The Green's function of a weighted n–cycle

A. Carmona, A.M. Encinas, S.Gago, M.J. Jiménez, M. Mitjana

Departament de Matemàtiques,

Universitat Politècnica de Catalunya.

Abstract

The periodicity of problems in mathematics and applied science leads to the solution of linear systems that involve circulant coefficient matrices. In this work, we analyze a type of circulant matrices namely $\mathsf{A} = \mathsf{Circ}(a, b, c, \ldots, c, b)$. It turns out that A is nothing but the combinatorial Laplacian of the n-cycle when $a = 2, b = -1$ and $c = 0$ or, more generally, for any $q \in \mathbb{R}$, the matrix associated with the Scrödinger operator on the cycle with constant potential $2(q-1)$. Hence, its inverse is the Green's function of the n-cycle. The inversion of circulant matrices strongly connects with the resolution of second order difference equations with constant coefficients. Using this approach, we can give a necessary and sufficient condition for the invertibility of matrix A. It is known that, when exists, the inverse is also a circulant matrix. In this case, we explicitly give a closed formula for the expression of the coefficients of A^{-1}.

Besides, we give conditions for the invertibility of circulant matrices associated with combinatorial structures such as $\mathsf{A} = \mathsf{Circ}(a, a + b(n-1), \ldots, a+jb(n-j), \ldots, a+b(n-1))$ or $\mathsf{A} = \mathsf{Circ}(a, a, b, b, a, a, \ldots, a, a, b, b, a)$.

The case $c = 0$ was solved by O. Rojo assuming the condition $|a| > 2|b| > 0$; that is when A is a strictly diagonally dominant matrix. In this work we derive the inverse of a general symmetric circulant tridiagonal matrix, without assuming the hypothesis of diagonally dominance.

1 Matrices $\mathrm{Circ}(a, b, c, \ldots, c, b)$

For any $a, b, c \in \mathbb{R}$, let $\mathsf{b}(a, b, c) \in \mathbb{R}^n$ defined as $\mathsf{b}(a, b, c) = (a, b, c, \ldots, c, b)$. Then, $\mathrm{Circ}(a, b, c, \ldots, c, b) = \mathrm{Circ}\big(\mathsf{b}(a, b, c)\big)$ and $\mathsf{b}_\tau(a, b, c) = \mathsf{b}(a, b, c)$, since matrix $\mathrm{Circ}(a, b, c, \ldots, c, b)$ is symmetric. Regarding the case $\mathsf{b}(a, b, b) = \mathsf{a}(a, b, b)$, matrix $\mathrm{Circ}(a, b, b, \ldots, b, b)$ has been analyzed in the previous section, so from now on we assume $c \neq b$. The case $c = 0$ has been analyzed in [1] under the name of symmetric circulant tridiagonal matrix, assuming the condition $|a| > 2|b| > 0$; that is, that $\mathrm{Circ}\big(\mathsf{b}(a, b, 0)\big)$ is a strictly diagonally dominant matrix.

Notice that $\mathrm{Circ}\big(\mathsf{b}(2, -1, 0)\big)$ is nothing but the so called *combinatorial Laplacian* of a n–cycle. More generally, for any $q \in \mathbb{R}$, $\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)$ is the matrix associated with the Schrödinger operator on the cycle with constant potential $2(q-1)$ and hence its inverse is the Green's function of a n–cycle; or equivalenty, it can be seen as the Green function associated with a path with periodic boundary conditions, see [2]. Since the inversion of matrices of type $\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)$ involves the resolution of second order difference equations with constant coefficients, we enumerate some of their properties.

A *Chebyshev sequence* is a sequence of polynomials $\{Q_n(x)\}_{n\in\mathbb{Z}}$ that satisfies the recurrence

$$Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x), \quad \text{for each } n \in \mathbb{Z}. \tag{1}$$

Recurrence (1) shows that any Chebyshev sequence is uniquely determined by the choice of the corresponding zero and one order polynomials, Q_0 and Q_1

respectively. In particular, the sequences $\{T_n\}_{n=-\infty}^{+\infty}$ and $\{U_n\}_{n=-\infty}^{+\infty}$ denote the *first and second kind Chebyshev polynomials* that are obtained when we choose $T_0(x) = U_0(x) = 1$, $T_1(x) = x$, $U_1(x) = 2x$.

Next we describe some properties of the Chebyshev polynomials of first and second kind that will be useful in the present work. See [3] for proofs and more details.

(i) For any Chebyshev sequence $\{Q_n\}_{n=-\infty}^{+\infty}$ there exists $\alpha, \beta \in \mathbb{R}$ such that $Q_n(x) = \alpha U_{n-1}(x) + \beta U_{n-2}(x)$, for any $n \in \mathbb{Z}$.

(ii) $T_{-n}(x) = T_n(x)$ and $U_{-n}(x) = -U_{n-2}(x)$, for any $n \in \mathbb{Z}$. In particular, $U_{-1}(x) = 0$.

(iii) $T_{2n+1}(0) = U_{2n+1}(0) = 0$, $T_{2n}(0) = U_{2n}(0) = (-1)^n$, for any $n \in \mathbb{Z}$.

(iv) Given $n \in \mathbb{N}^*$ then, $T_n(q) = 1$ iff $q = \cos\left(\frac{2\pi j}{n}\right)$, $j = 0, \dots, \lceil \frac{n-1}{2} \rceil$, whereas $U_n(q) = 0$ iff $q = \cos\left(\frac{\pi j}{n+1}\right)$, $j = 1, \dots, n$. In this case, $U_{n-1}(q) = (-1)^{j+1}$ and $U_{n+1}(q) = (-1)^j$.

(v) $T_n(1) = 1$ and $U_n(1) = n + 1$, whereas $T_n(-1) = (-1)^n$ and $U_n(-1) = (-1)^n(n + 1)$, for any $n \in \mathbb{Z}$.

(vi) $T_n(x) = xU_{n-1}(x) - U_{n-2}(x)$ y $T'_n(x) = nU_{n-1}(x)$, for any $n \in \mathbb{Z}$.

(vii) $2(x - 1)\sum\limits_{j=0}^{n} U_j(x) = U_{n+1}(x) - U_n(x) - 1$, for any $n \in \mathbb{N}$.

Chebyshev recurrence (1) encompasses all linear second order recurrences with constant coefficients, see [4], so we can consider more general recurrences. Let $\{H_n(r, s)\}_{n=0}^{\infty}$, where $r, s \in \mathbb{Z}$ and $s \neq 0$, the *Horadam numbers* defined as the solution of the recurrence

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 0, \ \ H_1 = 1. \tag{2}$$

Notice that for any $n \in \mathbb{N}^*$, $H_n(1, 1) = F_n$, the n–th *Fibonacci number*, $H_n(2, 1) = P_n$, the n–th *Pell number*, $H_n(1, 2) = J_n$, the n–th *Jacobsthal number* and $H_n(2, -1) = U_{n-1}(1) = n$.

The equivalence between any second order difference equation and Chebyshev equations leads to the following result, see [4, Theorem 3.1] and [5, Theorem 2.4].

Lemma 1.1. *Given $r, s \in \mathbb{Z}$ and $s \neq 0$, we have the following results:*

(i) If $s < 0$, then $H_n(r, s) = (\sqrt{-s})^{n-1}U_{n-1}\left(\frac{r}{2\sqrt{-s}}\right)$, $n \in \mathbb{N}^*$.

(ii) If $s > 0$, then $H_{2n}(r, s) = rs^{n-1}U_{n-1}\left(1 + \frac{r^2}{2s}\right)$, $n \in \mathbb{N}^*$.

In particular, for any $n \in \mathbb{N}^*$, $F_{2n} = U_{n-1}\left(\frac{3}{2}\right)$, $J_{2n} = 2^{n-1}U_{n-1}\left(\frac{5}{4}\right)$, and $P_{2n} = 2U_{n-1}(3)$. In addition, $H_{2n}(r, r) = r^n U_{n-1}\left(1 + \frac{r}{2}\right)$ when $r > 0$ and $H_n(r, r) = (\sqrt{-r})^{n-1}U_{n-1}\left(\frac{\sqrt{-r}}{2}\right)$ for $r < 0$.

In addition, for any $q \in \mathbb{R}$ we denote by $\mathsf{u}(q)$, $\mathsf{v}(q)$ and $\mathsf{w}(q)$ the vectors in $\mathbb{R}^n$ whose components are $u_j = U_{j-2}(q)$, $v_j = U_{j-1}(q)$ and $w_j = U_{j-2}(q) + U_{n-j}(q)$, respectively.

Lemma 1.2. *For any $q \in \mathbb{R}^n$, the following properties hold:*

(i) $\mathsf{w}_\tau(q) = \mathsf{w}(q)$ *and* $\langle \mathsf{w}(q), \mathsf{1}\rangle = \dfrac{T_n(q)-1}{q-1}$. *Moreover,* $\mathsf{w}(1) = n\mathsf{1}$.

(ii) $\mathsf{w}(q) = \mathsf{0}$ *iff* $q = \cos\left(\frac{2\pi j}{n}\right)$, $j = 1, \dots, \lceil \frac{n-1}{2} \rceil$. *In this case,* $\langle \mathsf{u}(q), \mathsf{1}\rangle = \langle \mathsf{v}(q), \mathsf{1}\rangle = 0$.

(iii) When n is even, then $w_{2j-1}(0) = 0$ *and* $w_{2j}(0) = (-1)^{j-1}\big[1 - (-1)^{\frac{n}{2}}\big]$, $j = 1, \dots, \frac{n}{2}$.

(iv) When n is odd, then $w_{2j-1}(0) = (-1)^{\frac{n+1}{2}+j}$, $j = 1, \dots, \frac{n+1}{2}$ *and* $w_{2j}(0) = (-1)^{j-1}$, $j = 1, \dots, \frac{n-1}{2}$.

(iv) When n is odd, then $w_j(-1) = (-1)^{j-1}(n + 2 - 2j)$, $j = 1, \dots, n$.

Proof. $\mathsf{w}(q) = \mathsf{0}$ iff $U_{n-j}(q) = -U_{j-2}(q)$ for any $j = 1, \dots, n$ and this equality holds iff $U_{n-1}(q) = 0$ and $U_{n-2}(q) = -1$. Moreover, $U_{n-1}(q) = 0$ iff $q = \cos\left(\frac{k\pi}{n}\right)$, $k = 1, \dots, n-1$, thus $U_{n-2}(q) = (-1)^{k+1}$, leads to $U_{n-2}(q) = -1$ iff $k = 2j$. □

Remark: The quotient $\frac{T_n(q)-1}{q-1}$ is well defined for $q = 1$, because $T_n(1) = 1$, $U_n(1) = n+1$, and $T_n'(q) = nU_{n-1}(q)$, using l'Hôpital's rule, $\lim\limits_{q\to 1}\langle \mathsf{w}(q), \mathsf{1}\rangle = nU_{n-1}(1) = n^2$. Moreover, for $q = 1$, is $\mathsf{w}(1) = n\mathsf{1}$ thus, $\langle \mathsf{w}(1), \mathsf{1}\rangle = n^2$.

Proposition 1.3. *For any $q \in \mathbb{R}$,*

$$\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)\mathsf{w}(q) = 2[T_n(q) - 1]\mathsf{e}.$$

and the following holds:

(i) $\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)$ *is invertible iff* $q \neq \cos\left(\frac{2\pi j}{n}\right)$, $j = 0, \dots, \lceil \frac{n-1}{2} \rceil$ *and,*

$$\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)^{-1} = \frac{1}{2[T_n(q) - 1]}\mathrm{Circ}\big(\mathsf{w}(q)\big).$$

(ii) If $q = 1$, the linear system $\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)\mathsf{h} = \mathsf{v}$ *is compatible iff* $\langle \mathsf{v}, \mathsf{1}\rangle = 0$ *in this case, for any $\gamma \in \mathbb{R}$ the only solution satisfying* $\langle \mathsf{h}, \mathsf{1}\rangle = \gamma$ *is given by*

$$h_j = \frac{\gamma}{n} - \frac{1}{2n}\sum_{i=1}^{n} |j - i|(n - |i - j|)v_i, \quad j = 1, \dots, n.$$

(iii) If $q = \cos\left(\frac{2\pi j}{n}\right)$, $j = 1, \dots, \lceil \frac{n-1}{2} \rceil$, *the linear system* $\mathrm{Circ}\big(\mathsf{b}(2q, -1, 0)\big)\mathsf{h} = \mathsf{v}$ *is compatible iff* $\langle \mathsf{h}, \mathsf{u}(q)\rangle = \langle \mathsf{h}, \mathsf{v}(q)\rangle = 0$.

Proof. To prove (i), notice that $\mathsf{w}(q)$ is the first column of the Green function for the Schrödinger operator for a n–cycle, or equivalently for a $(n+1)$–path with periodic boundary conditions, see [2, Proposition 3.12].

To prove (ii), it suffices to see that $\mathsf{G} = (g_{ij})$, where $g_{ij} = \dfrac{1}{12n}\big(n^2 - 1 - 6|i - j|(n - |i - j|)\big)$, $i, j = 1, \dots, n$ is the Green function of the Combinatorial Laplacian of the cycle, see for instance [6]. The third claim (iii), comes from

(ii) of Lemma 1.2 that states $\mathsf{w}(q) = \mathsf{0}$. In addition, in this case, $U_{n-1}(q) = 0$, $U_{n-2}(q) = -1$ and $U_n(q) = 1$. Besides, vectors $\mathsf{u}(q)$ and $\mathsf{w}(q)$ satisfy

$$\begin{aligned} 2qu_1 - u_2 - u_n &= -1 - U_{n-2}(q) = 0, \\ -u_1 - u_{n-1} + 2qu_n &= -U_{n-3}(q) + 2qU_{n-2}(q) = U_{n-1}(q) = 0, \\ 2qv_1 - v_2 - v_n &= 2q - 2q - U_{n-1}(q) = 0, \\ -v_1 - v_{n-1} + 2qv_n &= -1 - U_{n-2}(q) + 2qU_{n-1}(q) = 0, \end{aligned}$$

thus, $\mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{u}(q) = \mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{v}(q) = \mathsf{0}$. □

Next, the main result in this section is proved. We give necessary and sufficient conditions for the existence of the inverse of matrix $\mathrm{Circ}(a,b,c,\dots,c,b)$ and we explicitly obtain the coefficients of the inverse, when it exists.

Theorem 1.4. *For $a,b,c \in \mathbb{R}$, the circulant matrix $\mathrm{Circ}(a,b,c,\dots,c,b)$ is invertible iff*

$$\big(a + 2b + (n-3)c\big) \prod_{j=1}^{\lceil \frac{n-1}{2} \rceil} \Big[a - c + 2(b-c)\cos\Big(\frac{2\pi j}{n}\Big)\Big] \neq 0$$

and, in this case

$$\mathrm{Circ}(a,b,c,\dots,c,b)^{-1} = \mathrm{Circ}\big(\mathsf{g}(a,b,c)\big),$$

where if $a \neq 3c - 2b$

$$g_j(a,b,c) = \frac{U_{j-2}(q) + U_{n-j}(q)}{2(c-b)[T_n(q) - 1]} - \frac{c}{(a+2b-3c)\big(a+2b+(n-3)c\big)}, \quad j = 1,\dots,n,$$

with $q = \dfrac{c-a}{2(b-c)}$, whereas

$$g_j(3c-2b,b,c) = \frac{1}{12n(c-b)}\big(n^2 - 1 - 6(j-1)(n+1-j)\big) + \frac{1}{n^2 c}, \quad j = 1,\dots,n.$$

Proof. A necessary condition for the invertibility of $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)$ is $\langle \mathsf{b}(a,b,c), \mathsf{1}\rangle = a + 2b + (n-3)c \neq 0$, so, we will assume that this condition holds. Moreover, a necessary and sufficient condition to get $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)$ invertible is the compatibility of the linear system $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)\mathsf{g} = \mathsf{e}$, and in that case there is an only solution that satisfies $\langle \mathsf{g}, \mathsf{1}\rangle = \langle \mathsf{b}(a,b,c), \mathsf{1}\rangle^{-1}$.

Hence,

$$\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)\mathsf{g} = \mathsf{e} \text{ iff } \mathrm{Circ}\big(\mathsf{b}(a-c,b-c,0)\big)\mathsf{g} = \mathsf{e} - c\langle \mathsf{b}(a,b,c), \mathsf{1}\rangle^{-1}\mathsf{1}$$

and moreover, $\langle \mathsf{g}, \mathsf{1}\rangle = \langle \mathsf{b}(a,b,c), \mathsf{1}\rangle^{-1}$.

Since $\mathsf{b}(a-c,b-c,0) = (c-b)\mathsf{b}(2q,-1,0)$, the linear system

$$\mathrm{Circ}\big(\mathsf{b}(a-c,b-c,0)\big)\mathsf{g} = \mathsf{e} - c\langle \mathsf{b}(a,b,c), \mathsf{1}\rangle^{-1}\mathsf{1}$$

is equivalent to system

$$\mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{g} = \frac{1}{(c-b)\big(a+2b+(n-3)c\big)}\Big(\big(a+2b+(n-3)c\big)\mathsf{e} - c\mathsf{1}\Big).$$

If g is a solution of the above system, then

$$\begin{aligned}\frac{(a+2b-3c)}{(c-b)\big(a+2b+(n-3)c\big)} &= \langle \mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{g},\mathsf{1}\rangle = \langle \mathsf{g},\mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{1}\rangle \\ &= \langle \mathsf{b}(2q,-1,0),\mathsf{1}\rangle\langle \mathsf{g},\mathsf{1}\rangle = \frac{(a+2b-3c)}{(c-b)}\,\langle \mathsf{g},\mathsf{1}\rangle.\end{aligned}$$

As a consequence, if $a+2b-3c\neq 0$ then $\langle \mathsf{g},\mathsf{1}\rangle = \dfrac{1}{a+2b+(n-3)c} = \langle \mathsf{b}(a,b,c),\mathsf{1}\rangle^{-1}$. Under this assumption; that is, if $a\neq 3c-2b$ or equivalently $q\neq 1$, then $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)\mathsf{g}=\mathsf{e}$ iff

$$\mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)\mathsf{g} = \frac{1}{(c-b)\big(a+2b+(n-3)c\big)}\Big(\big(a+2b+(n-3)c\big)\mathsf{e}-c\mathsf{1}\Big).$$

In addition, if $\prod_{j=1}^{\lceil\frac{n-1}{2}\rceil}\Big[a-c+2(b-c)\cos\Big(\frac{2\pi j}{n}\Big)\Big]\neq 0$, then $q\neq\cos\Big(\frac{2\pi j}{n}\Big)$, for any $j=1,\dots,\lceil\frac{n-1}{2}\rceil$. Using claim (i) in Proposition 1.3, $\mathrm{Circ}\big(\mathsf{b}(2q,-1,0)\big)$ is invertible, and

$$\begin{aligned}\mathsf{g} &= \frac{1}{2(c-b)\big(a+2b+(n-3)c\big)[T_n(q)-1]}\mathrm{Circ}\big(\mathsf{w}(q)\big)\Big(\big(a+2b+(n-3)c\big)\mathsf{e}-c\mathsf{1}\Big) \\ &= \frac{1}{2(c-b)\big(a+2b+(n-3)c\big)[T_n(q)-1]}\Big(\big(a+2b+(n-3)c\big)\mathsf{w}(q)-c\langle \mathsf{w}(q),\mathsf{1}\rangle\,\mathsf{1}\Big).\end{aligned}$$

If there exists $j=1,\dots,\lceil\frac{n-1}{2}\rceil$, such that $a-c+2(b-c)\cos\Big(\frac{2\pi j}{n}\Big)=0$, i.e. $q=\cos\Big(\frac{2\pi j}{n}\Big)$, then, statement (ii) in Lemma 1.2 ensures

$$\langle\big(a+2b+(n-3)c\big)\mathsf{e}-c\mathsf{1},\mathsf{v}(q)\rangle = \big(a+2b+(n-3)c\big)v_1(q) = a+2b+(n-3)c\neq 0$$

so, by claim (iii) in Proposition 1.3, the linear system $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)\mathsf{g}=\mathsf{e}$ is incompatible and, $\mathrm{Circ}\big(\mathsf{b}(a,b,c)\big)$ is not invertible.

When $a=3c-2b$, this is $q=1$, then $a+2b+(n-3)c=nc$ and system

$$\mathrm{Circ}\big(\mathsf{b}(2,-1,0)\big)\mathsf{g} = \frac{1}{n(c-b)}(n\mathsf{e}-\mathsf{1})$$

is compatible. Moreover, using claim (ii) in Proposition 1.3, the vector $\mathsf{g}\in\mathbb{R}^n$ whose components are given for any $j=1,\dots,n$ by

$$g_j = \frac{1}{n^2c}-\frac{1}{2n(c-b)}(j-1)\big(n-(j-1)\big)+\frac{1}{2n^2(c-b)}\sum_{i=1}^{n}|j-i|(n-|i-j|),$$

is the only solution of the system satisfying $\langle \mathsf{g},\mathsf{1}\rangle=\dfrac{1}{nc}$. Last, we only have to take into account that $\sum\limits_{i=1}^{n}|j-i|(n-|i-j|)=\dfrac{n}{6}(n^2-1)$, for any $j=1,\dots,n$. □

The case $a=3c-2b$ in the above theorem, involves the Green function of a cycle. Cases related to this, raise as application in the analysis of problems associated with this combinatorial structures.

Corollary 1.5. *For a given $a, b \in \mathbb{R}$, matrix*

$$\mathsf{A} = \mathrm{Circ}\big(a, a+b(n-1), a+2b(n-2), \dots, a+jb(n-j), \dots, a+b(n-1)\big)$$

is invertible iff $\big(6a+b(n^2-1)\big)b \neq 0$ *and,*

$$\mathsf{A}^{-1} = \frac{6}{n^2\big(6a+b(n^2-1)\big)}\,\mathsf{J} - \frac{1}{2nb}\,\mathrm{Circ}\big(\mathsf{b}(2,-1,0)\big).$$

Corollary 1.6. *For a given $a, b \in \mathbb{R}$, the following results hold:*

(i) If $n = 1 \bmod(4)$, then $\mathsf{A} = \mathrm{Circ}(a, a, b, b, a, a, \dots, a, a, b, b, a)$ is invertible iff $(a-b)\big(a(n+1)+b(n-1)\big) \neq 0$ and then

$$\mathsf{A}^{-1} = \frac{1}{a-b}\mathrm{Circ}(\mathsf{b}(0,1,0)\big) - \frac{2(a+b)}{(a-b)\big(a(n+1)+b(n-1)\big)}\,\mathsf{J}$$

(ii) If $n = 2 \bmod(4)$, then $\mathsf{A} = \mathrm{Circ}\big(\frac{a+b}{2}, a, \frac{a+b}{2}, b, \frac{a+b}{2}, \dots, \frac{a+b}{2}, b, \frac{a+b}{2}, a)$ is invertible iff $(a-b)\big(a(n+1)+b(n-1)\big) \neq 0$ and then

$$\mathsf{A}^{-1} = \frac{1}{a-b}\mathrm{Circ}(\mathsf{b}(0,1,0)\big) - \frac{2(a+b)}{(a-b)\big(a(n+1)+b(n-1)\big)}\,\mathsf{J}$$

(iii) If $n = 3 \bmod(4)$, then $\mathsf{A} = \mathrm{Circ}(b, a, a, b, b, \dots, a, a, b, b, a, a)$ is invertible iff $(a-b)\big(a(n+1)+b(n-1)\big) \neq 0$ and then

$$\mathsf{A}^{-1} = \frac{1}{a-b}\mathrm{Circ}(\mathsf{b}(0,1,0)\big) - \frac{2(a+b)}{(a-b)\big(a(n+1)+b(n-1)\big)}\,\mathsf{J}$$

(iv) When n is odd, then

$$\mathsf{A} = \mathrm{Circ}\big(a+nb, a-(n-2)b, \dots, a+(-1)^{j-1}(n+2-2j)b, \dots, a-(n-2)b\big)$$

is invertible iff $b(an+b) \neq 0$ and then

$$\mathsf{A}^{-1} = \frac{1}{4b}\mathrm{Circ}\big(\mathsf{b}(2,1,0)\big) - \frac{a}{b(an+b)}\,\mathsf{J}.$$

We end up this paper by deriving the inverse of a general symmetric circulant tridiagonal matrix, without assuming the hypothesis of diagonally dominance. Notice the difference between our result and the methodology given in [1].

Corollary 1.7. *For $a, b \in \mathbb{R}$, $b \neq 0$, the circulant matrix $\mathrm{Circ}(a, b, 0, \dots, 0, b)$ is invertible iff*

$$\prod_{j=0}^{\lceil \frac{n-1}{2} \rceil} \Big[a + 2b\cos\Big(\frac{2\pi j}{n}\Big)\Big] \neq 0$$

and, in this case

$$\mathrm{Circ}(a, b, 0, \dots, 0, b)^{-1} = \mathrm{Circ}\big(\mathsf{g}(a, b, 0)\big),$$

where

$$g_j(a,b,0) = \frac{(-1)^j}{2b[1-(-1)^n T_n(\frac{a}{2b})]}\Big[U_{j-2}\Big(\frac{a}{2b}\Big) + (-1)^n U_{n-j}\Big(\frac{a}{2b}\Big)\Big], \quad j = 1, \dots, n.$$

Notice that the diagonally dominant hypothesis $|a| > 2|b|$ clearly implies that $a + 2b\cos\big(\frac{2\pi j}{n}\big) \neq 0$ for any $j = 0, \dots, n$.

Acknowledgments

This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under projects MTM2011-28800-C02-01 and MTM2011-28800-C02-02.

References

[1] O. Rojo, A new method for solving symmetric circulant tridiagonal systems of linear equations, Computers Math. Applic. 20 (1990), 61-67.

[2] E. Bendito, A. M. Encinas, A. Carmona, Eigenvalues, eigenfunctions and Green's functions on a path via Chebyshev polynomials, Appl. Anal. Discrete Math. 3 (2) (2009) 282–302. `doi:10.2298/AADM0902282B`.

[3] J. Mason, D. Handscomb, Chebyshev Polynomials, Chapman & Hall/CRC, 2003.

[4] D. Aharonov, A. Beardon, K. Driver, Fibonacci, Chebyshev, and orthogonal polynomials, Amer. Math. Monthly 112 (7) (2005) 612–630. `doi:10.2307/30037546`.

[5] A.M. Encinas, M.J. Jiménez, Floquet Theory for second order linear difference equations, submitted.

[6] E. Bendito, A. Carmona, A. Encinas, M. Mitjana, Generalized inverses of symmetric M-matrices, Linear Algebra Appl. 432 (9) (2010) 2438 – 2454. `doi:10.1016/j.laa.2009.11.008`.