Title: Study of the Laplacian eigenvalues of fractal sets

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Preface

This thesis presents an example of known discretization methods for spectral problems in partial differential equations and it is applied with some computations in planar domains with irregular (non-smooth) and self-similar boundary.

In particular we deal with the Dirichlet eigenvalue problem for the Laplace operator. An important property of the Laplace operator is that commutes with the elements of the isometry group of the domain (see page 387 proposition 2.1 [18]). In this work we compute the fundamental tone which essentially is the first eigenvalue $\lambda_1$ of the Dirichlet problem for a family of domains and our purpose is to get an intuition of how the geometry of the boundary (in this case the irregularity and self-similarity of the curves) affects $\lambda_1$.

In Section 1 we give an introduction of the properties and connections with other branch of mathematics and physics.

In Section 2 we introduce preliminary concepts about the analytical formulation of the problem.

In Section 3 we proof some known results on the discretization method (Finite Element approximation) to the Dirichlet eigenvalue problem.

In Section 4 we give some computations for a family of examples, such computations are carried with MATLAB programs.

In Section 5 we provide different examples, a non-simply connected one and its three-dimensional embedded surface analogue.

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1 Introduction

Let $M$ be a Riemannian manifold of dimension $n$ with Riemannian metric $g$, volume element $dv$ and $\Omega \subset M$ an open and connected subset. The Laplace-Beltrami (or also called the Laplacian when $M = \mathbb{R}^n$) operator on $M$ is defined as the divergence of the gradient, in local coordinates $(x_1, \ldots, x_n)$ it is given by the formula

$$\Delta = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{|\det g|} g^{ij} \frac{\partial}{\partial x_j} \right)$$

where $g^{ij}$ denotes the coefficients of the inverse matrix representing $g$. It is known that for the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

there exists a sequence $\{(\lambda_k, E_k)\}_{k=1}^{\infty}$ of real numbers $\lambda_k \in \mathbb{R}$ satisfying

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots < \infty$$

and of finite-dimensional subspaces $E_k \subset C^\infty(\Omega)$ where $(\lambda_k, u)$ solves the Dirichlet eigenvalue problem for each $u \in E_k$. The numbers $\lambda_k$ are the Lagrange multipliers of the minimization problem with Lagrangian

$$E(u) := \int_{\Omega} \frac{1}{2} |\nabla u|^2 - \lambda u^2 \, dv$$

restricted to the Hilbert manifold (see [38]) given by the equation $\int_{\Omega} u^2 \, dv = 1$. One has similar results for the Neumann eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \end{cases}$$

In some cases we will need to solve the mixed eigenvalue problem

$$\begin{cases} -\Delta u = \nu u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N \end{cases}$$

where $\Gamma_D, \Gamma_N \subset \partial \Omega$ with $\Gamma_D \cup \Gamma_N = \partial \Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

A precise study of eigenvalue problems is given in [9]. There are many branches in mathematics and physics showing applications of eigenvalue problems [37, 2, 21, 17, 34, 33, 35, 38, 39, 41, 22].

In this work we restrict to $M = \mathbb{R}^2$ and $\Omega$ is a planar domain, then in cartesian coordinates $g = dx^2 + dy^2$ is the Euclidean metric and the Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$
1.1 The Geometry of the Laplacian

Let $G$ be the symmetry group of $\Omega$ (if $\Omega = M$ we denote it by $G = \text{Isom}(M)$). Then it is known (see [18]) that the Laplace operator $\Delta$ is invariant under $G$ (note that $G$ acts in $\Omega$ and the action in $C^\infty(\Omega)$ is defined as $g \cdot u(x) = u(g^{-1} \cdot x)$, $\forall g \in G$). This fact suggests that the Laplace operator contains an amount of geometric properties, some clever examples are the trace formulas (Poisson summation formula and Selberg’s trace formula), where a sum over the eigenvalues is compared with a sum over the primitive and closed geodesics of the manifold (see [35]) via the Fourier transform. In the study of spectral invariants such as the counting function, the theta function (heat trace and wave trace) and the zeta function given respectively by

\[ H(\lambda) = \sum_{j: \lambda_j < \lambda} 1, \]

\[ \theta(t) = \text{tr}(e^{-t\Delta}) = \sum_{j=1}^{\infty} e^{-\lambda_j t}, \quad \mathcal{W}(t) = \text{tr}(e^{it\sqrt{\Delta}}) \quad t > 0, \]

\[ \zeta(s) = \text{tr}(\Delta^{-s}) = \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j^s} \quad \Re s > n \]

some constants with geometric data can be obtained [41, 34, 21, 24]

\[ H(\lambda) = \frac{\text{vol}(\Omega)}{6\pi^2} \lambda^3 \pm \frac{\text{vol}(\partial\Omega)}{16\pi} \lambda^2 + o(\lambda), \quad \text{as} \lambda \to +\infty, \]

\[ \theta(t) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^j, \]

where if $dv$ is the volume element induced by the Riemannian metric and $S$ is the scalar curvature, then

\[ a_0 = \text{vol}(\Omega), \quad a_1 = \frac{1}{6} \int_\Omega S \, dv, \quad \text{for} \partial\Omega = \emptyset \]

and

\[ a_0 = C_n \text{vol}(\Omega), \quad a_1 = C'_n |\partial\Omega|, \quad \text{for} \partial\Omega \neq \emptyset \]

for some constants $C_n, C'_n$. For fractals in $\mathbb{R}^2$ the exponents of the counting function are related with the dimension, see[39] for the details.

These invariants are constructed from the heat equation and the wave equation defined by the operators $\partial_t - \Delta, \partial_t^2 - \Delta$, respectively. Suppose that a perfectly elastic and infinitely thin membrane of solid matter, of uniform material and thickness, is given in a domain $\Omega$ with a fixed boundary. Then if the height of a point in the membrane $x$ at time $t$ is given by $u(x,t)$, $u$ must be a solution of the following problem

\[
\begin{align*}
(\partial_t^2 - c^2 \Delta) u &= 0 \quad \text{in} \ \Omega, \quad t > 0 \\
 u(\cdot, t) &= 0 \quad \text{on} \ \partial\Omega, \quad t > 0 \\
 u(x, 0) &= u_0(x) \quad x \in \Omega
\end{align*}
\]
where $u_0$ is the height of the membrane at the initial time $t = 0$, and $c$ is a constant depending on the material properties of the membrane \[33\]. The vibrations of the membrane decompose into natural vibrations, these natural vibrations $\{u_k\}_{k \geq 1}$ are time independent and they form a numerable system where each $u_k$ is a solution of the associated eigenvalue problem which is given by (1). The number $\lambda_k \in \mathbb{R}$ is called the Dirichlet eigenvalue of the eigenfunction $u_k$ and gives the associated frequency $f_k = \sqrt{\frac{\lambda_k}{2\pi}}$. The first frequency $f_1$ is called the fundamental tone. Then the solution $u$ can be expressed as a limit of a linear time-dependent combination of $\{u_k\}_{k \geq 1}$.

If $M = \mathbb{R}$ and $\Omega = (0, \pi)$ then the problem consists to find the height of a vibrating string and the natural vibrations are given by $u_k(x) = \sin(kx)$ for $k \geq 1$.

Other geometric properties can be obtained by the first eigenvalues instead of their asymptotics. Faber-Krahn isoperimetric inequality tells us that the minimizer of the first eigenvalue (over the open subsets of $M$ of fixed area) is the disk which in physical terms essentially tells us that the the gravest fundamental tone of all possible drums of fixed area is attained by the disk, the drum with circular boundary. If $\Omega \subset \mathbb{R}^2$, $\lambda_1(\Omega)$ denotes the first eigenvalue of $\Omega$, and $\text{vol}(\Omega) = 1$ then the Faber-Krahn isoperimetric inequality reads

$$\lambda_1(\Omega) \geq \frac{j_{0,1}^2}{\pi} \simeq 0.76548$$

where $j_{0,1}$ is the first zero of the Bessel function $J_0$ (the first eigenfunction of the disk).

### 1.2 The Finite Difference Method

Finite Differences is a pointwise method which approximates differential operators via a finite difference approximation of the derivative. The Finite Difference approach using the 5-point Laplacian is only a particular case of the Finite Difference methods. If we use central differences, i.e., we approximate the derivative of a function $u \in C^1(\mathbb{R})$ by

$$\frac{du}{dx}(x) = \frac{u(x + h/2) - u(x - h/2)}{h} + O(h^2)$$

for $h > 0$ sufficiently small then for a function $u \in C^2(\mathbb{R})$ the approximation for the second derivative is

$$\frac{d^2u}{dx^2}(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + O(h^2).$$

So for a bounded, open and connected set $\Omega \subset \mathbb{R}^2$ the uniform approximation for the Laplacian of a function $u \in C^2(\Omega)$ is given by

$$\Delta u(x) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x) =$$

$$\frac{u(x + h, y) - 2u(x, y) + u(x - h, y)}{h^2} + \frac{u(x, y + h) - 2u(x, y) + u(x, y - h)}{h^2} + O(h^2).$$

For a uniform pointwise discretization $\{(z_i)\}_{i=1}^N \subset \Omega$ the discretization of (1) reads

$$Au = \lambda u$$
where \( u = (u_i)_{i=1}^N \), \( u_i = u(z_i) \) and \( A \) is the matrix obtained by the discretization of the derivative, its coefficients depend on the boundary of the domain and of course of the labels of the points \( z_i \). So the problem is reduced to an ordinary and finite-dimensional eigenvalue problem. For further details of the application of Finite Difference methods see [27].

### 1.3 Spectral Methods

It is known that derivatives are reduced to multiplication via the Fourier transform. The discrete Fourier transform (DFT) of a vector \( v \in \mathbb{R}^m \) is defined to be the vector \( \hat{v} \) with components

\[
\hat{v}_k = \frac{2\pi}{m} \sum_{j=1}^{m} e^{-ikx_j} v_j, \quad k = \frac{m}{2} + 1, ..., \frac{m}{2},
\]

where \( x_j \) are the Fourier grid points (\( m \) equally spaced points in \([0, 2\pi]\))

\[
x_j = \frac{2\pi j}{m}, \quad j = 1, ..., m.
\]

The inverse DFT is given by

\[
v_j = \frac{1}{2\pi} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} e^{ikx_j} \hat{v}_k, \quad j = 1, ..., m.
\]

Given the vector of values of a function on the Fourier grid one can compute the \( n \)th spectral derivative as follows

1. Compute \( \hat{v} \) from \( v \).
2. Define \( \hat{w}_k = (ik)^n \hat{v}_k \). If \( n \) is odd, set \( \hat{w}_{m/2} = 0 \).
3. Compute \( w \) from \( \hat{w} \).

Since this operation is linear the second derivative can be represented by a matrix or it can be performed by the use of the Fast Fourier Transform (FFT). The above procedure is equivalent to first constructing the \textit{trigonometric interpolant}

\[
p(x) = \frac{1}{2\pi} \left( \frac{1}{2} e^{-ixm/2} \hat{v}_{-m/2} + \frac{1}{2} e^{ixm/2} \hat{v}_{m/2} + \sum_{k=-m/2+1}^{m/2-1} e^{ikx} \hat{v}_k \right),
\]

and then computing the \( n \)th derivative of the interpolant evaluating it

\[
w_k = p^{(n)}(x_k).
\]

This procedure can be applied in the unit disk \( \mathbb{D} \) using separation of variables (in polar coordinates), which reduces the partial differential eigenvalue problem to an ordinary differential eigenvalue problem and can be solved applying the discrete Fourier transform. For a bounded simply connected polygonal domain \( \Omega \in \mathbb{R}^2 \) is similar if we map \( \Omega \) to the disk via the inverse of the Schwarz-Christoffel mapping \( f \), where

\[
f'(\omega) = C \prod_{k=1}^{N} (\omega - \omega_k)^{\beta_k},
\]
where $\omega_k$ are the preimages of the corners $z_k$ of the polygon $\Omega$, i.e. $\omega_k = f^{-1}(z_k)$, see [4, 11, 12, 5] for the details. In the following two figures we show the preimage of equally spaced radii and of concentric circles in two examples used in Section 4, the computations are done with the MATLAB code given in [11].

![Figure 1](image1.png) ![Figure 2](image2.png)

Figure 1.3.1 Figure 1.3.2

Then, problem (1) is reduced to find $(\lambda, v)$ where

$$
\begin{cases}
-\Delta v = \lambda |f'|^2 v & \text{in } D \\
v = 0 & \text{on } \partial D
\end{cases}
$$

and $u = v \circ f^{-1}$.

Another example of spectral methods is the Method of Particular Solutions made by Fox, Henrici, and Moler (see [14, 6, 20]) which uses linear combinations of analytic expressions of the eigenfunctions of an unbounded sector of angle $\pi/\alpha$ given by

$$u^{(k)}(r, \theta) = J_{\alpha k}(\sqrt{\lambda} r) \sin \alpha k \theta$$

called Fourier-Bessel functions. On then tries to vary $\lambda$ until one can find a linear combination, i.e. a function

$$u^*(r, \theta) = \sum_{k=1}^{N} c^{(N)}_k u^{(k)}(r, \theta)$$

satisfying the boundary conditions. One approach to impose the boundary conditions is to require $u^*(r_i, \theta_i) = 0$ for $i = 1, ..., N$ where $(r_i, \theta_i)$ are $N$ given collocation points on the boundary. Then one considers the square system of nonlinear equations

$$A(\lambda)c = 0$$

with

$$a_{ik}(\lambda) = J_{\alpha k}(\sqrt{\lambda} r_i) \sin \alpha k \theta_i, \quad i, k = 1, ..., N.$$ 

The modified method given in [6] shows a better approach using collocations points not only at the boundary but at the interior of the domain.

1.4 On a concept of Fractal Geometry

Since we will deal with domains with fractal boundary we need to define what 'fractal' means. Some authors dedicate much work to get a definition and an intuition of what fractality is [29, ?], since our case is a simple one we only give a few definitions. Most of the examples
shown in Section 4 are open bounded subsets of $\mathbb{R}^2$ and the boundary is given by a piecewise continuous curve $\Gamma$. The curve $\Gamma$ has infinite Lebesgue measure and zero area. One then needs to define a measure "between" length and area.

For $\varepsilon > 0$ define the corresponding tubular neighborhood of $\partial \Omega$ to be the set $\{ x \in \Omega : d(x, \partial \Omega) < \varepsilon \}$, where $d(x, \partial \Omega) = \inf_{y \in \partial \Omega} d(x, y)$. If $\text{vol}_n$ is the $n$-dimensional Lebesgue measure then the volume of the tubular neighborhood is defined to be $V(\varepsilon) := \text{vol}_n \{ x \in \Omega : d(x, \partial \Omega) < \varepsilon \}$.

The Minkowski dimension of $\partial \Omega$ is

$$D = D_{\partial \Omega} := \inf \{ \alpha \geq 0 : V(\varepsilon) = O(\varepsilon^{n-\alpha}) \text{ as } \varepsilon \to 0^+ \}.$$ 

Then the Minkowski content of $\partial \Omega$ is

$$M = M(D; \partial \Omega) = \lim_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{-(n-D)},$$

and the set $\partial \Omega$ is Minkowski measurable if its Minkowski content exists in $(0, \infty)$. The upper and lower Minkowski content are respectively defined by

$$M^* = M^*(D; \partial \Omega) = \lim_{\varepsilon \to 0^+} \sup V(\varepsilon)\varepsilon^{-(n-D)}$$

and

$$M_* = M_*(D; \partial \Omega) = \lim_{\varepsilon \to 0^+} \inf V(\varepsilon)\varepsilon^{-(n-D)}.$$ 

Then we have that $0 \leq M_* \leq M^* \leq \infty$ and $\partial \Omega$ is Minkowski measurable if and only if $M_* = M^* = M \in (0, \infty)$.

In [29] a natural definition of fractality is given, a set is said to be fractal if its Hausdorff dimension is strictly greater than the topological dimension (in our case $n$ is the topological dimension). In self-similar cases the Hausdorff dimension and the Minkowski dimension coincide. For a deep analysis on the subject see [13, 24].
2 The Differential Eigenvalue Problem

In this section we will give the analytical tools to define formally the eigenvalue problem we want to solve. We are not dealing in the pointwise sense (which the original problem (1) is naturally stated) but in a variational sense. First of all, functional analytical tools are needed for the definition of eigenvalues since the essential property of a diagonalizable operator is compactness (in our case the Laplacian is not compact but its inverse it is). In order to reformulate the eigenvalue problem in a variational sense we need some tools based on weak derivatives and hence on Sobolev Spaces, a good reference on the subject is [1]. These two preliminary parts are the first two subsections, then we proceed by the Spectral Theory of linear operators only stating the results (see [7] for proofs) and then we prove some results on approximation theory given in [3, 36]. Finally we give the reformulation of the source and the eigenvalue problem. We restrict to the Dirichlet problem (for the Neumann or mixed problem one proceeds similarly).

2.1 Preliminaries on Functional Analysis

Let $X$ be a vector space and let $\|\cdot\|$ be a norm defined on $X$ (that is $\|x\| = 0 \iff x = 0$, $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$, $\|x + z\| \leq \|x\| + \|z\|$ for all $x, z \in X$) then the pair $(X, \|\cdot\|)$ is called a normed space. The linear span of a subset $\{x_n\}_{n \geq 1}$ is defined as

$$\langle \{x_n\}_{n \geq 1} \rangle := \left\{ x \in X \mid x = \sum_{n=1}^{\infty} a_n x_i : \exists I \subset \mathbb{N} : a_i \neq 0 \forall i \in I, |I| < \infty \right\}.$$

We say that $(X, \|\cdot\|)$ is a Banach space if it is complete with the given norm i.e., every Cauchy sequence is convergent. If the norm $\|\cdot\|_X$ is induced by a scalar product $\langle \cdot, \cdot\rangle_X$ (that is $\|x\|_X = \sqrt{\langle x, x\rangle_X}$ then the vector space $X$ (or $(X, \langle \cdot, \cdot\rangle_X)$) is called an inner product space. Note that in complex vector spaces a scalar product is an hermitian and non-degenerate bilinear form. A Hilbert space is a complete inner product space, in a Hilbert space the Cauchy-Schwarz inequality holds

$$|\langle x, y\rangle_X| \leq \|x\|_X \|y\|_X.$$

We write $X = Y \oplus Z$ (for $X, Y, Z$ vector spaces) is for each $x$ there is a unique $y \in Y$ and a unique $z \in Z$ such that $x = y + z$. In a Hilbert space $X$ the orthogonal complement of a closed subspace $Y$ is the closed subspace given by

$$Y^\perp := \{ x \in X \mid \langle x, y\rangle_X = 0 \ \forall y \in Y \}.$$

In an inner product space $X$ a subset $\{x_n\}_{n \geq 1}$ is called a complete orthonormal system if $(x_i, x_j) = \delta_{ij}$ and the system is called a complete orthonormal basis if for each $x \in X$ we can write $x = \sum_{n=1}^{\infty} a_n x_i$ where $a_i = \langle x, x_i \rangle$ i.e., $X = \langle \{x_n\}_{n \geq 1} \rangle$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be complex normed spaces. If $T : X \to Y$ is a linear operator (that is $T(\alpha x + \beta z) = \alpha T(x) + \beta T(z)$ for all $\alpha, \beta \in \mathbb{C}$, $\forall x, z \in X$) then the norm of $T$ is defined by

$$\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}.$$
The null space (or kernel) and the image space (or range) of are respectively defined by

\[ N(T) := \{ x \in X \mid Tx = 0 \}, \quad T(X) = \{ y \in Y \mid \exists x \in X : Tx = y \}. \]

The operator \( T \) is bounded if \( \| T \| < \infty \) and the space of bounded linear operators acting on \( X \) and \( Y \) is denoted by \( \mathcal{L}(X,Y) \) (which is a normed space with norm \( \| \cdot \|_{\mathcal{L}(X,Y)} = \| \cdot \| \)). The space of bounded linear functionals \( \mathcal{L}(X,C) := X' \) is called the dual space. For a bounded linear operator \( T : X \to Y \) between normed spaces the adjoint operator \( T' : Y' \to X' \) is defined by the property

\[ \forall g \in Y' \exists f \in X' : f(x) = (T'g)(x) = g(Tx). \]

Then \( \| T' \|_{\mathcal{L}(Y',X')} = \| T \|_{\mathcal{L}(X,Y)}. \) In most of cases \( X \) is a Hilbert space with a norm induced by a scalar product \(( \cdot, \cdot )_X. \) Let \( X \) and \( Y \) be Hilbert spaces and \( T \in \mathcal{L}(X,Y), \) then the adjoint operator \( T^* \) of \( T \) is defined as \( T^* : Y \to X \) such that \( \forall x \in X \) and \( \forall y \in Y \)

\[ (Tx, y)_Y = (x, T^*y)_X. \]

A bounded linear operator \( T : X \to X \) is said to be Hermitian if \( T^* = T, \) unitary if \( T \) is bijective and \( T^* = T^{-1} \) and normal if \( TT^* = T^*T. \) Let \( X \) be a Hilbert space and \( Y \) a closed subspace of \( X. \) Then the direct sum \( X = Y \oplus Y^\perp \) holds (see [?]) and it defines a mapping \( P : X \to Y \) such that \( y = Px. \) The mapping \( P \) is called a projection of \( X \) onto \( Y \) and satisfies \( P^2 = P, \) \( N(P) = Y^\perp. \) A bounded linear operator \( P : X \to X \) on a Hilbert space \( X \) is a projection if and only if \( P \) is self-adjoint and \( P^2 = P. \)

### 2.2 Sobolev spaces

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega. \) The weak derivative \( \partial \) of a function in \( L^p(\Omega), \) for \( p \geq 1, \) is defined as the linear map which respects the integration by parts formula. It is shown that this map behaves similarly as the derivative for functions that are not differentiable (that the weak derivative of a constant is 0 in a weak sense, Leibniz’s rule...). Let \( \alpha \) be a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \) we define

\[ \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial |\alpha| f}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \]

where \( |\alpha| = \sum_{i=1}^n \alpha_i. \) Let \( s \) be a non-negative integer and \( 1 \leq p < \infty. \) The Sobolev spaces are defined as

\[ W^{s,p}(\Omega) := \{ f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \forall |\alpha| \leq s \} \]

associated with the norm

\[ \| f \|_{W^{s,p}(\Omega)} := \left( \sum_{|\alpha| \leq s} \int_{\Omega} |\partial^\alpha f(x)|^p \, dx \right)^{1/p}. \]

The corresponding semi-norm is defined as
\[ |f|_{W^{s,p}(\Omega)} := \left( \sum_{|\alpha|=s} \int_{\Omega} |\partial^\alpha f(x)|^p \, dx \right)^{1/p}. \]

\( W^{s,p}_0(\Omega) \) denotes the closure of \( C^\infty_c(\Omega) \) in the \( W^{s,p} \) norm. If \( p = 2 \) then the Sobolev spaces \( W^{s,2}(\Omega) \) are also Hilbert spaces, they are denoted by \( H^s(\Omega) \) (similarly it is denoted \( H^s_0(\Omega) = W^{s,2}_0(\Omega) \)).

The Sobolev spaces of fractional order can be defined as follows. Let \( s \geq 0 \) and \( 1 \leq p < \infty \). Define \( \lfloor s \rfloor \) to be the greatest non-negative integer less or equal than \( s \) and \( \{s\} \in (0, 1) \) to be the corresponding fractional part of \( s \), i.e., \( s = \lfloor s \rfloor + \{s\} \). Then \( W^{s,p}(\Omega) \) is the space of distributions \( u \in C^\infty_c(\Omega)' \) such that

\[
\int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\lfloor s \rfloor p}} \, dxdy < \infty \quad \forall |\alpha| = \lfloor s \rfloor,
\]

with norm

\[
\|u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{\lfloor s \rfloor,p}(\Omega)} + \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\lfloor s \rfloor p}} \, dxdy.
\]

The negative norm for a function \( f \in L^2(\Omega) \) is defined as

\[
\|f\|_{-s} := \sup_{u \in H^s_0(\Omega), \|u\|_{H^s(\Omega)}} |(f, u)_{L^2(\Omega)}|.
\]

By Schwarz's inequality, we have

\[
|(f, u)_{L^2(\Omega)}| \leq \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \cdot \|u\|_{H^s(\Omega)} \implies \|f\|_{-s} \leq \|f\|.
\]

We denote by \( H^{-s}(\Omega) \) the completion of \( L^2(\Omega) \) with respect to the negative norm \( \|\cdot\|_{-s} \).

**Theorem 2.1.** The dual space \( H^s_0(\Omega)' \) of \( H^s_0(\Omega) \) may be identifies with the completion of \( L^2(\Omega) \) with respect to the negative norm, i.e.,

\[ H^s_0(\Omega) = H^{-s}(\Omega). \]

Furthermore, any continuous linear functional on \( H^{-s}(\Omega) \) can be represented by an element in \( H^s_0(\Omega) \), i.e.,

\[ H^{-s}(\Omega)' = H^s_0(\Omega). \]

If \( u \in C^\infty_c(\Omega) \), the restriction of \( u \) on \( \partial\Omega \), called the trace operator, is defined as

\[ \gamma_0(u) = u|_{\partial\Omega}. \]

**Theorem 2.2.** Let \( \Omega \) be a bounded Lipschitz domain and \( 1/2 < s \leq 1 \). The mapping \( \gamma_0 \) defined on \( C^\infty_c(\Omega) \) has a unique continuous extension as a linear operator from \( H^s(\Omega) \) onto \( H^{s-1/2}(\partial\Omega) \). In addition

\[ H^1_0(\Omega) = \{ u \in H^1(\Omega) : \gamma_0(u) = 0 \}. \]
Proposition 2.1. Poincaré-Friedrichs inequality
If $\Omega$ is bounded, then $\|\cdot\|_{H^m(\Omega)}$ is an equivalent norm to $\|\cdot\|_{H^m_0(\Omega)}$ in $H^m_0(\Omega)$.

As a particular case of The Rellich-Kondrachov Theorem (see [1]) we have the following result.

Theorem 2.3. Compact Embedding
Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded and locally Lipschitz domain. Then for $1 < p < \infty$, $2m \geq n$ and $j \geq 0$ we have that the following embeddings

1. $W^{j+m,p}(\Omega) \hookrightarrow C^j(\overline{\Omega})$ if $n > mp$

2. $W^{j+m,p}(\Omega) \hookrightarrow W^{j,p}(\Omega)$ if $n = mp$

are compact. The embeddings are also compact if we replace $W^{j+m,p}(\Omega)$ by $W^{j+m,p}_0(\Omega)$. In particular, for $m \geq p = n = 2$ and $j = 0$ we have

$$H^m(\Omega) \hookrightarrow C^0(\overline{\Omega}), \quad H^m_0(\Omega) \hookrightarrow C^0(\overline{\Omega})$$

and for $m = 1$

$$H^1(\Omega) \hookrightarrow L^2(\Omega), \quad H^1_0(\Omega) \hookrightarrow L^2(\Omega)$$

all embeddings being compact.

2.3 Spectral theory of Linear Operators

Let $T : X \to X$ be a bounded linear operator. Then the limit

$$r_\sigma(T) := \lim_{k \to \infty} \|T^k\|^{1/k}$$

exists and is called the spectral radius of $T$.

The resolvent operator of $T$, $R_z(T)$ is defined as

$$R_z(T) = (T - z1)^{-1}$$

provided that $T - z1$ has an inverse.

Definition 2.1. Let $X$ be a complex normed space and $T : X \to X$ a linear operator. A regular value $z$ of $T$ is a complex number such that

1. $R_z(T)$ exists,
2. $R_z(T)$ is bounded, and
3. $R_z(T)$ is defined on a set which is dense in $X$.

The resolvent set $\rho(T)$ of $T$ is the set of all regular values $z$ of $T$. Its complement $\sigma(T) := C \setminus \rho(T)$ is called the spectrum of $T$. The spectrum $\sigma(T)$ can be partitioned into three disjoint sets:
1. \( \sigma_p(T) \) which is called the point spectrum, is the set of \( z \) such that \( R_z(T) \) does not exist (or not well defined). If \( z \in \sigma_p(T) \) then \( z \) is called an eigenvalue of \( T \).

2. \( \sigma_c(T) \) which is called the continuous spectrum, is the set of \( z \) such that \( R_z(T) \) exists and is defined on a dense set in \( X \), but \( R_z(T) \) is unbounded.

3. \( \sigma_r(T) \) which is called the residual spectrum, is the set of \( z \) such that \( R_z(T) \) exists and the domain of \( R_z(T) \) is not dense in \( X \).

For \( z_1, z_2 \in \rho(T) \), the first resolvent equation is given by

\[
R_{z_1} - R_{z_2} = (z_2 - z_1)R_{z_1}R_{z_2} = (z_2 - z_1)R_{z_2}R_{z_1}.
\]

For \( z \in \rho(T_1) \cap \rho(T_2) \), the second resolvent equation is given by

\[
R_z(T_1) - R_z(T_2) = R_z(T_1)(T_2 - T_1)R_z(T_2) = R_z(T_2)(T_2 - T_1)R_z(T_1).
\]

Lemma 2.4. For \( T \in \mathcal{L}(X) \), the following properties hold

1. If \( |z| > r(\sigma(T)) \), \( R_z(T) \) exists and has the series expansion

\[
R_z(T) = \sum_{k=0}^{\infty} z^{-k-1}T^k.
\]

2. \( \rho(T) \) and \( \sigma(T) \) are nonempty. \( \sigma(T) \) is compact.

3. \( r(\sigma(T)) = \max_{z \in \sigma(T)} |z| \).

Definition 2.2. Let \( z \in \sigma_p(T) \) be an eigenvalue of \( T \). If

\[
T_z x := Tx - zx = 0
\]

for some \( x \neq 0 \), \( x \) is called an eigenfunction of \( T \) associated to \( z \).

A subspace \( M \) of \( X \) is called an invariant subspace under \( T \) if \( T(M) \subset M \). If \( X = M \oplus N \), where \( M, N \) are closed subspaces of \( X \) and invariant under \( T \), \( T \) is said to be completely reduced by \( (M, N) \).

Let \( \lambda \) be an isolated eigenvalue of \( T \) such that there exists simple closed curves \( \Gamma, \Gamma' \subset \rho(T) \) enclosing \( \lambda \). Furthermore, both \( \Gamma \) and \( \Gamma' \) enclose no other eigenvalues of \( T \). Define

\[
P := \frac{1}{2\pi i} \int_{\Gamma} R_z(T)dz.
\]

Then \( P \in \mathcal{L}(X) \) and by the first resolvent equation

\[
P^2 = \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} R_z(T)R_{z'}(T)dz'dz = \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Gamma'} \frac{R_z(T) - R_{z'}(T)}{z' - z} dz'dz.
\]
Without loss of generality we can suppose that $\Gamma'$ does not intersect $\Gamma$ since $\|R_z(T) - R_{z'}(T)\|_{|z' - z|}$ is bounded at $\Gamma \cap \Gamma'$ and the integral does not depend on different paths in the same homotopy class [23]. Let’s suppose that $\Gamma'$ encloses a bounded open set $U$ (so $\partial U = \Gamma'$) such that $\Gamma \subset U$, by this choice it follows that
$$\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z'} dz = 0, \quad \frac{1}{2\pi i} \int_{\Gamma'} \frac{1}{z'} dz' = 1.$$ 

Then by Fubini’s theorem we have
$$P^2 = \frac{1}{(2\pi i)^2} \int_{\Gamma} R_z(T) \int_{\Gamma'} \frac{1}{z'} dz' dz - \frac{1}{(2\pi i)^2} \int_{\Gamma'} R_{z'}(T) \int_{\Gamma} \frac{1}{z'} dz dz'.$$
$$= \frac{1}{2\pi i} \int_{\Gamma} R_z(T) dz = P.$$

Thus $P$ is a projection operator. For the given paths $P$ is the projection from $X$ to the generalized eigenspace associated with $\lambda$ when $T$ is a compact operator.

**Definition 2.3.** Let $X$ and $Y$ be normed spaces. An operator $T : X \to Y$ is called a compact linear operator if $T$ is linear and for every bounded subset $M \subset X$, $T(M)$ is relatively compact, i.e., $T(M)$ is compact.

**Proposition 2.2.** Let $X$ and $Y$ be normed spaces and $T : X \to Y$ be a linear operator. Then $T$ is compact if and only if for every bounded sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$, $\{Tx_n\}_{n \in \mathbb{N}}$ has a convergent subsequence.

Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If either $T$ or $S$ us compact, $TS$ is compact from $X$ to $Z$.

**Lemma 2.5.** Let $X$ and $Y$ be normed spaces. Then

1. Every compact linear operator $T : X \to Y$ is bounded, hence continuous.
2. If $\text{dim } X = \infty$, the identity operator $1 : X \to Y$ is not compact.

**Proposition 2.3.** Let $X$ and $Y$ be normed spaces and $T : X \to Y$ be a linear operator. Then

1. If $T$ is bounded and $\text{dim } T(X) < \infty$, $T$ is compact.
2. If $\text{dim } X < \infty$, $T$ is compact.

**Proposition 2.4.** Let $\{T_n : X \to Y\}_{n \in \mathbb{N}}$ be a sequence of compact operators. If $\{T_n\}_{n \in \mathbb{N}}$ is uniformly convergent, i.e., $\|T_n - T\| \to 0$, then the limit operator $T := \lim_{n \to \infty} T_n$ is compact.

**Proposition 2.5.** Let $T : X \to Y$ be a linear operator. If $T$ is compact, its adjoint operator $T' : Y' \to X'$ is compact.
Theorem 2.6. (Fredholm Alternative)
Let $X$ be a Banach space and $T : X \to X$ be compact. Then the equation
\[(z - T)u = f, \quad z \neq 0\]
has a unique solution $u \in X$ for any $f \in X$ if and only if the homogeneous equation
\[(z - T)u = 0\]
has only the trivial solution $u = 0$. In such a case, the operator $z - T$ has a bounded inverse.

Let $T : X \to X$ be a compact linear operator. The set of eigenvalues of $T$ is at most countable and 0 is the only possible accumulation point. Every $\lambda \in \sigma(T) \setminus \{0\}$ is an eigenvalue. If $X$ is infinite dimensional, then $0 \in \sigma(T)$.

For an eigenvalue $\lambda \neq 0$ the dimension of the corresponding eigenspace of $T$ is finite and the null spaces of $T_\lambda, T_\lambda^2, T_\lambda^3, ...$ are finite dimensional. There is a number $r \in \mathbb{N}$ depending on $\lambda \neq 0$ such that
\[X = N(T_\lambda^r) \oplus T_\lambda^r(X)\]
Furthermore, the null spaces satisfy
\[N(T_\lambda^r) = N(T_\lambda^{r+1}) = N(T_\lambda^{r+2}) = ...\]
and the ranges satisfy
\[T_\lambda^r(X) = T_\lambda^{r+1}(X) = T_\lambda^{r+2}(X) = ....\]
If $r > 0$
\[N(T_\lambda^0) \subset N(T_\lambda) \subset ... \subset N(T_\lambda)\]
and
\[T_\lambda^0(X) \supset T_\lambda(X) \supset ... \supset T_\lambda(X)\]

Definition 2.4. The space $N(T_\lambda^r)$ is called the generalized eigenspace of $T$ associated to the eigenvalue $\lambda$. The algebraic multiplicity of $\lambda$ is defined as $\dim N(T_\lambda^r)$. The geometric multiplicity is defined as $\dim N(T_\lambda)$.

Let $T : X \to X$ be a bounded self-adjoint operator on a complex Hilbert space $X$. Then

1. $\sigma_p(T) \subset \mathbb{R}$ (all eigenvalues of $T$ are real)
2. $N(T_\lambda) \perp N(T_\mu) \iff \lambda \neq \mu$ (the eigenfunctions corresponding to different eigenvalues of $T$ are orthogonal with respect to the inner product)
3. $\|T\| = \sup_{\|x\|=1} |(Tx, x)_X|$ 

Theorem 2.7. Let $T : X \to X$ be a compact, self-adjoint, linear operator in a Hilbert space $X$. Then there exist at most a countable set of real eigenvalues $\{\lambda_n\}_{n=1}^\infty$ and corresponding eigenfunctions $x_1, x_2, ...$ such that

1. $Tx_j = \lambda_j x_j$ and $x_j \neq 0$, $j = 1, 2, ...$, 

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2. \( (x_m, x_n)_X = 0 \), if \( n \neq m \),

3. \(|\lambda_1| \geq |\lambda_2| \geq \ldots \geq 0\),

4. if the sequence of eigenvalues is infinite, \( \lim_{j \to \infty} \lambda_j = 0 \)

5. \( T x = \sum_{j \geq 1} \lambda_j (x, x_j)_X x_j \) with convergence in \( X \) when the sum has infinitely many terms,

6. \( X = \langle \{x_n\}_{n=1}^\infty \rangle \oplus N(T) \).

Let \( X \) be a complex Banach space with norm \( \| \cdot \| \) and \( T \) be a compact operator on \( X \). Let \( \{X_h\} \) be a sequence of finite dimensional subspaces of \( X \) and \( \{T_h : X_h \to X_h\} \) be a sequence of linear operators.

Let \( Y \) be a closed subspace of \( X \). For \( x \in X \), the distance between \( x \) and \( Y \) is defined as \( d(x, Y) = \inf_{y \in Y} \|x - y\| \) and for \( Z \) a closed subspace of \( X \), \( d(Y, Z) := \sup_{y \in Y, \|y\|=1} d(y, Z) \). The gap between \( Y \) and \( Z \) is defined as

\[ \delta(Y, Z) = \max\{d(Y, Z), d(Z, Y)\} \]

Let \( E : X \to X \) be the spectral projection defined in (3) for a closed simple curve \( \Gamma \subset \rho(T) \cap \rho(T_h) \) (enclosing a domain \( D \subset C \) with maybe nonempty intersection with \( \sigma(T) \)). \( E_h : X_h \to X_h \) is defined similarly

\[ E_h := \frac{1}{2\pi i} \int_{\Gamma} R_z(T_h) \, dz. \]

If \( T_h \to T \) as \( h \to 0 \), \( E_h \) is well defined for \( h \) small enough.

The \( h \)-norm of an operator \( T \) us defined as

\[ \|T\|_h = \sup_{x \in X_h, \|x\|=1} \|Tx\|. \]

**Theorem 2.8.** The Descloux, Nassif, Rappaz (DNR) conditions for a sequence of operators \( \{T_h\} \) defined on \( X_h \) and for an operator \( T \) on \( X \) are

**P1.** \( \lim_{h \to 0} \|T - T_h\|_h = 0 \),

**P2.** \( \lim_{h \to 0} d(x, X_h) = 0, \ \forall x \in X \).

Assume that the first condition is satisfied. Then the following is satisfied

(a) For \( F \subset \rho(T) \) closed, there exists a constant \( C \) independent of \( h \) such that

\[ \|R_z(T_h)\|_h \leq C \ \forall z \in F \]

for \( h \) small enough.

(b) Let \( \Omega \subset C \) be an open set such that \( \sigma(T) \subset \Omega \). Then there exists \( h_0 > 0 \) such that

\[ \sigma(T_h) \subset \Omega, \ \forall h < h_0 \]
The differential eigenvalue problem

(c) One has that \( \lim_{h \to 0} \| E - E_h \|_h = 0 \) and \( \lim_{h \to 0} d(E_h(X_h), E(X)) = 0 \).

(d) If, in addition, the second condition is satisfied, we have that

\[
\lim_{h \to 0} d(x, E_h(X_h)) = 0 \quad \forall x \in E(X).
\]

Proof. (a) Let \( z \in F \subset \rho(T) \). Then for any \( x \in X \), there exists a constant \( C > 0 \) such that

\[
\|(z - T)x\| \geq 2C\|x\|.
\]

For \( h > 0 \) small enough, \( P1 \) implies

\[
\|(T - T_h)x\| \leq C\|x\| \quad \forall x \in X_h.
\]

Hence for \( x \in X_h \) and \( z \in F \), we have that

\[
\|(z - T_h)x\| \geq \|(z - T)x\| - \|(T - T_h)x\| \geq C\|x\|.
\]

Since \( X_h \) is finite dimensional, \( R_z(T_h) \) exists

\[
\|R_z(T_h)\|_h \leq C.
\]

(b) It is a direct consequence of (a).

(c) For \( h > 0 \) small enough, one has that

\[
\|E - E_h\|_h \leq \frac{1}{2\pi} \int_{\Gamma} \|R_z(T) - R_z(T_h)\|_h|dz| = \frac{1}{2\pi} \int_{\Gamma} \|R_z(T)(T - T_h)R_z(T_h)\|_h|dz| = \frac{1}{2\pi} \int_{\Gamma} \|R_z(T)\| \cdot \|(T - T_h)\|_h R_z(T_h)\|_h|dz|.
\]

Combination of \( P1 \) and (a) implies \( \lim_{h \to 0} \|E - E_h\|_h = 0 \). Then it follows that

\[
\lim_{h \to 0} d(E_h(X_h), E(X)) = 0.
\]

(d) Let \( x \in E(X) \). From \( P2 \), we conclude that there exists a sequence \( \{x_h \in X_h\}_{h>0} \) such that

\[
\lim_{h \to 0} \|x - x_h\| = 0.
\]

Thus we have that

\[
\|x - E_hx_h\| = \|E(x - x_h)\| \leq \|x - x_h\| + \|E - E_h\|_h \leq \|E\|\|x - x_h\| + \|E - E_h\| \|x\|.
\]

Since \( E \) is continuous we proved (d).
Assume that $T$ is a compact operator from $X$ to $X$ and \{${T_h}$\}$_{0 < h < 1}$ is a family of compact operators also from $X$ to $X$. In addition, $T_h \to T$ in norm as $h \to 0$.

Let $\lambda \in \sigma(T)$, i.e., $\lambda$ is an eigenvalue of $T$. Then there exists a smallest integer $r$, called the ascent of $\lambda I - T$, such that

$$N((\lambda I - T)^r) = N((\lambda I - T)^{r+1}).$$

Recall that the space $N((\lambda I - T)^r)$ is finite dimensional and its dimension $m = \dim N((\lambda I - T)^r)$ is called the algebraic multiplicity of $\lambda$. The geometric multiplicity $n$ of $\lambda$ is the dimension of $N(\lambda I - T)$ (note that $n \leq m$). A vector $u \in N((\lambda I - T)^r)$ is called a generalized eigenvector of $T$ and its order is the smallest integer such that $u \in N((\lambda I - T)^j)$.

If $X$ is a Hilbert space and $T$ is self-adjoint, the ascent of $\lambda - T$ is one and the algebraic multiplicity equals the geometric multiplicity (Theorem 2.7).

Since $T_h$ converges to $T$ in norm, $E_h$ converges to $E$ in norm and

$$\dim E_h(X_h) = \dim E(X) = m.$$

In addition, there exists exactly $m$ eigenvalues of $T_h$ inside $\Gamma$ if $h$ is small enough. We denote these values by $\lambda_{1,h}, ..., \lambda_{m,h}$. Consequently,

$$\lim_{h \to 0} \lambda_{j,h} = \lambda \quad \text{for } j = 1, ..., m.$$

Consider the adjoint operator $T^*$ on the dual space $X^*$. If $\lambda$ is an eigenvalue with algebraic multiplicity $m$, then $\lambda$ is an eigenvalue of $T^*$ with the same algebraic multiplicity $m$. The ascent of $\lambda - T^*$ is also $r$. Let $E^*$ be the projection operator associated with $T^*$ and $\lambda$ and $E_h^*$ be the discrete projection operator associated with $T_h$ and $\lambda_{1,h}, ..., \lambda_{m,h}$. Note that when $X$ is a Hilbert space, it is natural to work with the Hilbert adjoint $T^*$.

The following results we present are due to Babuška and Osborn [3]. Let $\lambda$ be a nonzero eigenvalue of $T$ with algebraic multiplicity $m$ and ascent $r$. Let $\lambda_{1,h}, ..., \lambda_{m,h}$ be the eigenvalues of $T_h$ that converge to $\lambda$. Let $\phi_1, ..., \phi_m$ be a basis for $R(E)$ and $\phi'_1, ..., \phi'_m$ be the dual basis to $\phi_1, ..., \phi_m$. The following theorem shows how $R(E)$ can be approximated by $R(E_h)$.

**Theorem 2.9.** There is a constant $C$ independent of $h$ such that, for $h$ small enough,

$$\delta(R(E), R(E_h)) \leq C\| (T - T_h) |_{R(E)} \|.$$

The average of the discrete eigenvalues

$$\hat{\lambda}_h = \frac{1}{m} \sum_{j=1}^{m} \lambda_{j,h}.$$

tends to $\lambda$ as $h \to 0$.

**Theorem 2.10.** Let $\phi_1, ..., \phi_m$ be a basis for $R(E)$ and $\phi'_1, ..., \phi'_m$ be the dual basis. Then there exists a constant $C$, independent of $h$ such that

$$|\lambda - \hat{\lambda}_h| \leq \frac{1}{m} \sum_{j=1}^{m} |\langle (T - T_h) \phi_j, \phi'_j \rangle| + C\| (T - T_h) |_{R(E)} \| \| (T^* - T_h^*) |_{R(E)} \|.$$
Proof. The operator $E_{h|R(E)} : R(E) \to R(E_h)$ is injective since
\[
\|E - E_h\| \to 0.
\]
In addition, $E_{h|R(E)} : R(E) \to R(E_h)$ is surjective since
\[
\dim R(E) = \dim R(E_h) = m.
\]
Hence $E_{h|R(E)}$ is well defined. For $h$ sufficiently small and $f \in R(E)$ with $\|f\| = 1$, we have that
\[
1 - \|E_h f\| = \|Ef\| - \|E_h f\| \leq \|Ef - E_h f\| \leq \|E - E_h\|\|f\| \leq \frac{1}{2},
\]
which implies $\|E_h f\| \geq \|f\|/2$. Hence $(E_{h|R(E)})^{-1}$ is bounded in $h$. Define
\[
\hat{T}_h = (E_{h|R(E)})^{-1} T_h E_{h|R(E)} : R(E) \to R(E)
\]
and $\hat{T} = T_{|R(E)}$.

$
\lambda_{j,h}, j = 1, ..., m, \text{ are eigenvalues of } \hat{T}_h. \text{ We have that}
\]
\[
\text{tr} \hat{T} = m \lambda, \quad \text{tr} \hat{T}_h = m \hat{\lambda}_h,
\]
and
\[
\lambda - \hat{\lambda}_h = \frac{1}{m} \text{tr}(\hat{T} - \hat{T}_h) = \frac{1}{m} \sum_{j=1}^{m} \langle (\hat{T} - \hat{T}_h) \phi_j, \phi_j' \rangle
\]
where $\{\phi_j\}_{j=1}^{m}$ is a basis for $R(E)$ and $\{\phi'_j\}_{j=1}^{m} \subset R(E)'$ is the corresponding dual basis.
Note that $\phi'_j$ can be extended to $X$ as follows. Since $X = R(E) \oplus N(E)$, for $f \in X$, we write $f = g + h$ with $g \in R(E)$ and $h \in N(E)$. Defines
\[
\langle f, \phi'_j \rangle = \langle g, \phi'_j \rangle.
\]
$\phi'_j$ is bounded on $X$ and thus $\phi'_j \in X'$. Since
\[
T_h E_h = E_h T_h \quad \text{and} \quad (E_{h|R(E)})^{-1} E_h = I_{|R(E)},
\]
on one has that
\[
\langle (\hat{T} - \hat{T}_h) \phi_j, \phi_j' \rangle = \langle T \phi_j - (E_{h|R(E)})^{-1} T_h E_h \phi_j, \phi_j' \rangle = \langle (E_{h|R(E)})^{-1} E_h (T - T_h) \phi_j, \phi_j' \rangle = \langle (T - T_h) \phi_j, \phi_j' \rangle + \langle ((E_{h|R(E)})^{-1} E_h - 1)(T - T_h) \phi_j, \phi_j' \rangle.
\]
Let $L_h = (E_{h|R(E)})^{-1} E_h$. $L_h$ is the projection on $R(E)$ along $N(E_h)$. Then $L'_h$ is the projection on $N(E_h) = R(E'_h)$ along $R(E)$. Consequently,
\[
\langle ((E_{h|R(E)})^{-1} E_h - 1)(T - T_h) \phi_j, \phi_j' \rangle = \langle (L_h - 1)(T - T_h) \phi_j, (E' - E'_h) \phi_j' \rangle.
\]
Thus the following holds
\[
\langle (L_h - 1)(T - T_h) \phi_j, (E' - E'_h) \phi_j' \rangle \leq \left( \sup_{h > 0} \|L_h - 1\| \right) \|T - T_h\|_{|R(E)} \|E' - E'_h\|_{|R(E)} \|\phi_j\| \|\phi_j'\| \leq C \cdot \left( \sup_{h > 0} \|L_h - 1\| \right) \|T - T_h\|_{|R(E)} \|E' - E'_h\|_{|R(E)}.
\]
\[\square\]
2.4 Variational Formulation for the Dirichlet Problem

Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open, connected set and \( f \in C(\Omega) \). The Dirichlet problem in \( \Omega \) for the Laplace equation is to find a function \( u \in C^2(\Omega) \) such that

\[
\begin{cases}
-\Delta u = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  

Multiplying the partial differential equation of (4) by a function \( v \in C^\infty_c(\Omega) \) and integrating we have

\[
-\int_{\Omega} v \Delta u \, dx = \int_{\Omega} v(x)f(x) \, dx
\]

using integration by parts formula (or the divergence theorem)

\[
\int_{\Omega} v \Delta u \, dx = -\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} v \, dy
\]

(where \( dx \) denotes the Lebesgue measure on \( \mathbb{R}^n \) and \( dy \) denotes the measure on \( \partial \Omega \)) we obtain

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} v(x)f(x) \, dx.
\]

Since the \( L^2 \) norm of the gradient of a function vanishing at the boundary \( \partial \Omega \) is also a norm and \( (C^\infty_c(\Omega), \| \cdot \|_{L^2(\Omega)}) = H^1_0(\Omega) \) it is natural to define a ”weak” notion of a solution.

**Definition 2.5.** A weak solution for the Dirichlet Problem (4) is a function \( u \in H^1_0(\Omega) \) such that for any \( v \in H^1_0(\Omega) \) and \( f \in L^2(\Omega) \)

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} v(x)f(x) \, dx.
\]

or equivalently

\[
(u, v)_{H^1_0(\Omega)} = (f, v)_{L^2(\Omega)}.
\]

**Theorem 2.11. Riesz-Fréchet**

Given \( (H, (\cdot, \cdot)) \) a Hilbert space. Each linear functional can be represented by an element of \( H \), i.e.,

\[
\forall \varphi \in H' \exists! u \in H : \varphi = (u, \cdot) = : \varphi_u
\]

or equivalently \( H \simeq H' \).

Given \( \varphi \in H' \), \( u \in H \) such that \( E(u) = \min_{v \in H} E(v) \), where

\[
E(v) := \frac{1}{2} \|v\|^2 - \varphi(v)
\]

is called the Lagrangian.

Then for \( H = H^1_0(\Omega) \) we have that the Riesz-Fréchet theorem implies the existence and uniqueness of a weak solution for the Dirichlet problem (4).

The second-order differential operator

\[
L = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + c(x), \quad a_{ij} = a_{ji}, \; b_i, \; c \in L^\infty(\Omega) \forall i, j = 1, \ldots, n
\]
2 THE DIFFERENTIAL EIGENVALUE PROBLEM

is called elliptic if there exists a constant $\lambda > 0$ such that

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \ x \in \Omega.$$ 

For elliptic partial differential operators the weak formulation works similarly as for the Laplace equation (replacing $\Delta$ by $L$ in the definition of the Dirichlet problem). The scalar product is then replaced by a bilinear form which in general is not symmetric. To obtain an equivalent result to Riesz-Fréchet called The Lax-Milgram Lemma (see [7]) the bilinear form must satisfy the following (for $Y = X$).

**Definition 2.6.**

1. Let $X$ and $Y$ be Hilbert spaces. A mapping $a : X \times Y \to \mathbb{C}$ is called a sesquilinear form if it is complex-bilinear, i.e.

$$a(\alpha_1 u + \alpha_2 v, \phi) = \alpha_1 a(u, \phi) + \alpha_2 a(v, \phi),$$

$$a(u, \alpha_1 \phi + \alpha_2 \psi) = \bar{\alpha}_1 a(u, \phi) + \bar{\alpha}_2 a(u, \psi),$$

for all $u, v \in X$, $\phi, \psi \in Y$, $\alpha_1, \alpha_2 \in \mathbb{C}$.

2. The sesquilinear form is bounded if there exists a constant $C > 0$ such that

$$|a(u, v)| \leq \|u\|_X\|v\|_Y$$

for all $u \in X$, $v \in Y$.

3. If $Y = X$ the sesquilinear form is called coercive if there exists a constant $\alpha > 0$ such that

$$a(u, u) \geq \alpha\|u\|_X^2$$

for all $u \in X$.

**Theorem 2.12. Lax-Milgram Lemma**

Let $a : X \times X \to \mathbb{C}$ be a bounded coercive sesquilinear form. There exists a unique solution $u \in X$ to

$$a(u, v) = f(v) \quad \forall v \in X$$

for $f \in X'$ satisfying

$$\|u\|_X \leq \frac{C}{\alpha}\|f\|_{X'},$$

where $C$ and $\alpha$ are the constants of boundedness and coercivity given in the above definition.

In our case we have that the bilinear form $a(\cdot, \cdot) = (\cdot, \cdot)_{H^1_0(\Omega)}$ is real and symmetric, defined on the Hilbert space $X = H^1_0(\Omega)$, Cauchy-Schwarz inequality gives boundedness of the $a$ and the coercivity is given by Poincaré inequality or equivalence of norms $(\cdot, \cdot)_{H^1_0(\Omega)} \sim \|\cdot\|_{H^1_0(\Omega)}^2$. One can obtain similar results using other boundary conditions, Neumann boundary conditions or Mixed boundary conditions, for example. The Neumann Problem for the Laplace equation is to solve

$$\begin{cases}
-\Delta u = f(x) & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega
\end{cases} \quad (5)$$

A similar procedure as in the Dirichlet Problem gives us a notion of what should satisfy a weak solution. In order to define the trace of $\frac{\partial u}{\partial n}$ we need to ask, to the solution $u$, to be in $H^2(\Omega)$ to have $\frac{\partial u}{\partial n} \in L^2(\Omega)$. 

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Definition 2.7. A weak solution for the Neumann Problem (5) is a function \( u \in H^2(\Omega) \) such that for any \( v \in H^1(\Omega) \) and \( f \in L^2(\Omega) \)
\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} v(x)f(x) \, dx.
\]

Note that to have uniqueness of the weak solution we need to impose some restrictions to \( f \) (see [7]).

2.5 Variational Formulation for the Dirichlet Eigenvalue Problem

Let \( \Omega \subset \mathbb{R}^n \) be a bounded, open, connected set. The Dirichlet Eigenvalue Problem in \( \Omega \) for the Laplace equation is to find a function \( u \in C^2(\Omega) \) and a number \( \lambda \in \mathbb{R} \) such that
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\] (6)

By the Riesz-Fréchet theorem we can define a monomorphism \( T : L^2(\Omega) \to L^2(\Omega) \) which maps \( f \) to the weak solution \( u \) of the Dirichlet problem (4), i.e., \( Tf = u \) and consequently
\[
(Tf,v)_{H^1_0(\Omega)} = (f,v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega).
\]

The compact embedding \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) implies that \( T \) is a compact operator, note that it is also self-adjoint.

Definition 2.8. A weak solution for the Dirichlet Eigenvalue Problem (4) is a function \( u \in H^1_0(\Omega) \) and a number \( \lambda \in \mathbb{R} \) such that for any \( v \in H^1_0(\Omega) \)
\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} vu \, dx
\]
or equivalently
\[
(u,v)_{H^1_0(\Omega)} = \lambda (u,v)_{L^2(\Omega)}.
\]

Using the operator \( T \), the problem is equivalent to the operator eigenvalue problem
\[
\lambda Tu = u.
\]

Thus, \( \lambda \) is a Dirichlet eigenvalue if and only if \( \mu := 1/\lambda \) is an eigenvalue of the compact self-adjoint operator \( T \).

As in the source problem one can obtain similar results for the Neumann Problem or the Mixed boundary Problem.
3 The Finite Element Method

Finite Element methods deal with variational formulation of (1) given in (4). To discretize the variational problem a mesh for the set Ω is usually needed. In our case (see Section 4) the mesh is a triangulation satisfying some restrictions and the analysis of discretization of the variational problem is reduced to an analysis on each triangle (in this case the triangle is called an element). In this section we follow [36].

3.1 Finite Elements

In what follows we assume that Ω is a Lipschitz bounded domain on $\mathbb{R}^2$.

Definition 3.1. A finite element is a triple $(K, P, N)$ such that

1. $K \subset \mathbb{R}^2$ is a Lipschitz polygon (e.g., triangle, square)
2. $P$ is a space of functions (e.g. polynomials) on $K$,
3. $N = \{N_1, ..., N_s\}$ is a set of linear functionals on $P$, called degrees of freedom.

Definition 3.2. Let $(K, P, N)$ be a finite element. The basis $\{\phi_1, ..., \phi_s\}$ of $P$ dual to $N$ (i.e., $N_i(\phi_j) = \delta_{ij}$) is called the nodal basis of $P$.

Definition 3.3. 1. A partition $T = \{K_1, ..., K_M\}$ of $\Omega$ into triangle elements is called admissible provided the following properties hold
   (a) $\Omega = \bigcup_{i=1}^M K_i$,
   (b) if $K_i \cap K_j$ consists of exactly one point, then it is a common vertex of $K_i$ and $K_j$,
   (c) If for $i \neq j$, $K_i \cap K_j$ consists of a line segment, then $K_i \cap K_j$ is a common edge of $K_i$ and $K_j$.

2. We write $T_h$, $h > 0$, implying every element has diameter at most $2h$.

3. A family of partitions $\{T_h\}_{h}$ is called shape regular provided there exists a constant $\kappa > 0$ independent of $h$ such that every $K \in T_h$ contains an circle of radius $\rho_K \geq h_K/\kappa$ where $h_K$ is the half diameter of $K$.

The third condition of the above definition it is equivalent (in two dimensions only) to impose that the minimal angle of each triangulation is bounded below uniformly in the shape regular class. A similar restriction to impose to the triangles $K$ of a given triangulation should involve the first eigenvalue (of the Laplacian) of the given triangle, requiring a minimization of the quantity $\frac{\lambda_1(K)}{|K|^2}$ it may be useful (at least intuitively) to get the desired regularity for the triangles.

The reference element in this case is defined to be the triangle $\hat{K}$ whose vertices are $(0,0)$, $(1,0)$, and $(0,1)$. For any $K \in T$, there is an affine mapping $F_K : \hat{K} \rightarrow K$ such that $F(\hat{K}) = K$ and such that it is given by

$$F_K \hat{x} = B_K \hat{x} + \hat{b}.$$

The reference element $(\hat{K}, \hat{P}, \hat{N})$ is affine equivalent to he finite element $(K, P, N)$ if the following hold
1. $F_K((\hat{K})) = K$
2. $F_K \circ \hat{P} = P$
3. $N \circ F_K = \hat{N}$

Let $K$ be a triangle in $T$ and $\mathcal{P}_k(K)$ denote the set of all polynomials of degree at most $k$. As a $\mathbb{R}$-vector space it has dimension $s := \dim_{\mathbb{R}} \mathcal{P}_k = \sum_{r=0}^{k} (r + 1) = \frac{(k+1)(k+2)}{2}$.

Take $z_1, ..., z_s \in K$ $s$ different points which lie on $k+1$ lines. The values $p(z_1), ..., p(z_s)$ uniquely determine $p \in \mathcal{P}_k$. The nodal basis of $\mathcal{P}_k$ is a subset of functions such that takes a nonzero value at exactly one point and it forms a basis of $\mathcal{P}_k$.

In the linear case we have $k = 1$ and $s = 3$. Let $\{z_1, z_2, z_3\}$ be the vertices of $K$ and $\mathcal{N}_1 = \{N_1, N_2, N_3\}$ such that $N_i(v) = v(z_i)$. For the reference element $\hat{K}$ we have that $z_1 = (0,0)$, $z_2 = (1,0)$, and $z_3 = (0,1)$ and then the linear basis functions for $\mathcal{P}_1$

$$L_1 = 1 - x - y, \quad L_2 = x, \quad L_3 = y,$$

satisfy $N_i(L_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

**Definition 3.4.** Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i\}$ be the nodal basis for $\mathcal{P}$ dual to $\mathcal{N}$. If $v$ is a function for which all $N_i \in \mathcal{N}$ are defined, the local interpolant on $K$ is given by

$$I_K v := \sum_{i=1}^{\dim_{\mathbb{R}} \mathcal{P}} N_i(v) \phi_i.$$

The global interpolant on $\Omega$ is given by

$$I_{\Omega} = I_K$$

It is proved that for every continuous function on a triangle Lagrange element $K$ there is a unique interpolation polynomial. We also have that any piecewise infinitely differentiable function $v : \Omega \to \mathbb{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\overline{\Omega})$.

**Definition 3.5.** The finite element space is

$$V_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_k \ \forall \ K \in T\} \subset H^1(\Omega),$$

which is a subset of the continuous functions on $\overline{\Omega}$ for $k \geq 1$.

### 3.2 Abstract Convergence Theory

**Lemma 3.1. Céa’s Lemma**

Let $\{X_h\}_{h>0}$ be a family of finite dimensional subspaces of a Hilbert space $X$. Let $\varphi \in X'$. Then the problem of finding $u_h \in X_h$ such that

$$(u_h, v_h) = \varphi(v_h) \ \forall \ v_h \in X_h$$

has a unique solution. In addition, if $u$ is the exact solution of finding $u \in X$ such that

$$(u, v) = \varphi(v) \ \forall \ v \in X,$$
then there is a constant $C$ independent of $u$ and $u_h$ such that
\[
\| u - u_h \| \leq C \inf_{v_h \in X_h} \| u - v_h \|.
\]

**Proof.** Since $X_h$ is a closed subspace of $X$, it is a Hilbert space with the same scalar product. We also have that $\varphi \in X'_h$. Then by the Riesz-Frèchet theorem there exists a unique solution $u_h \in X_h$.

We have Galerkin orthogonality
\[
(u - u_h, v_h) = (u, v_h) - (u_h, v_h) = \varphi(v_h) - \varphi(v_h) = 0 \quad \forall v_h \in X_h
\]
which implies
\[
(u - u_h, u_h - v_h) = 0 \quad \forall v_h \in X_h.
\]

Then by Cauchy-Schwarz inequality
\[
\| u - u_h \|^2 = (u - u_h, u - u_h) = (u - u_h, u - v_h) \leq \| u - u_h \| \| u - v_h \|.
\]

**Proposition 3.1.** Let $\rho h \leq \text{diam} K \leq h$, where $0 < h \leq 1$, and $\mathcal{P}$ be a finite dimensional subspace of $W^{t,p}(K) \cap W^{m,q}(K)$, where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $0 \leq m \leq l$. Then there exists $C = C(\hat{P}, \hat{K}, l, p, q, \rho)$ such that for all $v \in \mathcal{P}$, we have that
\[
\| v \|_{W^{t,p}(K)} \leq C h^{m-l+n/p-n/q} \| v \|_{W^{m,q}(K)}.
\]

**Definition 3.6.** Given a triangular mesh $\mathcal{T}_h = \{K_1, K_2, ..., K_M\}$ of $\Omega$, the mesh dependent norm is defined as
\[
\| v \|_{m,h} := \left( \sum_{K \in \mathcal{T}} \| v \|_{H^m(K)}^2 \right)^{1/2}, \quad m > 1.
\]

**Definition 3.7.** A Lipschitz domain is said to satisfy the cone condition if the interior angles are positive (the angles $\alpha_i$ satisfy $0 < \alpha_i < \pi$ for each $i$), so that a nontrivial cone can be positioned in $\Omega$ with its tip at the vertex.

**Proposition 3.2.** Let $\Omega \subset \mathbb{R}^2$ be Lipschitz domain satisfying the cone condition. Let $t \geq 2$ and suppose $z_1, ..., z_s$ are $s = t(t + 1)/2$ prescribed points in $\overline{\Omega}$ such that the interpolant operator $I : H^t \to \mathcal{P}_{t-1}$ is well defined for polynomials of degree $\leq t - 1$. Then there exists a constant $C$ depending on $\Omega$ and $z_i, i = 1, ..., s$, such that
\[
\| u - I u \|_{H^t(\Omega)} \leq C \| u \|_{H^t(\Omega)} \quad \forall u \in H^t(\Omega).
\]

**Proof.** Let’s define the norm
\[
\| v \| := |v|_{H^t(\Omega)} + \sum_{i=1}^s |v(z_i)|
\]
on $H^t(\Omega)$ to show that it is an equivalent norm to $\| \cdot \|_{H^t(\Omega)}$.

Since $H^t(\Omega) \hookrightarrow C(\Omega)$ we have
\[
|v(z_i)| \leq C \| v \|_{H^t(\Omega)}, \quad i = 1, 2, ..., s
\]
which implies
\[ \|v\| \leq (1 + Cs)\|v\|_{H^t(\Omega)}. \]

On the other hand, suppose that there is no constant \( C \) such that
\[ \|v\|_{H^t(\Omega)} \leq \|v\| \quad \forall v \in H^t(\Omega). \]

Then there exists a sequence \( \{v_n\}_{n \geq 1} \subset H^t(\Omega) \) such that
\[ \|v_n\| = 1, \quad \|v_n\| \leq 1/n, \quad n = 1, 2, \ldots. \]

Due to the compact embedding \( H^t(\Omega) \hookrightarrow H^{t-1}(\Omega) \) (note that \( t \geq 2 \)), there exists a subsequence of \( \{v_n\}_{n \geq 1} \), still denoted by \( \{v_n\}_{n \geq 1} \), that converges in \( H^{t-1}(\Omega) \). Since \( |v_n|_{H^t(\Omega)} \to 0 \) and
\[ \|v_m - v_n\|^2_{H^t(\Omega)} \leq \|v_m - v_n\|^2_{H^{t-1}(\Omega)} + \left( |v_m|_{H^t(\Omega)} + |v_n|_{H^t(\Omega)} \right)^2, \]
the sequence \( \{v_n\}_{n \geq 1} \) is Cauchy in \( H^t(\Omega) \). There exists a \( v^* \in H^t(\Omega) \) such that
\[ \|v^*\|_{H^t(\Omega)} = 1 \quad \text{and} \quad \|v^*\| = 0. \]

This leads to a contradiction. Hence \( \|\cdot\| \) is equivalent to \( \|\cdot\|_{H^t(\Omega)} \).

Since \( Iu(z_i) = u(z_i) \) and \( t \geq 2 \)
\[ \|u - Iu\|_{H^t(\Omega)} \leq C\|u - Iu\| = C|u - Iu|_{H^t(\Omega)} = C|u|_{H^t(\Omega)}. \]

Let \( T \) be a triangulation for \( \Omega \). Define
\[ V^{t-1} := V^{t-1}(T) = \{v \in L^2(\Omega) : v|_K \in P_{t-1} \forall K \in T\}. \]

Then there exists a unique interpolation operator
\[ I_h : H^t(\Omega) \to V^{t-1}, \quad t \geq 2. \]

Let \( K \) and \( \tilde{K} \) be affine equivalent, i.e., there exists a bijective affine mapping \( F : \tilde{K} \to K \) such that
\[ F(\tilde{x}) = B\tilde{x} + b \]
where \( B \) is a nonsingular matrix and \( \tilde{x} \in \tilde{K}, b \in K \). If \( v \in H^m(\Omega) \), then \( \tilde{v} := v \circ F \in H^m(\tilde{K}) \), and there exists a constant \( C \) depending only on the domain \( \tilde{K} \) and \( m \) such that
\[ |\tilde{v}|_{H^m(\tilde{K})} \leq C \frac{\|B\|^m}{\sqrt{|\det B|}} |v|_{H^m(K)}, \]
\[ |v|_{H^m(K)} \leq C\|B^{-1}\|^m \sqrt{|\det B|} |\tilde{v}|_{H^m(\tilde{K})} \]
where the norm \( \|B\| \) is taken in \( \mathbb{R}^{2 \times 2} \) and since all norms in a finite-dimensional vector space are equivalent the given inequalities does not depend on the chosen norm. Note that since the matrix norm is submultiplicative we have \( \|B^{-1}\| \leq C\|B\|^{-1}. \)
**Proposition 3.3.** Let $t \geq 2$, and suppose $\mathcal{T}_h$ is a shape regular triangulation of $\Omega$. Then there exists a constant $C = C(m, \hat{K}, \kappa)$ depending only on $m$, $\hat{K}$ and the constant $\kappa$ given by $\mathcal{T}_h$ (see Definition 3.3) such that
\[
\|u - I_h u\|_{m,h} \leq C(m, \hat{K}, \kappa) h^{t-m} \|u\|_{H^t(\Omega)}
\]
for $u \in H^t(\Omega), \ 0 \leq m \leq t$.

**Proof.** Let $\mathcal{T}_h$ be a shape-regular triangulation for $\Omega$. For $K \in \mathcal{T}_h$, let $R(K)$ be the radius of the largest circle inscribed in $K$ (the incircle) and $r(K)$ be the radius of the smallest circle containing $K$ (the circumcircle). Since $K \in \mathcal{T}_h$ we have $R(K) \leq h$. Let $F : \hat{K} \to K$ for $K \in \mathcal{T}_h$ be the affine mapping. On the reference triangle $\hat{K}$, by Proposition 3.2, we have
\[
|u - I_h u|_{H^m(K)} \leq C \sqrt{\det B} \|B^{-1}\|^m |\hat{u} - I_h \hat{u}|_{H^m(\hat{K})} \leq C \sqrt{\det B} \|B^{-1}\|^m \cdot C |\hat{u}|_{H^m(\hat{K})}
\]
\[
\leq C \sqrt{\det B} \|B^{-1}\|^m \cdot C \frac{|\hat{B}|^t}{\sqrt{\det B}} |\hat{u}|_{H^t(\hat{K})} \leq C(\|B\| \cdot \|B^{-1}\|)^m \|B\|^t \|u\|_{H^t(\hat{K})}.
\]
Since $\mathcal{T}_h$ is shape-regular, there exists some $\kappa > 0$ such that $r(K) \geq h_K / \kappa = \frac{R(K)}{\kappa}$ which implies that $\frac{R(K)}{r(K)} \leq \kappa$. In addition, if we choose the spectral norm $\|B\| := \sqrt{\lambda_{\max}(B^TB)}$ then
\[
\|B\| = R(K)/R(\hat{K}) \leq h/R(\hat{K})
\]
and
\[
\|B^{-1}\| = \frac{1}{\sqrt{\lambda_{\min}(B^TB)}} = \frac{1}{\sqrt{\lambda_{\min}(B^TB)}} = \frac{r(\hat{K})}{r(\hat{K})} = r(\hat{K}).
\]
This implies
\[
\|B\| \cdot \|B^{-1}\| = \frac{r(\hat{K})}{r(\hat{K})} \cdot \frac{R(K)}{r(K)} = c(\hat{K} \kappa).
\]
Thus the following holds
\[
|u - I_h u|_{H^t(K)} \leq Ch^{t-l} |u|_{H^t(K)}.
\]
Summing over $l$ from 0 to $m$, we obtain
\[
\|u - I_h u\|_{H^m(K)} \leq Ch^{t-m} |u|_{H^t(K)} \ \forall u \in H^t(\hat{K}), \ K \in \mathcal{T}_h.
\]
Finally summing over each element $K \in \mathcal{T}_h$ one obtains the result.  

\section{3.3 Lagrange Elements for the Source Problem}

By the Riesz-Frèchet theorem we obtain the following

**Proposition 3.4.** There exists a unique solution $u \in H^1_0(\Omega)$ solving the Dirichlet problem (4) such that
\[
\|u\|_{H^1(\Omega)} \leq C \|f\|_{-1}.
\]

The relevance of the following result is shown in the relative errors of examples of Section 4. For the details see [16, 26].
**Theorem 3.2.** Let $\Omega$ be a bounded Lipschitz polygon. There exists an $\alpha_0 > 1/2$ depending on the interior angles of $\Omega$ such that for all $1/2 \leq \alpha \leq \alpha_0$, the solution of (4) satisfies
\[
\|u\|_{H^{1+\alpha}} \leq C\|f\|_{H^{-1+\alpha}}.
\]
In particular, $\alpha_0 = 1$ when $\Omega$ is convex.

By Poincaré-Friedrichs inequality we have that
\[
\inf_{v_h \in V^h} \|u - v_h\|_{L^2(\Omega)} \leq Ch \min\{k+1,r\} \|u\|_{H^{r}(\Omega)},
\]
(7)
\[
\inf_{v_h \in V^h} \|u - v_h\|_{H^1(\Omega)} \leq Ch \min\{k,r-1\} \|u\|_{H^{r}(\Omega)},
\]
(8)

Let’s assume that $\Omega$ is covered by a regular triangular mesh $\mathcal{T}$. Let $V^h \subset H^1(\Omega)$ be the finite element space of the Lagrange elements of order $k$ with zero values at the nodes on $\partial \Omega$. For a Lipschitz polygon $\Omega \subset \mathbb{R}^2$ the discrete problem for the Dirichlet Problem (4) is to find $u_h \in V^h$ such that
\[
(u_h, v_h)_{H^1_0(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V^h.
\]
Since there exists a unique solution for the discrete problem we can define a discrete solution operator
\[
T_h : L^2(\Omega) \rightarrow V^h \subset L^2(\Omega)
\]
such that
\[
(T_h f, v_h)_{H^1_0(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \forall v_h \in V^h.
\]
It is clear that $T$ is self-adjoint and compact.

**Proposition 3.5.** Suppose $\{\mathcal{T}_h\}_{h>0}$ is a family of shape regular triangulations of $\Omega$. Let $u$ be the solution of the Dirichlet problem (4) such that $u \in H^s_0(\Omega)$, $s > 1$. Let $\tau := \min\{k, s-1\}$ where $k$ is the order of the Lagrange elements. Then the finite element approximation $u_h$ of $u$ satisfies
\[
\|u - u_h\|_{H^{1}(\Omega)} \leq C h^\tau \|f\|_{H^{-1}(\Omega)}.
\]
(9)
Equivalently, we have
\[
\|u - u_h\|_{H^s_0(\Omega)} \leq C h^\tau \|f\|_{H^{-1}(\Omega)}
\]

**Proof.** From Céa’s Lemma (Lemma 3.1),
\[
\|u - u_h\|_{H^{1}(\Omega)} \leq C \inf_{v_h \in V^h} \|u - v_h\|_{H^{1}(\Omega)}.
\]
Then (8) implies that
\[
\|u - u_h\|_{H^{1}(\Omega)} \leq C h^\tau \|u\|_{H^{r+1}(\Omega)} \leq C h^\tau \|f\|_{H^{r-1}(\Omega)},
\]
where we have used Theorem 3.2. By the result on negative norm (Theorem 2.1), we have that
\[
\|f\|_{H^{r-1}(\Omega)} \leq \|f\|_{H^{-1}(\Omega)},
\]
and thus
\[
\|u - u_h\|_{H^{1}(\Omega)} \leq C h^\tau \|f\|_{H^{-1}(\Omega)}
\]
\[\square\]
Corollary 3.2.1. Let $f \in H^1(\Omega)$ in (4). Then we have that
\[
\|Tf - T_h f\|_{H^1(\Omega)} \leq Ch^\tau \|f\|_{H^1(\Omega)}
\]

Lemma 3.3. Aubin-Nitsche Lemma
Let $H$ be a Hilbert space with the norm $\| \cdot \|_H$ and the scalar product $(\cdot, \cdot)$. Let $V$ be a subspace which is also a Hilbert space with norm $\| \cdot \|_V$. In addition, the embedding of $V$ to $H$ is continuous. Let $a$ be a bounded coercive sesquilinear form on $V \times V$. Given $f \in V^*$, let $u$ and $u_h$ be the solutions of
\[
a(u, v) = f(v) \quad \forall v \in V
\]
and
\[
a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h,
\]
respectively. Then the finite element solution $u_h \in V_h \subset V$ satisfies
\[
\|u - u_h\|_H \leq C \|u - u_h\|_V \sup_{g \in H, g \neq 0} \left\{ \frac{1}{\|g\|_H} \inf_{v \in V_h} \|\phi_g - v\|_V \right\},
\]
where, for every $g \in H$, $\phi_g \in V$ denotes the corresponding unique solution of this equation
\[
a(w, \phi_g) = (g, w) \quad \forall w \in V.
\]

Proof. By the Riesz Representation Theorem, the norm of an element in a Hilbert space can be defined as
\[
\|w\|_H = \sup_{g \in H, g \neq 0} \frac{(g, w)}{\|g\|_H},
\]
Letting $w = u - u_h$ in (10), we obtain by Galerkin orthogonality
\[
(g, u - u_h) = a(u - u_h, \phi_g) = a(u - u_h, \phi_g - v_h) \leq C \|u - u_h\|_V \|\phi_g - v_h\|_V.
\]
It follows that
\[
(g, u - u_h) \leq C \|u - u_h\|_V \inf_{v_h \in V_h} \|\phi_g - v_h\|_V.
\]
The duality argument (11) implies that
\[
\|u - u_h\|_H = \sup_{g \in H, g \neq 0} \frac{(g, u - u_h)}{\|g\|_H} \leq C \|u - u_h\|_V \sup_{g \in H, g \neq 0} \left\{ \inf_{v_h \in V_h} \frac{\|\phi_g - v_h\|_V}{\|g\|_H} \right\}.
\]

Corollary 3.3.1. Let $\mathcal{T}_h$ be a family of shape regular triangulation of $\Omega$ and $V_h$ be the Lagrange finite element space of order $k$ associated with $\mathcal{T}_h$. Let $u$ and $u_h$ be the solutions of (??) and (??), respectively. Assume that $u \in H^s(\Omega), 1 \leq s \leq 2$ and $\tau = \min\{k, s - 1\}$. Then
\[
\|u - u_h\|_{L^2(\Omega)} \leq Ch^\tau \|u - u_h\|_{H^1(\Omega)}.
\]
Furthermore, if $f \in H^{r-1}(\Omega)$ so that $u \in H^{1+r}(\Omega)$,
\[
\|u - u_h\|_{L^2(\Omega)} \leq Ch^{2r} \|f\|_{H^{r-1}(\Omega)} \leq Ch^{2r} \|f\|_{L^2(\Omega)}.
\]
Proof. Let \( H = L^2(\Omega) \) with norm \( \| \cdot \|_{L^2(\Omega)} \) and \( V = H^1_0(\Omega) \) with \( H^1(\Omega) \) norm. We have \( V \subset H \) and the embedding is continuous. Since \( \phi_g \) solves (10) the estimates in (9) implies

\[
\sup_{g \in H, \; g \neq 0} \left\{ \frac{\| \phi_g - v_h \|_{H^1(\Omega)}}{\| g \|_{L^2(\Omega)}} \right\} \leq C h^r.
\]

Applying the Aubin-Nitsche Lemma and (7), we obtain that

\[
\| u - u_h \|_{L^2(\Omega)} \leq C h^r \| u - u_h \|_{H^1(\Omega)}.
\]

\[\Box\]

### 3.4 Convergence Analysis

The discrete eigenvalue problem is to find \( u_h \in V_h \) and \( \lambda_h \in \mathbb{R} \) such that

\[
(u_h, v_h)_{H^1_0(\Omega)} = \lambda_h (u_h, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h.
\]

The problem is equivalent to the operator eigenvalue problem:

\[
\lambda_h T_h u_h = u_h.
\]

As in the continuous case, \( \lambda_h \) is an eigenvalue if and only if \( \mu_h := 1/\lambda_h \) is an eigenvalue of \( T \). From the last section we have that (note that \( T : L^2(\Omega) \to V_h \subset L^2(\Omega) \))

\[
\|T f - T_h f\|_{L^2(\Omega)} \leq C h^{2r} \| f \|_{L^2(\Omega)}
\]

which implies

\[
\|T - T_h\|_{L^2(\Omega)} \leq C h^{2r}.
\]

**Corollary 3.3.2.** Let \( u \) be an eigenfunction associated with the eigenvalue \( \lambda \) of multiplicity \( m \). Let \( w_1^h, \ldots, w_m^h \) be the eigenfunctions associated with the \( m \) discrete eigenvalues \( \lambda_1^h, \ldots, \lambda_m^h \) approximating \( \lambda \). Then there exists \( u_h \in \langle w_1^h, \ldots, w_m^h \rangle \) such that

\[
\| u - u_h \|_{L^2(\Omega)} \leq C h^{2r} \| u \|_{L^2(\Omega)}
\]

**Theorem 3.4.** Let \( \hat{\lambda}_h = \frac{1}{m} \sum_{j=1}^m \lambda_j^h \) where \( \lambda_1^h, \ldots, \lambda_m^h \) are the discrete eigenvalues approximating \( \lambda \). Then the following convergence rate holds

\[
|\lambda - \hat{\lambda}_h| \leq C h^{2r}.
\]

**Proof.** Due to the fact that both \( T \) and \( T_h \) are self-adjoint and in view of Theorem 2.10, we only need to approximate

\[
\sum_{j,k=1}^m |((T - T_h) \phi_j, \phi_k)|,
\]

where \( \{ \phi_1, \ldots, \phi_m \} \) is a basis for the generalized eigenspace \( R(E) \) corresponding to \( \lambda \).

Using the definition of \( T \) and \( T_h \), symmetry of \( a(\cdot, \cdot) \), Galerkin orthogonality and the estimate of \( T - T_h \), we have that

\[
|((T - T_h) u, v)| = |(v, (T - T_h) u)| = |a(T v, (T - T_h) u)| = |a((T - T_h) u, T v)|
\]

\[
= |a((T - T_h) u, (T - T_h) v)| \leq \|(T - T_h) u\|_{H^1(\Omega)} \|(T - T_h) v\|_{H^1(\Omega)} \leq C h^{2r},
\]

which holds for any \( u, v \in R(E) \) with \( \| u \|_{L^2} = \| v \|_{L^2} = 1 \). \[\Box\]
Theorem 3.5. The operator $T_{H^1_0(\Omega)}$ from $H^1_0(\Omega)$ to $H^1_0(\Omega)$ is compact.

Proof. Let $\{u_n\}_{n>0}$ be a bounded sequence in $H^1_0(\Omega)$. Due to the compact embedding of $H^1_0(\Omega)$ to $L^2(\Omega)$, there exists a convergent subsequence of $\{u_n\}_{n>0}$, still denoted by $\{u_n\}_{n>0}$, in $L^2(\Omega)$. Let $u := \lim_{n \to \infty} u_n \in L^2(\Omega)$. Then $Tu \in H^1_0(\Omega)$ and

$$(Tu,v)_{H^1_0(\Omega)} = (u,v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega).$$

On the other hand, we have that

$$(T_{H^1_0(\Omega)}u_n,v)_{H^1_0(\Omega)} = (u_n,v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega).$$

Therefore

$$(Tu - T_{H^1_0(\Omega)}u_n,v)_{H^1_0(\Omega)} = (u - u_n,v)_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty \quad \forall v \in H^1_0(\Omega).$$

In particular

$$\|Tu - T_{H^1_0(\Omega)}u_n\|^2_{H^1_0(\Omega)}(Tu - T_{H^1_0(\Omega)}u_n, Tu - T_{H^1_0(\Omega)}u_n)_{H^1_0(\Omega)} \to 0 \quad \text{as } n \to \infty$$

which implies

$$T_{H^1_0(\Omega)}u_n \to Tu \quad \text{as } n \to \infty$$

so $T_{H^1_0(\Omega)}$ is compact. \hfill \Box

3.5 Reduction to an Algebraic Eigenvalue Problem

For a given triangulation $T$, that is $N$ nodes (with coordinates in a $2 \times N$ matrix $p$) and $E$ triangles (with boundary nodes in a $3 \times E$ matrix $t$). Let $\{\phi_1, \phi_2, ..., \phi_N\}$ be the basis of the linear Lagrange element space $V_h \subset H^1(\Omega)$ associated to $T$. Let

$$u_h = \sum_{j=1}^{N} u_j \phi_j.$$

Substituting $u_h$ in (??) and choosing $v_h = \phi_i$ for each $i = 1, ..., N$ we obtain the following generalized algebraic eigenvalue problem, that is to find $u = (u_1, ..., u_N)^T$ such that

$$Au = \lambda_M u$$

where

$$A_{ij} = (\phi_i, \phi_j)_{H^1_0(\Omega)} = (\nabla \phi_i, \nabla \phi_j)_{L^2(\Omega)}, \quad M_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)}.$$

To compute the entries of the matrices we need a numerical quadrature to compute the integrals appearing in the bilinear products due to the variational formulation. Let $a^i$, $i = 1, 2, 3$ be the vertices of the triangle $K$ and $a^{ij}$ be the midpoint of the edge $a^i a^j$ where $1 \leq i < j \leq 3$. Then the quadrature used for the linear Lagrange elements is defined by the rule

$$\int_K f(x) \, dx \simeq \frac{|K|}{3} \sum_{1 \leq i < j \leq 3} f(a^{ij}).$$
This quadrature is exact for polynomials of order up to 2. The interpolation error can be estimated as the following

\[
\left| \int f(x) \, dx - \frac{|K|}{3} \sum_{1 \leq i < j \leq 3} f(a^{ij}) \right| \leq C h^3 \sum_{|\alpha|=3} \int |D^\alpha f| \, dx.
\]

Now, an effective method of computing the eigenvalues of a matrix is needed [40, 36, 15]. In our case, the matrix is sparse and Arnoldi method (which is already implemented in MATLAB) gives a good approximation of the eigenvalues via the Ritz values [36, 15].
4 Examples

In this section we show a few examples of some computed (with MATLAB [30]) eigenpairs with the finite element method with linear Lagrange elements. Our examples consist of two-dimensional simply connected sets with self-similar boundary (each set in our examples is determined by its boundary and they are computing using the code in [28]). The boundary is identified by the sequence of angles \( \{\theta_j\}_{j=1}^m \), \( m \geq 2 \) conforming the generating curve \( \gamma_1 : [0,1] \to \mathbb{C} \) (a polygonal curve made of segments of angle \( \theta_i \) with the abscissa and each segment with the same length \( l \)) and a symmetry number \( \omega \) which is the number of "sides" of the set having at each side the iterated curve, i.e., replacing to a regular polygon of \( \omega \) sides (of length 1) the iterated curve at each side. Then we write the identification procedure as

\[
\partial \Omega \sim (\{\theta_j\}_{j=1}^m, \omega), \quad \gamma \sim [\theta_j]^m_{j=1}.
\]

To obtain \( \partial \Omega \) from \( \gamma \) (and similarly \( \partial \Omega_k \) from \( \gamma_k \)) we only need to rotate \( \gamma \) multiplying it by \( e^{2\pi i s} \), \( s = 0, 1, ..., \omega - 1 \) and adding a constant in order to have a continuous boundary. Using the abuse of notation \( \gamma = \text{Im} \gamma = \gamma([0,1]) \) we have

\[
\partial \Omega = \gamma \cup \left( \gamma(1) + e^{\frac{2\pi i}{\omega}} \gamma \right) \cup \left( \gamma(1) + e^{\frac{2\pi i}{\omega}} \gamma(1) + e^{\frac{4\pi i}{\omega}} \gamma \right) \cup ...
\]

\[
\cup \left( \gamma(1) + e^{\frac{2\pi i}{\omega}} \gamma(1) + ... + e^{\frac{2\pi i(\omega - 2)}{\omega}} \gamma(1) + e^{\frac{2\pi i(\omega - 1)}{\omega}} \gamma \right).
\]

We assume that the given sequences of angles correspond to curves with no self-intersections (so the set \( \Omega \) is simply connected). The iteration of the curve is done replacing each segment of the polygonal curve to the generating curve scaled with a factor \( h^k = 1/r^k \) \( (h := 1/r) \)

where \( r = \sum_{j=0}^m \cos(\theta_j) \) is called the scaling factor and \( k \) is the number of required iterations (in this case the iterated curve is denoted by \( \gamma_k \), for \( k = 0 \) the curve is the segment \( \gamma_0 := [0,1] \) and for \( k = 1 \) the curve is the generating curve \( \gamma_1 \). We require to the sequence of angles (in order to obtain an "admissible" curve) to satisfy \( \theta_1 = 0 \) and \( l = 1/r \).

For example, the Koch snowflake is a set with a boundary generated by a curve of angles \([0,\pi/3, -\pi/3, 0]\) at each of the \( \omega = 3 \) sides of a triangle, we identify the set with the pair \(([0,\pi/3, -\pi/3, 0], 3) \). In this case we have \( r = 3 \). In Figure 4.1 it is shown the generating curve \( \gamma_1 \), in Figure 4.2 \( \partial \Omega_1 \), in Figure 4.3 \( \gamma_6 \) and in Figure 4.4 \( \partial \Omega_6 \).
If we use the same generating curve at 4 sides of a square the pair identifying then the set is 
\((0, \pi/3, -\pi/3, 0, 4)\) (see Figure 4.5). Figure 4.6 corresponds to the pair 
\((0, -\pi/3, \pi/3, 0, 4)\), we identify such pair with the pair 
\((0, \pi/3, -\pi/3, 0, -4)\).
We use polygonal approximations to the self-similar sets, that is, the set $\Omega$ is approximated by the monotone sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ (that is, $\Omega_k \subset \Omega_{k+1} \subset \Omega$) of polygons (which are locally Lipschitz and a finite element method can be applied) and for each $\varepsilon > 0$ there is a $k_0$ and $x_k \in \partial \Omega_k$ such that $d(x, x_k) < \varepsilon$ (where $d$ denotes the Euclidean distance) for each $k > k_0$ and for each $x \in \partial \Omega$. In this case we write

$$\lim_{k \to \infty} \Omega_k = \Omega,$$

using the same definition of convergence for $\{\gamma_k\}_{k \in \mathbb{N}}$ to $\gamma$ we have

$$\lim_{k \to \infty} \gamma_k = \gamma.$$

We define $\Omega_0$ to be a $\omega$-gon. We normalize the sets $\Omega$ to $\hat{\Omega} = c\Omega + b(\Omega) := \{y \in \mathbb{R}^2 : y = cx + b(\Omega), \ x \in \Omega\}$, where $b(\Omega) = b(\Omega_0)$ is the center of the $\omega$-gon, making the measure equal to 1, i.e., $\text{vol}_2(\hat{\Omega}) = |\hat{\Omega}| = 1$, $c = \frac{1}{\sqrt{|\Omega|}}$. Define $A_k = |\Omega_k|$ and $A = |\Omega|$, now the sequence $\{\hat{\Omega}_k\}_{k \in \mathbb{N}}$ (with $\hat{\Omega}_k = c_k \Omega$, $c_k = \frac{1}{\sqrt{|\Omega_k|}}$) is not monotone in general. This normalization is useful to compare the eigenvalues of some different sets to observe the dependence on the
regularity (or irregularity) of the boundary and in the following examples we compute the eigenvalues of $\Omega$ instead of $\hat{\Omega}$. By the Faber-Krahn inequality given in Section 1

$$\lambda_1(\hat{\Omega}) \geq \frac{j^2_0}{\pi}.$$ 

In each example we give the dimension (note that since the curves are self-similar the Hausdorff dimension is equal to the Minkowski dimension) of the boundary $\partial \Omega$ which is given by the following formula

$$\dim \partial \Omega = \log_r m = \frac{\log m}{\log r},$$

the area of $\Omega_k$, $1/c_k^2$ (computed via determinants of a triangulation covering $\Omega_k$, the triangulation does not need to satisfy any regularity condition), if $\gamma(1) = 1$ it can be computed via the formula, which holds for $\omega \in \mathbb{Z}$

$$|\Omega_k| = |\Omega_0| + \omega \sum_{j=1}^{k} m^{j-1} r^{-2j} = |\Omega_0| + \frac{1}{r^2} \sum_{j=1}^{k} m^{j-1} - \frac{m}{r^2} \sum_{j=1}^{k} m^{j-1} = \lim_{k \to \infty} |\Omega_k|$$

where

$$I := \int_0^1 \Im(\gamma(t)) \, dt = \frac{|\Omega_1| - |\Omega_0|}{\omega}$$

is the area under the generating curve.

One can compute also the perimeter (using the one dimensional Lebesgue measure)

$$|\partial \Omega_k| = |\omega| \left( \frac{m}{r} \right)^k, \quad |\partial \Omega| = \infty.$$ 

Not all the curves are Minkowski measurable (see [24]). In Table 1 we show some computations, they suggest that a greater dimension gives a greater first eigenvalue. The isometry group of $\Omega$ denoted as $\text{Isom}(\Omega)$ contains rotations and involutions only (since $\Omega \subset \mathbb{R}^2$) and it plays an important role since more symmetry minimizes the first eigenvalue (see [19]). One can observe in examples 4.5 and 4.6 that obstructions play an important role in the first eigenvalue, a way to formalize this fact is to compare the inradius, the radius of the largest circle inscribed in the polygon $\hat{\Omega}_k$, where even in the normalized examples a small inradius in the obstructed geometries is obtained. In the following subsections some eigenfunctions are shown, the size element $h$ of the triangulations are between 0.01 and 0.07. In some cases we compute also the relative error of the first eigenvalue $\lambda_1(\Omega_k)$ for some $k$. Since all cases are non-convex the error is not quadratic and one can observe how the error depends on the angles. In the following figures some approximations of the eigenfunctions are shown for the respective domains and an approximation of the associated eigenvalues are shown using the abuse of notation $\hat{\Omega}_k$. Only in the first example (known as the Koch snowflake) we give a particular attention due to symmetry, several authors computed approximations of its eigenvalues [25, 4, 31]. In the other cases the eigenfunctions are computed on a triangulation on all of $\Omega_k$, in each case it is specified the polygonal approximation $k$. 

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4.1 \([0, \pi/3, -\pi/3, 0], 3\)

In this case the symmetry group is isomorphic to the dihedral group of order 12 and with the use of symmetry [31] we can reduce the Dirichlet boundary problem (1) in \(\Omega\) to the mixed eigenvalue problem (2) in a portion of \(\Omega, \tilde{\Omega}\) shown in Figure 4.1.1. For the approximation of linear Lagrange elements in the mixed problem see [32]. The second eigenfunction satisfy other symmetries as it is shown in Figure 4.1.3 and Figure 4.1.4. It is important to note that this reduction simplifies the generation of the mesh (since we reduce the number of sides of the polygon) and gives a better approximation (at least in finite elements) to the solution.

In Figure 4.1.2 the approximation of the first eigenfunction \(u_1\) is represented. In this case, since \(u_1\) solves (2) and is normalized with respect to \(\tilde{\Omega}\)

\[
\|u_1\|_{L^2(\tilde{\Omega})}^2 = 1, \quad \|u_1\|_{L^2(\Omega)}^2 = 12\|u_1\|_{L^2(\tilde{\Omega})}^2 = 12.
\]

Then if we want to normalize \(u_1\) with respect to the \(L^2(\Omega)\) norm we need to consider \(u_1/sqrt(12)\). Then, \(\lambda_1(\Omega) = \|\nabla u_1\|_{L^2(\Omega)}^2\). The relative error of the first eigenvalue is shown in Figure 4.1.8. Computing other eigenfunctions of \(\tilde{\Omega}\) gives to other non-normalized symmetric eigenfunctions shown in Figures 4.1.5, 4.1.6, and 4.1.7.

Similarly, for the second eigenfunction \(u_2\) where the approximation is shown in Figure 4.1.4 we have that

\[
\|u_1\|_{L^2(\tilde{\Omega})}^2 = 1, \quad \|u_1\|_{L^2(\Omega)}^2 = 4\|u_1\|_{L^2(\tilde{\Omega})}^2 = 4.
\]
where now $\tilde{\Omega}$ is a quarter part of the polygon shown in Figure 4.1.3. The relative error of the approximation of $u_2$ is given in Figure 4.1.9.
Figure 4.1.7

Figure 4.1.8

Figure 4.1.9
4.2 \([0, \pi/3, 0], 3\)

Figure 4.2.1

Figure 4.2.2

Figure 4.2.3
4.3 \((0, \pi/2, 0], 4\)
4 EXAMPLES

Figure 4.3.3

\[ \lambda_{4}(\Omega_2) = 117.6175 \]

Figure 4.3.4

\[ \lambda_{14}(\Omega_2) = 261.1642 \]

Figure 4.3.5

Relative error \( \lambda_{4}(\Omega_2) \)
4.4 \([\left[0, \frac{\pi}{3}, 0, \frac{\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, 0\right], 3]\)

Figure 4.4.1

Figure 4.4.2

Figure 4.4.3
4.5 \([0, \pi/2 - 0.1, -\pi/2 + 0.1, 0]\)

In the first example the approximate inradius is 0.4571 and in the second is 0.0535.

4.5.1 \(([0, \pi/2 - 0.1, -\pi/2 + 0.1, 0], 4)\)

Figure 4.5.1

4.5.2 \(([0, \pi/2 - 0.1, -\pi/2 + 0.1, 0], -4)\)

Figure 4.5.2
4.6 \quad ([0, \pi/2 - 0.05, -\pi/2 + 0.05, 0], -4)

In this case the approximate inradius is 0.0261. A small element size $h$ is needed, otherwise the method does not see the first eigenvalue which means that the first eigenvalue of the reduced generalized eigenvalue problem (see 3.5) does not correspond with the first eigenvalue of (1).

Figure 4.6.1
5 Miscellaneous examples

In this section we only show two anomalous cases given in [29], both of two with dimension $\dim \Omega = \log_3 8 \simeq 1.8928$. The first one is known as the Sierpinsky carpet and it has zero area. If the eigenvalues are bounded then the normalized are zero since

$$\lambda_k(\hat{\Omega}) = \lambda_k(\Omega)|\Omega|.$$ 

The area of the Menger sponge (its surface analog embedded in $\mathbb{R}^3$) is 0 too. So in these cases it is not reasonable to speak about discrete spectrum of $\Omega$ in the two-dimensional sense but it is for the approximation cases $\Omega_k$ shown in the figures.

5.1 The Sierpinsky carpet

The area of $\Omega_k$ can be easily computed, $|\Omega_k| = 1 - \frac{1}{8} \sum_{r=1}^{k} \left( \frac{8}{3^2} \right)^r = \left( \frac{8}{9} \right)^k$. If we also use the abuse of notation $\Omega_k = \hat{\Omega}_k$ then the eigenvalues are given also in the figures.
5.2 The Menger sponge

In the following figure the eigenvalues are of the original set $\Omega_3$ instead of the normalized set.

\begin{align*}
\lambda_1(\Omega_3) &= 3.8304 \\
\lambda_2(\Omega_3) &= 8.76 \\
\lambda_3(\Omega_3) &= 16.8028 \\
\lambda_4(\Omega_3) &= 25.3101 \\
\lambda_5(\Omega_3) &= 25.312 \\
\lambda_6(\Omega_3) &= 34.1803 \\
\lambda_7(\Omega_3) &= 41.3762 \\
\lambda_8(\Omega_3) &= 46.4717 \\
\lambda_9(\Omega_3) &= 49.6892 \\
\lambda_{10}(\Omega_3) &= 52.7245 \\
\lambda_{11}(\Omega_3) &= 60.0028 \\
\lambda_{12}(\Omega_3) &= 63.6662
\end{align*}

Figure 5.2
References


