The Green’s function of a weighted $n$–cycle

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Abstract

The periodicity of problems in mathematics and applied science leads to the solution of linear systems that involve circulant coefficient matrices. In this work, we analyze a type of circulant matrices namely $A = \text{Circ}(a, b, c, \ldots, c, b)$. It turns out that $A$ is nothing but the combinatorial Laplacian of the $n$–cycle when $a = 2, b = -1$ and $c = 0$ or, more generally, for any $q \in \mathbb{R}$, the matrix associated with the Schrödinger operator on the cycle with constant potential $2(q-1)$. Hence, its inverse is the Green’s function of the $n$–cycle. The inversion of circulant matrices strongly connects with the resolution of second order difference equations with constant coefficients. Using this approach, we can give a necessary and sufficient condition for the invertibility of matrix $A$. It is known that, when exists, the inverse is also a circulant matrix. In this case, we explicitly give a closed formula for the expression of the coefficients of $A^{-1}$.

Besides, we give conditions for the invertibility of circulant matrices associated with combinatorial structures such as $A = \text{Circ}(a + b(n-1), \ldots, a + jb(n-j), \ldots, a + b(n-1))$ or $A = \text{Circ}(a, a, b, a, a, \ldots, a, a, b, a)$. The case $c = 0$ was solved by O. Rojo assuming the condition $|a| > 2|b| > 0$; that is when $A$ is a strictly diagonally dominant matrix. In this work we derive the inverse of a general symmetric circulant tridiagonal matrix, without assuming the hypothesis of diagonally dominance.

1 Matrices $\text{Circ}(a, b, c, \ldots, c, b)$

For any $a, b, c \in \mathbb{R}$, let $b(a, b, c) \in \mathbb{R}^n$ defined as $b(a, b, c) = (a, b, c, \ldots, c, b)$. Then, $\text{Circ}(a, b, c, \ldots, c, b) = \text{Circ}(b(a, b, c))$ and $b_r(a, b, c) = b(a, b, c)$, since matrix $\text{Circ}(a, b, c, \ldots, c, b)$ is symmetric. Regarding the case $b(a, b, b) = a(a, b, b)$, matrix $\text{Circ}(a, b, b, \ldots, b, b)$ has been analyzed in the previous section, so from now on we assume $c \neq b$. The case $c = 0$ has been analyzed in [1] under the name of symmetric circulant tridiagonal matrix, assuming the condition $|a| > 2|b| > 0$; that is, that $\text{Circ}(b(a, b, 0))$ is a strictly diagonally dominant matrix.

Notice that $\text{Circ}(b(2, -1, 0))$ is nothing but the so called combinatorial Laplacian of a $n$–cycle. More generally, for any $q \in \mathbb{R}$, $\text{Circ}(b(2q, -1, 0))$ is the matrix associated with the Schrödinger operator on the cycle with constant potential $2(q-1)$ and hence its inverse is the Green’s function of a $n$–cycle; or equivalently, it can be seen as the Green function associated with a path with periodic boundary conditions, see [2]. Since the inversion of matrices of type $\text{Circ}(b(2q, -1, 0))$ involves the resolution of second order difference equations with constant coefficients, we enumerate some of their properties.

A Chebyshev sequence is a sequence of polynomials $\{Q_n(x)\}_{n \in \mathbb{Z}}$ that satisfies the recurrence

$$Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x), \quad \text{for each } n \in \mathbb{Z}. \quad (1)$$

Recurrence (1) shows that any Chebyshev sequence is uniquely determined by the choice of the corresponding zero and one order polynomials, $Q_0$ and $Q_1$. 

1
respectively. In particular, the sequences \( \{T_n\}_{n=-\infty}^{+\infty} \) and \( \{U_n\}_{n=-\infty}^{+\infty} \) denote the \textit{first and second kind Chebyshev polynomials} that are obtained when we choose \( T_0(x) = U_0(x) = 1, T_1(x) = x, U_1(x) = 2x \).

Next we describe some properties of the Chebyshev polynomials of first and second kind that will be useful in the present work. See [3] for proofs and more details.

(i) For any Chebyshev sequence \( \{Q_n\}_{n=-\infty}^{+\infty} \) there exists \( \alpha, \beta \in \mathbb{R} \) such that \( Q_n(x) = \alpha U_{n-1}(x) + \beta U_{n-2}(x) \), for any \( n \in \mathbb{Z} \).

(ii) \( T_{-n}(x) = T_n(x) \) and \( U_{-n}(x) = -U_{n-2}(x) \), for any \( n \in \mathbb{Z} \). In particular, \( U_{-1}(x) = 0 \).

(iii) \( T_{2n+1}(0) = U_{2n+1}(0) = 0, T_{2n}(0) = U_{2n}(0) = (-1)^n \), for any \( n \in \mathbb{Z} \).

(iv) Given \( n \in \mathbb{N}^* \) then, \( T_n(q) = 1 \) iff \( q = \cos \left( \frac{j\pi}{n} \right), j = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \), whereas \( U_n(q) = 0 \) iff \( q = \cos \left( \frac{j\pi}{n+1} \right), j = 1, \ldots, n \). In this case, \( U_{n-1}(q) = (-1)^j U_j(q) \), for any \( n \in \mathbb{Z} \).

(v) \( T_n(1) = 1 \) and \( U_n(1) = n + 1 \), whereas \( T_n(-1) = (-1)^n \) and \( U_n(-1) = (-1)^n(n + 1) \), for any \( n \in \mathbb{Z} \).

(vi) \( T_n(x) = xU_{n-1}(x) - U_{n-2}(x) \) y \( T_n(x) = nU_{n-1}(x) \), for any \( n \in \mathbb{Z} \).

(vii) \( 2(x-1) \sum_{j=0}^{n} U_j(x) = U_{n+1}(x) - U_n(x) - 1 \), for any \( n \in \mathbb{N} \).

Chebyshev recurrence [1] encompasses all linear second order recurrences with constant coefficients, see [4], so we can consider more general recurrences. Let \( \{H_n(r,s)\}_{n=0}^{\infty} \), where \( r, s \in \mathbb{Z} \) and \( s \neq 0 \), the \textit{Horadam numbers} defined as the solution of the recurrence

\[
H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 0, \quad H_1 = 1.
\]

Notice that for any \( n \in \mathbb{N}^* \), \( H_n(1,1) = F_n \), the \( n \)-th Fibonacci number, \( H_n(2,1) = P_n \), the \( n \)-th Pell number, \( H_n(1,2) = J_n \), the \( n \)-th Jacobsthal number and \( H_n(2,-1) = U_{n-1}(1) = n \).

The equivalence between any second order difference equation and Chebyshev equations leads to the following result, see [4] Theorem 3.1 and [3] Theorem 2.4.

**Lemma 1.1.** Given \( r, s \in \mathbb{Z} \) and \( s \neq 0 \), we have the following results:

(i) If \( s < 0 \), then \( H_n(r,s) = (\sqrt{s})^{n-1}U_{n-1}(\frac{r}{2\sqrt{s}}) \), \( n \in \mathbb{N}^* \).

(ii) If \( s > 0 \), then \( H_{2n}(r,s) = rs^{n-1}U_{n-1}(1 + \frac{r^2}{2s}) \), \( n \in \mathbb{N}^* \).

In particular, for any \( n \in \mathbb{N}^* \), \( F_{2n} = U_{n-1}(\frac{3}{2}) \), \( J_{2n} = 2^{n-1}U_{n-1}(\frac{3}{2}) \), and \( P_{2n} = 2U_{n-1}(3) \). In addition, \( H_{2n}(r,r) = r^nU_{n-1}(1 + \frac{r^2}{2}) \) when \( r > 0 \) and \( H_n(r,r) = (\sqrt{-r})^{n-1}U_{n-1}(\frac{1}{\sqrt{-r}}) \) for \( r < 0 \).

In addition, for any \( q \in \mathbb{R} \) we denote by \( u(q), v(q) \) and \( w(q) \) the vectors in \( \mathbb{R}^n \) whose components are \( u_j = U_{j-2}(q), v_j = U_{j-1}(q) \) and \( w_j = U_{j-2}(q) + U_{n-j}(q) \), respectively.
Lemma 1.2. For any $q \in \mathbb{R}^n$, the following properties hold:

(i) $w_r(q) = w(q)$ and $\langle w(q), 1 \rangle = \frac{T_n(q) - 1}{q - 1}$. Moreover, $w(1) = n1$.

(ii) $w(q) = 0$ iff $q = \cos \left( \frac{2\pi j}{n} \right)$, $j = 1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil$. In this case, $\langle u(q), 1 \rangle = \langle v(q), 1 \rangle = 0$.

(iii) When $n$ is even, then $w_{2j-1}(0) = 0$ and $w_{2j}(0) = (-1)^{j-1} \left[ 1 - (-1)^{\frac{n}{2}} \right]$, $j = 1, \ldots, \frac{n}{2}$.

(iv) When $n$ is odd, then $w_{2j-1}(0) = (-1)^{\frac{n+1}{2} - j}$, $j = 1, \ldots, \frac{n+1}{2}$ and $w_{2j}(0) = (-1)^{j-1}$, $j = 1, \ldots, \frac{n-1}{2}$.

Proof. When $n$ is odd, then $w_j(-1) = (-1)^{j-1}(n - 2j), j = 1, \ldots, n$.

Remark: The quotient $\frac{T_n(q) - 1}{q - 1}$ is well defined for $q = 1$, because $T_n(1) = 1$, $U_n(1) = n + 1$, and $T_n(q) = nU_n(q)$, using l’Hôpital’s rule, $\lim_{q \to 1} \langle w(q), 1 \rangle = nU_n^{-1}(1) = n^2$. Moreover, for $q = 1$, is $w(1) = n1$ thus, $\langle w(1), 1 \rangle = n^2$.

Proposition 1.3. For any $q \in \mathbb{R}$,

$$\text{Circ}(b(2q, -1, 0)) w(q) = 2[T_n(q) - 1]e.$$ 

and the following holds:

(i) $\text{Circ}(b(2q, -1, 0))$ is invertible iff $q \neq \cos \left( \frac{2\pi j}{n} \right)$, $j = 0, \ldots, \left\lceil \frac{n-1}{2} \right\rceil$ and,

$$\text{Circ}(b(2q, -1, 0))^{-1} = \frac{1}{2[T_n(q) - 1]} \text{Circ}(w(q)).$$

(ii) If $q = 1$, the linear system $\text{Circ}(b(2q, -1, 0))d = \gamma$ is compatible iff $\langle \gamma, 1 \rangle = 0$ in this case, for any $\gamma \in \mathbb{R}$ the only solution satisfying $\langle d, 1 \rangle = \gamma$ is given by

$$h_j = \frac{\gamma}{n} - \frac{1}{2n} \sum_{i=1}^{n} |j - i|(n - |i - j|)v_i, \quad j = 1, \ldots, n.$$ 

(iii) If $q = \cos \left( \frac{2\pi j}{n} \right)$, $j = 1, \ldots, \left\lceil \frac{n-1}{2} \right\rceil$, the linear system $\text{Circ}(b(2q, -1, 0))d = \gamma$ is compatible iff $\langle d, \gamma(u(q)) \rangle = \langle \gamma, v(q) \rangle = 0$.

Proof. To prove (i), notice that $w(q)$ is the first column of the Green function for the Schrödinger operator for a $n$-cycle, or equivalently for a $(n + 1)$-path with periodic boundary conditions, see [2 Proposition 3.12].

To prove (ii), it suffices to see that $G = (g_{ij})$, where $g_{ij} = \frac{1}{12n}(n^2 - 1 - 6|i - j|(n - |i - j|))$, $i, j = 1, \ldots, n$ is the Green function of the Combinatorial Laplacian of the cycle, see for instance [6]. The third claim (iii), comes from
(ii) of Lemma 1.2 that states \( w(q) = 0 \). In addition, in this case, \( U_{n-1}(q) = 0 \), \( U_{n-2}(q) = -1 \) and \( U_n(q) = 1 \). Besides, vectors \( u(q) \) and \( w(q) \) satisfy

\[
2qu_1 - u_2 - u_n = -1 - U_{n-2}(q) = 0, \\
u_1 - u_{n-1} + 2qu_n = -U_{n-3}(q) + 2qU_{n-2}(q) = U_{n-1}(q) = 0, \\
2qv_1 - v_2 - v_n = 2q - 2q - U_{n-1}(q) = 0, \\
v_1 - v_{n-1} + 2qv_n = -1 - U_{n-2}(q) + 2U_{n-1}(q) = 0,
\]

thus, \( \text{Circ}(b(2q, -1, 0))u(q) = \text{Circ}(b(2q, -1, 0))v(q) = 0 \). \( \square \)

Next, the main result in this section is proved. We give necessary and sufficient conditions for the existence of the inverse of matrix \( \text{Circ}(a, b, c, \ldots, c, b) \) and we explicitly obtain the coefficients of the inverse, when it exists.

**Theorem 1.4.** For \( a, b, c \in \mathbb{R} \), the circulant matrix \( \text{Circ}(a, b, c, \ldots, c, b) \) is invertible iff

\[
(a + 2b + (n - 3)c) \prod_{j=1}^{[\frac{n+1}{2}]} [a - c + 2(b - c) \cos \left( \frac{2\pi j}{n} \right)] \neq 0
\]

and, in this case

\[
\text{Circ}(a, b, c, \ldots, c, b)^{-1} = \text{Circ}(g(a, b, c)),
\]

where if \( a \neq 3c - 2b \)

\[
g_j(a, b, c) = \frac{U_{j-2}(q) + U_{n-j}(q)}{2(c - b)[T_n(q) - 1]} - \frac{c}{(a + 2b - 3c)(a + 2b + (n - 3)c)}, \quad j = 1, \ldots, n,
\]

with \( q = \frac{c - a}{2(b - c)} \), whereas

\[
g_j(3c - 2b, b, c) = \frac{1}{12n(c - b)} \left( n^2 - 1 - 6(j - 1)(n + 1 - j) \right) + \frac{1}{n^2c}, \quad j = 1, \ldots, n.
\]

**Proof.** A necessary condition for the invertibility of \( \text{Circ}(b(a, b, c)) \) is \( (b(a, b, c), 1) = a + 2b + (n - 3)c \neq 0 \), so, we will assume that this condition holds. Moreover, a necessary and sufficient condition to get \( \text{Circ}(b(a, b, c)) \) invertible is the compatibility of the linear system \( \text{Circ}(b(a, b, c))g = e \), and in that case there is an only solution that satisfies \( (g, 1) = (b(a, b, c), 1)^{-1} \).

Hence,

\[
\text{Circ}(b(a, b, c))g = e \iff \text{Circ}(b(a - c, b - c, 0))g = e - c(b(a, b, c), 1)^{-1}
\]

and moreover, \( (g, 1) = (b(a, b, c), 1)^{-1} \).

Since \( b(a - c, b - c, 0) = (c - b)b(2q, -1, 0) \), the linear system

\[
\text{Circ}(b(a - c, b - c, 0))g = e - c(b(a, b, c), 1)^{-1}
\]

is equivalent to system

\[
\text{Circ}(b(2q, -1, 0))g = \frac{1}{(c - b)(a + 2b + (n - 3)c)} \left( (a + 2b + (n - 3)c)e - c1 \right).
\]
If $g$ is a solution of the above system, then
\[
\frac{(a + 2b - 3c)}{(c - b)(a + 2b + (n - 3)c)} = \langle \text{Circ}(b(2q, -1, 0))g, 1 \rangle = \langle g, \text{Circ}(b(2q, -1, 0))1 \rangle = \langle b(2q, -1, 0), 1 \rangle = \frac{(a + 2b - 3c)}{(c - b)} = \langle g, 1 \rangle.
\]

As a consequence, if $a + 2b - 3c \neq 0$ then $\langle g, 1 \rangle = \frac{1}{a + 2b + (n - 3)c} = \langle b(a, b, c), 1 \rangle^{-1}$. Under this assumption; that is, if $a \neq 3c - 2b$ or equivalently $q \neq 1$, then $\text{Circ}(b(a, b, c))g = e$ iff
\[
\text{Circ}(b(2q, -1, 0))g = \frac{1}{(c - b)(a + 2b + (n - 3)c)} \left((a + 2b + (n - 3)c)e - c1\right).
\]

In addition, if $\prod_{j=1}^{\lfloor n/2 \rfloor} \left[a - c + 2b - c\cos \left(\frac{2\pi j}{n}\right)\right] \neq 0$, then $g \neq \cos \left(\frac{2\pi j}{n}\right)$, for any $j = 1, \ldots, \lfloor n/2 \rfloor$. Using claim (i) in Proposition 1.3, $\text{Circ}(b(2q, -1, 0))$ is invertible, and
\[
g = \frac{1}{2(c - b)(a + 2b + (n - 3)c)[T_v(q) - 1]} \text{Circ}(w(q)) \left((a + 2b + (n - 3)c)e - c1\right)
\]
\[
= \frac{1}{2(c - b)(a + 2b + (n - 3)c)[T_v(q) - 1]} \left((a + 2b + (n - 3)c)w(q) - c(w(q), 1)\right).
\]

If there exists $j = 1, \ldots, \lfloor n/2 \rfloor$, such that $a - c + 2b \cos \left(\frac{2\pi j}{n}\right) = 0$, i.e. $q = \cos \left(\frac{2\pi j}{n}\right)$, then, statement (ii) in Lemma 1.2 ensures
\[
\langle (a + 2b + (n - 3)c)e - c1, v(q) \rangle = (a + 2b + (n - 3)c)v_1(q) = a + 2b + (n - 3)c \neq 0
\]
so, by claim (iii) in Proposition 1.3, the linear system $\text{Circ}(b(a, b, c))g = e$ is incompatible and, $\text{Circ}(b(a, b, c))$ is not invertible.

When $a = 3c - 2b$, this is $q = 1$, then $a + 2b + (n - 3)c = n c$ and system
\[
\text{Circ}(b(2, -1, 0))g = \frac{1}{n(c - b)} (ne - 1)
\]
is compatible. Moreover, using claim (ii) in Proposition 1.3, the vector $g \in \mathbb{R}^n$ whose components are given for any $j = 1, \ldots, n$ by
\[
g_j = \frac{1}{n^2 c} - \frac{1}{2n(c - b)} (j - 1)(n - (j - 1)) + \frac{1}{2n^2 (c - b)} \sum_{i=1}^{n} |j - i|(n - |i - j|),
\]
is the only solution of the system satisfying $\langle g, 1 \rangle = \frac{1}{nc}$. Last, we only have to take into account that $\sum_{i=1}^{n} |j - i|(n - |i - j|) = \frac{n}{6}(n^2 - 1)$, for any $j = 1, \ldots, n$.

The case $a = 3c - 2b$ in the above theorem, involves the Green function of a cycle. Cases related to this, raise as application in the analysis of problems associated with this combinatorial structures.
Corollary 1.5. For a given \( a, b \in \mathbb{R} \), matrix
\[
A = \text{Circ}(a, a + b(n - 1), a + 2b(n - 2), \ldots, a + jb(n - j), \ldots, a + b(n - 1))
\]
is invertible iff \((6a + b(n^2 - 1))b \neq 0\) and,
\[
A^{-1} = \frac{6}{n^2(6a + b(n^2 - 1))}J - \frac{1}{2nb} \text{Circ}(b(2, -1, 0)).
\]

Corollary 1.6. For a given \( a, b \in \mathbb{R} \), the following results hold:

(i) If \( n = 1 \mod(4) \), then \( A = \text{Circ}(a, a, b, b, a, a, \ldots) \) is invertible iff \((a - b)(a + b(n + 1) + b(n - 1)) \neq 0\) and then
\[
A^{-1} = \frac{1}{a - b} \text{Circ}(b(0, 1, 0)) - \frac{2(a + b)}{(a - b)(a + b(n + 1) + b(n - 1))} J.
\]

(ii) If \( n = 2 \mod(4) \), then \( A = \text{Circ}(\frac{a + b}{2}, a, a + b, b, a + b, \ldots) \) is invertible iff \((a - b)(a + b(n + 1) + b(n - 1)) \neq 0\) and then
\[
A^{-1} = \frac{1}{a - b} \text{Circ}(b(0, 1, 0)) - \frac{2(a + b)}{(a - b)(a + b(n + 1) + b(n - 1))} J.
\]

(iii) If \( n = 3 \mod(4) \), then \( A = \text{Circ}(b, a, a, b, b, \ldots) \) is invertible iff \((a - b)(a + b(n + 1) + b(n - 1)) \neq 0\) and then
\[
A^{-1} = \frac{1}{a - b} \text{Circ}(b(0, 1, 0)) - \frac{2(a + b)}{(a - b)(a + b(n + 1) + b(n - 1))} J.
\]

(iv) When \( n \) is odd, then
\[
A = \text{Circ}(a + nb, a - (n - 2)b, \ldots, a + (-1)^{j-1}(n + 2 - 2j)b, \ldots, a - (n - 2)b)
\]
is invertible iff \( b(2n + b) \neq 0 \) and then
\[
A^{-1} = \frac{1}{4b} \text{Circ}(b(2, 1, 0)) - \frac{a}{b(2n + b)} J.
\]

We end up this paper by deriving the inverse of a general symmetric circulant tridiagonal matrix, without assuming the hypothesis of diagonally dominance. Notice the difference between our result and the methodology given in [1].

Corollary 1.7. For \( a, b \in \mathbb{R}, b \neq 0 \), the circulant matrix \( \text{Circ}(a, b, 0, \ldots, 0, b) \) is invertible iff
\[
\prod_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \left[ a + 2b \cos \left( \frac{2\pi j}{n} \right) \right] \neq 0
\]
and, in this case
\[
\text{Circ}(a, b, 0, \ldots, 0, b)^{-1} = \text{Circ}(g(a, b, 0))
\]
where
\[
g_j(a, b, 0) = \frac{(-1)^j}{2b[1 - (-1)^nT_n(\frac{a}{2b})]} \left[ U_{j-2} \left( \frac{a}{2b} \right) + (-1)^nU_{n-j} \left( \frac{a}{2b} \right) \right], \quad j = 1, \ldots, n.
\]

Notice that the diagonally dominant hypothesis \(|a| > 2|b|\) clearly implies that \( a + 2b \cos \left( \frac{2\pi j}{n} \right) \neq 0 \) for any \( j = 0, \ldots, n \).
Acknowledgments

This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under projects MTM2011-28800-C02-01 and MTM2011-28800-C02-02.

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