Periodic points, Lie symmetries and non-integrability of planar maps

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We present a criterion for the $\mathcal{C}^m$-non-integrability near elliptic fixed points of smooth planar measure preserving maps.

What kind of integrability?

A planar map F is *$\mathcal{C}^m$-locally integrable at an elliptic fixed point p* if there exists a neighborhood $\mathcal{U}$ of p and a *first integral* $V \in \mathcal{C}^m(\mathcal{U})$ with $m \geq 2$,

i.e. $V(F(x)) = V(x)$ such that

all the level curves $\{V = h\} \cap \mathcal{U}$ are *closed curves* surrounding p, which is an isolated *non-degenerate* critical point of V in $\mathcal{U}$.

Remember that a map is a *measure preserving map* if

$$m(F^{-1}(B)) = m(B)$$

for any measurable set B, where $m(B) = \int_B \nu(x,y)\,dxdy$, and $\nu|_{\mathcal{U}} \neq 0$.

Birkhoff normal form at a k-resonant elliptic point

- A fixed point p of a $\mathcal{C}^1$-real planar map F is elliptic when the eigenvalues of $DF(p)$ have modulus one, but excluding ± 1.
- When the eigenvalues are not roots of unity of order ℓ for $0 < \ell \leq k$ we will say that p is not k-resonant.
- A $\mathcal{C}^{k+1}$-map, with a not k-resonant fixed point p, is locally $\mathcal{C}^{k+1}$-conjugate to its *Birkhoff normal form*:

$$F_B(z) = \lambda z \left(1 + \sum_{j=1}^{[(k-1)/2]} B_j (z\bar{z})^j\right) + O(|z|^{k+1}), \qquad (1)$$

where $z = x + iy$, and $[\cdot]$ denotes the integer part.

Lie Symmetries

A *vector field* X is said to be a *Lie symmetry* of F if it satisfies

$$X(F(\mathbf{x})) = (DF(\mathbf{x}))\,X(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{U}.$$

This implies that $\dot{\mathbf{x}} = X(\mathbf{x})$ is invariant under the change of variables given by F. From a dynamic viewpoint *F maps any orbit of $\dot{\mathbf{x}} = X(\mathbf{x})$, into another orbit of this system.* In the integrable case we have

Theorem 1. *[[1], see also [2],[3]] Let F be a $\mathcal{C}^2(\mathcal{U})$ orientation preserving map with an* invariant measure *with density $0 \neq \nu \in \mathcal{C}^1(\mathcal{U})$ and with a first integral $V \in \mathcal{C}^2(\mathcal{U})$. Then*

(a) The vector field $X = \frac{1}{\nu}\left(-V_y\frac{\partial}{\partial x} + V_x\frac{\partial}{\partial y}\right)$ is a Lie Symmetry of F.

(b) If a connected component γ_h of $\{V(x) = h\}$ without fixed points is invariant by F and $\gamma_h \cong \mathbb{S}^1$, then

$$F(x,y)|_{\gamma_h} = \varphi(\tau(h); x, y)$$

is conjugate to a rotation with rotation number $\theta(h) = \dfrac{\tau(h)}{T(h)}$.

(c) Therefore, if $F|_{\gamma_h}$ has rotation number $\theta(h) = q/p \in \mathbb{Q}$, with $\gcd(p,q) = 1$, then $\gamma_h \subset \mathcal{U}$ is a continuum of p-periodic points of F.

Main result

We present the following criterion for non-integrability of planar maps:

Theorem 2. *Let $F \in \mathcal{C}^{2n+2}$ be a measure preserving map with a non-vanishing density $\nu \in \mathcal{C}^{2n+3}$, and an elliptic fixed point p,* not $(2n+1)$-resonant, *with Birkhoff constant is $B_n = i\,b_n \in i\mathbb{R}$. Assume that there is an unbounded sequence $\{N_k\}_k$, s.t. F has* finitely many N_k-periodic points *in $\mathcal{U} \Rightarrow$ it is* ***NOT*** *$\mathcal{C}^{2n+4}$-locally integrable at* **p**.

Proof. Suppose that F has an first integral V, then:

- F possesses a smooth *Lie symmetry* $X = \frac{1}{\nu}\big(-V_y, V_x\big)$ because it preserves an *invariant measure* with a smooth density.
- Since it has an elliptic fixed point, with non-zero purely imaginary *Birkhoff constant*, the rotation number function $\theta(h)$ associated to each level $\{V = h\}$ is continuous and non-constant (see Proposition 3).
- If $\theta(h)$ is non constant, there should exist closed level sets such that on them F has rational rotation numbers with *all* denominators bigger that some N_0.
- Since there exists a Lie symmetry X, these levels *have continua* of real periodic points for all $N \geq N_0$ in a given neighborhood of the elliptic point. Indeed, recall that by Theorem 1:
  If $\{V = h\}$ is invariant by X and F, without singular points of X, and diffeomorphic to $\mathbb{S}^1$ then $F|_{\{V=h\}}$ is conjugated to a rotation. Moreover if its corresponding rotation number is rational, $\rho = p/q \in \mathbb{Q}$, then the map F has a continuum of q-periodic points.
- But we are assuming that for an unbounded sequence of natural numbers $\{N_k\}_k$, F has *finitely many N_k-periodic points* in $\mathcal{U}$, a contradiction.

□

Proposition 3. *If $F \in \mathcal{C}^{2n+2}$ is measure preserving, $\mathcal{C}^{2n+4}$-locally integrable at p, and $B_n = i\,b_n \Rightarrow$ the rotation number $\theta(h)$ associated to each curve $\{V = h\}$ is not constant (hence there exists continua of all periods $N \geq N_0$).*

Proof. Suppose that $\theta(h)$ is constant. We will prove that F is globally $\mathcal{C}^{2n+2}$-conjugate to the linear map $L(q) = DF(p)\,q$.

- F possesses a smooth *Lie symmetry* $X = \frac{1}{\nu}\big(-V_y, V_x\big)$ of class $\mathcal{C}^{2n+3}$ with a *non-degenerate center* at p, in fact

$$DF(p) = e^{\tau_p DX(p)}, \text{ where } \tau_p = \lim_{h \to h_p} \theta(h)T(h).$$

- The new vector field $Y(x,y) = T(x,y)\,X(x,y)$, is also a Lie Symmetry of F of class $\mathcal{C}^{2n+2}(\mathcal{U})$, having an *isochronous* center at p with period function $T(h) \equiv 1$. Hence,

$$F(q) = \varphi_Y(\tau, q).$$

with τ constant (not depending on h).

- The isochronous center Y linearizes. Since $DF(p) = e^{\tau DY(p)}$, we prove that the "Bochner"-type map

$$\Phi(q) = \int_0^1 \mathrm{e}^{-DY(p)\,s}\,\varphi_Y(s,q)\,ds,$$

is a $\mathcal{C}^{2n+2}$-conjugation between F and the linear map $L(q) = DF(p)\,q$.

But F is also $\mathcal{C}^{2n+2}$-conjugated to the Birkhoff normal form, which is non linear, a contradiction.

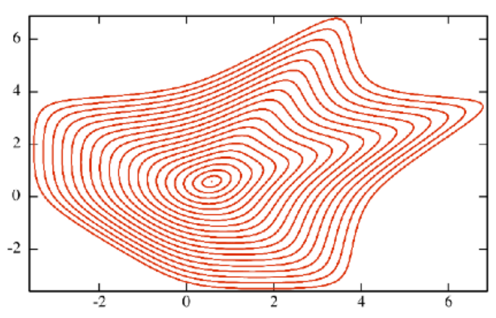
□

The Cohen map case

We have applied Theorem 2 to prove the local non-integrability of a variety of maps, in particular the Cohen's one

$$F(x,y) = \left(y, -x + \sqrt{y^2+1}\right).$$

It seems that the non-integrability of the Cohen map was first conjectured by Cohen and communicated by C. de Verdière to Moser in 1993. Rychlik and Torgesson shown that *it has not first integrals given by algebraic functions* [5]. Inspired by Lowther [4], we have:



Theorem 4. *The Cohen map is not $\mathcal{C}^6$-locally integrable at its fixed point $\left(\sqrt{3}/3, \sqrt{3}/3\right)$.*

We need to prove that for an unbounded sequence $\{N_k\}$ the Cohen map has *finitely many* N_k-periodic points. We will use the following result:

Theorem 5. *Let $G : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial map. Let G_d denote the homogenous map corresponding to the maximum degree d terms of G. If $\mathbf{y} = \mathbf{0}$ is the unique solution in $\mathbb{C}^N$ of the homogeneous system $G_d(\mathbf{y}) = \mathbf{0}$, then $G(\mathbf{y}) = \mathbf{0}$ has finitely many solutions.*

The Cohen map writes as the equation $x_{n+2} = -x_n + \sqrt{x_{n+1}^2 + 1}$. Their solutions are contained in

$$(x_n + x_{n+2})^2 - x_{n+1}^2 - 1 = 0.$$

Therefore the N-periodic orbits satisfy the system

$$\begin{cases} (x_1+x_3)^2 - x_2^2 - 1 = 0, \\ (x_2+x_4)^2 - x_3^2 - 1 = 0, \\ \quad\vdots \\ (x_{N-1}+x_1)^2 - x_N^2 - 1 = 0, \\ (x_N+x_2)^2 - x_1^2 - 1 = 0. \end{cases}$$

Applying Theorem 5, we will prove that, for some unbounded sequence of values of N, $\mathbf{x} = \mathbf{0}$ is the unique solution of the linear systems.

$$\begin{cases} x_1 + x_3 = \pm x_2, \\ x_2 + x_4 = \pm x_3, \\ \quad\vdots \\ x_{N-1} + x_1 = \pm x_N, \\ x_N + x_2 = \pm x_1, \end{cases} \Leftrightarrow \quad A_N\mathbf{x} = \mathbf{0} \text{ with } A_N(\varepsilon_1,\ldots,\varepsilon_N) = \begin{pmatrix} 1 & \varepsilon_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \varepsilon_2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \varepsilon_1 & 1 & \cdots & 0 \\ & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & \varepsilon_{N-2} & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & \varepsilon_{N-1} \\ \varepsilon_N & 1 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

with $\varepsilon_j \in \{-1,1\}$, for each $j = 1,\ldots,N$.

Lemma 6. *For every choice of $\varepsilon_j \in \{-1,1\}$, with $j = 1,\ldots,n$, and for all $N \neq \dot{3}$,*

$$\det(A_N(\varepsilon_1,\ldots,\varepsilon_N)) \equiv F_N \mod 2,$$

where F_N are the Fibonacci numbers.

Recall that F_N are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

and modulus 2,

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$$

Hence

Corollary 7. *For all $N \neq \dot{3}$, the Cohen map has* finitely many *N-periodic points.*

Proof of Theorem 4. Suppose that F is integrable, then:

- F possesses a *Lie symmetry* because it is *measure preserving*.
- At the fixed point $\left(\sqrt{3}/3, \sqrt{3}/3\right)$, $B_1 = 135/256i \neq 0$.
- Since there exists a Lie symmetry, there should exists level sets with continua periodic points for all $N \geq N_0$. But we have proved that for all $N \neq \dot{3}$, G has finitely many N-periodic points, a contradiction.

□

References

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