Assume now that all inequalities in (5.4) hold but (3). We proceed in the same way we did above, when computing the value of the game provided all inequalities in (5.4) held. As before, the game has a value, now for the associated pairs of strategies A1 and B1. However, this case differs from the previous one on the fact that it is not enough to just look at Figure 5.2 to decide the pairs of strategies associated to the value of the game in all running nodes (recall that the pairs of strategies does not depend on the starting position of the game, as we already justified in Remark 5.2).

More precisely, when all inequalities in (5.4) hold, the node B2=B6 has associated the largest expected value of the game both in Figure 5.2 and 5.3, between all compatible pairs of strategies for the players. But since inequality (3) of (5.4) now does not hold, some of the edges in the two graphs are reversed. Because of that, node A=B1 is now the node with largest associated expected value of the game, among all compatible pairs of strategies. On the other hand, node A1=B1 is the node with largest associated expected value of the game in Figure 5.3, after making all changes required in the edges and among all compatible pairs of strategies for the players.

From these, it follows that the pairs of strategies Ax, for $x = 2, \ldots, 6$, are associated to $u_1^I = u_1^{II}$ but not to the value of the game, since $u_i^I = u_i^{II}$ does not hold for all $i = 1, 2$ (it fails for $i = 2$) for any of these pairs of strategies. However, the pairs of strategies A1=B1 are associated to both $u_1^I = u_1^{II}$ (in view of Figure 5.2) and $u_2^I = u_2^{II}$ (in view of Figure 5.3), so that the game has a value associated to the pairs of strategies A1=B1.

Remark 5.4. *The case we have just described is a clear example on how important it is to consider the graph of strategies of all running nodes when trying to determine the pair (or pairs) of strategies associated to the value of the game. This fact motivates the question of whether it could be possible that the intersection set of pairs of strategies with largest expected value of the game associated to each running node x_i, is empty. The answer is negative due to the existence and uniqueness of the value of the game, which was already treated in Section 4.2.2.*

Therefore, the game will always have a value, since the existence (and uniqueness) of a common pair of strategies with largest expected value of the game for all running game nodes, is guaranteed by Theorem 4.8. And because of the existence of the value of the game (and therefore the existence of an associated unique pair of strategies), it is sufficient to look at as many different graphs of strategies of running nodes as necessary until we end up with a unique pair of strategies.

Just to conclude the discussion of this example, the pair of strategies associated to the value of the game are A1=B1 when the inequality (2) in (5.4) is not satisfied (and therefore inequality (3) is also violated), and the same result is obtained when inequality (1.5) does not hold neither. The pair of strategies is A4 when the inequality (1) is not fulfilled and also when the inequality (0) fails (hence just inequality (-1) holds in (5.4)). This discussion is summarized in Table 5.2.

Conditions satisfied	Pair of strategies
(3), (2), (1.5), (1), (0), (-1)	B2 (=B6)
~~(3)~~, (2), (1.5), (1), (0), (-1)	A1 (=B1)
~~(3)~~, ~~(2)~~, (1.5), (1), (0), (-1)	A1 (=B1)
~~(3)~~, ~~(2)~~, ~~(1.5)~~, (1), (0), (-1)	A1 (=B1)
~~(3)~~, ~~(2)~~, ~~(1.5)~~, ~~(1)~~, (0), (-1)	A4
~~(3)~~, ~~(2)~~, ~~(1.5)~~, ~~(1)~~, ~~(0)~~, (-1)	A4

Table 5.2: Pairs of strategies in Table 5.1 associated to the value of the game in terms of the conditions in (5.4) that are satisfied. The strategies in brackets are not optimal.

Figure 5.4: Game positions of the game on a segment with multiple running nodes.

Remark 5.5. *The failure of inequality (0) means that $F_0 \geq F_3$, but in practice it is sufficient to consider $F_0 = F_3$ since the case $F_0 > F_3$ leads to the symmetric situation to the one already analyzed.*

About the pairs of strategies which involve either action 3 or 5 in x_2, they are not in any case the ones associated to the value of the game. This is reasonable since we are assuming that the players are playing optimally, and these strategies are against the best interest for Player I. This is so since these pairs of strategies in x_2 imply that Player I chooses to play a round of classical random Tug-of-War game, in a situation where letting Player II move would yield the same game position plus an extra ϵ-payment, which Player I would not receive otherwise.

5.2 Game on a segment with multiple running nodes

After looking in detail into the game on a segment with two running nodes in the previous section, it is natural to consider the generalization of the game to the case with n running nodes (see Figure 5.4). As before, we assume $F_0 \leq F_{n+1}$.

Let us examine heuristically the choices available to both players to motivate the analysis below. On the one hand, at any given running node, Player I has three options: to move towards x_0, towards x_{n+1}, or let Player II decide. As in the previous section, moving towards x_0 is not reasonable, since $F_0 \leq F_{n+1}$ implies that the new game position chosen by Player I would actually be more favorable to Player II. Note that Player I would attain the same position and receive an ϵ-payoff by letting Player II decide next move, which will be towards x_0.

On the other hand, Player II has two options: to move towards x_0 or towards x_{n+1}. In principle, moving towards x_{n+1} (the terminal node with largest terminal payoff) seems against Player's II interest. However, there could be situations where it might be preferable for Player II to minimize the damage and end the the game right away paying F_{n+1}, in order to avoid ϵ-payments.

This heuristic reasoning yields three regions in the segment according to both player's choices, which can be described as follows in terms of two indices $j_1 < j_2$:

- at the nodes x_i, for $0 \leq i \leq j_1$, Player I allows Player II to move, who moves towards the

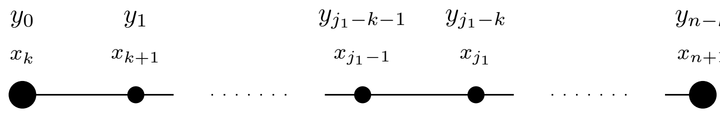

Figure 5.5: Game positions after relabeling the original nodes x_i as y_{i-k}, for $k \leq i \leq n+1$.

terminal node x_0;

- at the nodes x_i, for $j_1 + 1 \leq i \leq j_2 - 1$, Players I and II play a round of classical random Tug-of-War game, where the players want to move towards the terminal node x_{n+1} and x_0, respectively;
- at the nodes x_i, for $j_2 \leq i \leq n+1$, Player I allows Player II to move, who moves towards the terminal node x_{n+1}.

In fact, as we are going to see, the expected value of the game at each game position is completely determined by looking at the pairs of strategies S_I, S_{II} of the form just described. These pairs of strategies S_I, S_{II} are the only reasonable ones to be adopted by the players, since playing classical random Tug-of-War game with Players I and II wanting to move towards node x_0 and x_{n+1} respectively (that is, switching their roles), is not a reasonable strategy. Moreover, playing classical random Tug-of-War game when both players want to move towards the same terminal node, is not a reasonable situation either, in this case Player I will receive a better payoff by allowing Player II to move.

Next we present the auxiliary results Lemma 5.6 and 5.8, necessary in the analysis of this game, in terms of the indices j_1 and j_2 introduced above.

Lemma 5.6. *Consider $1 \leq j_1 \leq n$ and let S_I, S_{II} be any pair of strategies such that Player I allows Player II to move at node x_{j_1}, while Player II moves towards the terminal node x_0 at all game positions x_i for $1 \leq i \leq j_1$. Then, the expected value of the game at x_i for $0 \leq i \leq j_1$, is $u_i = F_0 + i\,\epsilon$.*

Proof. We argue by strong mathematical induction. Let P_i be the property stated in the lemma and assume it holds for $0 \leq i \leq j_1 - 1$. We aim to prove that P_{j_1} is also true. There are two different possible situations in terms of Player I's strategy at the game positions x_i for $1 \leq i \leq j_1 - 1$.

On the one hand, suppose that Player I allows Player II to move at node x_k for some $1 \leq k \leq j_1 - 1$. Then, since P_k is true by the strong induction hypothesis, the value of the game at the game positions x_i for $0 \leq i \leq k$, is $u_i = F_0 + i\,\epsilon$. According to this, we can relabel all game nodes (see Figure 5.5) so that the original node x_k is now a terminal node with associated expected payoff $F_k = F_0 + k\,\epsilon$.

The expected value of the game at node y_i for $0 \leq i \leq j_1 - k$, is $u_i = F_k + i\,\epsilon = F_0 + (k+i)\,\epsilon$, since P_{j_1-k} is true by hypothesis of the strong induction. In terms of the original game nodes, the computed value u_i is the expected value of the game at the node x_{i+k}, $u_i = F_0 + (k+i)\,\epsilon$, for $0 \leq i \leq j_1 - k$. In particular, when $i = j_1 - k$ we have that $u_{j_1} = F_0 + j_1\,\epsilon$, so that P_{j_1} is also true.

On the other hand, the other possible game situation is that Players I and II play a round of classical random Tug-of-War game with opposite terminal nodes as their respective objectives, in all the nodes x_i for $1 \leq i \leq j_1 - 1$. That is, we are assuming now that Player I does not allow Player II to move at any node x_i for $i < j_1$, on the contrary to the previous game situation considered. In this case we can compute directly the value u_{j_1}, which follows from the solution of the coupled linear system

$$\begin{cases} u_0 = F_0; \\ u_i = \frac{1}{2}(u_{i-1} + u_{i+1}), & \text{for } 1 \leq i \leq j_1 - 1; \\ u_{j_1} = u_{j_1 - 1} + \epsilon; \end{cases}$$

in view of (5.1) and (5.2). Thus, we obtain $u_i = F_0 + i\,\epsilon$ for $0 \leq i \leq j_1$, which gives us the result we wanted to prove. □

Remark 5.7. *Due to this lemma, provided that the strategy of Player I at the node x_{j_1} implies letting move Player II, it does not matter if the strategy in the game positions x_i for $1 \leq i \leq j_1 - 1$ involves, either the action of letting move Player II, or playing classical random Tug-of-War game trying to move towards the terminal node x_{n+1} (note that these two actions are the only reasonable ones for Player I). In other words, the expected value of the game at the node x_{j_1}, provided all the necessary hypothesis in Lemma 5.6 hold, is independent of the strategy of Player I at the game positions x_i, for $1 \leq i \leq j_1 - 1$.*

An analogous result to Lemma 5.6 is the following. Its proof follows analogously to the one for Lemma 5.6.

Lemma 5.8. *Consider $1 \leq j_2 \leq n$ and let S_I, S_{II} be any pair of strategies such that Player I allows Player II to move at node x_{j_2}, while Player II moves towards the terminal node x_{n+1} at all game positions x_i for $j_2 \leq i \leq n$. Then, the expected value of the game at the game position x_i for $j_2 \leq i \leq n+1$, is $u_i = F_{n+1} + (n+1-i)\,\epsilon$.*

Remark 5.9. *Just as it occurred with Lemma 5.6, the expected value of the game at the node x_{j_2}, provided all the necessary hypothesis in Lemma 5.8 hold, is independent of the strategy of Player I at the game positions x_i, for $j_2 + 1 \leq i \leq n$. Note that the only reasonable actions for Player I to choose at these nodes are, either let Player II move towards the terminal node x_{n+1}, or play a round of classical random Tug-of-War game trying to move towards the terminal node x_0.*

Remark 5.10. *Our last observation concerns the "coherence" of reasonable pairs of strategies S_I, S_{II}. That is, suppose that the player's strategy indicates at the running node x_i to move the game token towards a terminal node, say x_0, and the action at the node x_{i+1} is moving towards the opposite terminal node, x_{n+1}. Then, in all the game positions x_j for $1 \leq j \leq i$, the aim of the player is to attain the terminal node x_0, while in all the other running nodes x_j for $i+1 \leq j \leq n$, the player wants to arrive to the terminal node x_{n+1}. Note that a situation where the action to take at x_i is to move towards the terminal node x_0, while the action at x_{i-1} is to move towards the terminal node x_{n+1}, cannot be part of a reasonable strategy since both actions contradict each other.*

At this point, if we want to compute the expected value of the game for each pair of strategies S_I, S_{II}, it is enough to restrict ourselves to the reasonable strategies described above, which simplifies remarkably the number of computations. When $n = 3$, for instance, the number of pairs of strategies S_I, S_{II} to consider is reduced to 9 from a total of 72; when $n = 4$, just 13 strategies out of 432 have to be taken into account; etc. As one would expect, this reasoning also works in our previous example of the game on a segment with two running nodes: the only reasonable pairs of strategies S_I, S_{II} are A1, A2, A4, B2, B4, from the total of 12 appearing in Table 5.1.

With the previous considerations, the computation of the expected value of the game associated to each pair of strategies S_I, S_{II} and the existence of a game value, follow in the same way as in the game on a segment with two running nodes. Just to mention that the equivalent computation to (5.1) and (5.2) for the case of n running nodes, in terms of the indices j_1 of Lemma 5.6 and j_2 of Lemma 5.8, is

$$u_i = F_0 + i\,\epsilon, \qquad \text{for } 0 \leq i \leq j_1; \tag{5.7}$$

$$u_i = \frac{n+1-i}{n+1}(F_0 + j_1\,\epsilon) + \frac{i}{n+1}[F_{n+1} + (n+1-j_2)\,\epsilon], \qquad \text{for } j_1+1 \le i \le j_2-1; \tag{5.8}$$
$$u_i = F_{n+1} + (n+1-i)\,\epsilon, \qquad \text{for } j_2 \le i \le n+1. \tag{5.9}$$

The comparison of the expected values of the game we have just computed, gives us the relation

$$F_0 + \left(j_2 + 1 - \frac{n+1}{j_1+1}\right)\epsilon < F_{n+1} \tag{5.10}$$

when comparing (5.7) and (5.8), the relation

$$F_0 + \left(j_1 - (n+1) + \frac{(j_2-1)(n+1-j_2)}{n+2-j_2}\right)\epsilon < F_{n+1} \tag{5.11}$$

by comparing (5.8) and (5.9), and finally

$$F_{n+1} < F_0 + [2j_1 - (n+1)]\,\epsilon \tag{5.12}$$

when taking into account (5.7) and (5.9).

Concerning these inequalities, (5.10) holds while a round of classical random Tug-of-War game at the running node x_{j_1+1} (where Players I and II want to move towards the terminal node x_{n+1} and x_0, respectively) yields a larger expected value of the game at this node, than letting Player II move the game token (where both players aim towards the terminal node x_0). Similarly, condition (5.11) holds while a round of classical random Tug-of-War game at the running node x_{j_2-1} (where Players I and II want to move towards the terminal node x_{n+1} and x_0, respectively), has associated a smaller expected value of the game at this node than letting Player II move the game token towards the terminal node x_{n+1} (both players aim to attain the terminal node x_{n+1}). Finally, (5.12) is satisfied when the expected value of the game at the running node x_{j_1} is larger in the situation that both players aim to reach the terminal node x_{n+1}, than keep moving towards x_0 as their strategies indicate in all running nodes x_i for $1 \le i \le j_1 - 1$ (since both players always want to reach the same terminal node, Player I always allows Player II to move).

The relations (5.10), (5.11) and (5.12) are the equivalent version for the multiple running node case of (5.4) when $n = 2$. Thus, one has now all the necessary results to determine the pair of strategies S_I, S_{II} which brings associated the value of the game for any given initial data: n, F_0, F_{n+1} and $\epsilon > 0$. We insist on the fact that the value of the game exists (and is unique), according to Section 4.2.2.

5.3 Y-game with two running nodes

This game is a slightly modified version of the game on a segment with two running nodes. The only difference is that now there are not two terminal nodes but three, so that the game position x_2 is adjacent to three different nodes (Figure 5.6). We will assume $F_0 \le F_3 \le F_4$. Since the game is played following the same rules of the game on a segment, all possible pairs of strategies are the ones considered in Table 5.3. Note that $S_I^{x_3}(x_2)$ does not appear since it makes no sense because $S_I^{x_4}(x_2)$ is at least as good if not better.

The computation of the expected value of the game at each running node for each pair of strategies, is done in the same manner as in the game on a segment. The results obtained are the following:

$$A1: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + 2\epsilon \end{cases} \qquad A2: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = \frac{1}{2}(F_0 + F_4 + \epsilon) \end{cases} \qquad A3: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + \epsilon \end{cases}$$

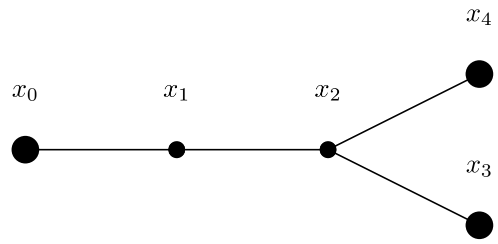

Figure 5.6: Game positions of the Y-game with two running nodes.

	$S_I^*(x_1)$	$S_I^{x_2}(x_1)$
$S_{II}^{x_0}(x_1)$	A	B

	$S_I^*(x_2)$	$S_I^{x_4}(x_2)$	$S_I^{x_1}(x_2)$
$S_{II}^{x_1}(x_2)$	1	2	3
$S_{II}^{x_3}(x_2)$	4	5	6

Table 5.3: Table of all possible pairs of strategies for the Y-game with two running nodes.

$$A4: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_3 + \epsilon \end{cases} \qquad A5: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = \frac{1}{2}(F_3 + F_4) \end{cases} \qquad A6: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = \frac{1}{2}(F_0 + F_3 + \epsilon) \end{cases}$$

$$B1: \begin{cases} u_1 = F_0 + \epsilon \\ u_2 = F_0 + 2\epsilon \end{cases} \qquad B2: \begin{cases} u_1 = \frac{1}{3}(2F_0 + F_4) \\ u_2 = \frac{1}{3}(F_0 + 2F_4) \end{cases} \qquad B3: \begin{cases} u_1 = F_0 \\ u_2 = F_0 \end{cases} \tag{5.13}$$

$$B4: \begin{cases} u_1 = \frac{1}{2}(F_0 + F_3 + \epsilon) \\ u_2 = F_3 + \epsilon \end{cases} \qquad B5: \begin{cases} u_1 = \frac{1}{2}\left(F_0 + \frac{F_3}{2} + \frac{F_4}{2}\right) \\ u_2 = \frac{1}{2}(F_3 + F_4) \end{cases} \qquad B6: \begin{cases} u_1 = \frac{1}{3}(2F_0 + F_3) \\ u_2 = \frac{1}{3}(F_0 + 2F_3) \end{cases}$$

By comparing all the expected values of the game obtained separately for u_1 and u_2, we can order the pairs of strategies from smallest associated expected value of the game to largest. The comparison of u_1 leads us to the graph of Figure 5.7, and Figure 5.8 is associated to the comparison of u_2. The conditions that arise when comparing the values u_1 and u_2 respectively, and which must hold so that the directed edges in Figures 5.7 and 5.8 are correct as displayed, are (5.14). The left-hand side inequalities in (5.14) already appeared in (5.4) for the example on a segment, while the inequalities on the right come from adding the extra terminal node, x_4, to the game on a segment.

$$\begin{cases} (-1): & \epsilon > 0, \\ (0): & F_0 < F_4, \\ (1): & F_0 + \epsilon < F_4, \\ (2): & F_0 + 2\epsilon < F_4, \\ (3): & F_0 + 3\epsilon < F_4, \end{cases} \quad + \quad \begin{cases} (0'): & F_0 < F_3, \\ (0''): & F_3 < F_4, \\ (1'): & F_0 + \epsilon < F_3, \\ (2''): & F_4 < F_3 + 2\epsilon, \\ (3'): & F_0 + 3\epsilon < F_3, \\ (\alpha): & F_0 + F_4 < 2F_3 + \epsilon, \\ (\beta): & 2F_0 + F_4 < 3F_3, \\ (10): & F_0 + 3F_4 + 3\epsilon < 4F_3, \\ (11): & F_0 + 3F_3 + 3\epsilon < 4F_4. \end{cases} \tag{5.14}$$

Since the same arguments used in the game on a segment apply here, the game always has a value and is independent of the node where the game starts. Note that this results are valid not

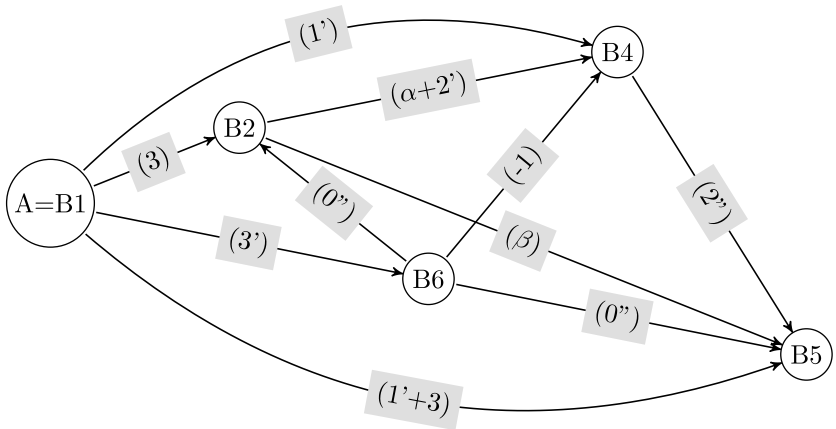

Figure 5.7: Directed graph where the nodes are the pairs of strategies described in Table 5.3 and computed in (5.13), and edges point towards the largest expected value of the game at the x_1-node. The letter A stands for all possible combinations Ax, $x = 1, \ldots, 6$.

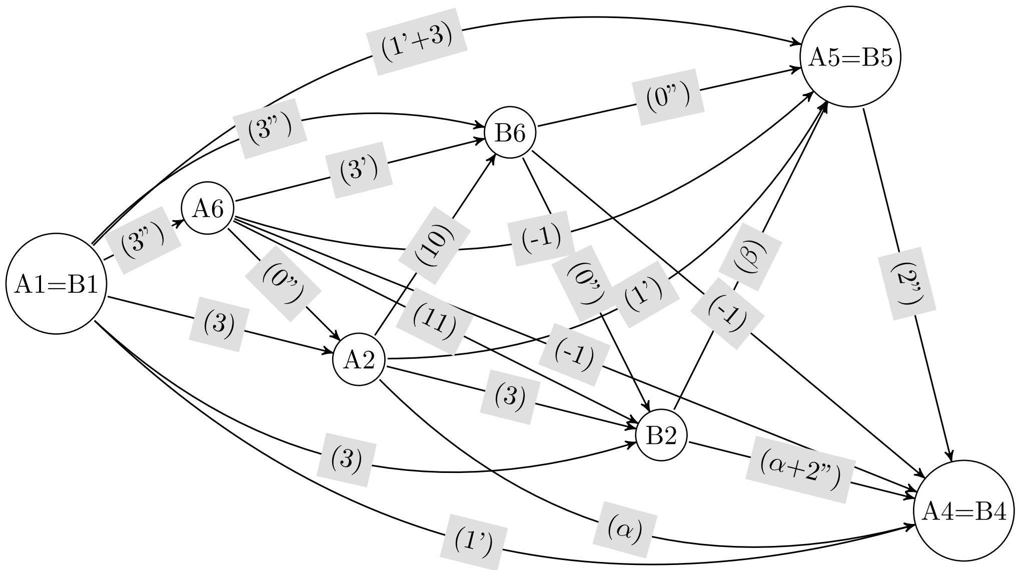

Figure 5.8: Directed graph where the nodes are the pairs of strategies described in Table 5.3 and computed in (5.13), and edges point towards the largest expected value of the game at the x_2-node.

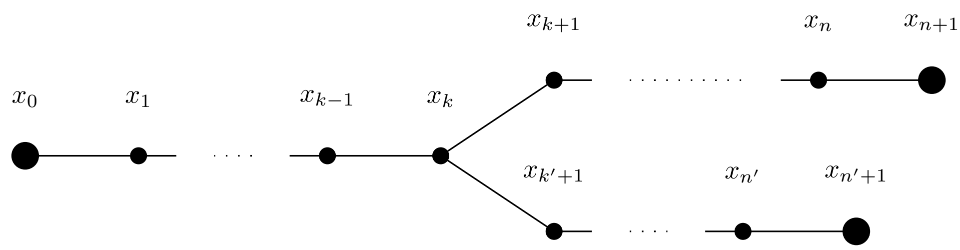

Figure 5.9: Game positions of the Y-game with multiple running nodes.

just for the graphs in Figures 5.7 and 5.8, but for any modified version of them, arising when any of the inequalities in (5.14) is not satisfied.

To finish the analysis of this game, we point out that strategies 2 and 6 at the game position x_2 bring associated the same issues we already commented in the game on a segment with two running nodes. That is, although any pair of strategies involving strategy 2 or 6 gives the same expected value of the game after a formal computation, strategy 6 is not reasonable from a point of view of players who play optimally. Another strategy at node x_2 which is not reasonable either is 3, something which was also discussed in the segment version of this game with two running nodes.

5.4 Y-game with multiple running nodes

The generalization of the Y-game with two running nodes to the case with multiple running nodes (Figure 5.9) can, in fact, be reduced to the game on a segment with multiple running nodes. The key observation is that at the node where all three sub-branches meet, which we will refer to as the Y-point, either player is going to choose to move towards one and only one of the terminal nodes x_0, x_{n+1}, $x_{n'+1}$. Then, once the pairs of strategies S_I, S_{II} for the players are fixed, the game is played, in practice, in a segment.

In other words, we just take into account the two sub-branches involved in the actions at the Y-point and solve a game on a segment with multiple running nodes, which results from restricting the game to this two particular sub-branches. More precisely, we are looking at a new graph which results from avoiding the sub-branch not considered in the action of the players at the Y-point in the original Y-game with multiple running nodes. As a result, we obtain the expected value of the game for all game positions in the two sub-branches considered, in particular in the Y-point.

The expected value of the game at the Y-point, say u_Y, depends on the expected value of the game at the terminal nodes of the two previous sub-branches and can be computed using only these two. Now u_Y, once computed, can be interpreted as the terminal node of a game on a segment with multiple running nodes, constituted by this node and the sub-branch not considered before. Hence, we are dealing again with a game on a segment with multiple running nodes. Solving it, the expected value of the game for all game positions of the original Y-game with multiple running nodes is determined.

Chapter 6

A PDE for the continuous game value of the Totalitarian Tug-of-War

As we mentioned in Section 3.4, the DPP of the classical random Tug-of-War game can be seen as a discretization of the infinity Laplacian, which emerges in the limit as $\epsilon \to 0$ of this DPP. Then, it is natural to ask whether this is the case for the DPP (4.10) associated to the Totalitarian Tug-of-War. It turns out that (4.10) can be seen as a discretization of the fully nonlinear second-order degenerate partial differential equation

$$\min\left\{|\nabla u(x)| - 1, -\Delta_\infty^N u(x)\right\} = 0 \quad \text{for} \quad x \in \Omega. \tag{6.1}$$

Note that due to the lack of regularity of u (recall that viscosity solutions are merely continuous according to Definition 1.3), we cannot expect that equation (6.1) holds in the classical sense. This motivates us to work in the framework of viscosity solutions, reviewed in Chapter 1.

The main result of this chapter is the derivation of (6.1) from (4.10), the DPP associated to the ϵ-Totalitarian Tug-of-War, and is formalized in the following theorem.

Theorem 6.1. *Let u_ϵ be the solution of the Dirichlet problem (4.12) and F the same boundary function in (4.12) but restricted to $\partial\Omega$. Then, $u = \lim_{\epsilon\to 0} u_\epsilon$ uniformly in $\overline{\Omega}$ is the unique viscosity solution to the Dirichlet problem*

$$\begin{cases} \min\left\{|\nabla u(x)| - 1, -\Delta_\infty^N u(x)\right\} = 0, & x \in \Omega; \\ u(x) = F(x), & x \in \partial\Omega. \end{cases} \tag{6.2}$$

Remark 6.2. *The gradient of a solution u to (6.2) has to be, at least formally, non-vanishing in Ω, since $|\nabla u(x)| \geq 1$ in the viscosity sense for all $x \in \Omega$ according to (6.1). This implies that we just have to consider (in the viscosity sense) the normalized ∞-Laplace operator as defined in (2.11) when $\nabla u(x) \neq 0$.*

The following comparison principle is necessary for the proof of Theorem 6.1, and is a particular case of the results in [16, Section 4]. Just to mention that for our convenience, from now on we will write (6.1) as

$$G[u](x) = \min\left\{|\nabla u(x)| - 1, -\Delta_\infty^N u(x)\right\}. \tag{6.3}$$

Theorem 6.3 (Comparison Principle). *Let the functions $u, v : \overline{\Omega} \to \mathbb{R}$ be respectively a viscosity subsolution and viscosity supersolution of (6.1), i.e., $G[u] \leq 0 \leq G[v]$ in Ω. Assume also that v is bounded from above in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.*

A result which follows easily from the comparison principle is the following bound for solutions of (6.2), which highlights the requirement for the viscosity supersolution of being bounded from above in Theorem 6.3. This assumption is required in Theorem 6.3 so that the auxiliary Lemma 6.10 below holds.

Proposition 6.4. *Let u be a viscosity solution to the Dirichlet problem (6.2) and $F : \partial\Omega \to \mathbb{R}$ bounded. Then, for all $x \in \overline{\Omega}$ and $K = \|F\|_{\infty,\partial\Omega}$,*

$$\operatorname{dist}(x, \partial\Omega) - K \leq u(x) \leq \operatorname{dist}(x, \partial\Omega) + K. \tag{6.4}$$

Proof. Consider, for $F(x)$ the same in (6.2), the Dirichlet problem

$$\begin{cases} \min\left\{|\nabla v(x)| - 1, -\Delta_\infty^N v(x)\right\} = 0, & x \in \Omega; \\ v(x) = K, & x \in \partial\Omega, \end{cases} \tag{6.5}$$

the unique viscosity solution of which is proved to be $v(x) = \operatorname{dist}(x, \partial\Omega) + K$ (see [16, Lemma 6.10]), a viscosity supersolution of (6.2). Moreover, it is continuous and bounded in $\overline{\Omega}$. Then, by Theorem 6.3 it follows that $u \leq v$ in $\overline{\Omega}$ and therefore the upper bound for u is proved.

About the lower bound of u, consider now the Dirichlet problem

$$\begin{cases} \min\left\{|\nabla v(x)| - 1, -\Delta_\infty^N v(x)\right\} = 0, & x \in \Omega; \\ v(x) = -K, & x \in \partial\Omega, \end{cases}$$

the solution of which is $v(x) = \operatorname{dist}(x, \partial\Omega) - K$, a viscosity subsolution of (6.2). On the other hand, since u is a viscosity solution to (6.2) by hypothesis, it is in particular a viscosity supersolution and bounded from above due to the first part of the proof. Then, by Theorem 6.3 it follows that $v \leq u$ in $\overline{\Omega}$ and the proof is finished. □

Another consequence of Theorem 6.3 is the uniqueness of solution to (6.2), which is the result that we really need in the proof of Theorem 6.1.

Corollary 6.5 (Uniqueness). *The Dirichlet problem (6.2) has a unique viscosity solution.*

Proof. Let u^1 and u^2 be two viscosity solutions of (6.2). In particular, they are both a viscosity subsolution and viscosity supersolution of (6.1) in Ω and bounded from above and below in $\overline{\Omega}$, according to Proposition 6.4. Thus, taking $u = u^1$ and $v = u^2$, by Theorem 6.3 we obtain $u^1 \leq u^2$ in $\overline{\Omega}$. On the other hand, if we now take $u = u^2$ and $v = u^1$, again by Theorem 6.3 we get $u^2 \leq u^1$ in $\overline{\Omega}$ and therefore $u^1 = u^2$ in $\overline{\Omega}$, that is, there is a unique viscosity solution to the Dirichlet problem (6.2). □

Since we are working in the framework of viscosity solutions, we include by convenience the definition of viscosity solution for equation (6.1), i.e., $G[u] = 0$ in Ω according to (6.3).

Definition 6.6. *A* viscosity subsolution *of the equation $G[u] = 0$ in Ω is an upper semicontinuous function $u : \Omega \to \mathbb{R}$ such that*

$$\min\left\{|\nabla\varphi(\hat{x})| - 1, -\left\langle D^2\varphi(\hat{x})\frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|}, \frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|}\right\rangle\right\} \leq 0, \tag{6.6}$$

whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ are such that $u(\hat{x}) = \varphi(\hat{x})$ and $u(x) \leq \varphi(x)$, for all x in a neighborhood of $\hat{x}$ (in other words, φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$, or equivalently, $u - \varphi$ has a local maximum at $\hat{x}$).

Similarly, a viscosity supersolution *of $G[u] = 0$ in Ω is a lower semicontinuous function $u : \Omega \to \mathbb{R}$ such that*

$$\min\left\{|\nabla\varphi(\hat{x})| - 1, -\left\langle D^2\varphi(\hat{x})\frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|}, \frac{\nabla\varphi(\hat{x})}{|\nabla\varphi(\hat{x})|}\right\rangle\right\} \geq 0, \tag{6.7}$$

whenever $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ are such that $u(\hat{x}) = \varphi(\hat{x})$ and $u(x) \geq \varphi(x)$, for all x in a neighborhood of $\hat{x}$ (in other words, φ touches u at $\hat{x}$ from below in a neighborhood of $\hat{x}$, or equivalently, $u - \varphi$ has a local minimum at $\hat{x}$).

Finally, a function $u : \Omega \to \mathbb{R}$ is a viscosity solution *of $G[u] = 0$ in Ω if it is both a viscosity subsolution and viscosity supersolution of $G[u] = 0$ in Ω.*

6.1 The limit PDE

In order to prove Theorem 6.1 we start showing that u is the uniform limit of u_ϵ in $\overline{\Omega}$ as $\epsilon \to 0$ and a viscosity solution to (6.2). Later, in Section 6.2, we will focus on proving the uniqueness of this u, that is, prove Theorem 6.3 so that Corollary 6.5 follows.

The following lemma, which relies on uniform convergence, provides us with the precise result for a rigorous proof of Theorem 6.1. We point out that this lemma is proved in [18, Lemma 4.5] for continuous functions, but this version does not apply in our case since u_ϵ is not continuous in general.

Lemma 6.7. *Let $u = \lim_{\epsilon\to 0} u_\epsilon$ uniformly in $\overline{\Omega} \subset \mathbb{R}^n$, $\hat{x} \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ such that $u(\hat{x}) = \varphi(\hat{x})$ and $u(x) < \varphi(x)$ for all x in a neighborhood $\mathcal{U}$ of $\hat{x}$, when $x \neq \hat{x}$ (in other words, φ touches u at $\hat{x}$ strictly from above in $\mathcal{U}$, or equivalently, $u - \varphi$ has a strict maximum at $\hat{x}$ in $\mathcal{U}$). Then, for any given $\eta_\epsilon > 0$ there exists a sequence of points $(x_\epsilon)_\epsilon \subset \mathcal{U}$ satisfying $\hat{x} = \lim_{\epsilon\to 0} x_\epsilon$ such that*

$$u_\epsilon(x) - \varphi(x) \leq u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon \quad \text{for all} \quad x \in \mathcal{U}. \tag{6.8}$$

Proof. Writing $u_\epsilon - \varphi = (u_\epsilon - u) + (u - \varphi)$, for a fixed ball $B_r(\hat{x})$ we have that

$$\begin{aligned}
\sup_{\mathcal{U}\setminus B_r(\hat{x})}(u_\epsilon - \varphi) &= \sup_{\mathcal{U}\setminus B_r(\hat{x})}\{(u_\epsilon - u) + (u - \varphi)\} \\
&\leq \sup_{\mathcal{U}\setminus B_r(\hat{x})}(u_\epsilon - u) + \sup_{\mathcal{U}\setminus B_r(\hat{x})}(u - \varphi) \\
&\leq 2\sup_{\mathcal{U}\setminus B_r(\hat{x})}(u - \varphi),
\end{aligned} \tag{6.9}$$

where the last inequality holds for ϵ small enough, since $u = \lim_{\epsilon\to 0} u_\epsilon$ uniformly in $\overline{\Omega}$ by hypothesis. Note also that u is bounded in $\overline{\Omega}$ according to Proposition 6.4 and then it follows that $\sup_{\mathcal{U}\setminus B_r(\hat{x})}(u - \varphi)$ is finite.

Moreover, for ϵ small enough (say $\epsilon < \epsilon_r$),

$$\sup_{\mathcal{U}\setminus B_r(\hat{x})}(u_\epsilon - \varphi) < u_\epsilon(\hat{x}) - \varphi(\hat{x}).$$

This is so because when $\epsilon \to 0$, on the one hand, the right-hand side approaches zero since $u_\epsilon \to u$ and $u(\hat{x}) = \varphi(\hat{x})$. But on the other hand, the left-hand side has a strictly negative limit according to (6.9) and because φ touches u at $\hat{x}$ strictly from above in $\mathcal{U}$, so that $\sup_{\mathcal{U}\setminus B_r(\hat{x})}(u - \varphi) < 0$. Therefore, there is a point $x_\epsilon \in B_r(\hat{x})$ such that, when $\epsilon < \epsilon_r$,

$$u_\epsilon(x) - \varphi(x) \leq \sup_{\mathcal{U}}(u_\epsilon - \varphi) \leq u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon \quad \text{for all} \quad x \in \mathcal{U}.$$

The proof finishes by letting $r \to 0$ via a sequence, say $r = 1, \frac{1}{2}, \frac{1}{3}, \ldots$. Hence there is a sequence of points $x_\epsilon \to \hat{x}$ which satisfy (6.8), as we wanted to prove. □

Remark 6.8. *Under the same hypothesis of Lemma 6.7 but for $\phi \in \mathcal{C}^2(\Omega)$ touching u at $\hat{x}$ strictly from below in $\mathcal{U}$, it is derived for any given $\eta_\epsilon > 0$ the existence of a sequence of points $(x_\epsilon)_\epsilon \subset \mathcal{U}$ satisfying $\hat{x} = \lim_{\epsilon\to 0} x_\epsilon$, such that*

$$u_\epsilon(x) - \phi(x) \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta_\epsilon \quad \text{for all} \quad x \in \mathcal{U}. \tag{6.10}$$

The proof of this last result follows similarly to the proof of Lemma 6.7.

Proof of Theorem 6.1. In order to prove that u_ϵ converges uniformly to u in $\overline{\Omega}$ as $\epsilon \to 0$, we invoke [3, Theorem 2.1], which provides uniform convergence in $\overline{\Omega}$. For our particular problem, the hypothesis required by this theorem are a comparison principle for equation (6.1), given by Theorem 6.3, and the following assumptions on the approximation scheme $\mathcal{G}[u_\epsilon]$, defined in (4.11) and that follows from the DPP for the ϵ-Totalitarian Tug-of-War game:

i) *Monotonicity*, i.e., $\mathcal{G}[u_\epsilon](x) \leq \mathcal{G}[v_\epsilon](x)$ whenever $u_\epsilon \geq v_\epsilon$ and $u_\epsilon(x) = v_\epsilon(x)$. This is so, as we already argued in Remark 4.14.

ii) *Stability*, which means that solutions u_ϵ of the Dirichlet problem (4.12) are bounded independently of ϵ (at least for ϵ small enough, since we are interested in the case $\epsilon \to 0$). It follows from Remark 4.12.

iii) *Consistency*, i.e., $\lim_{\epsilon\to 0} \mathcal{G}[\varphi](x_\epsilon) = G[\varphi](\hat{x})$ for $\lim_{\epsilon\to 0} x_\epsilon = \hat{x}$, where $G[\varphi]$ is defined in (6.3). For the proof of this fact, we consider the slightly modified approximation scheme

$$\min\left\{ \frac{1}{\epsilon}\left(\varphi(x_\epsilon) - \inf_{y\in\overline{B}_\epsilon(x_\epsilon)} \varphi(y) - \epsilon\right), \frac{1}{\epsilon^2}\left(2\varphi(x_\epsilon) - \sup_{y\in\overline{B}_\epsilon(x_\epsilon)} \varphi(y) - \inf_{y\in\overline{B}_\epsilon(x_\epsilon)} \varphi(y)\right)\right\}.$$

Note that the homogeneous equation associated to this new approximation scheme is equivalent to $\mathcal{G}[\varphi](x_\epsilon) = 0$, so that the result of [3, Theorem 2.1] still holds for the problem associated to u_ϵ, $\mathcal{G}[u_\epsilon] = 0$ in Ω.

At this point, the consistency follows with a similar argument to the one below, when proving that u is a viscosity solution to $G[u](\hat{x}) = 0$, so we skip it here.

In fact, the conditions above are required to be satisfied not just by the approximation scheme $\mathcal{G}[u_\epsilon]$, but the Dirichlet problem (4.12). It is the case, since $u_\epsilon(x) = F(x)$ for all $x \in \Gamma_\epsilon$ and $u(x) = F(x)$ for all $x \in \partial\Omega$. Then, we can apply [3, Theorem 2.1] and conclude that $u = \lim_{\epsilon\to 0} u_\epsilon$ uniformly in $\overline{\Omega}$.

It remains to check that u is a viscosity solution to (6.1). Recall that according to Definition 6.6, the fact that u is a viscosity solution is equivalent to be both a viscosity subsolution and viscosity supersolution, which is the way we proceed for its proof.

According to (4.10), either

$$u_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} u_\epsilon - \epsilon = 0 \qquad \text{or} \qquad u_\epsilon(x) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x)} u_\epsilon + \inf_{\overline{B}_\epsilon(x)} u_\epsilon\right) = 0 \tag{6.11}$$

or both, so that we have to check that u is a viscosity subsolution and viscosity supersolution in all three possible situations.

For the proof that u is a viscosity subsolution, consider a point $\hat{x} \in \Omega$ and a test function $\varphi \in \mathcal{C}^2(\Omega)$ such that φ touches u at $\hat{x}$ from above in a neighborhood of $\hat{x}$. Without loss of generality (see Remark 1.4) we can ask instead that φ touches u at $\hat{x}$ strictly from above in a neighborhood of $\hat{x}$. Then Lemma 6.7 holds and rearranging terms in (6.8) we have that

$$\inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon \leq \inf_{\overline{B}_\epsilon(x_\epsilon)} \{\varphi + u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon\}$$

$$= \inf_{\overline{B}_\epsilon(x_\epsilon)} \varphi + u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon$$
$$= \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi + u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon$$

and then

$$\inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi \le u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon. \tag{6.12}$$

From this it is clear that

$$\varphi(x_\epsilon) - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi - \eta_\epsilon \le u_\epsilon(x_\epsilon) - \inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon$$

and if we assume that $u_\epsilon(x) - \inf_{\overline{B}_\epsilon(x)} u_\epsilon - \epsilon = 0$, i.e., the first equation in (6.11) holds, it follows that

$$\varphi(x_\epsilon) - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi - \epsilon - \eta_\epsilon \le u_\epsilon(x_\epsilon) - \inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \epsilon = 0, \tag{6.13}$$

which can be equivalently written for $\epsilon > 0$ as

$$\frac{1}{\epsilon}\left(\varphi(x_\epsilon) - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi\right) - 1 - \eta_\epsilon \le 0. \tag{6.14}$$

At this point it is necessary to recall that the gradient of a solution to (6.1) has to be, at least formally, non-vanishing, as we already noted in Remark 6.2. Because of this, we can assume $|\nabla\varphi(\hat{x})| \ge 1$ since we are done otherwise. It follows that $\nabla\varphi(\hat{x}) \neq 0$ and since φ is in particular continuous and $\lim_{\epsilon\to 0} x_\epsilon = \hat{x}$ by Lemma 6.7, we also have that $\nabla\varphi(x_\epsilon) \neq 0$ for ϵ small enough.

On the other hand, let $x_\epsilon^{\min} \in \Omega$ be such that $\varphi(x_\epsilon^{\min}) = \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi$. In particular $x_\epsilon^{\min} \in \partial B_\epsilon(x_\epsilon)$, otherwise suppose to the contrary that there exists a subsequence $x_{\epsilon_k}^{min} \in B_{\epsilon_k}(x_{\epsilon_k})$ of minimum points of φ in Ω. Then $\nabla\varphi(x_{\epsilon_k}^{min}) = 0$ and since $\lim_{\epsilon_k\to 0} x_{\epsilon_k}^{min} = \hat{x}$, the continuity of φ implies that $\nabla\varphi(\hat{x}) = 0$, which is a contradiction with our previous discussion.

According to this,

$$x_\epsilon^{\min} = x_\epsilon - \epsilon\left[\frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|} + o(1)\right] \quad \text{as } \epsilon \to 0, \tag{6.15}$$

which can be deduced from the fact that, for ϵ small enough, φ is approximately the same as its tangent plane. More precisely, consider $x_\epsilon^{\min} = x_\epsilon - \epsilon\, v$ for $|v| = 1$, and ω any fixed direction with $|\omega| = 1$. Since $\varphi(x_\epsilon^{\min}) \le \varphi(x_\epsilon - \epsilon\,\omega)$, the Taylor expansion of $\varphi(x_\epsilon^{\min})$ at x_ϵ up to first order terms gives us

$$\varphi(x_\epsilon) - \langle\nabla\varphi(x_\epsilon), \epsilon\, v\rangle + o(\epsilon) = \varphi(x_\epsilon^{\min}) \le \varphi(x_\epsilon - \epsilon\,\omega) \quad \text{as } \epsilon \to 0$$

and after some manipulation,

$$\langle\nabla\varphi(x_\epsilon), v\rangle + o(1) \ge \frac{\varphi(x_\epsilon - \epsilon\,\omega) - \varphi(x_\epsilon)}{\epsilon} = \langle\nabla\varphi(x_\epsilon), \omega\rangle + o(1) \quad \text{as } \epsilon \to 0.$$

Since the previous argument holds for any direction ω, we can conclude that

$$v = \frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|} + o(1) \quad \text{as } \epsilon \to 0.$$

Now that expression (6.15) of $x_\epsilon^{\min}$ has been properly justified, we evaluate the Taylor expansion of φ around $x_\epsilon^{\min}$ at x_ϵ, i.e.,

$$\min_{\overline{B}_\epsilon(x_\epsilon)} \varphi = \varphi(x_\epsilon^{\min}) = \varphi\left(x_\epsilon - \epsilon\left[\frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|} + o(1)\right]\right)$$

$$= \varphi(x_\epsilon) - \left\langle \nabla\varphi(x_\epsilon),\, \epsilon\, \frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|} \right\rangle + o(\epsilon)$$
$$= \varphi(x_\epsilon) - \epsilon\, |\nabla\varphi(x_\epsilon)| + o(\epsilon) \quad \text{as} \quad \epsilon \to 0,$$

which after rearranging terms and dividing by ϵ, is equivalent to write

$$\frac{1}{\epsilon}\left(\varphi(x_\epsilon) - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi\right) = |\nabla\varphi(x_\epsilon)| - o(1) \quad \text{as} \quad \epsilon \to 0.$$

In combination with (6.14) we obtain the estimate

$$|\nabla\varphi(x_\epsilon)| - 1 \leq o(1) \quad \text{as} \quad \epsilon \to 0,$$

and taking $\epsilon \to 0$ one finally gets $|\nabla\varphi(\hat{x})| - 1 \leq 0$, as required according to (6.6).

Let us assume now that $u_\epsilon(x) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x)} u_\epsilon + \inf_{\overline{B}_\epsilon(x)} u_\epsilon\right) = 0$, i.e., the second equation in (6.11) holds. Consider the point $x_\epsilon^{\max} \in \Omega$ such that $\varphi(x_\epsilon^{\max}) = \max_{\overline{B}_\epsilon(x_\epsilon)} \varphi$ and its symmetric point in $\overline{B}_\epsilon(x_\epsilon)$, given by

$$\tilde{x}_\epsilon^{\max} = x_\epsilon - (x_\epsilon^{\max} - x_\epsilon)\,.$$

The evaluation of the Taylor expansion of φ up to second order terms around these two points at x_ϵ is, respectively,

$$\varphi\left(x_\epsilon^{\max}\right) = \varphi(x_\epsilon) + \langle \nabla\varphi(x_\epsilon),\, (x_\epsilon^{\max} - x_\epsilon)\rangle$$
$$+ \frac{1}{2}\left\langle D^2\varphi(x_\epsilon)\,(x_\epsilon^{\max} - x_\epsilon),\, (x_\epsilon^{\max} - x_\epsilon)\right\rangle + o\left(\epsilon^2\right) \quad \text{as} \quad \epsilon \to 0$$

and

$$\varphi\left(\tilde{x}_\epsilon^{\max}\right) = \varphi(x_\epsilon) - \langle \nabla\varphi(x_\epsilon),\, (x_\epsilon^{\max} - x_\epsilon)\rangle$$
$$+ \frac{1}{2}\left\langle D^2\varphi(x_\epsilon)\,(x_\epsilon^{\max} - x_\epsilon),\, (x_\epsilon^{\max} - x_\epsilon)\right\rangle + o\left(\epsilon^2\right) \quad \text{as} \quad \epsilon \to 0.$$

Adding both expressions we obtain

$$\varphi\left(x_\epsilon^{\max}\right) + \varphi\left(\tilde{x}_\epsilon^{\max}\right) - 2\varphi(x_\epsilon) = \left\langle D^2\varphi(x_\epsilon)\,(x_\epsilon^{\max} - x_\epsilon),\, (x_\epsilon^{\max} - x_\epsilon)\right\rangle + o\left(\epsilon^2\right) \quad \text{as} \quad \epsilon \to 0. \tag{6.16}$$

Furthermore, by our choice of $x_\epsilon^{\max}$ and $\tilde{x}_\epsilon^{\max}$, it holds

$$\varphi\left(x_\epsilon^{\max}\right) + \varphi\left(\tilde{x}_\epsilon^{\max}\right) - 2\varphi(x_\epsilon) \geq \max_{\overline{B}\epsilon(x_\epsilon)} \varphi + \min_{\overline{B}\epsilon(x_\epsilon)} \varphi - 2\varphi(x_\epsilon). \tag{6.17}$$

On the other hand, from (6.8) we deduce that

$$\sup_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \max_{\overline{B}_\epsilon(x_\epsilon)} \varphi \leq u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon$$

and

$$\inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi \leq u_\epsilon(x_\epsilon) - \varphi(x_\epsilon) + \eta_\epsilon.$$

The second of these two expressions is exactly (6.12), which was derived above. The first expression follows in a similar way. Adding these two inequalities, rearranging terms and multiplying by $\frac{1}{2}$, we obtain

$$\varphi(x_\epsilon) - \frac{1}{2}\left(\max_{\overline{B}_\epsilon(x_\epsilon)} \varphi + \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi\right) - \eta_\epsilon \leq u_\epsilon(x_\epsilon) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon + \inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon\right).$$

And because we are assuming that $u_\epsilon(x) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x)} u_\epsilon + \inf_{\overline{B}_\epsilon(x)} u_\epsilon\right) = 0$, i.e., the second equation in (6.11) holds, it follows that

$$\varphi(x_\epsilon) - \frac{1}{2}\left(\max_{\overline{B}_\epsilon(x_\epsilon)} \varphi + \min_{\overline{B}_\epsilon(x_\epsilon)} \varphi\right) \leq \eta_\epsilon.$$

The combination of this last inequality with (6.16) and (6.17) gives us for $\eta_\epsilon = o\left(\epsilon^2\right)$

$$-\left\langle D^2\varphi(x_\epsilon)\,(x_\epsilon^{\max} - x_\epsilon),\,(x_\epsilon^{\max} - x_\epsilon)\right\rangle \leq o\left(\epsilon^2\right) \quad \text{as} \ \ \epsilon \to 0,$$

which is more suitable to write as

$$-\,\epsilon^2\left\langle D^2\varphi(x_\epsilon)\,\frac{x_\epsilon^{\max} - x_\epsilon}{\epsilon},\,\frac{x_\epsilon^{\max} - x_\epsilon}{\epsilon}\right\rangle \leq o\left(\epsilon^2\right) \quad \text{as} \ \ \epsilon \to 0. \tag{6.18}$$

The reason for that is detailed next. In the same way we deduced (6.15), it is derived that the point $x_\epsilon^{\max} \in \Omega$ such that $\varphi(x_\epsilon^{\max}) = \max_{\overline{B}_\epsilon(x_\epsilon)} \varphi$ can be expressed as

$$x_\epsilon^{\max} = x_\epsilon + \epsilon\left[\frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|} + o(1)\right] \quad \text{as} \ \ \epsilon \to 0. \tag{6.19}$$

Hence, due to (6.19), (6.18) is equivalent to

$$-\epsilon^2\left\langle D^2\varphi(x_\epsilon)\,\frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|},\,\frac{\nabla\varphi(x_\epsilon)}{|\nabla\varphi(x_\epsilon)|}\right\rangle \leq o\left(\epsilon^2\right) \quad \text{as} \ \ \epsilon \to 0,$$

which according to the definition of the normalized infinity Laplacian in (2.11), is the same as writing

$$-\epsilon^2\Delta_\infty^N\varphi(x_\epsilon) \leq o\left(\epsilon^2\right) \quad \text{as} \ \ \epsilon \to 0.$$

After dividing by ϵ^2 and taking $\epsilon \to 0$ we get $-\Delta_\infty^N\varphi(\hat{x}) \leq 0$, as required according to (6.6).

When both equations in (6.11) hold, the proof that u is a viscosity subsolution, i.e., that condition (6.6) is satisfied, is clear by taking into account the entire argument for viscosity subsolutions developed above. This finishes the proof that u is a viscosity subsolution to the Dirichlet problem (6.2).

In order to prove that u is a viscosity supersolution to (6.2), we proceed analogously to the case of viscosity subsolution. We consider now the point $\hat{x} \in \Omega$ and the test function $\phi \in \mathcal{C}^2(\Omega)$ such that ϕ touches u at $\hat{x}$ strictly from below in a neighborhood of $\hat{x}$.

When assuming that the first equation in (6.11) holds, we get

$$\phi(x_\epsilon) - \min_{\overline{B}_\epsilon(x_\epsilon)} \phi - \epsilon + \eta_\epsilon \geq u_\epsilon(x_\epsilon) - \inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \epsilon = 0,$$

which is just the viscosity supersolution version of (6.13). From this it is derived as before that

$$|\nabla\phi(\hat{x})| - 1 \geq 0,$$

so that (6.7) is satisfied.

Now, let the second equation in (6.11) hold. Consider the point $x_\epsilon^{\min} \in \Omega$ be such that $\phi(x_\epsilon^{\min}) = \min_{\overline{B}_\epsilon(x_\epsilon)} \phi$ and its symmetric point in $\overline{B}_\epsilon(x_\epsilon)$, say $\tilde{x}_\epsilon^{\min}$. By adding the evaluation around these two points at x_ϵ of the Taylor expansion of ϕ up to second order terms, we get

$$\phi(x_\epsilon^{\min}) + \phi(\tilde{x}_\epsilon^{\min}) - 2\phi(x_\epsilon) = \left\langle D^2\phi(x_\epsilon)\,(x_\epsilon^{\min} - x_\epsilon),\,\left(x_\epsilon^{\min} - x_\epsilon\right)\right\rangle + o(\epsilon^2) \quad \text{as} \ \ \epsilon \to 0. \tag{6.20}$$

Furthermore, by our choice of $x_\epsilon^{\min}$ and $\tilde{x}_\epsilon^{\min}$, it holds

$$\phi(x_\epsilon^{\min}) + \phi(\tilde{x}_\epsilon^{\min}) - 2\phi(x_\epsilon) \leq \max_{\overline{B}\epsilon(x_\epsilon)} \phi + \min_{\overline{B}\epsilon(x_\epsilon)} \phi - 2\phi(x_\epsilon). \tag{6.21}$$

On the other hand, from (6.10) we deduce (in the same way as we already did in the subsolution case) that

$$\sup_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \max_{\overline{B}_\epsilon(x_\epsilon)} \phi \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta_\epsilon$$

and

$$\inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon - \min_{\overline{B}_\epsilon(x_\epsilon)} \phi \geq u_\epsilon(x_\epsilon) - \phi(x_\epsilon) - \eta_\epsilon.$$

Adding these two inequalities, rearranging terms and multiplying by $\frac{1}{2}$, we obtain

$$\phi(x_\epsilon) - \frac{1}{2}\left(\max_{\overline{B}_\epsilon(x_\epsilon)} \phi + \min_{\overline{B}_\epsilon(x_\epsilon)} \phi\right) + \eta_\epsilon \geq u_\epsilon(x_\epsilon) - \frac{1}{2}\left(\sup_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon + \inf_{\overline{B}_\epsilon(x_\epsilon)} u_\epsilon\right)$$

and because we are under the assumption that the second equation in (6.11) is satisfied, it follows that

$$\phi(x_\epsilon) - \frac{1}{2}\left(\max_{\overline{B}_\epsilon(x_\epsilon)} \phi + \min_{\overline{B}_\epsilon(x_\epsilon)} \phi\right) \geq -\eta_\epsilon.$$

The combination of this last inequality with (6.20) and (6.21) gives us

$$-\epsilon^2 \left\langle D^2\phi(x_\epsilon)\,\frac{x_\epsilon^{\min} - x_\epsilon}{\epsilon}, \frac{x_\epsilon^{\min} - x_\epsilon}{\epsilon}\right\rangle \geq o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0,$$

which in view of (6.15) can be written as

$$-\epsilon^2 \left\langle D^2\phi(x_\epsilon)\,\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}, \frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|}\right\rangle \geq o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0.$$

After dividing by ϵ^2 and taking $\epsilon \to 0$ it is derived that $-\Delta_\infty^N \phi(\hat{x}) \geq 0$, as required according to (6.7).

When both equalities in (6.11) hold, the proof that u is a viscosity supersolution, i.e., that condition (6.7) is satisfied, is clear by taking into account the entire argument for viscosity supersolutions developed above. This finishes the proof that u is a viscosity supersolution to (6.2) and then the proof of u being a viscosity solution to (6.2) is also complete. □

6.2 Comparison and uniqueness for the limit equation

In order to motivate the arguments that will appear in the proof of Theorem 6.3, let u and v be respectively a subsolution and supersolution of the equation $G = 0$ in Ω. Assume that both functions are twice differentiable everywhere and $u - v$ has a local maximum at $\hat{x}$, so that $\nabla u(\hat{x}) = \nabla v(\hat{x})$ and $D^2u(\hat{x}) \leq D^2v(\hat{x})$. Then, since equation (6.1) satisfies monotonicity condition (1.3), we have that

$$\begin{aligned} G(\hat{x}, u(\hat{x}), \nabla u(\hat{x}), D^2u(\hat{x})) \leq 0 &\leq G(\hat{x}, v(\hat{x}), \nabla v(\hat{x}), D^2v(\hat{x})) \\ &\leq G(\hat{x}, v(\hat{x}), \nabla u(\hat{x}), D^2u(\hat{x})). \end{aligned} \tag{6.22}$$

In the event that $G(x, r, p, X)$ is strictly non-decreasing in r (a simple but illustrative case), suppose that $u(\hat{x}) - v(\hat{x}) > 0$. According to this,

$$G(\hat{x}, v(\hat{x}), \nabla u(\hat{x}), D^2u(\hat{x})) < G(\hat{x}, u(\hat{x}), \nabla u(\hat{x}), D^2u(\hat{x})),$$

which is a contradiction in view of (6.22). It then follows that $u - v$ is non-positive at an interior maximum. Provided that $u \leq v$ on $\partial\Omega$, one can conclude that $u \leq v$ in $\overline{\Omega}$.

We seek to extend this argument to the case u upper semicontinuous in $\overline{\Omega}$ and v lower semicontinuous in $\overline{\Omega}$. Note that we are unable to simply plug $(\nabla u(\hat{x}), D^2u(\hat{x}))$ and $(\nabla v(\hat{x}), D^2v(\hat{x}))$

into G since these expressions must be replaced by the set-valued functions $J_{\Omega}^{2,+}u(\hat{x})$ and $J_{\Omega}^{2,-}v(\hat{x})$, the values of which might be empty at many points, including maximum points of $u(x) - v(x)$ in Ω. Then, in order to use $J_{\Omega}^{2,+}u(\hat{x})$ and $J_{\Omega}^{2,-}v(\hat{x})$, we employ a method that doubles the number of variables and then penalizes this doubling. More precisely, instead of maximizing $u(x) - v(x)$, we maximize the function $u(x) - v(y) - \tau/2|x-y|^2$ over $\overline{\Omega} \times \overline{\Omega}$, for $\tau > 0$ a parameter. As $\tau \to \infty$, we closely approximate maximizing $u(x) - v(x)$ over $\overline{\Omega}$.

In view of the use of the set-valued functions $J_{\Omega}^{2,+}u(\hat{x})$ and $J_{\Omega}^{2,-}v(\hat{x})$, it is derived that the framework of viscosity solutions is the natural one for our current analysis. According to this, one has to work in terms of Definition 6.6. We also include now the particular case of Lemma 1.8 formulated in terms of (6.1), which provides an equivalent definition to Definition 6.6 (as it was already proved for the general result, Lemma 1.8).

Lemma 6.9. *A function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (6.1) in Ω if and only if u is upper semicontinuous and*

$$\min\left\{|p| - 1, -\left\langle X\,\frac{p}{|p|}, \frac{p}{|p|}\right\rangle\right\} \leq 0, \quad \text{for all } \hat{x} \in \Omega \text{ and all } (p, X) \in J_{\Omega}^{2,+}u(\hat{x}),\ p \neq 0.$$

Similarly, a function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (6.1) in Ω if and only if u is lower semicontinuous and

$$\min\left\{|p| - 1, -\left\langle X\,\frac{p}{|p|}, \frac{p}{|p|}\right\rangle\right\} \geq 0, \quad \text{for all } \hat{x} \in \Omega \text{ and all } (p, X) \in J_{\Omega}^{2,-}u(\hat{x}),\ p \neq 0.$$

The proof of Theorem 6.3 (inspired in [7, Section 3] and [16, Section 4]) is presented now. We start with a lemma which is crucial for the proof of the theorem (it plays the analogous role of Lemma 4.13 for the proof of Theorem 4.10).

Lemma 6.10. *Let $v : \overline{\Omega} \to \mathbb{R}$ be a viscosity supersolution of (6.1) and bounded from above in $\overline{\Omega}$. Then, for every $\gamma > 0$ there exists a viscosity supersolution $\tilde{v} : \overline{\Omega} \to \mathbb{R}$ of the equation*

$$\min\{|\nabla \tilde{v}(x)| - 1, -\Delta_\infty^N \tilde{v}(x)\} = \mu \tag{6.23}$$

for some constant $\mu = \mu(\gamma, v) > 0$. Moreover, $\tilde{v} - v \leq \gamma$ in $\overline{\Omega}$ and $\tilde{v} - v \geq -\gamma$ on $\partial\Omega$.

Proof. Since v is bounded from above, $v \leq C$ for some constant $C > 0$. We look for a function $\tilde{v}$ in the form $\tilde{v} = g(v)$, where $g : (-\infty, C] \to \mathbb{R}$ is a smooth and increasing function such that g^{-1} is also smooth. Let $\hat{x} \in \Omega$ and $\tilde{\varphi} \in \mathcal{C}^2(\Omega)$ be such that $\tilde{\varphi}$ touches $\tilde{v}$ at $\hat{x}$ from below in a neighborhood of $\hat{x}$. It then follows that $\varphi = g^{-1}(\tilde{\varphi}) \in \mathcal{C}^2(\Omega)$ touches v from below at $\hat{x}$ in a neighborhood of $\hat{x}$. According to this and since v is a viscosity supersolution of (6.1) by hypothesis, in view of (6.7) we have that

$$\min\{|\nabla \varphi(\hat{x})| - 1, -\Delta_\infty^N \varphi(\hat{x})\} \geq 0. \tag{6.24}$$

This gives us the two next estimates. On the one hand,

$$|\nabla \tilde{\varphi}(\hat{x})| = |g'(\varphi(\hat{x}))||\nabla \varphi(\hat{x})| = g'(\varphi(\hat{x}))\,|\nabla \varphi(\hat{x})| \geq g'(\varphi(\hat{x})) \tag{6.25}$$

since g is increasing and $|\nabla \varphi(\hat{x})| \geq 1$ by (6.24). On the other hand,

$$\begin{aligned}
-\Delta_\infty^N \tilde{\varphi}(\hat{x}) &= -\left\langle D^2\tilde{\varphi}(\hat{x})\,\frac{\nabla\tilde{\varphi}(\hat{x})}{|\nabla\tilde{\varphi}(\hat{x})|}, \frac{\nabla\tilde{\varphi}(\hat{x})}{|\nabla\tilde{\varphi}(\hat{x})|}\right\rangle \\
&= -\left\langle g''(\varphi(\hat{x}))\,(\nabla\varphi(\hat{x}) \otimes \nabla\varphi(\hat{x}))\,\frac{g'(\varphi(\hat{x}))\nabla\varphi(\hat{x})}{|g'(\varphi(\hat{x}))||\nabla\varphi(\hat{x})|}, \frac{g'(\varphi(\hat{x}))\nabla\varphi(\hat{x})}{|g'(\varphi(\hat{x}))||\nabla\varphi(\hat{x})|}\right\rangle \\
&\qquad - \left\langle g'(\varphi(\hat{x}))\,D^2\varphi(\hat{x})\,\frac{g'(\varphi(\hat{x}))\nabla\varphi(\hat{x})}{|g'(\varphi(\hat{x}))||\nabla\varphi(\hat{x})|}, \frac{g'(\varphi(\hat{x}))\nabla\varphi(\hat{x})}{|g'(\varphi(\hat{x}))||\nabla\varphi(\hat{x})|}\right\rangle
\end{aligned}$$

$$= -g''(\varphi(\hat{x}))\,|\nabla\varphi(\hat{x})|^2 - g'(\varphi(\hat{x}))\,\Delta_\infty^N\varphi(\hat{x}). \tag{6.26}$$

We now consider, for $\varepsilon > 0$ (the constant γ in the statement will be chosen later as a function of ε) and $\alpha \leq C$,

$$g(\alpha) = (1+\varepsilon)\,\alpha - \frac{\varepsilon}{4C}\,\alpha^2.$$

In particular, $g'(\alpha) \geq 1 + \varepsilon/2$ for $\alpha \leq C$ and $g''(\alpha) = -\varepsilon/(2C)$. Using this in the previous estimates (6.25) and (6.26), one gets that

$$|\nabla\tilde{\varphi}(\hat{x})| \geq g'(\varphi(\hat{x})) \geq 1 + \frac{\varepsilon}{2}, \tag{6.27}$$

and taking also into account the estimates $|\nabla\varphi(\hat{x})| \geq 1$ and $-\Delta_\infty\varphi(\hat{x}) \geq 0$ from (6.24), it holds

$$\begin{aligned} -\Delta_\infty^N\tilde{\varphi}(\hat{x}) &= -g''(\varphi(\hat{x}))\,|\nabla\varphi(\hat{x})|^2 - g'(\varphi(\hat{x}))\,\Delta_\infty^N\varphi(\hat{x}) \\ &\geq -g''(\varphi(\hat{x}))\,|\nabla\varphi(\hat{x})|^2 \\ &\geq \frac{\varepsilon}{2C}. \end{aligned} \tag{6.28}$$

Finally, the combination of (6.27) and (6.28) leads us to

$$\min\left\{|\nabla\tilde{\varphi}(\hat{x})| - 1, -\Delta_\infty^N\tilde{\varphi}(\hat{x})\right\} \geq \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2C}\right\} = \frac{\varepsilon}{2}\min\left\{1, \frac{1}{C}\right\} = \mu > 0,$$

as we wanted to prove.

About the second part of the lemma, since $g(\alpha) - \alpha \leq \frac{3}{4}\,\varepsilon C$ for $\alpha \leq C$ and because $\tilde{v} = g(v)$, it follows that $\tilde{v} - v \leq \frac{3}{4}\,\varepsilon C$ in Ω. Similarly, since $g(\alpha) - \alpha \geq -\varepsilon D\left(1 + \frac{D}{4C}\right)$ for $\alpha \geq -D = -\left|\min_{\partial\Omega} v\right|$, it follows that $\tilde{v} - v \geq -\varepsilon D\left(1 + \frac{D}{4C}\right)$ on $\partial\Omega$. The results hold for every

$$\gamma = \varepsilon \max\left\{\frac{3}{4}\,C,\, D\left(1 + \frac{D}{4C}\right)\right\} > 0,$$

provided ε is small enough. □

The following lemma is also an auxiliary result in the proof of Theorem 6.3 and formalizes the ideas commented before on how to proceed with the proof of the comparison principle.

Lemma 6.11 (Ishii's Lemma)**.** *Let the real-valued functions u and v be respectively upper and lower semicontinuous in $\overline{\Omega} \subset \mathbb{R}^n$ and denote*

$$M_\tau = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \frac{\tau}{2}|x-y|^2\right)$$

for $\tau > 0$. If $(x_\tau, y_\tau) \in \overline{\Omega}\times\overline{\Omega}$ is such that

$$M_\tau = u(x_\tau) - v(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2,$$

then

i) $\lim_{\tau\to\infty} \tau|x_\tau - y_\tau|^2 = 0$;

ii) $\lim_{\tau\to\infty} M_\tau = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\overline{\Omega}}(u(x) - v(x))$, *whenever $x_\tau \to \hat{x} \in \overline{\Omega}$ as $\tau \to \infty$.*

Proof. Since $\tau|x_\tau - y_\tau|^2 \geq 0$, M_τ is non-increasing as τ increases. According to this,

$$M_{\tau/2} \geq u(x_\tau) - v(y_\tau) - \frac{\tau}{4}|x_\tau - y_\tau|^2$$

$$= u(x_\tau) - v(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2 + \frac{\tau}{4}|x_\tau - y_\tau|^2$$
$$= M_\tau + \frac{\tau}{4}|x_\tau - y_\tau|^2,$$

which is equivalent to say

$$4(M_{\tau/2} - M_\tau) \geq \tau|x_\tau - y_\tau|^2. \tag{6.29}$$

We use the fact that $\lim_{\tau\to\infty} M_\tau$ is finite, which holds since for all $x, y \in \overline{\Omega}$ compact, $u(x)-v(y)$ is upper semicontinuous and $\tau/2\,|x-y|^2$ is continuous (this is just the extension to semicontinuous functions of the extreme value theorem). Hence, by taking the limit $\tau \to \infty$ in (6.29) we can conclude that $\lim_{\tau\to\infty} \tau|x_\tau - y_\tau|^2 = 0$ since $\tau|x_\tau - y_\tau|^2 \geq 0$ and $\lim_{\tau\to\infty} 4(M_{\tau/2} - M_\tau) = 0$.

For the second result of the lemma, consider

$$u(x_\tau) - v(y_\tau) - \frac{\tau}{2}|x_\tau - y_\tau|^2 = M_\tau = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(u(x) - v(y) - \frac{\tau}{2}|x-y|^2\right)$$
$$\geq \sup_{x\in\overline{\Omega}} \left(u(x) - v(x)\right). \tag{6.30}$$

Because $\overline{\Omega}$ is compact we can always find a sequence $(x_\tau)_\tau$ and a point $\hat{x} \in \overline{\Omega}$ such that $\lim_{\tau\to\infty} x_\tau = \hat{x} \in \overline{\Omega}$. Moreover, from the first result of the lemma we also have that $\lim_{\tau\to\infty} y_\tau = \hat{x}$. According to this,

$$u(\hat{x}) - v(\hat{x}) \geq \limsup_{\tau\to\infty} \left(u(x_\tau) - v(y_\tau)\right) \geq \lim_{\tau\to\infty} \left(u(x_\tau) - v(y_\tau)\right) = \lim_{\tau\to\infty} M_\tau, \tag{6.31}$$

where in the leftmost inequality we used the upper semicontinuity of $u - v$ in $\overline{\Omega}$, and the rightmost equality follows from the first result of the lemma. Then, taking $\tau \to \infty$ in (6.30) and in combination with (6.31), one obtains

$$u(\hat{x}) - v(\hat{x}) \geq \lim_{\tau\to\infty} M_\tau \geq \sup_{x\in\overline{\Omega}} \left(u(x) - v(x)\right). \tag{6.32}$$

But $\sup_{x\in\overline{\Omega}} \left(u(x) - v(x)\right) \geq u(\hat{x}) - v(\hat{x})$, so that all the inequalities in (6.32) become equalities, which gives us the second result of the lemma. □

We also include a theorem (see [16, Theorem 4.8] for details) involving the set-valued functions $J_\Omega^{2,+}u(\hat{x})$ and $J_\Omega^{2,-}v(\hat{x})$, which are crucial when proving Theorem 6.3 due to the lack of regularity of viscosity subsolutions and viscosity supersolutions, as mentioned above.

Theorem 6.12. *Let u and v be real-valued functions respectively upper and lower semicontinuous in Ω and $(x_\tau, y_\tau) \in \Omega \times \Omega$ a local maximum point of the function $u(x) - v(y) - \tau/2\,|x-y|^2$, for $\tau > 0$. Then there exist $X_\tau, Y_\tau \in \mathcal{S}^n$ such that*

$$(\tau(x_\tau - y_\tau), X_\tau) \in \overline{J}_\Omega^{2,+}u(x_\tau); \qquad (\tau(x_\tau - y_\tau), Y_\tau) \in \overline{J}_\Omega^{2,-}v(y_\tau)$$

and

$$-3\tau \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\tau & 0 \\ 0 & -Y_\tau \end{pmatrix} \leq 3\tau \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{6.33}$$

Remark 6.13. *In particular, from the rightmost inequality in (6.33), we get*

$$\left\langle \begin{pmatrix} X_\tau & 0 \\ 0 & -Y_\tau \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ \xi \end{pmatrix} \right\rangle \leq 3\tau \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ \xi \end{pmatrix} \right\rangle \implies \langle X_\tau\, \xi,\, \xi\rangle - \langle Y_\tau\, \xi,\, \xi\rangle \leq 0,$$

and since this holds for any $\xi \in \mathbb{R}^n$ we can conclude that $X_\tau \leq Y_\tau$.

Proof of Theorem 6.3. Arguing by contradiction, we suppose that $\sup_{\overline{\Omega}}(u - v) > 0$. Since $\overline{\Omega}$ is compact and u and $-v$ are upper semicontinuous by hypothesis, the supremum is attained (this is just the extension to semicontinuous functions of the extreme value theorem). And because

$u \le v$ on $\partial\Omega$, it follows that there is a point $\hat{x} \in \Omega$ such that $u(\hat{x}) - v(\hat{x}) = \max_{\overline{\Omega}}(u - v) > 0$. On the other hand, by Lemma 6.10, for every $\gamma > 0$ there exists a lower semicontinuous function $\tilde{v}$ such that $\tilde{v} - v \le \gamma$ in Ω. As a result, $u(\hat{x}) - v(\hat{x}) > \gamma \ge \tilde{v}(\hat{x}) - v(\hat{x})$ for γ small enough and therefore $u(\hat{x}) > \tilde{v}(\hat{x})$.

This implies that there is a point $\tilde{x} \in \overline{\Omega}$ such that $u(\tilde{x}) - \tilde{v}(\tilde{x}) = \max_{\overline{\Omega}}(u - \tilde{v}) > 0$ (recall that $-\tilde{v}$ is upper semicontinuous by construction, as it is v). In fact $\tilde{x} \in \Omega$ since by Lemma 6.10, we can assume $\tilde{v} - v \ge -\gamma$ on $\partial\Omega$ and therefore $u(\tilde{x}) - \tilde{v}(\tilde{x}) > \gamma \ge v(x) - \tilde{v}(x) \ge u(x) - \tilde{v}(x)$ for all $x \in \partial\Omega$. For the sake of simplicity let us assume $\tilde{x}$ to be $\hat{x}$.

Consider then

$$M_\tau = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \left(u(x) - \tilde{v}(y) - \frac{\tau}{2}|x-y|^2 \right)$$

for $\tau > 0$, which is finite since $u(x) - \tilde{v}(y)$ is upper semicontinuous, $\tau/2\,|x-y|^2$ is continuous and $\overline{\Omega}$ is compact. This also implies that the supremum is achieved in $\overline{\Omega} \times \overline{\Omega}$. We denote this maximum point by (x_τ, y_τ), so that $M_\tau = u(x_\tau) - \tilde{v}(y_\tau) - \tau/2\,|x_\tau - y_\tau|^2$. In particular $M_\tau \ge u(\hat{x}) - \tilde{v}(\hat{x})$, so that from Lemma 6.11 it follows that $(x_\tau, y_\tau) \in \Omega \times \Omega$ for large τ. Then, Theorem 6.12 implies that there exist $X_\tau, Y_\tau \in \mathcal{S}^n$ such that

$$(\tau(x_\tau - y_\tau),\, X_\tau) \in \overline{J}_\Omega^{2,+} u(x_\tau) \quad \text{and} \quad (\tau(x_\tau - y_\tau),\, Y_\tau) \in \overline{J}_\Omega^{2,-} \tilde{v}(y_\tau) \quad \text{and} \quad X_\tau \le Y_\tau.$$

By Lemma 6.10, $\tilde{v}$ is a viscosity supersolution of equation (6.23) and a strict viscosity supersolution of (6.1), in fact. This is equivalent to saying that

$$\min\left\{ |\nabla \tilde{v}(x)| - 1, -\Delta_\infty^N \tilde{v}(x) \right\} \ge \mu > 0$$

holds in the viscosity sense. According to this, in view of Lemma 6.9 and the following numerical inequality

$$\min\{a,b\} - \min\{a,c\} \le \max\{0, b-c\} \quad \text{for} \quad a,b,c \in \mathbb{R},$$

we arrive to the following contradiction,

$$\begin{aligned} 0 < \mu &\le \min\left\{ \tau|x_\tau - y_\tau| - 1,\, -\left\langle Y_\tau \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|}, \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|} \right\rangle \right\} \\ &\qquad - \min\left\{ \tau|x_\tau - y_\tau| - 1,\, -\left\langle X_\tau \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|}, \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|} \right\rangle \right\} \\ &\le \max\left\{ 0,\, -\left\langle (Y_\tau - X_\tau) \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|}, \frac{x_\tau - y_\tau}{|x_\tau - y_\tau|} \right\rangle \right\} \\ &\le 0, \end{aligned}$$

where the last inequality follows from $X_\tau \le Y_\tau$. □

Consider now the following maximum principle which, although it does not play any role in the proof of Theorem 6.1, follows directly from Theorem 6.3 (the proof is inspired by [7, Section 5.B]).

Corollary 6.14 (Maximum Principle)**.** *Under the same assumptions of Theorem 6.3, it holds*

$$\sup_{x\in\overline{\Omega}} (u(x) - v(x)) = \sup_{x\in\partial\Omega} (u(x) - v(x)).$$

Proof. Let K be a constant such that $u - K \le v$ on $\partial\Omega$. Since $J_\Omega^{2,+}(u-K)(x) = J_\Omega^{2,+}u(x)$ and $G(x, u(x) - K, p, X) = G(x, u(x), p, X) \le 0$ for all $x \in \Omega$, it follows from Lemma 6.9 that $u - K$ is a viscosity subsolution of equation (6.1). Moreover, since u is upper semicontinuous in $\overline{\Omega}$, so it is $u - K$. At this point, by Theorem 6.3 we have that $u - K \le v$ in $\overline{\Omega}$, i.e., $u - v \le K$ in $\overline{\Omega}$. The result then follows by taking $K = \sup_{\partial\Omega}(u - v)$, which is finite since $\partial\Omega$ is compact and u and $-v$ are upper semicontinuous by hypothesis (this is just the extension to semicontinuous functions of the extreme value theorem). □

Appendix A

Second-order elliptic operators

The following definitions are taken from [9, Chapter 6], where a deep look into second-order elliptic equations is considered. Let G denote a second-order partial differential operator in either the so called *divergence form*,

$$Gu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{i,j}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u, \tag{A.1}$$

or in *non-divergence form*,

$$Gu = -\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u, \tag{A.2}$$

for a given function $u = u(x) : \Omega \to \mathbb{R}$ and real coefficient functions $a_{i,j}(x)$, $b_i(x)$, $c(x)$ for $i, j = 1, \ldots, n$, defined in Ω. Consider also the symmetry condition $a_{i,j}(x) = a_{j,i}(x)$ for all $i, j = 1, \ldots, n$.

Remark A.1. *Both forms are equivalent provided the coefficients $a_{i,j}(x)$ $(i, j = 1, \ldots, n)$ are differentiable and u is $\mathcal{C}^2$. Then, an operator G written in divergence form can be rewritten into non-divergence structure and vice versa. Indeed the divergence form (A.1), under the required regularity assumptions, becomes*

$$Gu = -\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_j \partial x_i} + \sum_{i=1}^{n} \tilde{b}_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u$$

for $\tilde{b}_i(x) = b_i(x) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_{i,j}(x)$ $(i = 1, \ldots, n)$, which is non-divergence form. In other words, we can go from (A.1) to (A.2) and vice versa whenever

$$\frac{\partial}{\partial x_j} \left(a_{i,j}(x) \frac{\partial u}{\partial x_i} \right) = a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j} a_{i,j}(x)$$

holds for all $i, j = 1 \ldots, n$. The divergence form is most natural for energy methods, based upon integration by parts, and the non-divergence form is most appropriate for maximum principle techniques. Observe that, in the case of solutions that might not be regular enough for (A.1) and (A.2) to hold in the classical sense, (A.1) and (A.2) are no longer equivalent.

Definition A.2 (Elliptic operator)**.** *The second-order partial differential operator G is said to be* elliptic *if there exists a constant $\lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\, \xi_i\, \xi_j \geq \lambda\, |\xi|^2$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

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