Title: Ohba’s conjecture and beyond for generalized colorings

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Abstract

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Let $G$ be a graph. Ohba’s conjecture states that if $|V(G)| \leq 2\chi(G) + 1$, then $\chi(G) = \chi^L(G)$. Noel, West, Wu and Zhu extended this result and proved that for any graph, $\chi^L(G) \leq \max\{\chi(G), \lceil(|V(G)| + \chi(G) - 1)/3\rceil\}$. Ohba, Kierstead and Noel proved that this bound is sharp for the ordinary chromatic number. In this work we prove that both results hold for generalized colorings as well, and find examples that prove the sharpness of the second one for the acyclic and star chromatic numbers.
# Contents

Introduction ........................................ 1

1 Colorings and list colorings of graphs ................. 3
   1.1 Graph coloring and list coloring .................. 3
   1.2 The Ohba conjecture and beyond .................. 4
   1.3 Generalised colorings ............................. 5

2 Proof of Theorem 1.6 ................................ 9
   2.1 Preliminaries ...................................... 9
   2.2 Near-acceptable colorings .......................... 11
   2.3 Counting ........................................... 15

3 Proof of Theorem 1.7 ................................ 19
   3.1 Preliminaries ...................................... 19
   3.2 Sufficient condition for an $L$-coloring ........... 21
   3.3 Merges .............................................. 22
       Parts of size 3 ...................................... 24
       Parts of size 4 ...................................... 25
       Remaining merges .................................... 26

4 Tightness ............................................ 29
   4.1 Ordinary chromatic number ......................... 29
   4.2 Acyclic chromatic number ........................... 31
   4.3 Star chromatic number ................................ 32

Conclusions ............................................ 35
One of the most studied topics in Graph Theory is graph coloring. The classical question in graph coloring is what is the minimum number of colors needed to color the vertices of a graph so that two adjacent vertices do not get the same color. This number is known as the chromatic number of a graph. The appeal of graph coloring is probably the simplicity of the statements of the problems. However, it is often the case that the solution to coloring problems is not simple. Probably the most famous example of this fact is the four color theorem: it states that any map in a plane can be colored using four colors in such a way that regions sharing a border do not share a color. In terms of graphs, this means that the chromatic number of a planar graph is at most four. This result was first conjectured by Guthrie in 1852, but it was not until 1977 that Appel and Haken constructed a computer-assisted proof of the theorem.

A variant of the classical coloring of graphs is the list coloring of graphs. Introduced by Vizing [17] and Erdős, Rubin and Taylor [4] independently, it is the same problem but restricting the set of possible colors that can be used for each of the vertices. This version of graph coloring has several motivations, such as completing the coloring of a graph that has already been partially colored, or the frequency assignment problem in mobile communications. There is a large literature on list colorings, also known as graph choosability. Some extensive surveys on the topic can be found in Alon [1] and Woodall [18].

The list chromatic number of a graph is the minimum number \( t \) such that assigning any \( t \) colors to each vertex there is a coloring of the graph in which each vertex receives a color from its list. The gap between the chromatic number and the list chromatic number can be really large in some graphs. For example, the complete bipartite graphs \( K_{n,n} \) (all pairs with one element in each of two sets of size \( n \) as edges) have chromatic number two while their list chromatic number is an increasing function of \( n \). There are several results and conjectures about under which conditions these two parameters are equal. Arguably the most famous one is the List Coloring Conjecture, that states that line graphs have equal chromatic and list chromatic numbers. This thesis concerns another such conjecture, Ohba’s conjecture, which states that the list chromatic number and the ordinary chromatic number coincide when the latter is larger than half the order of the graph. Ohba’s conjecture attracted a lot of interest. After several attempts it was proved asymptotically by Reed and Sudakov [16] by probabilistic methods, deserving an invited address at the International Congress of Mathematics in...

Motivated by many different theoretical and practical reasons, there is a large amount of generalized chromatic numbers, including acyclic chromatic numbers, $k$-distance chromatic numbers among many others. A large family of them can be described in a general framework stating the minimum colors a subgraph must have. Another family are related to graph homomorphisms. In this work we focus precisely on the acyclic and star chromatic numbers, first introduced by Grünbaum [7], as well as the ordinary chromatic number.

Ohba’s conjecture gives the list chromatic number when the chromatic number is half the order of the graph. Another result by Noel, West, Wu and Zhu [12] gives a bound on the list chromatic number when the chromatic number is close to half the order of the graph. The main contribution of this Thesis is to provide a proof of Ohba conjecture for all classes of chromatic numbers. We focus on acyclic and star chromatic numbers, where every pair of color classes induces a forest or a star forest respectively, but our proof applies to any conceivable definition of chromatic number which induces a partition on the vertex set of a graph.

In the first chapter we will introduce the necessary concepts of graph coloring needed to understand the results, and present all the conditions under which it is known for the chromatic and list chromatic numbers of a graph to coincide. We will restate the theorems that we want to prove in terms of partitions of the vertex set of a graph, which will allow us to prove them for any generalized coloring.

In the second chapter we will prove Ohba’s conjecture for generalized colorings. The proof is adapted from Noel, Reed and Wu’s proof [11], and the key idea is to define a bipartite graph that encodes a list assignment of a graph in which an acceptable coloring corresponds to a matching that saturates the graph. The proof then relies on Hall’s theorem on matchings to find this.

In the third chapter we will prove the other result by Noel, West, Wu and Zhu that deals with the case of graphs with a chromatic number close to half the order of the graph. The key idea is the same as for the proof of Ohba’s conjecture, and in the end the problem is again reduced to finding a matching in an adequately defined bipartite graph.

In the fourth chapter we will see examples that show that Ohba’s conjecture gives the best possible bounds for the ordinary, acyclic and star chromatic numbers. In the case of the ordinary chromatic number we also give an example that shows that the second result that we prove is also the best possible bound.

The last Chapter of the Thesis contains some final conclusions and comments on further work on the area.
Chapter 1

Colorings and list colorings of graphs

The goal of this work is to prove a theorem on list colorings for a more general notion of vertex-coloring of a graph. This chapter presents the basics of graph colorings and list colorings and defines the concept of $f$-colorings. We give some context to the results that we aim to prove, and state them in terms of $f$-colorings, or partitions of the vertex set of a graph.

1.1 Graph coloring and list coloring

Let $G$ be a graph. A vertex coloring of $G$ is a map

$$\chi : V(G) \to [k].$$

The coloring is proper if no edge is monochromatic, that is if adjacent vertices receive different colors. The chromatic number of $G$, denoted $\chi(G)$, is the minimum number of colors in a proper vertex coloring.

Let $L(v)$ be a list of colors associated to each vertex $v \in V(G)$. A list coloring of $G$ is a proper coloring $\chi$ such that

$$\chi(v) \in L(v) \text{ for every } v \in V(G).$$

$G$ is $k$-choosable if for every family of lists $\{L(v) : v \in V(G)\}$ with $|L(v)| \geq k$, there is a list coloring. The list chromatic number $\chi^L(G)$ of $G$ is the minimum integer $k$ such that $G$ is $k$-choosable.

In general, the list chromatic number of a graph is larger than its chromatic number. An easy example of this fact is the complete bipartite graph $K_{3,3}$, shown in 1.1. Let $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be the two stable sets of $K_{3,3}$, and define the list assignment $L$ as

$$L(u_i) = L(v_i) = \{1, 2, 3\} \setminus \{i\}.$$
Then there is no proper vertex coloring that assigns to each vertex a color from its list. In fact, there are bipartite graphs with arbitrarily high list chromatic number. Alon [1] shows that for bipartite graph $G$, $\chi_L(G)$ is bounded from below by a function of the minimum degree $\Delta = \Delta(G)$ which goes to infinity as $\Delta \to \infty$. In view of this gap between the parameters $\chi(G)$ and $\chi_L(G)$, the following question arises: under which conditions is there equality of chromatic and list chromatic numbers?

There are many conjectures that give conditions on a graph so that $\chi(G) = \chi_L(G)$. The most famous of these is known as the List Coloring conjecture, suggested by various researchers, that was published for the first time by Bollobás and Harris [2].

**Conjecture 1.1** (List Coloring Conjecture). If $G$ is a line graph, then $\chi(G) = \chi_L(G)$.

The line graph $L(G)$ of a graph $G$ is the graph that has a vertex for each edge of $G$, and in which two vertices are adjacent if the corresponding edges of $G$ shared a common endpoint. A special case of this conjecture was proved by Galvin [5].

**Theorem 1.2** (Galvin). If $G$ is the line graph of a bipartite graph, then $\chi(G) = \chi_L(G)$.

A graph is claw-free if it does not contain $K_{1,3}$ as an induced subgraph. In particular, line graphs are claw-free, so the following conjecture (see [6]) would imply the List Coloring Conjecture.

**Conjecture 1.3** (Gravier and Maffray). If $G$ is a claw-free graph, then $\chi(G) = \chi_L(G)$.

This thesis concerns the Ohba conjecture, which is another particular case in which $\chi(G) = \chi_L(G)$ and which we discuss in the next section.

## 1.2 The Ohba conjecture and beyond

Erdos, Rubin and Taylor [4] obtained in a paper an interesting example of a graph $G$ such that $\chi(G) = \chi_L(G) = k$. That was the case of the $k$-partite graph
CHAPTER 1. COLORINGS AND LIST COLORINGS OF GRAPHS

$K(2, 2, ..., 2)$, consisting of $k$ sets of cardinality two and all edges joining vertices in distinct sets. This example inspired Ohba [13] to conjecture the following.

**Conjecture 1.4 (Ohba).** If $|V(G)| \leq 2\chi(G) + 1$, then $\chi(G) = \chi^L(G)$.

The intuition behind this statement is that if the chromatic number of a graph is large (larger than half the number of vertices) then the list chromatic number will not be larger. If the conjecture proved true it would be the best possible bound, since there are examples of graphs on $2\chi(G) + 2$ vertices such that $\chi^L(G) > \chi(G)$.

We show an example of this fact in Chapter 4.

In the paper where this result first appeared, Ohba managed to prove that $\chi(G) = \chi^L(G)$ for graphs with $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$. Reed and Sudakov [15] improved this result and proved via a probabilistic argument that $\chi(G) = \chi^L(G)$ for graphs with $|V(G)| < \frac{3}{4}\chi(G) - \frac{4}{3}$, and later, the same authors obtained in [16] an asymptotic result by showing that $\chi^L(G) = \chi(G)$ as long as $n = (2-o(1))\chi(G)$.

The conjecture was finally proved by Noel, Reed and Wu [11] some ten years later. Their argument uses only Hall’s theorem on matchings, and as we will show in the second chapter it can be adapted to prove that Ohba’s conjecture is true for other chromatic numbers apart from the ordinary chromatic number.

When the chromatic number of a graph is not larger than half the number of vertices but it is close, intuitively the list chromatic number should not grow too large. Noel, West, Wu and Zhu [12] proved the following theorem.

**Theorem 1.5.** For any graph $G$,

$$\chi^L(G) \leq \max\{\chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil \}.$$  

For graphs such that $|V(G)| \leq 2\chi(G) + 1$ this statement is exactly Ohba’s conjecture. The interesting case is when $|V(G)|$ is not much larger than $2\chi(G) + 2$; in this case the theorem says that the gap between the chromatic number and the list chromatic number is not very big. More precisely, if $n = \chi(G) + t$ then $\chi^L(G) \leq \chi(G) + \left\lceil (t - 1)/3 \right\rceil$.

The bound is also sharp for some graphs with $|V(G)| \geq 2\chi(G) + 2$. We will see some examples in Chapter 4.

As it happens with 1.4, the proof of this theorem given by Noel, West, Wu and Zhu can also be adapted to prove the result for other notions of chromatic numbers. We will show this fact in Chapter 3.

### 1.3 Generalised colorings

Let $G$ denote the class of graphs and $f : G \rightarrow \mathbb{N}$. An $f$–coloring of $G$ is a vertex coloring $c$ such that, for every subgraph $H \subseteq G$,

$$|c(H)| \geq f(H).$$
In other words, every subgraph uses at least \( f(H) \) colors.

Several generalizations of the chromatic number fit into this framework. We are interested in:

(i) the ordinary chromatic number corresponds to the function \( f(K_2) = 2 \) and \( f(H) = 0 \) for all \( H \neq K_2 \): edges must receive at least two colors and nothing else is specified.

(ii) the acyclic chromatic number corresponds to the function \( f(K_2) = 2 \), \( f(C_n) = 3 \) for every cycle \( C_n, n \geq 3 \) and \( f(H) = 0 \) for every other graph: every two colors induce a graph which can not contain a cycle, and the coloring is proper.

(iii) the star chromatic number corresponds to the function \( f(K_2) = 2 \), \( f(P_4) = 3 \) and \( f(H) = 0 \) for every other graph: every two colors induce a graph with no \( P_4 \), namely a forest of stars, and the coloring is proper.

Of course the formulation allows for a very large class of colorings, for example

(iv) Distance \( k \) chromatic number: \( f(P_n) = \min(k, n), k \geq 2 \); here vertices at distance at most \( k - 1 \) receive distinct colors.

(v) Generalized colorings for LowTreeWidth decompositions: \( f(H) = tw(H) \) where \( tw(H) \) is the treewidth of \( H \).

(vi) Generalized colorings for LowTreeDepth decompositions: \( f(H) = td(H) \) where \( td(H) \) is the treedepth of \( H \).

We wish to write a statement similar to Ohba’s conjecture for these general notions of chromatic numbers. Note that an \( f \)-coloring of a graph \( G \) defines a partition of the set of vertices \( V \) into color classes.

Let \( \mathcal{P} = \{V_1, ..., V_k\} \) be a partition of a set \( V \). We say that \( \mathcal{P} \) is \( L \)-choosable for a given set \( L = \{L(v) \in \mathbb{N} : v \in V\} \) of lists if there is a map

\[
c : V \to \mathbb{N}
\]

such that

(i) \( c(v) \in L(v) \) for each \( v \in V \), and

(ii) the partition \( \{c^{-1}(i) : i \in C\} \) is a refinement of \( \mathcal{P} \), where \( C = \bigcup_{v \in V} L(v) \) denotes the list of possible colors.

We say that \( \mathcal{P} \) is \( t \)-choosable if it is \( L \)-choosable for any set of lists, each of cardinality \( t \). When \( t = |\mathcal{P}| \) we simply say that \( \mathcal{P} \) is choosable.
Thus, if a graph admits a chromatic partition which is $\chi(G)$-choosable then its list chromatic number coincides with its chromatic number. This adapts to any generic notion of chromatic number.

The proof for Ohba’s conjecture extends to a proof of the following result.

**Theorem 1.6.** A partition $\mathcal{P}$ of a set $V$ such that $|V| \leq 2|\mathcal{P}| + 1$ is choosable.

This theorem implies that any kind of chromatic number coincides with the corresponding list chromatic number as long as it has value at least $(|V| - 1)/2$. This means that the Ohba conjecture holds for all these other notions of chromatic numbers. We give a proof of this theorem in Chapter 2.

For the case $|V| \geq 2|\mathcal{P}| + 2$ we can also produce a statement similar to 1.5 in terms of a partition of the set of vertices.

**Theorem 1.7.** Let $\mathcal{P}$ be a partition of a set $V$ and $t = \max\{|\mathcal{P}|, \left\lceil \frac{|V|+|\mathcal{P}|-1}{3} \right\rceil \}$. Then $\mathcal{P}$ is $t$-choosable.

Again since any $f$-coloring induces a partition of the set of vertices, this result implies that Theorem 1.5 is true for any $f$-coloring. We give a proof of this theorem in Chapter 3.
Chapter 2

Proof of Theorem 1.6

In this chapter we adapt a proof by Noel, Reed and Wu [11] to prove Theorem 1.6, which implies Ohba’s conjecture for our generalized definition of $f$-colorings. We use the notation $\chi(G)$ and $\chi^L(G)$ for the chromatic and list chromatic numbers associated to these $f$-colorings, independently of the chosen $f$. We do this because the proof is valid for all of them, since the list $f$-coloring found respects the original $f$-coloring partition, or is a refinement of it.

2.1 Preliminaries

Let $C = |\bigcup_{v \in V(G)} L(v)|$ be the set of colors in our list assignment. We define the following bipartite graphs that we wil use throughout the whole proof. Let $B$ be the bipartite graph with vertex set $C \cup V(G)$ and an edge between a color $c \in C$ and a vertex $v \in G$ if and only if $c \in L(v)$. In this graph a matching corresponds to a partial list coloring of $G$. Let $f : V(G) \to \mathbb{N}$ be a coloring of $G$ and $V_f = \{f^{-1}(c) : c \in C\}$ the color classes under $f$. We define $B_f$ as the bipartite graph with vertex set $C \cup V_f$ and an edge between a color $c' \in C$ and a color class $f^{-1}(c)$ if and only if $c' \in \bigcap_{c \in f^{-1}(c)} L(v)$. A matching in this graph corresponds to a partial list coloring of $G$ such that its color classes are contained in $V_f$.

To find this matching the proof uses Hall’s theorem.

**Theorem 2.1 (Hall).** Let $B = (X, Y)$ be a bipartite graph. $B$ has a matching that saturates $X$ if and only if for every $S \subseteq X$,

$$|S| \leq |N_B(S)|.$$  

We consider $G$ to be a minimal counterexample of the Ohba conjecture, and in the end we will reach a contradiction. This $G$ is a graph with $n = 2k + 1$ vertices such that it admits a proper coloring $f : V(G) \to [k]$. $G$ has $\chi(G) = k$ and $\chi^L(G) > k$, and Ohba’s conjecture is true for all graphs on fewer vertices. $L$
is a list assignment of $G$ such that $|L(G)| \geq k$ for all $v \in V(G)$ and $G$ is not $L$-colorable.

The first step is to show that it suffices to prove the Theorem for lists which span less than $n$ colors. This lemma was proved independently by Kierstead [8] and Reed and Sudakov [15] and is a standard reduction for minimal counterexamples in choosability problems.

**Lemma 2.2** (Small Pot Lemma). Suppose that, for every collection $\{L(v) : v \in V(G)\}$ of lists, each with $|L(v)| \geq t$ elements and with a total number of colors $|\bigcup_{v \in V(G)} L(v)| < n$, there is an $f$-coloring where each vertex gets a color in its list. Then $\chi^L(G) \leq t$.

**Proof.** Suppose $\chi^L_a(G) > t$. Then there exists a list assignment $\{L(v) : v \in V(G)\}$, $|L(v)| \geq t$, for which there is no coloring where each vertex gets a color in its list. Take one such assignment that minimizes $|C|$. If $|C| < n$ we are done, so suppose $|C| \geq n$.

Let $H \subset C$ be the minimum subset of colors that cannot be covered by a matching in $B$. $H$ is non empty, since we are assuming $|C| \geq n$, and by minimality of $H$, $|H| \leq |V(G)|$ otherwise there would be a matching in $B$ saturating $V(G)$ that would correspond to a coloring of $G$ in which each vertex gets a color in its list. Again by minimality of $H$ there is a matching $M$ of size $|H| - 1$ with all edges ending in $H$. Let $W \subset V(G)$ be the subset of endpoints of $M$ in $V(G)$. Note that $|W| = |N(H)|$, so we have that for all $v \in V(G) \setminus W$, $L(v) \cap H = \emptyset$.

We define a new list assignment $L'$ in the following way. Let $u \in V(G) \setminus W$. For $v \in W$ $L'(v) = L(u)$, and for $v \notin W$ $L'(v) = L(v)$. Let $C' = \bigcup_{v \in V(G)} L'(v)$. Since the list $L(u)$ used to define the new list assignment was disjoint with $H$ we have that $|C'| < |C|$, and so by minimality there is a coloring that is acceptable with respect to the list assignment $L'$. That means we have a coloring of $V(G) \setminus W$ with the original list assignment $L$ that does not use the colors in $H$, which can be extended to a coloring of $G$ with the assignments for $W$ given by the matching $M$.

**Corollary 2.3.** Let $G$ be our minimal counterexample, $f$ the given coloring of $G$ with $k$ colors. Suppose that there is a monochromatic class with two elements. Then $\chi(G) \leq k$.

**Proof.** The proof is by induction on $n$. By the Lemma we may assume that the total number of colors is at most $n - 1$. Suppose that one monochromatic set has two elements $\{x, y\}$. Then $L(x) \cup L(y) < n$ and $L(x), L(y) \geq k$ implies that $L(x) \cap L(y) \neq \emptyset$. Let $c$ be a color common to the two lists. Remove this part and $c$ from all lists of remaining lists. By induction, the resulting graph can be colored with the given lists. This coloring can be extended to the pair $\{x, y\}$ by assigning the color $c$ to them.

So in the given coloring of our minimal counterexample for the Ohba conjecture
there are no color classes of size two. We can also see that in a non-singleton part of $G$ the lists of all vertices have empty intersection.

Lemma 2.4. If $P$ is a part of $G$ such that $|P| \geq 3$, then $\bigcap_{v \in P} L(v) = \emptyset$.

Proof. Suppose that a non-singleton part $P$ has a common color $c$ in all its lists. Color the vertices of $P$ with $c$. Let $G' = G - P$, and define the new list assignments $L'(v) = L(v) - \{c\}$ for every $v \in G'$. Then:

$$
|V(G')| \leq 2(k-1) + 1
$$

$$
\chi(G') \leq k - 1
$$

$$
|L'(v)| \geq k - 1 \text{ for all } v \in V(G')
$$

so by minimality of our counterexample there is an acceptable coloring for $G'$ that does not use $c$, so together with coloring $P$ with $c$ there is an acceptable coloring for $G$. \qed

2.2 Near-acceptable colorings

We see that we can find a ”near-acceptable” coloring for $G$, that is, we allow a vertex to get a color that is not on its list as long it is the only vertex to receive that color and the color appears on many lists. We quantify this notion, introduced in [11].

Definition 2.5. Let $A$ be the set of singleton parts in the partition of $V(G)$ induced by $f$. We say that a color $c \in C$ is frequent if one of the following is true:

- $N_B(c) \geq k + 1$.
- $|N_B(c) \cap A| \geq |V(G)| - |C| = \gamma$.

Definition 2.6. We say a coloring $f$ is near-acceptable for a list assignment $L$ if for every vertex $v \in V(G)$, either:

- $f(v) \in L(v)$, or
- $f(v)$ is frequent and $f^{-1}(f(v)) = \{v\}$.

That is, a coloring is near-acceptable if a vertex either gets a color in its list, or gets a frequent color that no other vertex uses.

We will show that if we have enough frequent colors then we have a near-acceptable coloring for $L$. In the graph $B_f$ this is going to be a matching in which there might be some edges that do not exist, but such missing edges join singleton classes with frequent colors.
Proposition 2.7. If $C$ contains at least $k$ frequent colors, then there is a near-acceptable coloring for $L$.

Proof. Let $F$ be a set of $k$ frequent colors. We construct a near acceptable coloring applying a greedy procedure in three steps.

Step 1: Take $V_1 \subseteq V(G)$ and $f_1 : V_1 \rightarrow C \setminus F$ an acceptable partial coloring such that $V_1$ has the maximum number of vertices possible, and subject to this, such that it has vertices of the maximum number of parts possible.

Step 2: For every part $P$, define $R_P := P \setminus V_1$. Order the parts $P_1, ..., P_k$ so that $|R_{P_1}| \geq ... \geq |R_{P_k}|$. For each part $P_i$ we try to color the vertices yet uncolored, namely the ones in $R_{P_i}$, with a frequent color that we have not used yet and such that it is in the lists of all the vertices in $R_{P_i}$. We stop at $i + 1$ when we cannot do this anymore. This gives us the sets:

$$V_2 := \bigcup_{j=1}^{i} R_{P_j},$$
$$V_3 := \bigcup_{j=i+1}^{k} R_{P_j},$$

and a partial acceptable coloring $f_2 : V_2 \rightarrow F$. If $i = k$ we have found an acceptable coloring. Suppose $i < k$ and there is no frequent color available for all vertices of $R_{P_{i+1}}$.

Step 3: Let $U \subset F$ be the set of frequent colors that have not been used in step 2. We have $|U| = k - i$. If $|V_3| \leq k - i$ we can map each of the vertices of $V_3$ to $U$ injectively and this mapping $f_3 : V_3 \rightarrow U$ is a near-acceptable coloring, since each vertex gets a frequent color that is only used for that vertex.

Suppose that $|V_3| \geq k - i + 1$. Then by the ordering of the $P_i$’s we have that $|R_{P_{i+1}}| \geq 2$ and this implies $|V_2| \geq |R_{P_{i+1}}| \geq 2i$. Since $|V(G)| = 2k + 1$, we get

$$|V_1 \cup V_2| = |V(G)| - |V_3| \leq (2k + 1) - (k - i + 1) = k + i,$$

and so

$$|V_1| \leq (k + i) - |V_2| \leq (k + i) - 2i = k - i.$$

Let us show that $V_1$ has precisely $k - i$ vertices. For this we use the fact that every color of $U$ is not in $L(v)$ for at least one vertex $v \in R_{P_{i+1}}$. Every vertex of $R_{P_{i+1}}$ has $k$ colors in its list, and $|F| = k$, so the colors of $C \setminus F$ appear at least $|U| = k - i$ times in the lists of the vertices in $R_{P_{i+1}}$. If $c \in C \setminus F$ is in the list of $j > 0$ vertices of $R_{P_{i+1}}$, then:

• $c$ was not used in step 1 to color any vertices of $P_{i+1}$, otherwise we would have colored these $j$ vertices in step 1 and would have got a larger $V_1$. 


At least $j$ vertices received the color $c$ in step 1, otherwise we could change
those vertices for our $j$ vertices of $R_{P_{i+1}}$ in step 1 and would have got a
larger $V_1$.

Since the colors of $C \setminus F$ appear at least $k - i$ times in the lists of $R_{P_{i+1}}$, we
have that at least $k - i$ vertices of $V(G) - P_{i+1}$ have been colored in step 1. So
$|V_1| \geq k - i$. So indeed $|V_1| = k - i$ and $V_1 \cap P_{i+1} = \emptyset$.

Since $R_{P_{i+1}} = P_{i+1}$ we have that $|R_{P_{i+1}}| = |P_{i+1}| \geq 3$ by Corollary 2.1. Then:

\[ |V_2| \geq i|R_{P_{i+1}}| \geq 3i, \]
\[ |V_1| \leq (k + i) - |V_2| \leq (k + i) - 3i = k - 2i, \]

and this implies $i = 0$, and $|V_1| = k$, $|V_2| = 0$, $U = F$ and $R_{P_1} = P_1$ (because
$V_1 \cap P_{i+1} = \emptyset$). Now, we have $k$ frequent colors that we have not used and $k + 1$
vertices to color. If we can find one color that is in the lists of two vertices of the
same part, we can map the rest of the vertices injectibly to the remaining colors
and we obtain a near-acceptable coloring.

We want to find $c \in F$ available for $u, v \in P_1$. The colors of $C \setminus F$ appear in the
lists of $P_1$ at most $|V_1| = k$ times by the maximality of $V_1$. Since $P_1$ has at least
size 3 and each list has at least size $k$, we have that the colors of $F$ appear in the
lists of $P_1$ at least $2k$ times. But $|F| = k$, so by the pigeonhole principle there is
a color $c \in F$ which appears at least in two lists of $P_1$.

We next show that if we have a near-acceptable coloring, then we can find an
acceptable coloring.

**Proposition 2.8.** If there exists a near-acceptable coloring for $L$, then there
exists an acceptable coloring for $L$.

**Proof.** If we have a matching in $B_f$ that saturates $V_f$ then we have an acceptable
coloring for our graph, so we suppose that we do not have such a matching. By
Hall, this implies that there exists $S \subseteq V_f$ such that $|N_{B_f}(S)| < |S|$.

This means that there is a vertex in $S$, that is, $f^{-1}(c^*) \in S$ such that $c^* \notin N_{B_f}(S)$
In particular $c^* \notin N_{B_f}(f^{-1}(c^*))$. So there exists a vertex $v \in V(G)$ such that
$f(v) = c^*$ and $c^* \notin L(v)$. Since the coloring is near-acceptable, this means $c^*$ is a
frequent color and $v$ is the only vertex colored with $c^*$.

Case 1: $c^*$ appears in at least $k + 1$ lists. Then:

\[ L(v) = N_{B_f}(f^{-1}(c^*)) \subseteq N_{B_f}(S), \]

so $|N_{B_f}(S)| \geq k$. By our choice of $S$, we have that $|S| > |N_{B_f}(S)| \geq k$. We see
that $|S| \leq k$ to reach a contradiction.
Since \( c^* \notin N_{B_f}(S) \), every color class of \( S \subseteq V_f \) must contain a vertex that does not have \( c^* \) in its list. But \( c^* \) appears in at least \( k + 1 \) lists, so there are at most \( |V(G)| - (k + 1) = (2k + 1) - (k + 1) = k \) vertices that can have a list in which \( c^* \) does not appear, so \( |S| \leq k \).

Case 2: \( c^* \) appears in more than \( \gamma = |V(G)| - |C| \) lists of singletons.

We assume that \( f \) is surjective and that our choice of \( S \) maximizes \( |S| - |N_{B_f}(S)| \). By maximality of \( S \) we then have that for every \( T \subseteq V_f \setminus S \), \(|N_{B_f}(T) - N_{B_f}(S)| \geq |T| \). By Hall there is a matching in \( B_f - N_{B_f}(S) \) that saturates \( V_f \setminus S \).

We are assuming that \( c^* \) appears in the lists of at least \( \gamma \) singletons and that \( c^* \notin N_{B_f}(S) \). This implies that the singletons that have \( c^* \) in their list are all in \( V_f \setminus S \).

Our near acceptable coloring \( g : V(G) \to C \) is a partial acceptable coloring when restricted to \( V_f \setminus S \to C \setminus N_{B_f}(S) \), since we have a matching, and also when restricted to the classes with more than one element of \( S \), because if \( f(v) \notin L(v) \) then \( v \) is a singleton. Combining these two we have that the only thing that remains is to color the singletons of \( S \). We define \( G' \) as the singletons of \( S \), and \( l \) as the number of classes with more than one element. Then we have:

\[
|V(G')| \leq 2(k - l) + 1,
\]

\[
\chi(G') \leq k - l,
\]

\[
|L'(v)| \geq k - l \text{ for all } v \in V(G'),
\]

because we are taking from \( G \) at least all elements of classes of more than one element, so \(|V(G')| \leq (2k + 1) - 2l = 2(k - l) + 1 \) and \( \chi(G') \leq k - l \). For the last inequality, in which \( L'(v) \) is the new list obtained by removing all colors used in the partial acceptable coloring of \( G \setminus G' \), we only need to see that \( g(G' \setminus G') \) uses at most \( l \) colors of \( N_{B_f}(S) \). But this is true because

\[
\frac{\text{#classes with more than one element or classes of } V_f \setminus S - \text{#classes of } V_f \setminus S}{\text{#classes with more than one element}} \leq l,
\]

and this tells us that the total number of colors used in the partial coloring, minus the colors that are not in \( N_{B_f}(S) \) is at most \( l \).

By minimality of our counterexample \( G \) we then have that there exists an acceptable coloring for \( G' \), which combined with the partial acceptable coloring that we have for \( G \setminus G' \) gives an acceptable coloring for \( G \). \( \Box \)

Later in the proof we will use again the last argument we used in 2.8. We write it as a lemma.

**Lemma 2.9.** Let \( S \subseteq V_f \) that maximizes \( |S| - |N_{B_f}(S)| \) and assume \( f \) is surjective. If there is a matching \( M \) in \( B_f - N_{B_f}(S) \) saturating \( V_f - S \) and \( V_f - S \) contains at least \( \gamma \) singletons of \( G \), then there is an acceptable coloring for \( L \).
2.3 Counting

The previous section tells us that our minimal counterexample $G$ must have less than $k$ frequent colors. Let $b = \#\text{non-singleton parts of } G$. In this section we see that $G$ actually has less than $b$ frequent colors and this will lead us to a contradiction. First we need the following proposition:

**Proposition 2.10.** $c \in C$ is frequent if and only if $c \in L(v)$ for every singleton $v$ of $G$.

*Proof.* Let $c$ be a frequent color and suppose $c$ is not in the list of a singleton $v$. Then we add $c$ to $L(v)$. By minimality of $G$ then there is an acceptable coloring with this new list, and this acceptable coloring must assign $c$ to $v$. So for the original lists there is a near-acceptable coloring, and by 2.8 this means there is an acceptable coloring.

Suppose now that $c$ is in the list of every singleton. If we see that $G$ has more than $\gamma$ singletons then $c$ is frequent. Suppose that there are less than $\gamma$ singleton parts, that is, suppose that $k - b < \gamma$. Let $F'$ be the set of colors that appear on at least $k + 1$ lists (a subset of frequent colors). Then, a color from $C \setminus F'$ can appear at most in $k$ lists, and a color from $F'$ can appear at most in $|V(G)| - b = 2k + 1 - b$ lists, because the intersection of all the lists of a non-singleton part is empty, so we remove every color from $F'$ from the list of at least one vertex for each non-singleton part. Therefore:

$$k|V(G)| \leq \sum_{v \in V(G)} L(v) = \sum_{c \in C} |N_B(c)| \leq k|C \setminus F'| + (2k + 1 - b)|F'| = k|C| + (k + 1 - b)|F'|.$$  

And so:

$$|F'| \geq \frac{k(|V(G)| - |C|)}{k + 1 - b} = \frac{k\gamma}{k + 1 - b} \geq k.$$  

But this is a contradiction because by 2.7 $G$ cannot have $k$ frequent colors. \qed

**Proposition 2.11.** There are less than $b$ frequent colors.

*Proof.* Suppose there exist $b$ frequent colors $c_1, \ldots, c_b$, that by 2.10 are available for every singleton. Label the singletons as $v_{b+1}, \ldots, v_k$ and let $A_b = \{c_1, \ldots, c_b\}$. Now for every $i \in \{b+1, \ldots, k\}$ we choose a color $c_i \in L(v_i) \setminus A_{i-1}$, where $A_i := A_{i-1} \cup \{c_i\}$. We define the new set of lists:

$$L'(v) = \begin{cases} A_k & \text{if } v \text{ is a singleton} \\ L(v) & \text{otherwise} \end{cases}$$

in this new list assignment $L'$ there are $k$ colors that are available for every singleton, so there are $k$ frequent colors, and by 2.7 and 2.8 there is an acceptable coloring $f'$ for $L'$. 


We construct an acceptable coloring \( f \) for the original list assignment \( L \). For every non-singleton vertex, we define \( f(v) = f'(v) \).

Note that \( f' \) assigns to the singleton parts of \( G \) a set of \( k-b \) colors disjoint from the colors it assigns to the non-singleton parts of \( G \). We denote the set of singletons by \( S \) and this set of colors by \( f'(S) \). Let \( S' := \{ v_i \text{ singleton} : c_i \in f'(S) \} \). Then:

\[
|S'| = k - b - |f'(S) \cap A_b|,
\]

and so

\[
|S - S'| = |f'(S) \cap A_b|
\]

Now, for every singleton \( v_i \in S' \) we define \( f(v_i) = c_i \). For the rest of the singletons, that is for singletons \( v \in S \setminus S' \), we can map them bijectively to the colors in \( f'(S) \cap A_b \), which gives a near-acceptable coloring of \( G \).

To reach a contradiction we only need to see that \( G \) has \( b \) colors that appear in the lists of all singletons. In our graph \( B \) this corresponds to finding a large set \( X \) of singletons such that \( N_B(X) \) is small. In particular we see the following:

**Proposition 2.12.** If \( c^* \in C \) is not available for every singleton, then there exists a set of singletons \( X = X_{c^*} \) such that:

- \( |X| \geq k + 1 - b - \gamma \), and
- \( |\bigcup_{v \in X} L(v)| \leq 2k - |N_B(c^*)| \).

**Proof.** Let \( x \) be a singleton such that \( c^* \notin L(x) \). We define the following list assignment:

\[
L^*(v) = \begin{cases} L(x) \cup \{c^*\} & \text{if } v = x \\ L(v) & \text{otherwise} \end{cases}
\]

where \( L \) is the original list assignment. The by maximality of the lists there is an acceptable list-coloring \( g \), in which \( x \) must be the only vertex to receive the color \( c^* \) and for any other vertex \( g(v) \in L(v) \).

If there is a matching in \( B_f \) that saturates \( V_f \) then \( G \) is \( L \)-colorable, so there exists a set \( S \subseteq V_f \) such that \( |S| > |N_{B_f}(S)| \). We choose one such set that maximizes \( |S| - |N_{B_f}| \). Then, there is a matching \( M \) in \( B_f - N_{B_f}(S) \) that saturates \( V_f - S \).

Since \( |N_{B_f}(S)| < |S| \) and for every color \( c \neq c^* \) we have \( c \sim f^{-1}(c) \) in \( B_f \), then \( f^{-1}(c^*) \in S \) and \( c^* \notin N_{B_f}(S) \), that is, every class of \( S \) contains at least one vertex that does not have \( c^* \) in its list. Therefore:

\[
|N_{B_f}(S)| < |S| \leq |V(G)| - |N_{B_f}(c^*)| \leq 2k + 1 - |N_{B_f}(c^*)|.
\]

We take \( X \) the set of singletons such that they are in \( S \). Then

\[
\bigcup_{v \in X} L(v) \subseteq N_{B_f}(S)
\]
and so
\[ \left| \bigcup_{v \in X} L(v) \right| \leq 2k - |N_{B_f}(c^*)|. \]

Suppose $|X| < k + 1 - b - \gamma$. This means there are at least $\gamma$ singletons in $V_f - S$. Then by 2.9 there is an acceptable coloring for $L$. \hfill \Box

We take $c^*$ a color that is not frequent, that is, it is not in the list of every singleton, and such that it maximizes $|N_B(c^*)|$, and we take the set $X = X_{c^*}$ given by the last proposition. Let $Z$ be the set of the $b - 1$ colors that appear more frequently in the lists of $X$, and let $Y = N_B(X) - Z$. Notice that $|Z| = b - 1$ and that if $c_1 \in Z$ and $c_2 \in Y$, then $|N_B(c_1) \cap X| \geq |N_B(c_2) \cap X|$. Since a frequent color is in the list of every singleton, we are assuming that all (at most $b$) frequent colors are in $Z$. Let $c' \in Y$ the a color that maximizes $|N_B(c') \cap X|$. We want to see that $|N_B(c') \cap X| \geq \gamma$, which would make $c'$ frequent, and which would mean that not all frequent colors are in $Z$, so there are more than $b$ frequent colors and we have reached a contradiction. The remaining part of the proof aims to see that $|N_B(c') \cap X| \geq \gamma$ and it is a counting argument.

**Definition 2.13.** $\beta := k - |N_B(c^*)|$.

Note that $\beta \geq 0$ because $c^*$ is not a frequent color. First we need to prove some inequalities:

**Lemma 2.14.** The following inequalities are satisfied:

- $\gamma + b \leq k$.
- $b \leq \frac{k+1}{2}$.
- $2\gamma < k + 1 - b$.

**Proof.** In 2.10 we proved that $k - b \geq \gamma$, so $\gamma + b \leq k$.

Recall that non-singleton parts have size at least three, so $(k - b) + 3b \leq |V(G)| \leq 2k + 1$, and rearranging this expression we get $b \leq \frac{k+1}{2}$.

Recall that in the proof for 2.10 we saw that the set $F'$ of colors that appear in at least $k + 1$ lists satisfies
\[ \frac{k\gamma}{k + 1 - b} \leq |F'| \leq b - 1, \]

ad by the previous inequality, $b - 1 < k/2$. Together these imply that $2\gamma < k + 1 - b$. \hfill \Box

**Proposition 2.15.** If $\beta \leq 2(k + 1 - b - 2\gamma$ then $|N_B(c') \cap X| \geq \gamma$. 

Proposition 2.16. Recall that \( c' \) maximizes \( |N_B(c') \cap X| \) and note that since \( |Z| = b - 1 \) then for every vertex \( v \in X \) we have \( |L(v) \cap Y| \geq k - (b - 1) = k + 1 - b \). Therefore:

\[
|Y||N_B(c') \cap X| \geq \sum_{c \in Y} |N_B(c) \cap X| = \sum_{v \in X} |L(v) \cap Y| \geq |X|(k + 1 - b).
\]

Reordering this expression and using 2.12 we get:

\[
|N_B(c') \cap X| \geq \frac{|X|(k + 1 - b)}{|Y|} = \frac{|X|(k + 1 - b)}{|N_B(X)| + 1 - b} \geq \frac{(k + 1 - b - \gamma)(k + 1 - b)}{2k - |N_B(c')| + 1 - b} = \frac{(k + 1 - b - \gamma)(k + 1 - b)}{\beta + k + 1 - b}
\]

Since by 2.9 \( 2\gamma < k + 1 - b \) and \( \beta \geq 0 \), if \( \beta \leq 2(k + 1 - b - 2\gamma) \) then \( 0 \leq \beta \gamma < (k + 1 - b - 2\gamma)(k + 1 - b) \). Therefore we take the last inequality and multiply in the numerator and denominator by \( \gamma \) and we get:

\[
|N_B(c') \cap X| \geq \frac{\gamma(k + 1 - b - \gamma)(k + 1 - b)}{\gamma \beta + \gamma(k + 1 - b)} > \frac{\gamma(k + 1 - b - \gamma)(k + 1 - b)}{(k + 1 - b - 2\gamma)(k + 1 - b) + \gamma(k + 1 - b)} = \gamma.
\]

\( \square \)

The following proposition completes the proof with a factor of four to spare:

**Proposition 2.16.** \( \beta < \frac{1}{2}(k + 1 - b - 2\gamma) \).

Proof. Recall that \( c^* \) maximizes \( |N_B(c^*)| \) and that there are less than \( b \) frequent colors. Note that, for every color \( c \in C \), \( |N_B(c)| \leq |V(G)| - b \) since by 2.4 every non-singleton part has empty intersection of lists. Let \( F \) be the set of frequent colors. Then:

\[
|V(G)|k = (2k + 1)k \leq \sum_{v \in V(G)} |L(v)| = \sum_{c \in C} |N_B(c)| \leq |C - F||N_B(c^*)| + |F|(2k + 1 - b) \leq (2k + 1 - \gamma - b + 1)|N_B(c^*)| + (b - 1)(2k + 1 - b).
\]

Substituting \( |N_B(c^*)| = k - \beta \) and rearranging we have:

\[
(2k + 2 - \gamma - b)\beta \leq (-\gamma - b + 1)k + (b - 1)(2k + 1 - b) = (b - 1)(k + 1 - b) - k\gamma.
\]

By 2.14 we have that \( \gamma + b \leq k \), so \( \gamma + b < 2k + 2 \), and that \( b \leq \frac{k+1}{2} \). Dividing both sides of the inequality by \( (2k + 2 - \gamma - b) \) we get:

\[
\beta \leq \frac{(b - 1)(k + 1 - b) - k\gamma}{2k + 2 - \gamma - b} < \frac{\frac{1}{2}k(k + 1 - b - 2\gamma)}{2k + 2 - \gamma - b} \leq \frac{\frac{1}{2}k(k + 1 - b - 2\gamma)}{k + 2} < \frac{1}{2}k(k + 1 - b - 2\gamma).
\]

\( \square \)
Chapter 3

Proof of Theorem 1.7

In this chapter we adapt a proof by Noel, West, Wu and Zhu [12] to prove Theorem 1.7, which implies Theorem 1.5 for our generalized definition of $f$-colorings. We use the notation $\chi(G)$ and $\chi^L(G)$ for the chromatic and list chromatic numbers associated to these $f$-colorings, independently of the chosen $f$. Similarly to the proof for Theorem 1.6, the list coloring found in the proof respects the parts of an original coloring, and so the proof is valid for any general notion of $f$-coloring.

3.1 Preliminaries

We use the notation $n = |V(G)|$ and $k = \chi(G)$. The first remark is that we only need to prove Theorem 1.7 for graphs with $n \geq 2\chi(G) + 2$. For smaller values of $n$ the statement is equivalent to Ohba’s conjecture. In this case, we suppose that Theorem 1.7 fails, and take a minimal counterexample $G$ on $n \geq 2k + 2$, and a list assignment $L$ with lists of size greater than $k + 1$ such that $G$ does not admit an $L$-coloring.

We use the same notation from Chapter 2. Similarly to the proof for Ohba’s conjecture, the first observation to make is that we may assume $|C| < n$ by Lemma 2.2. Next, we find more specific restrictions on $G$ and $L$ for this problem, beginning with the following proposition.

**Proposition 3.1.** Let $A$ be a stable set in $G$ whose lists have a common color. Then

$$\left\lceil \frac{|V(G - A)| + \chi(G - A) - 1}{3} \right\rceil = \left\lceil \frac{n + k - 1}{3} \right\rceil.$$

**Proof.** Let $c$ be a color that is in every list of $A$. For every vertex $v \in G - A$ let $L'(v) = L(v) - \{c\}$. Since $|L(v)| \geq k + 1$ and $G$ is minimal, then $|L'(v)| \geq k \geq \chi(G - A)$, and $|L'(v)| \geq \left\lceil \frac{n + k - 1}{3} \right\rceil$. 

19
If \( \left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil \) is an integer, then \( |L'(v)| \geq \max\{\chi(G-A), \left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil \} \), and by the minimality of \( G \) this means there is an \( L' \)-coloring of \( G - A \). We can extend this to a coloring of \( G \) by giving the color \( c \) to \( A \), which is a contradiction. So \( \left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil \geq \left\lfloor \frac{n + k - 1}{3} \right\rfloor \), and in fact there is equality because \( (G - A) \subseteq G \).

**Corollary 3.2.** The lists on a part of size two are disjoint.

**Proof.** Let \( A \) be a part of size two with a shared color. Then

\[
\left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil = \left\lceil \frac{(n-2) + (k-1) - 1}{3} \right\rceil < \left\lfloor \frac{n + k - 1}{3} \right\rfloor,
\]

which contradicts 3.1.

**Corollary 3.3.** Each color appears in at most two lists in each part in \( G \).

**Proof.** If there is a part \( A \) in which three vertices have a common color, then

\[
\left\lceil \frac{|V(G-A)| + \chi(G-A) - 1}{3} \right\rceil \leq \left\lceil \frac{(n-3)+k-1}{3} \right\rceil < \left\lfloor \frac{n + k - 1}{3} \right\rfloor,
\]

contradicting 3.1.

**Lemma 3.4.** \( \alpha(G) \leq 4 \).

**Proof.** Let \( A \) be a stable set in \( G \). By 3.3, each color appears in at most two lists of \( A \), and 2.2 bounds the number of colors. So we have:

\[
\sum_{v \in A} |L(v)| \leq 2 \left| \bigcup_{v \in V(G)} L(v) \right| \leq 2(n-1).
\]

Also, \( \sum_{v \in A} |L(v)| \leq |A| \left\lceil \frac{n + k - 1}{3} \right\rceil \). Together, these inequalities yield \( |A| \leq 6 \frac{n-1}{n+k-1} \), so \( |A| \leq 5 \). If there was inequality then \( n \geq 5k + 1 \) and there would be a part of size 6, but this is a contradiction with \( |A| \leq 5 \), so \( |A| \leq 4 \).

**Lemma 3.5.** \( \frac{n + k - 1}{3} \) is an integer.

**Proof.** Take \( A \) a largest part of \( G \), so \( n \leq k|A| \). Suppose the lists of \( A \) are disjoint. Then:

\[
n \leq k|A| \leq \sum_{v \in A} |L(v)| = \left| \bigcup_{v \in A} L(v) \right| \leq \left| \bigcup_{v \in V(G)} L(v) \right| < n.
\]

Therefore \( A \) must contain two vertices with lists with non-empty intersection, and we know it cannot be the whole part. Let this two vertices be \( A' \subseteq A \). Then:

\[
\left\lceil \frac{|V(G-A')| + \chi(G-A') - 1}{3} \right\rceil \leq \left\lceil \frac{(n-2) + k - 1}{3} \right\rceil = \left\lfloor \frac{n + k - 1}{3} \right\rfloor.
\]

If \( \frac{n + k - 1}{3} \) is not an integer, then it is a contradiction with 3.1.
Let $k_i$ be the number of parts of $G$ of size $i$. Note that $\sum_{i=1}^{4} k_i = k$ and $\sum_{i=1}^{4} ik_i = n$.

**Corollary 3.6.** The following relations are satisfied:

(a) $\frac{n+k-1}{3} = k + k_4 - \frac{k_1-k_3+k_4+1}{3}$ and both fractions are integers.

(b) $\frac{n+k-1}{3} + \frac{k}{3} \geq k + k_4 + \frac{2k_3-1}{3}$.

(c) $\frac{2(n+k-1)}{3} = n + \frac{k_1-k_3-2k_4-2}{3} = k + k_3 + 2k_4 + \frac{k+2k_2+k_3-2}{3}$.

**Proof.** (a) $n + k = 2k_1 + 3k_2 + 4k_3 + 5k_4 = 3k - k_1 + k_3 + 2k_4$. Both fractions are integers because of the previous lemma.

(b) We have $k \geq k_1 + k_3 + k_4$, so $\frac{k}{3} \geq \frac{k_1+k_3+k_4}{3}$. Using (a) we have $\frac{n+k-1}{3} + \frac{k}{3} \geq k + k_4 - \frac{k_1-k_3+k_4+1}{3} + \frac{k_1+k_3+k_4}{3} = k + k_4 + \frac{2k_3-1}{3}$.

(c) $2(n + k) = 4k_1 + 6k_2 + 8k_3 + 10k_4 = 4k + 2k_2 + 4k_3 + 6k_4$.  

## 3.2 Sufficient condition for an $L$-coloring

We have seen that $G$ has parts of size at most four, and that a color appears in at most two lists of one part, so if we find an $L$-coloring that is a refinement of the partition of $V(G)$, it has to be into parts of size at most two. We need to see which pairs will be our parts of size two. For this we define what it means to merge two vertices.

**Definition 3.7.** Merging two non-adjacent vertices $u$ and $v$ in $G$ means replacing them by a merged vertex $w$ to which we assign the list $L(w) = L(u) \cap L(v)$.

We consider the bipartite graph $B$ defined in Chapter 2, and we call $B^*$ the graph that we obtain in the same way restricted to merged vertices. The next section explains which mergings we want to perform, and in this section we see that under certain conditions there is a matching in $B^*$ after the mergings. We specify these conditions.

Let $A$ be a part of $G$, then we denote $A^*$ as the resulting part after performing the merges. Let $t_3$ be the number of parts of size 3 having merged vertices. Let $Z_3$ be a fixed set of $\lceil \frac{2}{3}k_3 \rceil$ parts of size 3. Let $Z_4$ be a fixed set of $\max\{0, \frac{k_1-k_3+k_4+1}{3}\}$ parts of size 4. We define the following properties:

(P1) $t_3 \geq k_3/3$.

(P2) At least one merge occurs in every part of size 4.

(P3) For $x, y, z \in A^*$ distinct vertices, $|L(x) \cup L(y) \cup L(z)| \geq n - t_3 - k_4$ (this property applies to parts of size 3 that have not merged and to parts of size 4 with one merge).
(P4) If $|A^*| = |A| = 3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + k_3 + k_4$.

(P5) If $A \in Z_3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + t_3 + k_4$.

(P6) If $|A| = 3$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + k_3' + k_4$.

(P7) If $A \in Z_4$ and $x, y \in A^*$, then $|L(x) \cup L(y)| \geq k + k_4$.

(P8) $B^*$ has a matching.

We want to use Hall’s theorem to find a matching in $B^*$. This requires that every set of vertices $S$ satisfies $|L(S)| \geq |S|$. The intuition behind these properties is that a large set $S$ must contain vertices with large lists, but when the sets get smaller we can eliminate such vertices (by merging them) and smaller bounds for $|L(S)|$ are enough. This is why the bounds get smaller when going through properties (P3)-(P7). Property (P8) simply says that when all the other properties are satisfied, then we have a matching in $B^*$, which gives us a matching of $B$.

**Lemma 3.8.** When the merges satisfy (P1)-(P8), the resulting $B$ has a matching.

**Proof.** By Hall’s theorem, we only need to see that every set of vertices $S$ (after the merges) satisfies $|L(S)| \geq |S|$.

We an restrict $S$ to only merged vertices by (P1)-(P7), and then (P8) guarantees a matching in $B^*$ that we can extend to a matching of $B$.

By (P2), the mergings leave at most $n - t_3 - k_4$ vertices, so $|S| \leq n - t_3 - k_4$. Then (P3) gives us $|L(S)| \geq |S|$ for any $S$ that contains three vertices from one part. So we may consider $S$ to have at most 2 vertices from each part, which means that $|S| \leq k + k_2 + k_3 + k_4 \leq 2k$.

If $S$ contains both (unmerged) vertices from a part of size two, by 3.2 their lists are disjoint, so $|L(S)| \geq 2k + 2 > |S|$. So we can assume $|S| \leq k + k_3 + k_4$. If $S$ contains two vertices from a part of size 3 with no merged vertices, then (P4) gives us $|L(S)| \geq k + t_3 + k_4$. So we can assume $|S| \leq k + t_3 + k_4$. If $S$ contains two vertices from a parts of $Z_3$, then (P5) gives us $|L(S)| \geq k_3 + t_4$. So we can assume $|S| \leq k + [k_3/3] + k_4$ (since $[k_3/3] = k_3 - |Z_3|$). If $S$ contains two vertices from any part of size 3, then (P6) gives us $|L(S)| \geq k + [k_3/3] + k_4$. So we can assume $|S| \leq k + k_4$. If $S$ contains two vertices from a part in $Z_4$, then (P7) gives us $|L(S)| \geq k + k_4$. So we can assume $|S| \leq k + k_4 - |Z_4| \leq \frac{n+k_4-1}{4}$ by 3.6 and the definition of $Z_4$. Now $|L(S)| \geq |S|$ if $S$ contains any unmerged vertex, and (P8) applies.

### 3.3 Merges

In this section we describe the merges that we perform so that they satisfy the properties (P1)-(P8). To guarantee (P3)-(P8) we want to merge vertices whose lists have large intersection. We define what we consider a good merge.
**Definition 3.9.** Let $A$ be a part of $G$ and $l(A) = \max\{|L(u) \cap L(v)| : u, v \in A\}$. If $|A| \geq 3$, a pair $\{u, v\} \subset A$ is good if:

- $|A| = 3$ and $|L(u) \cap L(v)| \geq \frac{k_1+k_4+1}{3}$, or
- $|A| = 4$ and $|L(u) \cap L(v)| \geq |L(w) \cap L(z)|$, where $\{w, z\} = A - \{u, v\}$.

A good merge is the merge of a good pair. $A$ is a good part if we can make a good merge in it.

By definition, a part of size 4 is always a good part. The next few results show that the parts of size 3 are also good.

**Lemma 3.10.** If $A$ is a part of size 3, then $\sum_{\{u,v\} \in A} |L(u) \cap L(v)| \geq k$.

**Proof.** By 3.3 a color appears at most twice in $A$, and by 2.2 $|C| = |\bigcup_{v \in V(G)} L(v)| < n$, so we have:

$$\sum_{\{u,v\} \in A} \ |L(u) \cap L(v)| = \sum_{u \in A} |L(u)| - |L(A)| \geq 3 \frac{n+k-1}{3} - (n-1) = k.$$

□

**Corollary 3.11.** If $|A| \geq 3$, then $l(A) \geq k/3$.

**Corollary 3.12.** If $|A| \geq 3$, a pair $\{u, v\} \subset A$ that maximizes $|L(u) \cap L(v)|$ is good.

**Proof.** By definition this is true when $|A| = 4$. When $|A| = 3$, we have $k_3 \geq 1$, and by the previous corollary $|L(u) \cap L(v)| = l(A) \geq \frac{k}{3} \geq \frac{k_1+k_4+1}{3}$. □

**Lemma 3.13.** A good part of size 3 $A$ satisfies (P6).

**Proof.** Let $A^* = \{x, y\}$ be the part after merging a good pair, and let $y$ be the merged vertex. By 3.3 $L(u) \cap L(v) \neq \emptyset$, so

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq \frac{n+k-1}{3} + \frac{k_1+k_4+1}{3} = k + \frac{k_3}{3} + k_4$$

by 3.6(a). □

**Lemma 3.14.** A good part of size 4 $A$ satisfies (P3) is (P1) holds.

**Proof.** Let $A^* = \{x, y, z\}$ be the part after merging a good pair, and let $x$ be the merged vertex. We know $|L(x)| \geq |L(y) \cap L(z)|$ and $L(x) \cap L(y) = L(x) \cap L(z) = \emptyset$. Therefore:

$$|L(x) \cup L(y) \cup L(z)| = |L(x)| + |L(y)| + |L(z)| - |L(y) \cap L(z)| \geq$$

$$\geq |L(y)| + |L(z)| \geq \frac{2(n+k-1)}{3} \geq n + \frac{k_1 - k_3 - 2k_4 - 2}{3}$$
by 3.6(c). Since (P1) holds, we have \( t_3 \geq k_3/3 \). We also have \( k_1 \geq 0 \) and \( 2k_4/3 \geq k_4 \), so

\[
|L(y)| + |L(z)| \geq n + \frac{k_1 - k_3 - 2k_4 - 2}{3} \geq n - t_3 - k_4.
\]

\( \square \)

We now need to specify which merges to make so that (P1)-(P7) are satisfied. For this we fix two sets \( Z_3 \) and \( Z_4 \) of parts of size 3 and 4 respectively and make good merges in them. Finally we see that we can make merges outside \( Z_3 \cup Z_4 \) so that (P8) is also satisfied.

### Parts of size 3

We fix a set \( Z_3 \) of \( \left\lfloor \frac{2k_3}{3} \right\rfloor \) parts of size 3. Exactly \( \left\lceil \frac{k_3}{3} \right\rceil \) parts of size 3 lie outside \( Z_3 \). We will take care of those parts later on. Recall that we have not yet given a value \( t_3 \) of parts of size 3 that will merge.

We set \( t_3 \) to be the largest integer for which there exists \( Z'_3 \subseteq Z_3 \) of size \( t_3 - \left\lceil \frac{k_3}{3} \right\rceil \) such that \( l(A) \geq \frac{k + t_3 - 1}{3} \) for all \( A \in Z'_3 \). In \( Z_3 \) we merge a pair in a part \( A \) that achieves \( l(A) \) in its list intersection if and only if \( A \in Z'_3 \).

By 3.12 all these merges are good because they have maximum intersection. It could be that \( Z'_3 \) is the empty set because no part has large enough list intersections, then \( t_3 = \left\lceil \frac{k_3}{3} \right\rceil \). In any case, we always have \( t_3 \geq k/3 \).

**Lemma 3.15.** If one merge is made in each part of size 3 outside \( Z_3 \), then (P1) holds, (P3) holds for parts of size 3, (P4) and (P5) hold, and (P6) holds for parts of size 3.

**Proof.** If one merge is made in each of the \( \left\lceil \frac{k_3}{3} \right\rceil \) parts of size 3 outside \( Z_3 \) then the total number of parts of size 3 that have a merge is \( t_3 \), so (P1) holds.

For parts of size 3, (P3) and (P4) apply only to parts that have not merged, that is parts in \( Z_3 - Z'_3 \). Those are the parts \( A \) such that \( l(A) \leq \frac{k + t_3 - 2}{3} \). Each color appears in at most two lists in a part \( A \), so we have that:

\[
|L(A)| = \sum_{v \in A} |L(v)| - \sum_{(u,v) \in A} |L(u) \cap L(v)| \geq \\
\geq \frac{3(n + k - 1)}{3} - \frac{3(k + t_3 - 2)}{3} > n - t_3 \geq n - t_3 - k_4
\]

proving (P3). For (P4) we take two vertices \( x, y \in A \) and we have:

\[
|L(x) \cup L(y)| = |L(x)| + |L(y)| - |L(x) \cap L(y)| \geq \\
\geq \frac{2(n + k - 1)}{3} - \frac{k + t_3 - 2}{3} \geq k + k_3 + 2k_4 + \frac{2k_2}{3} \geq k + k_3 + k_4
\]
using $k_3 \geq t_3$ and 3.6(c).

Since $k_3 \geq t_3$ and $k_3 \geq k_3/3$, (P4) implies (P5) and (P6) for parts that do not have a merge. Since (P5) only applies to parts in $Z_3$, we only need to prove it for parts in $Z_3$. Let $A = \{x, y\} \in Z_3$ and suppose $y$ is the merged vertex. By 3.3 $L(x) \cap L(y) = \emptyset$, so we have:

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq \frac{n + k - 1}{3} + \frac{k + t_3 - 1}{3} \geq k + \frac{k_2 + 2k_3 + t_3}{3} + k_4 - \frac{2}{3} \geq k + t_3 + k_4 - \frac{2}{3},$$

using 3.6(b) and $k_3 \geq t_3$. So (P5) is satisfied. We have already seen that (P6) is satisfied for parts with a merge in 3.13. In fact to complete the proof of (P6) for all parts we only need to specify a good merge in every part of size 3 outside $Z_3$, which we do later on.

\[\Box\]

### Parts of size 4

Corollary 3.6(a) and $n \geq 2k + 2$ imply $\frac{k_1 - k_3 + k_4 + 1}{3}$ is an integer smaller than $k_4$, so we can fix a set $Z_4$ of $\max\{0, \frac{k_1 - k_3 + k_4 + 1}{3}\}$ parts of size 4. We describe the merges in the parts that lie outside $Z_4$ in the next subsection. In this one we describe the merges in parts of $Z_4$.

For $A \in Z_4$ merge a pair $\{u, v\}$ such that $|L(u) \cap L(v)| = l(A)$. Merge the remaining pair $\{w, z\}$ if $|L(w) \cap L(z)| \geq s$, where $s = \frac{2n - k + 1}{3} - k_4$. This number is an integer by 3.5, and it is also larger than $\frac{k}{3}$ since $s - \frac{k}{3} > \frac{2(n - k)}{3} - k_4 \geq \frac{6k_4}{3} - k_4$.

We have seen $l(A) \geq k_3$ in 3.11, so the list of any merged vertex in a part of $Z_4$ has size at list $\frac{k}{3}$.

**Lemma 3.16.** These mergings satisfy (P7).

**Proof.** (P7) applies only to parts $A \in Z_4$. Let $x, y \in A^*$. We will show that $|L(x) \cup L(y)| \geq k + k_4$. We consider three cases.

Case 1: neither $x$ nor $y$ are merged vertices. By the construction of the merges, this implies $|L(x) \cap L(y)| \leq s - 1$. Therefore:

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| - |L(x) \cap L(y)| \geq \frac{2(n + k - 1)}{3} - \frac{2n - k - 2}{3} + k_4 = k + k_4.$$

Case 2: $x$ is not merged and $y$ is merged. By 3.3, $L(x) \cap L(y) = \emptyset$. Therefore:

$$|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq \frac{n + k - 1}{3} + \frac{k}{3} \geq k + k_4 - \frac{1}{3},$$

by 3.6(b) and the observations made about $s$. Since $|L(x) \cup L(y)|$ is an integer, we have $|L(x) \cup L(y)| \geq k + k_4$. 

Case 3: both \(x\) and \(y\) are merged vertices. We assume \(l(A) = |L(x)| \geq |L(y)| \geq s\). Since \(L(x) \cap L(y) = \emptyset\) by 3.3, we have

\[
|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq 2s = n + \frac{n - 2k + 2}{3} - 2k_4 > \sum_{i=1}^{4} ik_i - 2k_4 \geq k + k_4.
\]

\[\square\]

**Remaining merges**

The following lemma tells us the work we have left to prove Theorem 1.5.

**Lemma 3.17.** If in addition to the merges already specified we can specify one good merge in each part outside \(Z_3 \cup Z_4\) in such a way that (P8) holds, then Theorem 1.5 is true.

**Proof.** Specifying any merge in each part of size 3 outside \(Z_3\) completes the proofs of (P1),(P4) and (P5), and it completes the proof of (P3) for parts of size 3 by 3.15. If those merges are good, by 3.13 we also prove (P6).

We proved (P7) in the last subsection, since it only involves parts in \(Z_4\). Specifying any merge in each part of size 4 outside \(Z_4\) completes the proof for (P2). If these merges are good and we have specifies merges outside \(Z_3\), then by Corollary 3.14 we have (P3) for parts of size 4.

If the merges satisfy (P8) then Theorem 1.5 holds. \[\square\]

So we have reduced the proof of the theorem to merging a good pair in each part outside \(Z_3 \cup Z_4\) in such a way that (P8) holds. We can use (P7) to find this since we have already proved it completely.

Let \(T\) be the set of all merged vertices in \(Z_3 \cup Z_4\). Let \(Y\) be the set of parts outside \(Z_3 \cup Z_4\). We need to find distinct color for every vertex in \(T\) and every part in \(Y\) such that for a part \(A \in Y\) its assigned color belongs to both lists of a good pair in \(A\), and for \(w \in T\) the color is in its list.

For \(A \in Y\) let \(L_A\) be the set of colors that are in \(L(u) \cap L(v)\) for some good pair \(\{u,v\}\). Let \(X\) be the family of sets \(L_A\) and \(L(w)\) for \(A \in Y\) and \(w \in T\). What we need to do is find a matching in the bipartite graph \((T \cup Y, X)\). To do this we again use Hall’s theorem.

**Lemma 3.18.** If \(A \in Y\) and \(|A| = 3\), then \(|L_A| \geq k_3 + \frac{k_1 + k_4}{3}\).

**Proof.** We have seen that every part has at least one good pair. If a pair \(\{u,v\}\) is not good, then by definition \(|L(u) \cap L(v)| \leq \frac{k_1 + k_4}{3}\). At most two pairs are not
good, so we have
\[ |L_A| \geq k - \frac{2(k_1 + k_4)}{3} \geq k + \frac{k_1 + k_4}{3} \]
by 3.10 and \( k = \sum_{i=1}^{4} k_i \).

**Lemma 3.19.** If \( A \in Y \) and \( |A| = 4 \), then \( |L_A| \geq k_3 + k_4 \).

**Proof.** By the definition of a good pair and the fact that a color appears at most in two lists in a part we have
\[ |L_A| \geq \frac{1}{2} \sum_{(u,v) \in A} |L(u) \cap L(v)| = \frac{\sum_{u \in A} |L(u)|}{2} - \frac{|\bigcup_{u \in A} L(u)|}{2}. \]
Since there are less than \( n \) colors, this means
\[ |L_A| \geq \frac{1}{2} \left( \frac{4(n + k - 1)}{3} - (n - 1) \right) = \frac{n + 4k - 1}{3} > k \geq k_3 + k_4 \]
since \( n > 2k + 1 \).

The following lemma finds the desired matching and completes the proof.

**Lemma 3.20.** There is a matching in \( (T \cup Y, X) \).

**Proof.** By Hall’s theorem, we only need to see that for every set \( S \subseteq X \), the the union of the lists indexed by \( S \) has size at least \( |S| \).

By construction, each part of size 3 contributes at most one list to \( X \), each part of size 4 outside \( Z_4 \) also contributes at most one, and each part in \( Z_4 \) contributes at most two. So \( |S| \leq |X| \leq k_3 + k_4 + |Z_4| \).

If \( S \) contains two lists for a part \( A \) then \( A \in Z_4 \). By (P7), the union of these two lists has size at least \( k + k_4 \geq |X| \), so \( S \) contains at most one list from each part and \( |S| \leq k_3 + k_4 \). If \( S \) contains a list from a part of size 4 outside \( Z_4 \), then by 3.19 we are done. So we assume it does not and so \( |S| \leq k_3 + |Z_4| \).

If \( S \) contains a list from a part \( A \) of size 3 outside \( Z_3 \), then by 3.18 \( |L_A| \geq k_3 + \frac{k_1 + k_4}{3} \). Since \( A \) has size 3, \( k_3 \geq 1 \) and so \( \frac{k_1 + k_4}{3} \geq \max\{0, \frac{k_1 - k_3 + k_4 + 1}{3}\} = |Z_4| \) so \( |L_A| \geq |S| \).

So we can assume that \( S \) contains only lists of parts in \( Z_3 \cup Z_4 \), and at most one list for each part. In \( Z_3 \) we only merged vertices from \( Z_3' \), so
\[ |S| \leq |Z_3'| + |Z_4| = t_3 - \left\lceil \frac{k_3}{3} \right\rceil + \max\{0, \frac{k_1 - k_3 + k_4 + 1}{3}\}. \]

By construction, the list of any merged vertex in \( Z_3' \) has size at least \( \left\lceil \frac{k_1 + t_3 - 1}{3} \right\rceil \). Since \( t_3 \leq k_3 \), we have \( \left\lceil \frac{k_1 + t_3 - 1}{3} \right\rceil \geq |Z_3'| + |Z_4| \).
So $S$ contains lists from at most one merged vertex in each part of $Z_4$, so $|S| \leq |Z_4| = \max\{0, \frac{k_1-k_3+k_4+1}{3}\}$. We have seen that a list of a merged vertex in $Z_4$ has size at least $\frac{k}{3}$. Since $k \geq k_1 - k_3 + k_4$ and $\frac{k_1-k_3+k_4+1}{3}$ is an integer by 3.6(a), we have that the size of the union of the lists in $S$ is always at least $|S|$.
Chapter 4

Tightness

In this chapter we see examples showing that Ohba’s conjecture is tight for the ordinary, acyclic and star chromatic numbers. For this we see examples of graphs of \( n = 2k + 2 \) vertices, chromatic number \( k \) and a list assignment \( L \) with lists of size \( k \) such that the graphs are not \( L \)-colorable.

The bound for the list chromatic number in Theorem 1.5 is also sharp. For the particular case of \( k \)-partite graphs \( K_{m*k} \) a lower bound for the list chromatic number is also known. We show a construction that gives this lower bound.

We will first review the examples which show the sharpness of the results for the ordinary chromatic number. We next prove that these examples show also the sharpness of our results for generalized colorings in the cases of the acyclic and star chromatic numbers.

4.1 Ordinary chromatic number

Enomoto, Ohba, Ota and Sakamoto show in [3] that Ohba’s conjecture is the best possible bound, giving a list assignment to the complete \( k \)-partite graph \( K(4,2,2,...,2) \) that does not admit a coloring. In particular they prove the following statement.

**Theorem 4.1.** The list chromatic number of the complete \( k \)-partite graph \( K(4,2,...,2) \) is \( k + 1 \) when \( k \) is even.

**Proof.** To prove this theorem give an explicit list assignment \( L \) to \( G = K(4,2,...,2) \) and see that \( G \) is not \( L \)-colorable.

Let \( V_1,...,V_k \) denote the parts of \( G \), where \( V_1 = \{x_1, x_2, x_3, x_4\} \) and \( V_i = \{u_i, v_i\} \) for \( i = 2,...,k \).

Let \( A \) and \( B \) be disjoint lists of colors such that \( |A| = |B| = k \). Partition these sets as \( A = A_1 \cup A_2 \cup A_3 \cup A_4 \) and \( B = B_1 \cup B_2 \) so that \( |A_1| = |A_2|, |A_3| = |A_4|, \) and \( |B_1| = |B_2| = \frac{k}{2} \). We define the list assignment \( L \):
• In the parts of size 2: \( L(u_i) = A \) and \( L(v_i) = B \).

• In \( V_1 \): \( L(x_1) = A_1 \cup A_3 \cup B_1, L(x_2) = A_1 \cup A_4 \cup B_2, L(x_3) = A_2 \cup A_4 \cup B_1 \) and \( L(x_4) = A_2 \cup A_3 \cup B_2 \).

Figure 4.1 shows this list assignment.

In an \( L \)-coloring of \( G \) we must use \( k - 1 \) colors of \( A \) to color \( u_2, \ldots, u_k \) and \( k - 1 \) colors of \( B \) to color \( v_2, \ldots, v_k \). Thus only one color \( a \in A \) and one \( b \in B \) remain to color \( V_1 \). Suppose without loss of generality that \( b \in B_1 \). Then, if \( a \in A_1 \cup A_4 \) the vertex \( x_4 \) cannot be colored. If \( a \in A_2 \cup A_3 \) instead, the vertex \( x_2 \) cannot be colored. In any case, we have that \( \chi^L(G) > k \).

Noel [10] conjectured that the only complete \( k \)-partite graphs on \( 2k + 2 \) vertices such that \( \chi^L(G) > k \) are \( K(4, 2, \ldots, 2) \) and \( K(k/2 + 1, k/2 + 1, k/2 + 1, k/2 + 1, k/2 + 1) \) for even \( k \).

For Theorem 1.5, Kierstead [8] showed that the complete \( k \)-partite graph with parts of size 3 \( K(3, 3, \ldots, 3) = K_{3,k} \) has list chromatic number \( \chi^L(K_{3,k}) = \lceil \frac{4k - 1}{3} \rceil \), proving that the bound given by the theorem is the best possible. Ohba [14] generalized this result and proved that this is true for any complete \( k \)-partite graph with parts of size 1 and 3. Concretely:

**Theorem 4.2** (Ohba). Let \( k_1, k_3 \) be integers, and define \( k = k_1 + k_3 \) and \( n = k_1 + 3k_3 \). If \( G \) is the complete \( k \)-partite graph with \( k_1 \) parts of size 1 and \( k_3 \) parts of size 3, then

\[
\chi^L(K_{3,k}) = \max\{k, \left\lceil \frac{n + k - 1}{3} \right\rceil \}.
\]

**Proof.** We have to prove the lower bound, since the upper bound is a consequence of Theorem 1.5. If \( n \leq 2k + 1 \) it is proved by Theorem 1.4. So suppose \( \left\lceil \frac{n + k - 1}{3} \right\rceil > k \). We find a list assignment with lists of size \( \left\lceil \frac{n + k - 1}{3} \right\rceil - 1 \) for which there is no acceptable coloring of \( G \).
Let \( s = \frac{k_1 + 2k_3 - 1}{3} \) and let \( X_1, X_2, X_3 \) be disjoint sets of colors so that for \( i, j \in \{1, 2, 3\} \) we have
\[
|X_i \cup X_j| \geq |2s| = \left\lceil \frac{n + k - 1}{3} \right\rceil - 1,
\]
and
\[
|X_1 \cup X_2 \cup X_3| \leq k_1 + 2k_3 - 1.
\]
We assign the lists \( X_1 \cup X_2, X_1 \cup X_3 \) and \( X_2 \cup X_3 \) to the vertices of parts of size 3, and \( X_1 \cup X_2 \cup X_3 \) to the vertices of parts of size 1. An acceptable coloring of \( G \) must use at least 2 colors on each part of size 3, and 1 color on each part of size 1. This means it must use at least \( k_1 + 2k_3 \) colors. This contradicts the fact that \( |X_1 \cup X_2 \cup X_3| \leq k_1 + 2k_3 - 1 \).

Not all \( k \)-partite graphs have list chromatic number equal to \( \left\lceil \frac{n + k - 1}{3} \right\rceil \). Kierstead, Salmon and Wang [9] proved that the complete \( k \)-partite graph \( K_{4^*k} \) has list chromatic number \( \chi^L(K_{4^*k}) = \left\lfloor \frac{3k}{2} \right\rfloor \).

In general, the list chromatic number of complete \( k \)-partite graphs with parts of equal size \( m \) is not known, but Noel, West, Wu and Zhu found a lower bound [12]:
\[
\left\lceil \frac{2k(m-1)}{m} \right\rceil \leq \chi^L(K_{m^*k}) \leq \left\lfloor \frac{k(m+1) - 1}{3} \right\rfloor.
\]
The upper bound is Theorem 1.5, and the lower bound is useful for small \( m \) and arises from the following construction.

Let \( C \) be a set of \( 2k - 1 \) colors, and split \( C \) into sets \( X_1, \ldots, X_m \) of sizes \( \left\lfloor \frac{2k-1}{m} \right\rfloor \) and \( \left\lceil \frac{2k-1}{m} \right\rceil \). We define the list assignment \( L \) as follows. In each part, assign list \( C \setminus X_i \) to the \( i \)-th vertex. An \( L \)-coloring must use at least two colors in each part, since the intersection of all lists of one part is empty. These pairs of colors from every part are disjoint. Hence, the list coloring must use at least \( 2k \) colors, but \( |C| = 2k - 1 \), so there is no \( L \)-coloring. Each list has size at least \( 2k - 1 - \left\lceil \frac{2k-1}{m} \right\rceil \geq \left\lfloor \frac{2k(m-1)}{m} \right\rfloor - 1 \), so \( \chi^L(K_{m^*k}) \geq \left\lfloor \frac{2k(m-1)}{m} \right\rfloor \).

\( K_{4^*k} \) attains this lower bound, and for \( m = 5 \) and larger the list chromatic numbers have not been determined yet.

### 4.2 Acyclic chromatic number

A proper coloring of a graph is acyclic if every bichromatic subgraph induced by this coloring is a forest. The acyclic chromatic number of a graph is the minimum integer such that the graph admits a proper acyclic coloring.

We adapt the construction from the previous section to prove the tightness of Ohba’s conjecture in the case of the acyclic chromatic number. We want a graph \( G \) on \( n = 2k + 2 \) vertices and acyclic chromatic number \( k \), and a list assignment \( L \) such that \( G \) is not \( L \)-colorable.
Starting from the complete $k$-partite graph $K(4, 2, ..., 2)$ we remove some edges until the acyclic chromatic number of the remaining graph is $k$, and so that we can use the list assignment from the last section to prove that its list acyclic chromatic number is larger than $k$.

Again, we denote $G = V_1 \cup V_2 \cup ... \cup V_k$ the parts of this graph, where $V_1 = \{x_1, x_2, x_3, x_4\}$ and $V_i = \{u_i, v_i\}$ for $0 = 2, ..., k$. The subgraph induced by two color classes cannot have cycles. For any two parts of size 2 of $G$, we remove the edge $u_iv_j$ for $i < j$. For $V_1$ and a part of size 2 $V_i$, we remove the edges $v_ix_2$, $v_ix_3$, and $v_ix_4$. Figure 4.2 shows the edges remaining between parts.

It is clear that with the list assignment $L$ defined in the previous section, and $L$-coloring of $G$ must use $k - 1$ colors from $A$ and $k - 1$ colors from $B$ to color $u_2, ..., u_k$ and $v_2, ..., v_k$, since $G$ has all the $u_iu_j$ and $v_iv_j$ edges. Finally, with the two remaining colors $a \in A$ and $b \in B$ we find that one vertex of $V_1$ cannot be colored.

In the last section we also saw that Theorem 1.7 was tight for complete $k$-partite graphs with parts of size 1 or 3. For the acyclic chromatic number the same proof adapts to see examples for which Theorem 1.7 is tight. One example could be the following. Let $G$ be the graph obtained by a complete $k$-partite graph with $k_3$ parts of size 3 and $k_1$ parts of size 1 in which between any two parts of size 3 $P_i = \{a_i, b_i, c_i\}$ and $P_j = \{a_j, b_j, c_j\}$ the only edges are $a_ia_j, b_ib_j, c_ic_j$, $a_ib_j, b_ic_j$ for $i < j$. To each part of size 3 assign the lists $L(a_i) = X_1 \cup X_2$, $L(b_i) = X_1 \cup X_3$, $L(c_i) = X_2 \cup X_3$, and to each part of size 1 the list $X_1 \cup X_2 \cup X_3$. Then the same argument as in the last section proves that $\chi^L(G) \geq \lceil \frac{n+k-1}{3} \rceil$ and the lower bound is given by Theorem 1.7. Figure 4.2 shows the edges between parts and the list assignment.

4.3 Star chromatic number

A proper coloring of a graph is a star coloring if every bichromatic subgraph induced by this coloring is a forest of stars. The star chromatic number of a
graph is the minimum integer such that the graph admits a proper star coloring.

We see that Ohba’s conjecture is also a tight bound for the case of the star chromatic number. Our example is again based on the complete $k$-partite graph $K(4,2,...,2)$. First we define $G$ maximal in edges such that its star chromatic number is $k$.

Let $G = V_1 \cup V_2 \cup ... \cup V_k$ be the partition of this graph into color classes, where $V_i = \{x_1, x_2, x_3, x_4\}$ and $V_i = \{u_i, v_i\}$ for $0 = 2, ..., k$. The subgraph induced by two color classes must be a forest of stars. For any two parts of size 2 $V_i$ and $V_j$, $G$ has edges $u_i u_j$ and $v_i v_j$ between them. For $V_1$ and any part of size 2 $V_i$, $G$ has edges $x_1 u_i$, $x_2 u_i$, $x_3 u_i$ and $x_4 u_i$. Figure 4.3 shows these edges.

Again, we consider the list assignment $L$ defined in section 4.1. To color the vertices $u_2, ..., u_k$ we must use $k - 1$ colors of $A$, since $G$ contains all edges $u_i u_j$. To color the vertices $v_2, ..., v_k$ we must use $k - 1$ colors of $B$, since $G$ contains all edges $v_i v_j$. Only two colors $a \in A$ and $b \in B$ remain, and by the argument used in 4.1 it is clear that one vertex of $V_1$ cannot be colored with a color in its list.

Theorem 1.7 is also tight for the star chromatic number. This means there are graphs $G$ such that $\chi_s^L(G) = \lceil \frac{n+k-1}{3} \rceil$. Like for the acyclic chromatic number, we describe a graph for which the list assignments given in the proof for the ordinary chromatic number are still valid to see that $\chi_s^L(G) \geq \lceil \frac{n+k-1}{3} \rceil$. One example could be the following. Let $G$ be the graph obtained by a complete $k$-partite graph with $k_3$ parts of size 3 and $k_1$ parts of size 1 in which between any two parts of size 3 $P_i = \{a_i, b_i, c_i\}$ and $P_j = \{a_j, b_j, c_j\}$ the only edges are $a_i a_j, b_i b_j, c_i c_j$ for $i < j$. To each part of size 3 assign the lists $L(a_i) = X_1 \cup X_2$, $L(b_i) = X_1 \cup X_3$, $L(c_i) = X_2 \cup X_3$, and to each part of size 1 the list $X_1 \cup X_2 \cup X_3$. Then the same argument as in section 4.1 proves that $\chi_s^L(G) \geq \lceil \frac{n+k-1}{3} \rceil$ and the lower bound is given by Theorem 1.7. Figure 4.3 shows the edges between parts and the list assignment.

Figure 4.4: Edges between parts in the graph $G$.

Again, we consider the list assignment $L$ defined in section 4.1. To color the vertices $u_2, ..., u_k$ we must use $k - 1$ colors of $A$, since $G$ contains all edges $u_i u_j$. To color the vertices $v_2, ..., v_k$ we must use $k - 1$ colors of $B$, since $G$ contains all edges $v_i v_j$. Only two colors $a \in A$ and $b \in B$ remain, and by the argument used in 4.1 it is clear that one vertex of $V_1$ cannot be colored with a color in its list.

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Figure 4.5: Edges between parts in the graph $G$. 
Conclusions

We conclude this work by summarizing the obtained results and presenting problems that remain unsolved and are a natural continuation to this work.

We have seen that any graph $G$ such that $|V(G)| \leq 2\chi(G) + 1$ is chromatic choosable for any concept of coloring that can be expressed as an $f$-coloring.

Ohba’s conjecture gives the best possible bound for the ordinary, acyclic and star chromatic numbers. However, the critical example fails for other chromatic numbers, such as the distance $k$ chromatic number, since any graph with large such number cannot have many edges. The proof for Ohba takes a partition and refines it, but it does not use at any point any information that the chromatic number gives about this partition. It is possible that by using this information it can be proved that for the distance $k$ chromatic number and maybe other kinds of more restricted chromatic numbers the bound is not tight. In other words, it is possible that the equality between their list and plain versions is satisfied for larger values of $n$ than the one given by Theorem 1.6.

As for Noel, West, Wu and Zhu’s Theorem 1.5, we have found that the bound is also true for generalized colorings. Again, it is a consequence of Theorem 1.7 and so it only depends on an initial partition of the vertex set of a graph.

Ohba [14] proved the tightness of Theorem 1.5 for the ordinary chromatic number. More specifically, it is known to be tight for the complete $k$-partite graph with parts of size 1 or 3. A lower bound is known for complete $k$-partite graphs with fixed part size, and $K_{1+k}$ achieves this lower bound. For larger part size the list chromatic number should be found. For the acyclic and star chromatic number the situation is similar to the examples of tightness for the Ohba conjecture: the example for the ordinary chromatic number can be adapted to get one that proves tightness for these generalized colorings. Again, for other colorings the proof is no longer valid, and it is possible that Theorem 1.7 could be improved by using properties of the coloring itself in the proof.

Ohba [14] in fact proved that equality in Theorem 1.5 holds only for complete $k$-partite graphs with part sizes 1 or 3. This implies that the list chromatic number of a graph on at most $3\chi(G)$ vertices is bounded above by the list chromatic number of the complete $\chi(G)$-partite graph with part size 3. A similar result has been conjectured by Noel [10] for graphs on at most $m\chi(G)$ vertices for fixed $m$.

**Conjecture 4.3** (Noel). For $m, k \geq 2$, let $G$ be a $k$-chromatic graph on at most
mk vertices. Then $\chi^L(G)$ is bounded above by the list chromatic number of the complete k-partite graph with part size m.

Noel [10] also conjectures that if $G$ is a complete k-partite graph on $2k + 2$ vertices, then $\chi^L(G) > k$ if and only if $k$ is even and either every part of $G$ has size 1 or 3, or every part of $G$ has size 2 or 4. It has been proved in [3] that these graphs are not chromatic choosable. Any of these problems that have not been solved yet can also be formulated for generalized colorings.

Let $\chi_2(G)$ be the 2-distance chromatic number, that is, the ordinary chromatic number of the square $G^2$ of $G$. Every two color classes of a 2-distance coloring of $G$ induce a matching. We were not able to show that for the 2-distance chromatic number $\chi_2(G) = \chi(G^2)$ the bound in Theorem 1.6 is not sharp. However we conjecture that this is the case:

**Conjecture 4.4.** Let $G$ be a graph with $n = 2\chi_2(G) + 2$. Then $\chi^L_2(G) = \chi_2(G)$.
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