POINCARÉ SERIES OF MULTIPLIER IDEALS IN TWO-DIMENSIONAL LOCAL RINGS WITH RATIONAL SINGULARITIES

MARIA ALBERICH-CARRAMIÑANA, JOSEP ÀLVAREZ MONTANER, FERRAN DACHS-CADEFAU, AND VÍCTOR GONZÁLEZ-ALONSO

Abstract. We study the multiplicity of the jumping numbers of an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a two-dimensional local ring with a rational singularity. The formula we provide for the multiplicities leads to a very simple and efficient method to detect whether a given rational number is a jumping number. We also give an explicit description of the Poincaré series of multiplier ideals associated to $\mathfrak{a}$ proving, in particular, that it is a rational function.

1. Introduction

Let X be a complex surface with a rational singularity at a point $O \in X$ and $\mathcal{O}_{X,O}$ its corresponding local ring. Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal where $\mathfrak{m} = \mathfrak{m}_{X,O}$ is the maximal ideal of $\mathcal{O}_{X,O}$. Then, for any real exponent $c > 0$, we may consider its corresponding *multiplier ideal* $\mathcal{J}(\mathfrak{a}^c)$. It turns out that the multiplier ideal becomes smaller as the parameter c grows and, whenever we have an strict inclusion $\mathcal{J}(\mathfrak{a}^{c-\varepsilon}) \supsetneq \mathcal{J}(\mathfrak{a}^c)$ for arbitrarily small $\varepsilon > 0$, we say that c is a jumping number.

Since $\mathfrak{a}$ is $\mathfrak{m}$-primary, its associated multiplier ideals are $\mathfrak{m}$-primary as well so they have finite codimension, as $\mathbb{C}$-vector spaces, in $\mathcal{O}_{X,O}$. This fact prompted Ein-Lazarsfeld-Smith-Varolin [6] to define the *multiplicity* of a jumping number as the codimension as $\mathbb{C}$-vector spaces of two consecutive multiplier ideals. In general, for any positive real number c we can define its multiplicity as

$$m(c) := \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^c)}$$

where ε is small enough. In particular, c is a jumping number whenever $m(c) > 0$. In order to gather all the information given by all jumping numbers and their corresponding multiplicities, Galindo-Monserrat [8] introduced the so-called *Poincaré series of multiplier*

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ideals associated to $\mathfrak{a}$ as the series with fractional exponents

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^c.$$

The main result in [8] is the fact that the Poincaré series of a simple complete $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, for a smooth point O, is rational in the sense that it belongs to the field of fractional functions $\mathbb{C}(z)$ where the indeterminate z corresponds to a fractional power $t^{1/e}$ for a suitable $e \in \mathbb{N}_{>0}$. They also provided a closed formula for $P_{\mathfrak{a}}(t)$ that relies in Järviletho's formula [11] for the set of jumping numbers.

One of the goals of this paper is to extend their result to the case of any $\mathfrak{m}$-primary ideal in a surface with a rational singularity at O. To do so we provide first a systematic study of the multiplicities using the theory of *jumping divisors* introduced in [1]. Another goal that we achieve is to give a simple numerical criterion (see Theorem 5.2) which characterizes whether any given rational number is a jumping number.

The paper is organized as follows: First we briefly recall the basics on the theory of multiplier ideals and the aspects on the theory of singularities that we will use throughout this work.

In Section §3 we review the notion of *jumping divisors* introduced in [1]. In fact we will be mainly interested in the maximal jumping divisor since it satisfies a nice periodicity property. In particular we will give a geometrical description of this divisor. We also point out that, *en passant*, we provide several technical results that will be crucial in the rest of the paper.

The core of the paper can be found in Section §4. We provide two different formulas to describe the multiplicity for any $c \in \mathbb{R}_{>0}$. The first one (see Theorem 4.1) is described in terms of the maximal jumping divisor associated to c. The periodicity of this divisor leads to Proposition 4.5 that provides a very clean description of the growth of multiplicities in terms of dicritical components of the maximal jumping divisor. This is the key result that we will use in the description of the Poincaré series associated to $\mathfrak{a}$ in the final section. The second formula for the multiplicity (see Proposition 4.10) is given using the notion of *virtual codimension* introduced in [5] and [17].

In Section §5 we provide a very simple (and efficient) algorithm to compute the set of jumping numbers of $\mathfrak{a}$. It boils down to compute the multiplicities of the rational numbers in the set of *candidate jumping numbers*. This relies on a simple numerical criterion to characterize jumping numbers (see Theorem 5.2). Another consequence of the formulas for the multiplicities is that we can describe those jumping numbers contributed by dicritical divisors. In particular we give in Theorem 5.5 a full description of the jumping numbers in the interval $(1, 2]$.

The main result of Section §6 is a description of the Poincaré series of multiplier ideals for any $\mathfrak{m}$-primary ideal $\mathfrak{a}$. As a consequence, we can easily recover the case of simple ideals obtained by Galindo-Monserrat [8] in the smooth case. Finally we relate the Poincaré series to the Hodge spectrum of a generic element $f \in \mathfrak{a}$. In particular we recover an old result of Lê Văn Thành-Steenbrink [15] describing the Hodge spectrum of a plane curve.

2. Preliminaries

Let (X, O) be a germ of complex surface with at worst a rational singularity. Let $\mathcal{O}_{X,O}$ denote the local ring at O, $\mathfrak{m} = \mathfrak{m}_{X,O} \subseteq \mathcal{O}_{X,O}$ the maximal ideal, and let $\mathfrak{a} \subseteq \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal. Recall that a *log-resolution* of the pair $(X, \mathfrak{a})$ (or of $\mathfrak{a}$, for short) is a birational morphism $\pi : X' \to X$ such that

i) X' is smooth (in particular, π is a resolution of the singularity),
ii) the exceptional locus $E = Exc\,(\pi)$ is a divisor with simple normal crossings (the irreducible components $E_1, \ldots, E_r$ of E are all smooth and intersect transversely), and
iii) the preimage of $\mathfrak{a}$ is locally principal, that is, $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for some effective divisor F supported on E.

The theory of rational singularities was introduced by Artin in [3] and further developed by Lipman in [16]. We recall that the point O being (at worst) a rational singularity means that $R^1\pi_*\mathcal{O}_{X'} = 0$ for some (hence any) desingularization. A first consequence of Artin's results is that the exceptional divisor of any desingularization is a tree of rational curves. Indeed, according to [3, Proposition 1] a singularity is rational if and only if any effective divisor D with exceptional support has arithmetic genus (see [2, Page 486])

$$p_a(D) = 1 + \frac{1}{2}(K_{X'} + D) \cdot D \leqslant 0.$$

Since the components E_i of the exceptional divisor are smooth, we have $p_a(E_i) \geq 0$, hence $p_a(E_i) = 0$, which means that they are rational. Furthermore, there cannot be a *cycle* $E_1, \ldots, E_k$ of exceptional components (i.e., such that $E_1 \cdot E_2 = E_2 \cdot E_3 = \cdots = E_1 \cdot E_k = 1$ and $E_i \cdot E_j = 0$ for any other $i \neq j$), since the formula $p_a(A+B) = p_a(A) + p_a(B) + A \cdot B - 1$ would give $p_a(E_1 + \cdots + E_k) = 1$.

The above numerical characterization [3, Proposition 1] of rational singularities is not satisfying enough, since it involves testing every effective exceptional divisor. In the same work, Artin proved in [3, Theorem 3] that it is enough to check the *fundamental cycle*, the unique smallest non-zero effective divisor Z (with exceptional support) such that

$$Z \cdot E_i \leqslant 0 \qquad \text{for every } \ i = 1, \ldots, r.$$

Another important property of the fundamental cycle is that $\mathfrak{m} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-Z)$, hence any desingularization is a log-resolution of the maximal ideal $\mathfrak{m}$. .

Since rational singularities are $\mathbb{Q}$-factorial, it is possible to define a *relative canonical divisor* K_π of π, which can be characterized as the unique divisor $K_\pi = \sum_{i=1}^r k_i E_i$ supported on the exceptional divisor and such that

$$(K_\pi + E_j) \cdot E_j = \left(\sum_{i=1}^r k_i E_i \cdot E_j \right) + E_j^2 = -2 \tag{2.1}$$

for every exceptional component E_j (because of the adjunction formula). Note that the coefficients k_i are uniquely determined because the intersection matrix $(E_i \cdot E_j)_{i,j}$ is negative-definite, but they are not necessarily integral nor positive. Moreover, due to this numerical characterization, $K_{X'}$ can be replaced by K_π to compute the arithmetic genus as $p_a(Z) = 1 + \frac{1}{2}(K_\pi + Z) \cdot Z$.

The ideal $\mathfrak{a}$ being $\mathfrak{m}$-primary, F is supported on the exceptional locus, hence it can be written as $F = \sum_{i=1}^r e_i E_i$ for some positive integers e_i. For any component E_i, the *excess* of $\mathfrak{a}$ at E_i is defined as

$$\rho_i = -F \cdot E_i \geqslant 0. \tag{2.2}$$

If C is a curve through O defined by a general element in $\mathfrak{a}$, then ρ_i is the number of branches of the strict transform $\widetilde{C}$ that intersect E_i. The *total excess* is defined as $\rho = \sum_{i=1}^r \rho_i$, and is therefore the number of branches at O of a general curve of the linear system defined by $\mathfrak{a}$. In particular, $\rho > 0$.

For any $\mathbb{R}$-divisor $D = \sum_i d_i D_i$ in X', where the D_i are pairwise different prime divisors, its *round-down* $\lfloor D \rfloor$, *round-up* $\lceil D \rceil$ and *fractional part* $\{D\}$ are defined by applying the corresponding operation to the coefficients d_i.

The *multiplier ideal (sheaf)* associated to $\mathfrak{a}$ and some real number $c \in \mathbb{R}$ is defined as

$$\mathcal{J}(\mathfrak{a}^c) = \pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil).$$

Since $\mathfrak{a}$ is $\mathfrak{m}$-primary, any multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is also $\mathfrak{m}$-primary. Furthermore, for any $\varepsilon > 0$ it holds $\mathcal{J}(\mathfrak{a}^c) \supseteq \mathcal{J}(\mathfrak{a}^{c+\varepsilon})$, with equality for ε small enough. Hence the multiplier ideals form a discrete nested sequence

$$\mathcal{O}_{X,O} \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \dots$$

indexed by an increasing sequence of rational numbers $0 < \lambda_1 < \lambda_2 < \dots$ such that $\mathcal{J}(\mathfrak{a}^{\lambda_i}) = \mathcal{J}(\mathfrak{a}^c) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}})$ for any $c \in [\lambda_i, \lambda_{i+1})$. The λ_i are the so-called *jumping numbers* of the ideal $\mathfrak{a}$. We point out now two properties that will be useful in the sequel:

- (local vanishing) for any $c \in \mathbb{R}$, it holds $R^1\pi_* \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) = 0$, and
- (Skoda's theorem) $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{c-1})$ for any $c > 2$.

For further properties and some applications of multiplier ideals, we refer the reader to the book of Lazarsfeld [13].

Being $\mathfrak{m}$-primary, the multiplier ideals have finite $\mathbb{C}$-codimension in $\mathcal{O}_{X,O}$. This fact prompted Ein, Lazarsfeld and Varolin [6] to define the *multiplicity* of λ_i as

$$m(\lambda_i) = \dim_{\mathbb{C}} \frac{\mathcal{J}\left(\mathfrak{a}^{\lambda_{i-1}}\right)}{\mathcal{J}(\mathfrak{a}^{\lambda_i})}.$$

Since $\mathcal{J}\left(\mathfrak{a}^{\lambda_{i-1}}\right) = \mathcal{J}\left(\mathfrak{a}^{\lambda_i - \varepsilon}\right)$ for small ε, we can extend this definition to any $c \in \mathbb{R}$ as

$$m(c) := \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^c)} \tag{2.3}$$

With this definition, it is clear that c is a jumping number if and only if $m(c) > 0$.

In order to describe the behavior of the jumping numbers and its multiplicities, Galindo and Montserrat [8] introduced the *Poincaré series of multiplier ideals* associated to $\mathfrak{a}$, which after our definition of multiplicity can be written as

$$P_{\mathfrak{a}}(t) = \sum_{c\in\mathbb{R}_{>0}} m(c)\, t^{c}. \tag{2.4}$$

We introduce now some technical notation. Given any exceptional component E_i, define

$$\operatorname{Adj}(E_i) = \{E_j \mid E_i \cdot E_j = 1\} \quad \text{and} \quad a(E_i) = \#\operatorname{Adj}(E_i) = E_i \cdot (E - E_i),$$

the set of exceptional components adjacent to E_i and its number. More generally, for any reduced exceptional divisor $D = E_{i_1} + \cdots + E_{i_m}$ define

$$\operatorname{Adj}_D(E_i) = \{E_j \leqslant D \mid E_i \cdot E_j = 1\} \quad \text{and} \quad a_D(E_i) = \#\operatorname{Adj}_D(E_i),$$

the set of components adjacent to E_i *inside* D. Define also the set of components adjacent to D as

$$\operatorname{Adj}(D) = \{E_j \mid E_j \nleqslant D \,\text{and}\, D \cdot E_j = 1\}.$$

Finally, denote by $v_D = m$ (resp. a_D) the number of irreducible components of D (resp. intersections between two components of D). Since the exceptional set is a tree of rational curves, any D as before is a collection of trees of rational curves, and it is then clear that

$$\sum_{E_i\leqslant D} a_D(E_i) = 2a_D$$

and that $v_D - a_D$ equals the number of connected components of D. We also say that E_i is an *end* of D if $E_i \leqslant D$ and $a_D(E_i) = 1$.

Finally we mention that there are two kinds of exceptional divisors that will play a special role throughout this work:

- An exceptional component E_i is a *rupture* component if $a(E_i) \geqslant 3$, that is, it intersects at least three more components of E (different from E_i).
- We say that E_i is *dicritical* if $\rho_i > 0$. Dicritical components correspond to *Rees valuations* by [16].

3. Jumping divisors

Recall from [1, Definition 4.1] that a *jumping divisor* for a jumping number λ is a reduced exceptional divisor G such that $\lambda e_i - k_i \in \mathbb{Z}$ for every irreducible component $E_i \leqslant G$, and for small $\varepsilon > 0$ satisfies

$$\mathcal{J}\left(\mathfrak{a}^{\lambda-\varepsilon}\right) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda F \rceil + G\right). \tag{3.1}$$

That is, G gives a jump from the multiplier ideal with exponent λ to the previous one. In [1] it was proved that, given a jumping number λ, every jumping divisor G satisfies $G_\lambda \leqslant G \leqslant H_\lambda$ for some special jumping divisors G_λ and H_λ. These divisors are called respectively *minimal* and *maximal* jumping divisor, and the former is extensively studied in [1]. The aim of this section is to study the maximal one, which can be defined for any positive real number c and will play a prominent role in the rest of the paper.

Definition 3.1. Given any real number $c \in \mathbb{R}$, we define its associated *maximal jumping divisor* as

$$H_c = \lceil K_\pi - (c - \varepsilon) F \rceil - \lceil K_\pi - cF \rceil \tag{3.2}$$

for a sufficiently small $\varepsilon > 0$. Alternatively, it can be defined as the reduced divisor whose components are the exceptional curves E_i such that $k_i - ce_i \in \mathbb{Z}$.

It follows immediately from the definition that the maximal jumping divisors satisfy the following periodicity property.

Lemma 3.2. *For any real number $c \in \mathbb{R}$, we have $H_c = H_{c+1}$.*

Remark 3.3. The definition of minimal jumping divisors given in [1, Definition 4.3] is more involved and is closely related to the algorithm given in loc. cit. for the computation of the chain of multiplier ideals. Is for this reason that minimal jumping divisors are only defined for jumping numbers in [1]. However one may extend the definition to any positive real number c if we consider $G_c = 0$ for any non-jumping number $c > 0$. Notice that the equality (3.1) is still trivially satisfied for any divisor G such that $G_c \leqslant G \leqslant H_c$. Regarding the periodicity of the minimal jumping divisor, we only have $G_c = G_{c+1}$ for $c > 1$ (see [1, Proposition 4.8]) and there are examples where this equality does not hold for $c \leqslant 1$.

We focus now on the structure of H_c. We first prove some formulas to compute its intersection with its irreducible and connected components.

Lemma 3.4. *Fix $c \in \mathbb{R}_{\geqslant 0}$ and consider a component E_i of the jumping divisor H_c. Then*

$$(\lceil K_\pi - cF \rceil + H_c) \cdot E_i = -2 + c\rho_i + a_{H_c}(E_i) + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}.$$

Proof. For any $E_i \leqslant H_c$ we have

$$\begin{aligned}(\lceil K_\pi - cF \rceil + H_c) \cdot E_i &= ((K_\pi - cF) + \{-K_\pi + cF\} + H_c - E_i + E_i) \cdot E_i = \\ &= (K_\pi + E_i) \cdot E_i - cF \cdot E_i + (H_c - E_i) \cdot E_i + \{cF - K_\pi\} \cdot E_i.\end{aligned}$$

Let us now compute each summand separately. The first three terms are easy: $(K_\pi + E_i) \cdot E_i = -2$ follows from the adjunction formula, $-cF \cdot E_i = c\rho_i$ holds by definition, and clearly $a_{H_c}(E_i) = (H_c - E_i) \cdot E_i$ because $E_i \leqslant H_c$. It only remains to prove that

$$\{cF - K_\pi\} \cdot E_i = \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}, \tag{3.3}$$

which is also quite immediate. Indeed, writing $\{cF - K_\pi\} = \sum_{j=1}^{r} \{ce_i - k_i\} E_j$, (3.3) follows by observing that, for $j \neq i$, $E_j \cdot E_i = 1$ if and only if $E_j \in \mathrm{Adj}(E_i)$, and the term corresponding to $j = i$ vanishes because we assumed $E_i \leqslant H_c$, hence $ce_i - k_i \in \mathbb{Z}$. $\square$

Corollary 3.5. *For any $c \in \mathbb{R}_{>0}$ and any $E_i \leqslant H_c$, the sum*

$$c\rho_i + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\}$$

is an integer.

Proposition 3.6. *Fix any $c \in \mathbb{R}_{>0}$, and let H_c be its associated maximal jumping divisor. Then the following inequalities hold:*

- *$(\lceil K_\pi - cF \rceil + H_c) \cdot E_i \geqslant -1$ for all $E_i \leqslant H_c$, and*
- *$(\lceil K_\pi - cF \rceil + H_c) \cdot H \geqslant -1$ for any connected component $H \leqslant H_c$.*

Proof. From Lemma 3.4 we already know that $(\lceil K_\pi - cF \rceil + H_c) \cdot E_i \geqslant -2$ for all $E_i \leqslant H_c$. If equality holds, then it must also hold

- $a_{H_c}(E_i) = 0$, that is, E_i is an isolated component in H_c,
- $\{ce_j - k_j\} = 0$ for all $E_j \in \mathrm{Adj}(E_i)$, that is, every exceptional component E_j intersecting E_i is also contained in H_c, and
- $\rho_i = 0$.

The first two conditions imply that E_i is the only exceptional curve of the log-resolution. But in this case $\rho_i = \rho > 0$ and the third condition is not satisfied.

As for the second part, using Lemma 3.4 for all $E_i \leqslant H$ and summing up we obtain

$$\begin{aligned}(\lceil K_\pi - cF \rceil + H_c) \cdot H &= -2v_H + \sum_{E_i \leqslant H} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i \right) + 2a_H \\ &= -2 + \sum_{E_i \leqslant H} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} \right) + c \sum_{E_i \leqslant H} \rho_i \geqslant -2,\end{aligned}$$

where $a_H - v_H = 1$ due to the tree structure of the exceptional divisor and the connectedness of H. Equality holds if and only if

$$\sum_{E_i \leqslant H} \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} = \sum_{E_i \leqslant H} c\rho_i = 0.$$

The first condition implies that H is the whole exceptional divisor, and then the second condition implies that $\rho = 0$, which is impossible. Hence the inequality must be strict, and since $(\lceil K_\pi - cF \rceil + H_c) \cdot H \in \mathbb{Z}$, the claim follows. □

We will now get some insight on the topology of the H_c.

Theorem 3.7. *Fix any $c \in \mathbb{R}_{>0}$, and let H_c be the corresponding maximal jumping divisor. Then:*

- *The isolated components of H_c must be either a rupture divisor, a dicritical divisor or a divisor E_i with $a(E_i) = 2$ such that*

$$\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} = 1.$$

- *An end of a reducible connected component of H_c must be either a rupture divisor, a dicritical divisor or an end of the whole exceptional divisor.*

Proof. Let E_i be an isolated component of H_c. Assume that it is neither a rupture nor a dicritical component. Then it only has one or two adjacent components in the exceptional divisor. In the first case, if E_j is the only exceptional component in $\mathrm{Adj}\,(E_i)$, then the formula given in Lemma 3.4 reduces to $(\lceil K_\pi - cF \rceil + H_c) \cdot E_i = -2 + \{ce_j - k_j\}$. Since $\{ce_j - k_j\} < 1$, we would get $(\lceil K_\pi - cF \rceil + H_c) \cdot E_i < -1$, contradicting Proposition 3.6. The only possible remaining case is $a\,(E_i) = 2$. If $\mathrm{Adj}\,(E_i) = \{E_j, E_l\}$, then we have $(\lceil K_\pi - cF \rceil + H_c) \cdot E_i = -2 + \{ce_j - k_j\} + \{ce_l - k_l\}$. Since

$$0 \leqslant \{ce_j - k_j\} + \{ce_l - k_l\} < 2$$

must be an integer by Corollary 3.5 (we assumed E_i to be non-dicritical, i.e. $\rho_i = 0$), it must equal 0 or 1. But the former contradicts Proposition 3.6, hence the only possibility is that $\{ce_j - k_j\} + \{ce_l - k_l\} = 1$, which is the last possibility given in the statement.

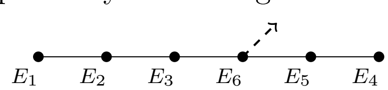
As for the second assertion, let E_i be an end of a reducible connected component of H_c that is neither a rupture divisor, nor a dicritical divisor nor an end of the whole exceptional divisor. Then it has two adjacent components in the whole exceptional divisor, say E_j and E_l, but only one of them, say E_j, is in H_c. Then we have

$$(\lceil K_\pi - cF \rceil + H_c) \cdot E_i = -2 + \{ce_l - k_l\} + 1 \notin \mathbb{Z},$$

which is impossible. □

There are examples where any of these cases is achieved, in particular we may find isolated components of H_c that are neither a rupture nor a dicritical divisor.

Example 3.8. Consider the ideal $\mathfrak{a} = (x^3, y^{10}) \subseteq \mathbb{C}\{x, y\}$. Its minimal log-resolution has six exceptional components $E_1, \ldots, E_6$ indexed according to the order in which they are obtained by successive blow-ups. They are arranged as the following dual graph shows



where the dashed arrow indicates that E_6 is the only dicritical component, with excess $\rho_6 = 1$. The relative canonical divisor is $K_\pi = E_1 + 2E_2 + 3E_3 + 4E_4 + 8E_5 + 12E_6$ and the divisor F such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ is $F = 3E_1 + 6E_2 + 9E_3 + 10E_4 + 20E_5 + 30E_6$.

The maximal jumping divisor associated to $c = \frac{3}{2}$ is $H_{\frac{3}{2}} = E_2 + E_4 + E_5 + E_6$. It has two connected components, one of which (E_2) is as predicted at the first statement of Theorem 3.7.

4. Multiplicities of Jumping Numbers

Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The aim of this section is to describe the multiplicity

$$m(c) = \dim_{\mathbb{C}} \frac{\mathcal{J}(\mathfrak{a}^{c-\varepsilon})}{\mathcal{J}(\mathfrak{a}^{c})}$$

for any real exponent $c > 0$, where ε is small enough. In Theorem 4.1 we will give a formula described in terms of the maximal jumping divisor associated to c. This formula and Proposition 4.5 will be the key ingredients for the description of the Poincaré series associated to $\mathfrak{a}$ that we will give in Theorem 6.1.

We will also provide a second formula for the multiplicity in Proposition 4.10 that is based on the concept of *virtual codimension* considered by Casas-Alvero [5] and Reguera [17] for the smooth and the rational singularities case respectively.

We start with the first formula.

Theorem 4.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$m(c) = (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \#\{\text{connected components of } H_c\}.$$

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) \longrightarrow \mathcal{O}_{X'}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow 0$$

Pushing it forward to X and applying local vanishing for multiplier ideals we get the short exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\mathcal{O}_{X'}(\lceil K_\pi - cF \rceil) \longrightarrow \pi_*\mathcal{O}_{X'}(\lceil K_\pi - cF \rceil + H_c) \longrightarrow \\ \longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \otimes \mathbb{C}_O \longrightarrow 0 \end{aligned}$$

or equivalently, since $H_c = \lceil K_\pi - (c-\varepsilon) F \rceil - \lceil K_\pi - cF \rceil$ for ε small enough,

$$0 \longrightarrow \mathcal{J}(\mathfrak{a}^c) \longrightarrow \mathcal{J}(\mathfrak{a}^{(c-\varepsilon)}) \longrightarrow H^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \otimes \mathbb{C}_O \longrightarrow 0$$

Therefore the multiplicity of c is just

$$\begin{aligned} m(c) &= h^0(H_c, \mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)) \\ &= \sum_{E_i \leqslant H_c} h^0(E_i, \mathcal{O}_{E_i}(\lceil K_\pi - cF \rceil + H_c)) - a_{H_c}, \end{aligned}$$

where in the second equality we have used that H_c has simple normal crossings, and hence the sections of the line bundle $\mathcal{O}_{H_c}(\lceil K_\pi - cF \rceil + H_c)$ correspond to sections over each component that agree on the a_{H_c} intersections. Indeed, we can consider the twist by $\mathcal{O}_{X'}(\lceil K_\pi - cF \rceil + H_c)$ of the following exact sequence

$$0 \longrightarrow \mathcal{O}_{H_c} \longrightarrow \bigoplus_{E_i \leqslant H_c} \mathcal{O}_{E_i} \longrightarrow \bigoplus_{E_i, E_j \leqslant H_c} \mathcal{O}_{E_i \cap E_j} \longrightarrow 0,$$

where the summands in the last term are length-one skyscraper sheaves (due to the simple normal crossings condition), the first map is the direct sum of the restrictions $\mathcal{O}_{H_c} \to \mathcal{O}_{E_i}$ and the second map is given by the differences at the intersections $E_i \cap E_j$.

Recall now that each exceptional component E_i is isomorphic to $\mathbb{P}^1$, and that the sections of a line bundle on $\mathbb{P}^1$ are determined by its degree (namely, $h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = d + 1$

if $d \geqslant -1$ and zero otherwise). Then, using that

$$\deg \mathcal{O}_{E_i}\left(\lceil K_\pi - cF\rceil + H_c\right) = \left(\lceil K_\pi - cF\rceil + H_c\right)\cdot E_i \geqslant -1$$

by Proposition 3.6, we get

$$\begin{aligned} m\left(c\right) &= \sum_{E_i \leqslant H_c} \left(\left(\lceil K_\pi - cF\rceil + H_c\right)\cdot E_i + 1\right) - a_{H_c} \\ &= \left(\lceil K_\pi - cF\rceil + H_c\right)\cdot H_c + v_{H_c} - a_{H_c} \\ &= \left(\lceil K_\pi - cF\rceil + H_c\right)\cdot H_c + \#\left\{\text{connected components of } H_c\right\}. \end{aligned}$$

□

Remark 4.2. When $c = \lambda$ is a jumping number, the same formula for the multiplicity can be described using the associated minimal jumping divisor G_λ. Namely,

$$m(\lambda) = (\lceil K_\pi - \lambda F\rceil + G_\lambda)\cdot G_\lambda + \#\{\text{connected components of } G_\lambda\}$$

The proof of this result holds verbatim to the one given for Theorem 4.1 but we have to refer to [1, Proposition 4.16] instead of Proposition 3.6.

For reduced divisors in the interval $G_\lambda < G < H_\lambda$ we may have $E_i \leqslant G$ such that

$$(\lceil K_\pi - \lambda F\rceil + G)\cdot E_i = -2 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{\lambda e_j - k_j\} + \lambda\rho_i + a_G(E_i) = -2.$$

Namely, this happens when E_i is a non-dicritical isolated component of G with all adjacent divisors in H_λ. However, these divisors can also provide a formula for the multiplicity of a jumping number as follows. Refining the arguments used in the proof of Theorem 4.1 we obtain:

$$m(\lambda) = (\lceil K_\pi - \lambda F\rceil + G)\cdot G + \#\{\text{c.c. of } G\} + \#\left\{E_i \;\middle|\; (\lceil K_\pi - \lambda F\rceil + G)\cdot E_i = -2\right\}.$$

In some cases it will be more convenient to use the following reinterpretation of the formula given in Theorem 4.1.

Corollary 4.3. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$m\left(c\right) = \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{c e_j - k_j\} + c\rho_i\right) - \#\left\{\textit{connected components of } H_c\right\}.$$

Proof. Using Lemma 3.4 we have:

$$\begin{aligned} m(c) &= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \# \{\text{connected components of } H_c\} \\ &= \sum_{E_i \leqslant H_c} \left(-2 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i + a_{H_c}(E_i) \right) + \# \{\text{c.c. of } H_c\} \\ &= -2v_H + \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i \right) + 2a_{H_c} + \# \{\text{c.c. of } H_c\} \\ &= \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j - k_j\} + c\rho_i \right) - \# \{\text{c.c. of } H_c\} \end{aligned}$$

□

As an immediate consequence of this we obtain the following slight generalization of a result of Tucker [21, Proposition 7.3]. We point out that Järviletho already proved in [11] that 1 is not a jumping number for simple $\mathfrak{m}$-primary ideals.

Corollary 4.4. *Suppose that O is a smooth point, and let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The multiplicity of $c = 1$ is*

$$m(1) = \rho - 1.$$

In particular, $c = 1$ is a jumping number if and only if $\mathfrak{a}$ is not simple.

Proof. The maximal jumping divisor for $c = 1$ has the same support as F, so the result follows from Corollary 4.3. □

From the formula given above and the periodicity of the maximal jumping divisor H_c, it is easy to control the growth of the multiplicities in terms of the excesses at dicritical components. This result is a key point in the proof of Theorem 6.1.

Proposition 4.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and H_c the maximal jumping divisor associated to some $c \in \mathbb{R}_{>0}$. Then,*

$$m\,(c+1) - m\,(c) = \sum_{E_i \leqslant H_c} \rho_i.$$

In particular, $0 \leqslant m\,(c+1) - m\,(c) \leqslant \rho$.

Proof. Recall that c and $c+1$ have the same jumping divisor H_c (see Lemma 3.2). Therefore, by Theorem 4.1, we have

$$m\,(c+1) - m\,(c) = -F \cdot H_c = \sum_{E_i \leqslant H_c} \rho_i.$$

□

4.1. **Virtual codimensions.** Given an effective $\mathbb{R}$-divisor $D = \sum d_i E_i$ with exceptional support we may consider its associated ideal (sheaf) $\pi_* \mathcal{O}_{X'}(-D) := \pi_* \mathcal{O}_{X'}(-\lceil D \rceil)$. Its stalk at O is an $\mathfrak{m}$-primary complete ideal of $\mathcal{O}_{X,O}$ that we will simply denote as I_D. We say that two divisors are *equivalent* if they define the same ideal. In the equivalence class of a given divisor D one may find a unique maximal representative, its so-called *antinef closure* $\widetilde{D}$ (see [16, §18]). First, recall that an effective divisor with integer coeficients D' is called *antinef* if $-D' \cdot E_i \geqslant 0$, for every exceptional prime divisor E_i.

The antinef closure of D can be computed using an inductive procedure called *unloading* that was already described in the work of Enriques [7, IV.II.17] (see also [12], [5, §4.6] and [17]). Here we will consider the version given by the first three authors in [1]. Unloading values to any D is to consider the new divisor

$$D' = \lceil D \rceil + \sum_{E_i \in \Theta} n_i E_i,$$

where Θ is the set of components $E_i \leqslant D$ with negative excesses, i.e.

$$\Theta := \{E_i \leqslant D \mid \rho_i = -\lceil D \rceil \cdot E_i < 0\}$$

and $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$. We say that the unloading is *tame* if $\rho_i = -1$ for all $E_i \in \Theta$ and there are no adjacent divisors in Θ. This is a mild generalization of the notion of tameness introduced in [5]. The antinef closure $\widetilde{D}$ of D is achieved after finitely many unloading steps.

Given a divisor D with exceptional support, we will define its *virtual codimension* or *virtual number of conditions* as

$$\mathcal{C}(D) := -\frac{\lceil D \rceil \cdot (\lceil D \rceil + K_\pi)}{2}.$$

The main feature of this invariant is that it coincides with the codimension of the associated ideal when D is antinef. For a proof of this result one may consult [5, Proposition 4.7.1] for the smooth case and [17, Proposition 3.7] for the rational singularities case.

Proposition 4.6. *Let D be an antinef divisor and I_D its associated ideal. Then:*

$$\mathcal{C}(D) = \dim_{\mathbb{C}} \mathcal{O}_{X,O}/I_D$$

This result is no longer true for arbitrary divisors. However, there are some non-antinef divisors for which this equality holds.

Proposition 4.7. *Assume that a divisor D' is obtained from a divisor D by performing a single unloading step. Then $\mathcal{C}(D) \geqslant \mathcal{C}(D')$ and the equality holds if and only if the unloading step is tame.*

Proof. Notice that, in order to compute the virtual codimension, we may always assume $D = \lceil D \rceil$. Hence, $D' = D + \sum_{E_i \in \Theta} n_i E_i$, where Θ and $n_i = \left\lceil \frac{\rho_i}{E_i^2} \right\rceil$ are defined as above.

Therefore:

$$\begin{aligned}
\mathcal{C}(D)-\mathcal{C}(D') &= -\tfrac{1}{2}\left(D^2-D'^2+K_\pi\cdot(D-D')\right)\\
&= -\tfrac{1}{2}\left(-2\left(\textstyle\sum_i n_iE_i\right)D-\left(\textstyle\sum_i n_iE_i\right)^2-K_\pi\cdot\left(\textstyle\sum_i n_iE_i\right)\right)\\
&= -\tfrac{1}{2}\left(-2\left(\textstyle\sum_i n_iE_i\right)D-\left(\textstyle\sum_i n_iE_i\right)^2+2\textstyle\sum_i n_i+\textstyle\sum_i n_iE_i^2\right)\\
&= \textstyle\sum_i \frac{n_i}{2}\left(-2\rho_i+(n_i-1)E_i^2-2\right)+\textstyle\sum_i\sum_{j>i} n_in_jE_i\cdot E_j
\end{aligned}$$

We are assuming $n_i \geqslant 1$ for all $E_i \in \Theta$ so the summands $\frac{n_i}{2}\left(-2\rho_i+(n_i-1)E_i^2-2\right)$ are always $\geqslant 0$. Notice that they are zero if and only if $\rho_i = -1$ for all $E_i \in \Theta$. On the other hand, $\sum_i\sum_{j>i} n_in_jE_i\cdot E_j \geqslant 0$ and equality holds if and only if $E_i\cdot E_j = 0$ for all $E_i \neq E_j \in \Theta$, i.e. there are no adjacent divisors in the set Θ. □

Corollary 4.8. *Let $\widetilde{D}$ be the antinef closure of a divisor D and I_D their associated ideal, then:*

$$\mathcal{C}(D) \geqslant \mathcal{C}(\widetilde{D}) = \dim_{\mathbb{C}} \mathcal{O}_{X,O}/I_D$$

and the equality holds if and only if all the unloading steps performed to obtain $\widetilde{D}$ are tame.

When we deal with multiplier ideals we can extract a very simple formula for the multiplicity of any real number.

Proposition 4.9. *Let D_c and $D_{c-\varepsilon}$ be the antinef closures of $\lfloor cF-K_\pi\rfloor$ and $\lfloor (c-\varepsilon)F-K_\pi\rfloor$ respectively, for any $c\in\mathbb{R}_{\geqslant 0}$ and ε small enough. Then, the multiplicity of c is*

$$m(c)=\mathcal{C}(D_c)-\mathcal{C}(D_{c-\varepsilon})=\frac{D_{c-\varepsilon}\cdot(D_{c-\varepsilon}+K_\pi)}{2}-\frac{D_c\cdot(D_c+K_\pi)}{2}.$$

Proof. We have

$$m(c)=\dim_{\mathbb{C}}\mathcal{O}_{X,O}/\mathcal{J}\left(\mathfrak{a}^c\right)-\dim_{\mathbb{C}}\mathcal{O}_{X,O}/\mathcal{J}\left(\mathfrak{a}^{c-\varepsilon}\right)$$

and, using Proposition 4.6, the virtual codimensions coincide with the codimension for antinef divisors so $m(c)=\mathcal{C}(D_c)-\mathcal{C}(D_{c-\varepsilon})$ and the result follows.

□

Actually there is no need to compute the antinef closure of the aforementioned divisors to obtain the same result.

Proposition 4.10. *For any $c\in\mathbb{R}_{\geqslant 0}$ and ε small enough we have*

$$m(c)=\mathcal{C}(\lfloor cF-K_\pi\rfloor)-\mathcal{C}(\lfloor (c-\varepsilon)F-K_\pi\rfloor)=$$

$$=\frac{\lfloor (c-\varepsilon)F-K_\pi\rfloor\cdot(\lfloor (c-\varepsilon)F-K_\pi\rfloor+K_\pi)}{2}-\frac{\lfloor cF-K_\pi\rfloor\cdot(\lfloor cF-K_\pi\rfloor+K_\pi)}{2}.$$

Proof. Recall that $\lceil K_\pi-(c-\varepsilon)F\rceil=\lceil K_\pi-cF\rceil+H_c$. Then:

$$\begin{aligned}
&\mathcal{C}(\lfloor cF-K_\pi\rfloor)-\mathcal{C}(\lfloor cF-K_\pi\rfloor-H_c)=\\
&\qquad=\tfrac{1}{2}(\lfloor cF-K_\pi\rfloor-H_c)\cdot(\lfloor cF-K_\pi\rfloor-H_c+K_\pi)-\tfrac{1}{2}(\lfloor cF-K_\pi\rfloor)\cdot(\lfloor cF-K_\pi\rfloor+K_\pi)\\
&\qquad=-\lfloor cF-K_\pi\rfloor\cdot H_c+\tfrac{H_c\cdot H_c}{2}-\tfrac{K_\pi\cdot H_c}{2}
\end{aligned}$$

$$= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c - \tfrac{(H_c + K_\pi) \cdot H_c}{2}$$
$$= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \#\{\text{connected components of } H_c\} = m(c).$$

Here we used the fact that

$$\frac{1}{2}(K_\pi + H_c) \cdot H_c = -v_{H_c} + a_{H_c} = -\#\{\text{connected components of } H_c\}$$

and Theorem 4.1.

□

Let $\lambda' < \lambda$ be two consecutive jumping numbers of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Despite the fact that $\lfloor \lambda' F - K_\pi \rfloor$ and $\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor$ have the same antinef closure their virtual codimensions may differ. However, we still have the following description of the multiplicity

Proposition 4.11. *Let $\lambda' < \lambda$ be two consecutive jumping numbers of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Then, the multiplicity of λ is*

$$m(\lambda) = \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor \lambda' F - K_\pi \rfloor) =$$
$$= \frac{\lfloor \lambda' F - K_\pi \rfloor \cdot (\lfloor \lambda' F - K_\pi \rfloor + K_\pi)}{2} - \frac{\lfloor \lambda F - K_\pi \rfloor \cdot (\lfloor \lambda F - K_\pi \rfloor + K_\pi)}{2}.$$

Proof. Consider all the rational numbers $\gamma \in (\lambda', \lambda)$ for which there exists at least one component E_i such that $\gamma e_i - k_i \in \mathbb{Z}$. We order them to form a finite sequence of rational numbers $\lambda' < \gamma_1 < \cdots < \gamma_r < \lambda$. Notice that these are the only rational numbers in this interval where the virtual codimension of $\lfloor \gamma F - K_\pi \rfloor$ may increase.

We have

$$m(\lambda) = \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor (\lambda - \varepsilon) F - K_\pi \rfloor) = \mathcal{C}(\lfloor \lambda F - K_\pi \rfloor) - \mathcal{C}(\lfloor \gamma_r F - K_\pi \rfloor)$$

and, at every step of the sequence, $m(\gamma_i) = \mathcal{C}(\lfloor \gamma_i F - K_\pi \rfloor) - \mathcal{C}(\lfloor \gamma_{i-1} F - K_\pi \rfloor)$. Therefore

$$m(\lambda) = m(\lambda) + \sum_{i>0} m(\gamma_i) = \mathcal{C}\left(\lfloor \lambda F - K_\pi \rfloor\right) - \mathcal{C}\left(\lfloor \lambda' F - K_\pi \rfloor\right)$$

due to the fact that $m(\gamma_i) = 0$ as these rational numbers are not jumping numbers. □

Remark 4.12. In the case that X is smooth we can check that the unloading steps needed to compute the antinef closure of $\lfloor cF - K_\pi \rfloor$ for any $c \in \mathbb{R}_{\geqslant 0}$ are tame. Indeed, repeating the same arguments considered in the proof of Proposition 4.11 we may end up with the case $c = 0$. It is then easy to check that $\mathcal{C}\left(\lfloor -K_\pi \rfloor\right) = \mathcal{C}(D_0) = 0$ so we get

$$\mathcal{C}\left(\lfloor cF - K_\pi \rfloor\right) = \mathcal{C}(D_c).$$

This concludes the remark thanks to Corollary 4.8.

5. Jumping Numbers via multiplicities

Fix a log-resolution $\pi : X' \longrightarrow X$ of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$. Consider the relative canonical divisor $K_\pi = \sum_{i=1}^r k_i E_i$, and the divisor $F = \sum_{i=1}^r e_i E_i$ such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. The jumps between multiplier ideals must occur at rational numbers that belong to the set of *candidate jumping numbers*

$$\left\{ \frac{k_i + m}{e_i} \mid m \in \mathbb{Z}_{>0} \right\}.$$

Not every candidate jumping number is necessarily a jumping number. Using the formulas for the multiplicity given in the previous section we can easily extract the set of jumping numbers since we have:

Proposition 5.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and $c \in \mathbb{R}_{>0}$. Then, c is a jumping number if and only if $m(c) > 0$.*

In addition, we have the following simple criterion

Theorem 5.2. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal and $c \in \mathbb{R}_{>0}$. Then, there exists a connected component $H \leqslant H_c$ such that*

$$(\lceil K_\pi - cF \rceil + H_c) \cdot H > -1$$

if and only if $m(c) > 0$.

Proof. Theorem 4.1 states that the multiplicity of c is:

$$\begin{aligned} m(c) &= (\lceil K_\pi - cF \rceil + H_c) \cdot H_c + \#\{\text{connected components of } H_c\} \\ &= \sum_{H \leqslant H_c} \left((\lceil K_\pi - cF \rceil + H_c) \cdot H + 1 \right), \end{aligned}$$

where the sum is taken over all the connected components $H \leqslant H_c$. Then, the result follows since $(\lceil K_\pi - cF \rceil + H_c) \cdot H \geqslant -1$ by Proposition 3.6. □

Therefore we have a simple algorithm to compute the set of jumping numbers of $\mathfrak{a}$ that boils down to compute the multiplicity of the rational numbers in the set of candidate jumping numbers by means of the formula given in Theorem 4.1 or the one given in Proposition 4.10. We have implemented this algorithm in the Computer Algebra system `Macaulay 2` [9]. The scripts of the source codes as well as the output in full detail of some examples will be available at the web page

`www.pagines.ma1.upc.edu/~jalvz/multiplier.html`

It turns out that this algorithm is more efficient than the algorithms considered by Tucker in [21] and the first three authors in [1].

5.1. **Jumping numbers contributed by dicritical divisors.** Another interesting consequence of the methods developed in the previous sections is the fact that we can describe a big chunk of the set of jumping numbers by means of an inspection of dicritical divisors. In the sequel we will consider a dicritical divisor E_i with excess $\rho_i = -F \cdot E_i > 0$ and value $v_i(F) = e_i$.

Theorem 5.3. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. Let $k \in \mathbb{N}$ be a non-negative integer such that $\frac{k}{e_i} > \frac{1}{\rho_i}$. Then, $\lambda = \frac{k}{e_i}$ is a jumping number.*

Proof. Let $H \leqslant H_\lambda$ be the connected component that contains the dicritical divisor E_i. For $\lambda = \frac{k}{e_i} > \frac{1}{\rho_i}$ we have

$$\begin{aligned}(\lceil K_\pi - \lambda F \rceil + H_\lambda) \cdot H &= \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{E_j \leqslant H_\lambda} \lambda \rho_j - 2 \\ &> \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j + 1 - 2 \ \geqslant \ -1\end{aligned}$$

and the result follows from Theorem 5.2. ∎

For the boundary case $\lambda = \frac{1}{\rho_i}$ we have the following criteria.

Proposition 5.4. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. Let $k \in \mathbb{N}$ be a non-negative integer such that $\frac{k}{e_i} = \frac{1}{\rho_i}$. Then, the following are equivalent:*

i) *$\lambda = \frac{1}{\rho_i}$ is not a jumping number.*
ii) *$H_\lambda = E$ is the whole exceptional component, and E_i is the only dicritical divisor.*

Proof. Let $H \leqslant H_\lambda$ be the connected component that contains the dicritical divisor E_i. For $\lambda = \frac{k}{e_i} = \frac{1}{\rho_i}$ we have

$$(\lceil K_\pi - \lambda F \rceil + H_\lambda) \cdot H = \sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} + \sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j + 1 - 2$$

By Theorem 5.2, $\lambda = \frac{k}{e_i} = \frac{1}{\rho_i}$ is not a jumping number when this intersection multiplicity is -1. Notice that a divisor E_j satisfies $\{\lambda e_j - k_j\} = 0$ if and only if $E_j \leqslant H_\lambda$. Thus

$$\sum_{E_j \in \mathrm{Adj}(H)} \{\lambda e_j - k_j\} = 0$$

if and only if $\mathrm{Adj}(H) = \emptyset$, or equivalently when $H_\lambda = E$. On the other hand

$$\sum_{\substack{E_j \leqslant H_\lambda \\ j \neq i}} \lambda \rho_j = 0$$

if and only if $\rho_j = 0$ for all $j \neq i$, i.e. when there are no dicritical divisors besides E_i.

∎

Notice that the result above also generalizes the fact that 1 is not a jumping number for simple $\mathfrak{m}$-primary ideals. We can also extend to our setting Järviletho's result on the behavior of the jumping numbers in the interval $(1,2]$ given in [11, Theorem 9.9] for simple complete ideals in a smooth surface.

Theorem 5.5. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The only jumping numbers in the interval $(1,2]$ are the following:*

- *$\lambda + 1$, where $\lambda \in (0,1]$ is a jumping number.*
- *$\lambda = \frac{k}{e_i}$, for $e_i < k \leqslant 2e_i$ with E_i dicritical divisor.*

Proof. Assume that a jumping number $\lambda \in (1,2]$ is not of the announced types and consider its associated maximal jumping divisor H_λ. If λ is not of the first type then $m(\lambda) - m(\lambda - 1) > 0$. If it is not of the second type, then $\rho_i = 0$ for any $E_i \leqslant H_\lambda$. Both conditions cannot be satisfied simultaneously by Proposition 4.5 so we get a contradiction. □

Remark 5.6. Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. A generic element $f \in \mathfrak{a}$ satisfies $\mathcal{J}(f^c) = \mathcal{J}(\mathfrak{a}^c)$ for any $c \in (0,1)$ so Theorem 5.5 says, roughly speaking, that the jumping numbers of $\mathfrak{a}$ are governed by the jumping numbers of a generic element $f \in \mathfrak{a}$ and the dicritical divisors of $\mathfrak{a}$.

6. Poincaré series of multiplier ideals

Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. In this section we will give a very simple description of the Poincaré series of multiplier ideals.

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c)\, t^c = \sum_{c \in (0,1]} \sum_{k \in \mathbb{N}} m(c+k)\, t^{c+k}$$

To such purpose we only need to control the following two issues: First we have to describe the multiplicities of the jumping numbers in the interval $(0,1]$. This can be done using the formulas given in Theorem 4.1 or Proposition 4.10. Secondly, and equally important, we have to control the recurrence that these multiplicities satisfy. As shown in Proposition 4.5, dicritical components in the maximal jumping divisor allow us to describe the recurrence.

The main result of this section is the fact that the Poincaré series of multiplier ideals is rational in the sense that it belongs to the field of fractional functions $\mathbb{C}(z)$, where the indeterminate z corresponds to a fractional power $t^{1/e}$ for $e \in \mathbb{N}_{>0}$ being the least common multiple of the denominators of all jumping numbers. The formula for the Poincaré series that we obtain is the following:

Theorem 6.1. *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be an $\mathfrak{m}$-primary ideal. The Poincaré series of $\mathfrak{a}$ can be expressed as*

$$P_{\mathfrak{a}}(t) = \sum_{c \in (0,1]} \left(\frac{m(c)}{1-t} + \rho_c \frac{t}{(1-t)^2} \right) t^c$$

where $\rho_c = -F \cdot H_c$ and H_c is the maximal jumping divisor associated to c.

Proof. Let $c \in (0,1]$ be a real number. For any $k \in \mathbb{N}$ we have, applying Proposition 4.5

$$m(c+k) = m(c) + k\rho_c,$$

where $\rho_c = m(c+1) - m(c) = -F \cdot H_c$. It follows that

$$\begin{aligned} \sum_{k \geqslant 0} m(c+k)\, t^{c+k} &= m(c)\, t^c + (m(c)+\rho_c)\, t^{c+1} + (m(c)+2\rho_c)\, t^{c+2} + \cdots \\ &= \left(\frac{m(c)}{1-t} + \rho_c \frac{t}{(1-t)^2} \right) t^c \end{aligned}$$

Thus we get the desired result. ☐

For the case of simple $\mathfrak{m}$-primary ideals we can easily recover the extension to the case where X has rational singularities of the main result of Galindo-Monserrat [8]. Our formulation slightly differs from theirs because we collect jumping numbers by the growth of the multiplicities instead of its critical divisors.

Corollary 6.2. [8, Theorem 2.1] *Let $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$ be a simple $\mathfrak{m}$-primary ideal. The Poincaré series of $\mathfrak{a}$ can be expressed as*

$$P_{\mathfrak{a}}(t) = \sum_{\substack{c \in (0,1] \\ \rho_c = 0}} \frac{m(c)}{1-t} t^c + \sum_{\substack{c \in (0,1] \\ \rho_c = 1}} \left(\frac{m(c)}{1-t} + \frac{t}{(1-t)^2} \right) t^c$$

Proof. Simple $\mathfrak{m}$-primary ideals only have one dicritical divisor with excess 1 so the result follows. ☐

6.1. **Hodge Spectrum.** Let X be a smooth complex variety of dimension d and consider an hypersurface with an isolated singularity at O defined by $f \in \mathcal{O}_{X,O}$. The Hodge spectrum $Sp(f)$ associated to f was introduced by Steenbrink [19] using the canonical mixed Hodge structure of the cohomology groups of the Milnor fiber of f. It is a fractional polynomial

$$Sp(f) = \sum_{c \in [0,d]} n(c)\, t^c,$$

where the rational number $c \in \mathbb{Q}$ is an *exponent* or *spectral number* if its associated multiplicity $n(c)$ is strictly positive. It is also known that the sum of all spectral numbers, counted with multiplicity, is equal to the Milnor number of f and that they are symmetric with respect to $\frac{d}{2}$, i.e. $n(c) = n(d-c)$

Budur [4] established a nice relation between the Hodge spectrum and the set of multiplier ideals. More precisely, the multiplicity of spectral numbers and the multiplicity of the so-called *inner* jumping numbers coincide in the interval $(0,1]$. We point out that the usual jumping numbers are inner jumping numbers whenever they are not integer numbers in the case of hypersurfaces with isolated singularities.

In the case where X has dimension two we can make a closer relationship between the Hodge spectrum of a plane curve $f \in \mathcal{O}_{X,O}$, that we assume as a generic element of an

$\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, and the Poincaré series of multiplier ideals of $\mathfrak{a}$. Roughly speaking, the information given by the Hodge spectrum is equivalent, taking into account the symmetry with respect to 1, to the information given by the terms of the Poincaré series in the interval $(0, 1)$. The aim of this section is to strengthen this relationship recovering some old results on the Hodge spectrum of a plane curve by using our methods.

The spectrum of a plane curve has been described by Lê Văn Thành and Steenbrink in [15] (see also [14], [18]). For the convenience of the reader we will reformulate their result using the terminology we are considering in this paper. To this aim, we consider a partial order on the exceptional components of the log-resolution. Since we are assuming that O is a smooth point, the exceptional divisor is naturally a *rooted* tree of rational curves, where the root E_1 is the (strict transform of) the exceptional divisor of the blow-up of O. The partial order is then defined by the paths from E_1, i.e. E_i precedes E_j if E_i belongs to the chain of components connecting E_1 and E_j. For any $i \neq 1$, we denote by $p\,(i)$ the index of the exceptional component immediately preceding E_i, so that $E_{p(i)}$ belongs to the chain connecting E_1 and E_i, and $E_i \cdot E_{p(i)} = 1$. The set of rupture or dicritical divisors different from the root E_1 will be denoted $\mathcal{R}$, i.e.

$$\mathcal{R} = \{ i \,|\, E_i \neq E_1 \text{ is a rupture or dicritical divisor} \}.$$

Theorem 6.3. [15, Theorem 1.5] *Let $f \in \mathcal{O}_{X,O}$ be the equation of a plane curve with an isolated singularity at the origin O. Let $c \in \mathbb{Q}$ be a rational number. Then, its associated multiplicity $n(c)$ in the Hodge spectrum of f is $n(c) = n'(c) + n''(c)$, where:*

- $n'(c) = \# \left\{ E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c \right\}$
- $n''(c) = \displaystyle\sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R} \cup \{1\}}} \left(-1 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ c e_j \} + c \rho_i \right)$

If we assume f as a generic element of an $\mathfrak{m}$-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,0}$ we can recover this result using the formula given in Theorem 4.1.

Proposition 6.4. *Let $f \in \mathcal{O}_{X,O}$ be the equation of a plane curve with an isolated singularity at the origin O. For any $c \in (0, 1)$ we have $n(c) = m(c)$.*

Proof. Lê Văn Thành and Steenbrink's formula states that:

$$\begin{aligned} n(c) &= \#\left\{E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c\right\} + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(-1 + \sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \\ &= \#\left\{E_i \,|\, i \in \mathcal{R} \text{ and } E_i + E_{p(i)} \leqslant H_c\right\} - \#\left\{E_i \,|\, i \in \mathcal{R}\cup\{1\} \text{ and } E_i \leqslant H_c\right\} \\ &\quad + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \\ &= -\#\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\} - \delta + \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) \end{aligned}$$

where $\delta = 1$ if $E_1 \leqslant H_c$ and $\delta = 0$ otherwise. Due to the rooted tree structure of the exceptional divisor, every connected component of H_c has exactly one minimal component E_i (the closest to E_1), and clearly $E_{p(i)} \nleqslant H_c$ if $i \neq 1$. There is therefore a bijection between the set $\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\}$ and the connected components of H_c that contain some rupture or dicritical component but do not contain E_1. Hence we have proved

$$\#\left\{E_i \,|\, i \in \mathcal{R}, E_i \leqslant H_c \text{ and } E_{p(i)} \nleqslant H_c\right\} + \delta = \#\left\{ \substack{\text{connected components of } H_c \\ \text{containing a divisor } E_i,\ i\in\mathcal{R}\cup\{1\}} \right\},$$

which gives the following expression for $n\,(c)$:

$$n(c) = \sum_{\substack{E_i \leqslant H_c \\ i \in \mathcal{R}\cup\{1\}}} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) - \#\left\{ \substack{\text{connected components of } H_c \\ \text{containing a divisor } E_i,\ i\in\mathcal{R}\cup\{1\}} \right\} \tag{6.1}$$

On the other hand, Corollary 4.3 gives (recall that $k_i \in \mathbb{Z}$ because O is a smooth point)

$$m\,(c) = \sum_{E_i \leqslant H_c} \left(\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i\right) - \#\left\{\text{connected components of } H_c\right\}. \tag{6.2}$$

To prove that both formulas coincide, we have to consider the terms

$$\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} + c\rho_i$$

for the $E_i \leqslant H_c$ with $i \notin \mathcal{R}\cup\{1\}$, as well as the connected components of H_c containing only components of this kind.

Consider first an E_i which is not an isolated component of H_c. On the one hand, by Theorem 3.7, all its adjacent components are contained in H_c, and hence $\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} = 0$. Since it is not dicritical, $\rho_i = 0$, and therefore E_i does not contribute to the first summand of $m(c)$. On the other hand, the connected component H of H_c containing E_i contains also either a rupture or dicritical component (again by Theorem 3.7), and hence its contribution to the second summand of (6.2) is already taken into account in (6.1).

To finish the proof, it remains to consider the E_i which are isolated components of H_c. In this case, Theorem 3.7 says that the contribution of E_i to the first term of (6.2) is $\sum_{E_j \in \mathrm{Adj}(E_i)} \{ce_j\} = 1$, which cancels with the contribution to the number of connected components. □

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Deptartament de Matemàtiques, Universitat Politècnica de Catalunya, Av. Diagonal 647, Barcelona 08028, Spain

E-mail address: Maria.Alberich@upc.edu, Josep.Alvarez@upc.edu

KU Leuven, Department of Mathematics, Celestijnenlaan 200B box 2400, BE-3001 Leuven, Belgium

E-mail address: Ferran.DachsCadefau@wis.kuleuven.be

Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

E-mail address: gonzalez@math.uni-hannover.de