Stark-Heegner points and p-adic L-functions

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Contents

Introduction

The theory of L-functions plays a key role in modern number theory. After Iwasawa's new insight on this theory, p-adic L-functions became an interesting alternative to classical ones.

A central feature of the theory of L-functions (classical or p-adic) is the study of their special values: this is where the analytic and the algebraic data meet.

Let us describe the most basic example. Let K be a number field of discriminant D_K and denote by $\mathcal{O}_K$ its ring of integers. The seminal example is the Dedekind zeta function of the field K, denoted by $\zeta_K(s)$. It is defined for $\Re(s) > 1$ as an *Euler product* indexed by all prime ideals in $\mathcal{O}_K$ as follows:

$$\zeta_K(s) = \prod_{\wp \subset \mathcal{O}_K} \frac{1}{1 - \mathbf{N}_{\mathbb{Q}}^K(\wp)^s}.$$

As was first proved by Hecke, this function admits a meromorphic continuation, and satisfies a functional equation of the form:

$$|D_K|^{\frac{s}{2}} \cdot \zeta_{K,\infty}(s) \cdot \zeta_K(s) = |D_K|^{\frac{1-s}{2}} \cdot \zeta_{K,\infty}(1-s) \cdot \zeta_K(1-s)$$

where $\zeta_{K,\infty}(s)$ is an adequate product of Gamma factors. The integral point $s = 0$ lies outside the domain of definition and mathematicians were interested in understanding the nature of this value.

Let us recall Dirichlet's unit theorem: the group of units $\mathcal{O}_K^\times$ is a finitely generated abelian group, whose free part has rank $r_1 + r_2 - 1$, where r_1 is the number of real embeddings $K \hookrightarrow \mathbb{R}$ and r_2 is the number of complex embeddings $K \hookrightarrow \mathbb{C}$ up to conjugation, so that $r_1 + 2r_2 = [K : \mathbb{Q}]$.

The behaviour of $\zeta_K(s)$ at $s = 0$ is the content of the so-called *analytic class number formula*, which we state here as follows:

Theorem 0.1 (Class number formula)**.** *The following are true:*

CNF0 The zeta function $\zeta_K(s)$ admits an analytic continuation and a functional equation relating values at s and $1 - s$;

CNF1 The order of vanishing of $\zeta_K(s)$ at $s = 0$ equals the rank of $\mathcal{O}_K^\times$, i.e. $\mathrm{ord}_{s=0}\zeta_K(s) = r$ *(analytic rank = algebraic rank);*

CNF2 The following formula holds true:

$$\lim_{s \to 0} s^{-r} \zeta_K(s) = -\frac{h_K R_K}{\omega_K}.$$

where:

- *h_K is the class number of K;*
- *ω_K is the number of roots of unity of $\mathcal{O}_K$;*
- *R_K is the complex regulator involving the logarithm of units of $\mathcal{O}_K$.*

One generally refers to the analytic class number formula when speaking about the following result:

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{\omega_K\sqrt{|D_K|}}$$

which can be easily proven to be equivalent to CFN1 and CFN2 together.

In the 1970's Stark developed in [Stark] an equivariant conjectural refinement of the above formula, now known as *Stark's conjectures*. More precisely, for any field extension $H \mid K$, the Artin formalism allows one to decompose the zeta function $\zeta_H(s)$ as a product of L-functions over K twisted by irreducible Artin representations ρ of $\mathrm{Gal}\,(H|K)$. The Stark conjectures aim to describe vanishing properties and special values of these twists at $s = 0$ in terms of invariants linked to the objects involved in a $\mathrm{Gal}\,(H|K)$-equivariant way.

Not surprisingly, one can formulate a similar conjecture for elliptic curves, which will be the main motivation for the present thesis. More precisely, we consider:

- an elliptic curve E defined over K. We denote by $V(E)$ the compatible system of Galois representations induced by its Tate module;
- an Artin representation $\rho : \mathrm{Gal}\,(H|K) \to \mathrm{GL}_n(L)$, where $L \subset \mathbb{C}$ is the field of coefficients of ρ. We associate to it a compatible system of Galois representations $V(\rho)$.

Our main object of interest is the Hasse-Weil-Artin L-function of E twisted by ρ:

$$L(E, \rho, s) := L(V(E) \otimes V(\rho), s),$$

This function is defined as an Euler product similar to that of $\zeta_K(s)$, which converges for $\Re(s) > 3/2$. When $\rho = \mathbb{1}_K$, the twisted L-function $L(E, \rho, s)$ is equal to the original one $L(E/K, s)$.

According to the Mordell-Weil theorem, $E(K)$ is a finitely generated abelian group. The rank of its free part is called the algebraic rang of E over K, that we denote by $r = r(E/K)$. In the early 60's, Birch and Swinnerton-Dyer formulated in [BSD] the following:

Conjecture 0.2 (Birch and Swinnerton-Dyer)**.** *The following are true:*

BSD0 The zeta function $L(E/K, s)$ admits an analytic continuation and a functional equation relating values at s and $2 - s$;

BSD1 The order of vanishing at $s = 1$ equals the rank E over K, i.e. $\mathrm{ord}_{s=1}L(E/K, s) = r$ *(analytic rank = algebraic rank);*

BSD2 The following formula holds true:

$$\lim_{s\to 0} s^{-r}L(E/K, s) = \Omega_{E/K} \cdot \Big(\prod_{\wp\subset\mathcal{O}_K} c_\wp\Big)\frac{\#\mathrm{Sha}_{E/K}}{\#T^2} \cdot R_{E/K}$$

where:

- *$\Omega_{E/K}$ is the complex period attached to E over K;*
- *$c_\wp$ is the Tamagawa number of E at $\wp$;*
- *$\mathrm{Sha}_{E/K}$ is the Shafarevich-Tate group;*
- *$R_{E/K}$ is the complex regulator involving Néron-Tate heights of a basis of the free part of the Mordell-Weil group.*

It is remarkable how theorem 0.1 and conjecture 0.2 share many similarities. Moreover there are equivariant refinements of the BSD conjecture, too. This sets up the basis for a vast generalization known as the *equivariant Tamagawa number conjecture* (ETNC) which includes, as special cases, both the Stark conjectures and the equivariant Birch and Swinnerton-Dyer conjecture.

In this dissertation we are mainly motivated by the equivariant BSD conjecture as an elliptic curve analogue of the Stark conjecture for number fields.

Let E/K be an elliptic curve and let $\rho : \mathrm{Gal}\,(H|K) \to \mathrm{GL}_n(L)$ and consider the twisted L-function $L(E, \rho, s)$. Assuming the BSD0 we can define the analytic rank associated with E twisted by ρ as:

$$r_{\mathrm{an}}(E, \rho) := \mathrm{ord}_{s=1} L(E, \rho, s).$$

On the algebraic side we consider the vector space V_ρ underlying the representation and the set of Galois equivariant homomorphisms $\psi : V_\rho \to E(H)_L$, where $E(H)_L := E(H) \otimes L$. In other words, we consider the possible ways to embed the representation V_ρ inside the Mordell-Weil group. In particular, the algebraic rank measures how many times the representation V_ρ occurs in $E(H)_L$:

$$r_{\mathrm{alg}}(E, \rho) := \dim_L \mathrm{Hom}_{G_K}(V_\rho, E(H)_L).$$

We also define the ρ-isotypical component of the Mordell-Weil group as follows:

$$E(H)^\rho := \sum_{\substack{\psi : V_\rho \to E(H) \\ \psi \text{ basis}}} \psi(V_\rho).$$

The rank part of the equivariant BSD conjecture predicts that the analytic rank and the algebraic rank are equal, so that:

$$r_{\mathrm{an}}(E, \rho) \stackrel{?}{=} r_{\mathrm{alg}}(E, \rho).$$

The equality has been established in few cases only: there are results when the base field is $\mathbb{Q}$, $r_{\mathrm{an}}(E, \rho) \leq 1$ and ρ has dimension at most 2, under some additional specific hypothesis. The common feature of these results is the use of Euler systems.

In order to overcome the rank part of the problem and give a unified vision of the Euler systems, Darmon, Lauder and Rotgers formulated in [DLR15] a p-adic analogue of the equivariant BSD conjecture which they call the *elliptic Stark conjecture*. We now briefly describe the result of their article. Consider a self-dual representation $\rho : \mathrm{Gal}\,(H|\mathbb{Q}) \to \mathrm{SL}_4(L)$ which is the tensor product of two odd representations ρ_1 and ρ_2 of dimension 2. By the modularity results of Khare and Winterberger (see [KW09]) we know that such representations arise from modular forms of weight one, so that $\rho_1 = \rho_g$, $\rho_2 = \rho_h$ and $\rho = \rho_{gh} = \rho_g \otimes \rho_h$, where:

$$g \in M_1(N_g, \overline{\chi}) \qquad \text{and} \qquad h \in M_1(N_h, \chi).$$

In this sense we can write $L(E, \rho_{gh}, s) = L(f \otimes g \otimes h, s)$, so that the L-function can be viewed as the Garrett's triple product L-function. In order to describe the elliptic Stark conjecture in detail, we first need to discuss the hypothesis under which it is formulated.

Hypotheses A and B of [DLR15] say that the L-function $L(E, \rho, s)$ vanishes to order at least two at $s = 1$ and that the order of vanishing is even. In particular it is assumed that the local signs of the functional equation of $L(E, \rho, s)$ are all positive.

Hypotheses C and C' of loc.cit. are important for the very definition of p-adic iterated integrals that we now describe. Given a module M on which the good Hecke operators T_n act and a common eigenform ϕ, we define $M[\phi]$ to be the subspace of M composed by vectors which have the same eigenvalues for all of the T_n's. For a test vector

$$(\breve{\gamma}, \breve{f}, \breve{h}) \in M_1(Np, \chi)_L^\vee[g_\alpha] \times S_2(Np, \mathbb{1})_L[f] \times M_1(Np, \chi)_L[h]$$

the associated p-adic iterated integral is defined as follows:

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} := \breve{\gamma}\big(e_{g_\alpha^*} e_{\mathrm{ord}}(\breve{F} \cdot \breve{h})\big) \tag{1}$$

where $\breve{F}$ is the overconvergent primitive of $\breve{f}$, $e_{g_\alpha *}$ is the projector onto the g_α^*-isotypical component and e_{ord} is Hida's ordinary projector. Hypothesis C and C' ensure us that the definition is meaningful by declaring that the g_α^*-isotypical component of the space of overconvergent modular forms in weight one is made solely of classical forms, so that, in particular, $e_{g_\alpha^*} e_{\mathrm{ord}}(\breve{F} \cdot \breve{h})$ is classical, too. Then we have the:

Conjecture 0.3 (Elliptic Stark conjecture). *Let $p \nmid N_g N_h$ be a prime at which E has ordinary reduction, let N_f be the tame part of N_E and $N = \mathrm{lcm}(N_f, N_g, N_h)$. Assume hypothesis A, B, and C,C' of [DLR15]. Then:*

- *if $r_{\mathrm{an}}(E, \rho) = 2$, then there exists a test vector:*

  $$(\breve{\gamma}, \breve{f}, \breve{h}) \in M_1(Np, \chi)_L^\vee[g_\alpha] \times S_2(Np, \mathbb{1})_L[f] \times M_1(Np, \chi)_L[h]$$

  for which:

  $$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = \frac{R_p(E, \rho)_{g_\alpha}}{\log_p(u_{g_\alpha})}$$

  is nonzero, where:

  - *u_{g_α} is the Gross-Stark unit associated with the adjoint representation associated with g_α;*
  - *$R_p(E, \rho)_{g_\alpha}$ is a p-adic ρ-equivariant regulator which is defined as the determinant of a 2 by 2 matrix involving elliptic logarithm of points in $E(H)_L^\rho$.*

- *if $r_{\mathrm{an}}(E, \rho) > 2$, then $\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = 0$ for every possible choice of test vector $(\breve{\gamma}, \breve{f}, \breve{h})$.*

Let us point out the presence of g_α: this is a choice of a p-stabilization of g which is needed to formulate the conjecture—this will be discussed later.

The proposition 2.6 of loc.cit. motivates the terminology of "elliptic Stark conjecture". More precisely, the p-adic iterated integral can be interpreted as the special value of the Garrett-Hida p-adic L-function, which is a p-adic avatar of $L(E, \rho, s)$, and the conjecture contains information about both the order of vanishing of this L-function and its leading term.

In [DLR15] the authors provide several heuristics in support of the conjecture, and they prove it in some special cases. More precisely, they consider the case in which g and h are theta series associated with Galois characters of a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$ and (f, K) satisfies the Heegner hypothesis. In this setting, the Heegner hypothesis ensures us that hypothesis A and B are satisfied and also, in most cases, that theta series satisfy hypothesis C and C' (see the discussion of section 4.2). Then the representation ρ splits as the direct sum of two representations ρ_1 and ρ_2 (not necessarily irreducible), and the availability of Heegner points and elliptic units allows the construction, respectively, of canonical elements $P_i \in E(H)_L^{\rho_i}$ which satisfy the already existent *p-adic Gross-Zagier formula* studied in [BDP12], and of global units $u_{g_\alpha} \in (\mathcal{O}_H^\times)_L^{\mathrm{Ad}(g_\alpha)}$ satisfying the Katz-Kronecker limit formula. Using those tools the authors achieve the following:

Theorem 0.4 (Darmon, Lauder, Rotger). *Assume that $N = \mathrm{lcm}(N_E, N_g, N_h)$ is square-free and that $p \nmid 2N$. Assume also that hypothesis C and C' hold. Then:*

- *if $r_{\mathrm{an}}(E, \rho_i) = 1$, then there exists a test vector:*

  $$(\breve{\gamma}, \breve{f}, \breve{h}) \in M_1(Np, \chi)_L^\vee[g_\alpha] \times S_2(Np, \mathbb{1})_L[f] \times M_1(Np, \chi)_L[h]$$

and a scalar $\lambda \in L^\times$ *for which:*

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = \lambda \cdot \frac{\log_{E,p}(P_1)\log_{E,p}(P_2)}{\log_p(u_{g_\alpha})}$$

- *if* $r_{\mathrm{an}}(E, \rho_i) > 1$ *for* $i = 1$ *or* $i = 2$*, then* $\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = 0$ *for any choice of test vector* $(\breve{\gamma}, \breve{f}, \breve{h})$*.*

Proof. This is achieved using the Garrett-Hida method, by p-adically interpolating the triple product L-function and comparing it with other p-adic L-functions (cf. [DLR15, §3]). □

Let us point out that this formula is qualitative: no computation is made to make λ explicit. It would be interesting to see what happens to the constant λ if one makes a specific choice for the test vector $(\breve{\gamma}, \breve{f}, \breve{h})$. In particular, it is worth to make here the following:

Remark 0.5. The quantity on the left depends on f (in fact $\breve{f}$) while the quantity on the right contains the elliptic logarithm, which is usually taken in the literature to be the logarithm associated with the Néron differential ω_E. In particular, the Néron differential is pulled back to a multiple of $\omega_f := 2\pi i f(\tau)d\tau$, via the modular parametrization $\pi : X_0(N_E) \to E$, i.e. $\pi^*(\omega_E) = C_E \cdot \omega_f$. The constant C_E would be a product of the Manin constant and the degree of the isogeny from the elliptic curve E_f, associated with f via Eichler-Shimura construction, and our elliptic curve E.

We use the result [BDP12, Theorem 3.12] and we want to avoid the constant C_E. For this reason, we consider the logarithm associated with the differential ω on E defined by:

$$\pi^*(\omega) = \omega_f.$$

This is a non-canonical choice but the results that we obtain can be easily translated in terms of the usual logarithm by adding the constant C_E. For a qualitative result this fact does not matter, but for a precise computation of λ it is important.

In [DLR15, remark 3.4], assuming that $N = D$ is square-free, that $p \nmid N$ and that $g = h = E_{1,\chi_K} \in M_1(D, \chi_K)$ is the Eisenstein series associated with the quadratic character χ_K attached to the extension $K|\mathbb{Q}$, the authors verified experimentally that:

$$\int_\gamma f \cdot h = \frac{\#E(\mathbb{F}_p)^2}{p(p-1)h_K} \cdot \frac{\log_{E,p}(P_K)^2}{\log_p(u_\wp)},$$

where h_K is the class number of K, $(u_\wp) = \wp^{h_K}$ and $P_K \in E(K) = E(H)_L^{\mathbb{1}_K}$. In this specific case the condition $N = D$ ensures us that the choice (γ, f, h) is canonical, where γ is the dual of g.

This raises two main questions:

(i) are we able to uncover some nice arithmetical meaning of the scalar λ, possibly after making some *canonical choice* of test vector $(\breve{\gamma}, \breve{f}, \breve{h})$?

(ii) can we establish the conjecture in other cases?

The first question was answered using the Rankin method instead of the Garrett method, but restricting to the case in which $h = E_{1,\chi_K}$. In this case $\rho_1 = \rho_2 = \rho_g$, so that $\rho = \rho_g^2$ and we have a factorization of L-functions:

$$L(E, \rho, s) = L(E, \rho_g, s)^2$$

The function $L(E, \rho_g, s) = L(f \otimes g, s)$ can be interpolated p-adically with Hida-Rankin's method and its special value can also be interpreted as a p-adic iterated integral. Since g has CM by χ_K, it can be seen as the theta series associated with a finite order Hecke character $\psi : G_K \to \mathbb{C}^\times$. In this setting, $P_1 = P_2 = P_{\bar{\psi}}$, $u_{g_\alpha} = u_{\bar{\psi}^2}$ and the main result of [CR1] is the following:

Theorem 0.6 (C.-Rotger). *Assume that* $(N_E, N_g/D) = 1$ *and take* $N = \mathrm{lcm}(N_E, N_g)$ *and a prime* $p \nmid 2N$. *Assume also that hypothesis C and C' hold and that* $r_{\mathrm{an}}(E, \rho_g) = 1$. *Let us call* $h = E_{1,\chi_K}$ *and let us choose the test vector:*

$$(\breve{\gamma}, \breve{f}, \breve{h}) \in M_1(Np, \chi)_L^\vee[g_\alpha] \times S_2(Np, \mathbb{1})_L[f] \times M_1(Np, \chi)_L[h]$$

as follows: $\breve{f}$ *and* $\breve{g}$ *are normalized eigenforms for all the Hecke operators,* $\breve{\gamma}$ *is the dual of* $\breve{g}$ *and* $\breve{h} = E_{1,\chi_N}$ *as defined in equation* (1.10). *Then there exists an explicit non-zero complex number* $\lambda(\breve{f}, \breve{g}) \in \mathbb{Q}_\psi(\breve{f})$ *for which*

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = \lambda(\breve{f}, \breve{g}) \cdot \frac{\log_{E,p}(P_{\bar{\psi}})^2}{\log_p(u_{\bar{\psi}^2})}.$$

In particular, in the case $N_E = D$ *the following formula holds true for* $\breve{f} = f$, $\breve{g} = g$*:*

$$\lambda(f, g) = \frac{(p - a_p(f)\psi(\bar{\wp}) + \psi^2(\bar{\wp}))^2}{p(p-1)} \cdot \frac{\lambda_0 \psi(\sqrt{-D})}{h_K g_K},$$

$$\lambda_0 = \begin{cases} 1 & \text{if } \psi^2 = 1, \text{ that is to say, if } g \text{ is Eisenstein} \\ \frac{12(p-1)}{p-(p+1)\psi^{-2}(\bar{\wp})+\psi^{-4}(\bar{\wp})} & \text{if } \psi^2 \neq 1, \text{ that is to say, if } g \text{ is cuspidal.} \end{cases}$$

Note that in this result we do not assume that N is square-free. In fact, the choice of h forces N_g/D to be a non-trivial square, unless $N_g = D$. In this sense the use of Hida-Rankin's method, although restrictive for the choice of h, also proves new cases of the elliptic Stark conjecture in a more explicit way than it is done by Garrett's method, answering both questions (i) and (ii).

The above theorem, together with theorem 0.4, furnishes good theoretical evidence for the elliptic Stark conjecture, but it is far from proving it in general. This leaves spaces for the examination of other cases. The first and natural idea is to see what happens in the case of bad reduction for E at p. For this reason, we decided to prove the case of multiplicative reduction for E, i.e. when $p||N_E$. Define $N_f = N_E/p$, then we have the following:

Theorem 0.7. *The results of theorem 0.4 and 0.6 hold still if we assume that* $p||N_E$, $p \nmid N_g N_h$. *The explicit formula of theorem 0.6 is true replacing* N_E *by* N_f.

Although this result is very similar to the previous theorems, we need to substitute the Bertolini-Darmon-Prasanna p-adic L-function with a two variables extension due to Castella in order to overcome some technical problem.

Let us review briefly the contents of this thesis.

In the first chapter we introduce some background material about modular forms, p-adic modular forms and various operators acting on these spaces. We introduce Coleman's classicality result: overconvergent modular forms of small slope are classical. We also discuss complex multiplication points and elliptic units in a way that fulfills our needs.

The second chapter is dedicated to the basics of complex L-functions. We introduce the L-function associated with a compatible system of Galois representations, then we treat several special L-functions of this kind. In particular, we introduce the notation and some results which will be used in the subsequent chapters.

Chapter three is the central part: it contains background material on all the p-adic L-functions that we need. In particular, for each of them we introduce the region of interpolation, the relation with the complex L-function and the results on special values. Most results discussed in this section are already present in the literature but we slightly generalize a few of them and we adapt the notation for our treatment.

The fourth chapter contains the proof of theorems 0.6 and 0.7 that are discussed in the introduction. We start by explaining the elliptic Stark conjecture in the cases we treat and we list the hypotheses for the comfort of the reader. We also discuss in more detail hypothesis C and C' which are needed to formulate the result. These results are the content of the articles [CR1] and [CR2].

Chapter 1

Background material

1.1 Modular forms for $\Gamma_1(N)$

Consider the Poincaré upper half plane:

$$\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

and the space of real analytic functions $\mathcal{C}^\infty(\mathcal{H}, \mathbb{C})$. For any integer $k \in \mathbb{Z}$ we define the *weight-k-action* of $\mathrm{GL}_2(\mathbb{Q})^+$ on $f : \mathcal{H} \to \mathbb{C}$ as follows:

$$f|_k\gamma(z) := \det(\gamma)^{k-1}(cz+d)^{-k}f(\gamma z). \tag{1.1}$$

We say that f is:

(i) *slowly increasing* (or that f has *moderate growth*) at infinity if for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there exist positive numbers c, e such that $|f|_k\gamma| \leq c(1 + y^{-e})$ as $y \to \infty$.

(ii) *rapidly decreasing* (or that f has *rapid decay*) at infinity if for any $e \in \mathbb{R}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there exists a positive c such that $|f|_k\gamma| \leq c(1 + y^e)$ as $y \to \infty$;

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup. The set of *real analytic modular forms* $M_k^{\mathrm{an}}(\Gamma)$ (resp. *cusp forms* $S_k^{\mathrm{an}}(\Gamma)$) of *weight* k and *level* Γ is the subset of real analytic functions $f \in \mathcal{C}^\infty(\mathcal{H}, \mathbb{C})$ having moderate growth (resp. rapid decay) at infinity such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

For a couple $(f, g) \in M_k^{\mathrm{an}}(\Gamma) \times S_k^{\mathrm{an}}(\Gamma)$, the product $f(z)\overline{g(z)}y^k$ is both Γ-invariant and rapidly decreasing at infinity. Since $y^{-2}dxdy$ is also Γ-invariant, we can define the *Petersson scalar product* on the space $S_k^{\mathrm{an}}(\Gamma) \times M_k^{\mathrm{an}}(\Gamma)$ as follows:

$$\langle f, g\rangle_N := \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}y^{k-2}dxdy. \tag{1.2}$$

Let $N \geq 1$ be an integer and consider the standard congruence subgroups:

$$\Gamma_1(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a - 1 \equiv c \equiv d - 1 \equiv 0 \pmod N\right\},$$
$$\Gamma_0(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N\right\}.$$

We are mainly interested in modular forms for $\Gamma = \Gamma_1(N)$, hence we will write $M_k^{\mathrm{an}}(N) := M_k^{\mathrm{an}}(\Gamma_1(N))$, $S_k^{\mathrm{an}}(N) := S_k^{\mathrm{an}}(\Gamma_1(N))$.

Definition 1.1. The set of modular forms $M_k(N)$ (resp. cusp forms $S_k(N)$) of weight k and level N is the finite dimensional sub-vector space consisting of the holomorphic functions in $M_k^{\mathrm{an}}(N)$ (resp. $S_k^{\mathrm{an}}(N)$).

Let $f \in M_k(N)$. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1(N),$$

we have $f(z+1) = f(z)$ and we can consider the Fourier expansion of f at infinity, which is given by:

$$f(q) = \sum_{n \geq 0} a_n(f) q^n, \qquad q = e^{2\pi i z},$$

where $a_n(f) \in \mathbb{C}$, because a modular form has moderate growth at infinity. In particular:

- $f \in S_k(N)$ if and only if $a_0(f|_k\gamma) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (in particular, $a_0(f) = 0$);
- we say that $f \in M_k(N)$ is *normalized* if $a_1(f) = 1$;
- we define the modular forms with coefficients in some subring A of $\mathbb{C}$, as the set of modular forms whose q-expansion has coefficients in A, and we write $M_k(N)_A$ and $S_k(N)_A$ for those sets;
- we can define an action of $\mathrm{Aut}(\mathbb{C})$ on the q-expansion.

Since a modular form is determined by its q-expansion, we will often define a modular form and work with it using its Fourier coefficients $a_n(f)$. This will be very important in the rest of the dissertation.

On the space of real analytic modular forms we define the *Shimura-Maass derivative operator* as

$$\delta_k := \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{k}{2iy}\right) : M_k^{\mathrm{an}}(N) \longrightarrow M_{k+2}^{\mathrm{an}}(N) \tag{1.3}$$

and we also consider its iterate $\delta_k^t := \delta_{k+2t} \cdot \cdots \cdot \delta_k : M_k^{\mathrm{an}}(N) \longrightarrow M_{k+2t}^{\mathrm{an}}(N)$ (impose $\delta_k^0 := \mathbb{1}$). The Shimura-Maass operator does not respect the spaces of modular forms: given $f \in M_k(N)$, $\delta_k f$ does not belong to $M_k(N)$ in general, although it is still a real analytic modular form. For an integer $0 \leq t \leq k/2$ we define the spaces of nearly overconvergent modular forms and cuspforms as follows:

$$M_k^{\mathrm{nh}}(N) := \bigoplus_{j=0}^{t} \delta_{k-2j}^j M_{k-2j}(N), \qquad S_k^{\mathrm{nh}}(N) := \bigoplus_{j=0}^{t} \delta_{k-2j}^j S_{k-2j}(N).$$

This is not the most general definition, but it will suit our needs. In particular, we can define the *holomorphic projection* as the map:

$$\Pi^{\mathrm{hol}} : M_k^{\mathrm{nh}}(N) \to M_k(N)$$

induced by the above decomposition. It clearly respects the subspace of cuspforms and, as shown in [Hid93, §10.1], it is both $\mathrm{SL}_2(\mathbb{Z})$-equivariant under the weight-k-action and $\mathrm{Aut}(\mathbb{C})$-equivariant for the action defined on q-expansions.

Proposition 1.2. *Suppose that $f \in S_k(N)$ and $g \in M_k^{\mathrm{nh}}(N)$. Then:*

$$\langle f, g \rangle_N = \langle f, \Pi^{\mathrm{hol}}(g) \rangle_N$$

Proof. See [Hid93, Theorem 10.2]. □

1.2 Nebentype decomposition and Hecke operators

We have a short exact sequence:

$$1 \to \Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$$

which defines the action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $M_k^{\mathrm{an}}(N)$ via the weight-k-action. For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ we call this operator the *Diamond operator* and we denote it by $\langle d \rangle$. This operator induces a decomposition of the various spaces of modular forms in χ-component, for every Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. If denote by $M_k^{\mathrm{an}}(N,\chi)$ the set of elements of $M_k^{\mathrm{an}}(N)$ on which $\langle d \rangle$ acts as $\chi(d)$, then we can write:

$$M_k^{\mathrm{an}}(N) = \bigoplus_{\chi} M_k^{\mathrm{an}}(N,\chi)$$

When $\chi = \mathbb{1}$ we recover $M_k^{\mathrm{an}}(\Gamma_0(N))$. If $f \in M_k^{\mathrm{an}}(N,\chi)$, we say that f has *nebentype character* χ. The nebentype decomposition respects:

- the subspace of real analytic cusp forms;
- the subspaces of modular forms and cusp forms;
- the subspaces of nearly holomorphic modular forms;

We can summarize the situation in the following diagram of inclusions:

$$\begin{array}{ccccc} S_k(N,\chi) & \subset & S_k^{\mathrm{nh}}(N,\chi) & \subset & S_K^{\mathrm{an}}(N,\chi) \\ \cap & & \cap & & \cap \\ M_k(N,\chi) & \subset & M_k^{\mathrm{nh}}(N,\chi) & \subset & M_K^{\mathrm{an}}(N,\chi). \end{array} \tag{1.4}$$

Given a modular form $f \in M_k(N,\chi)_A$ we can consider the modular form f^* obtained by applying the complex conjugation to the coefficients, i.e.:

$$f^* = \sum_{n \geq 0} \overline{a_n(f)} q^n.$$

We observe that $f^* \in M_k(N,\overline{\chi})_A$.

If we take $(f,g) \in M_k^{\mathrm{an}}(N,\chi) \times S_k^{\mathrm{an}}(N,\chi)$, then $f(z)\overline{g(z)}y^k$ and $y^{-2}dxdy$ are not only $\Gamma_1(N)$-invariant but also $\Gamma_0(N)$-invariant. For this reason we consider the following normalization of the Petersson scalar product:

$$\langle f,g \rangle_N := \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^{k-2}dxdy. \tag{1.5}$$

Notice that if $N \mid M$, then $\langle f,g \rangle_M = [\Gamma_0(N) : \Gamma_0(M)] \cdot \langle f,g, \rangle_N$. Define $\Im(N) := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, then:

$$\langle f,g \rangle_M = \frac{\Im(M)}{\Im(N)} \langle f,g \rangle_N. \tag{1.6}$$

We now present a short exposition of the Hecke operators and we refer to [DS05, §5] and [Hid93, §5.3] for more details. Given a prime number p we consider the double coset operator:

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \left\{ \gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \gamma' \mid \gamma,\gamma' \in \Gamma_1(N) \right\}.$$

This double coset can be written as a disjoint union:

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \sum_{i=0}^{p-1} \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ i & p \end{pmatrix} \cup \Gamma_1(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{if } p \nmid N$$

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \sum_{i=0}^{p-1} \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ i & p \end{pmatrix}, \qquad \text{if } p \mid N.$$

This decomposition allows us to define the Hecke operators as the double coset operator acting on the space of modular forms via the weight k action defined in equation 1.1. We denote them by T_p when $p \nmid N$ and U_p when $p \mid N$ and we call them, respectively, *good* and *bad* Hecke operators. For $f \in M_k(N, \chi)$ we have:

$$a_n(T_p(f)) = a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f).$$

where $a_{n/p}(f) = 0$ whenever $p \nmid n$. In particular, if f is normalized we have:

$$a_1(T_p(f)) = a_p(f)$$

The Hecke operators commute with each other and with the Diamond operator. This allows us to define operators T_n for all n which satisfy $a_1(T_n(f)) = a_n(f)$.

For a vector space V on which T_n are acting, we write $h(V)$ for the subset of $\mathrm{End}(V)$ generated by the Hecke operators and $h(V)_N$ for the subset generated by good Hecke operators. We use the notation $\mathfrak{h}_k(A) := h(S_k(N)_A)$ and $\mathfrak{h}_k(A)_N$, since this does not create ambiguities on the level. We have the following:

Theorem 1.3. *For every subring $A \subset \mathbb{C}$ we have a perfect pairing:*

$$\begin{aligned} \mathfrak{h}_k(A) \times S_k(N)_A &\to A \\ (\ T\ ,\ \ f\ ) &\mapsto a_1(T(f)) \end{aligned}$$

Proof. See [Hid93, §5.3, Theorem 1]. □

We say that $f \in M_k(N, \chi)$ is a *normalized eigenform* if it is normalized ($a_1(f) = 1$) and it is an eigenform for all the Hecke operators, i.e. for $\mathfrak{h}_k$ (or, if specified, for all good Hecke operators, i.e. for $\mathfrak{h}_{k,N}$). Normalized eigenforms have algebraic integer coefficients and we can define:

$$\mathbb{Q}(f) := \mathbb{Q}(a_n(f) \mid n \geq 0).$$

This field is called the *field of rationality* of f. It is known to be a number field. Using theorem 1.3 we see that every normalized eigenform f arises as a morphism of A-algebras $\lambda_f : \mathfrak{h}_k(A) \to A$, hence it defines a prime ideal $I_f \subset \mathfrak{h}_{k,N}(A)$ as follows:

$$I_f := \ker(\lambda_f) = \langle T_n - a_n(f) \mid \forall n \in \mathbb{Z}_{\geq 1}, (n, N) = 1\rangle$$

If $f \in M_k(N, \chi)$ is a normalized eigenform for all good Hecke operators, we define the f-isotypical component of $M_k(N, \chi)$ as:

$$M_k(N, \chi)[f] := \{g \in M_k(N, \chi) \mid T_p g = a_p(f)g, \forall p \nmid N\}.$$

Then $M_k(N, \chi)[f] = \ker I_f$ and it is the set of eigenforms $g \in M_k(N, \chi)$ which have the same system of eigenvalues as f for all good Hecke operators.

1.3 Oldforms, newforms and basis

For each integer $d \mid N$ and each modular form $f \in S_k(N/d)$ we can consider the map:

$$f(q) \mapsto f(q^d) \in S_k(N)$$

This gives rise to a dichotomy: in $S_k(N)$ some modular forms are linear combination of modular forms of lower level and some others are not. To be more precise, we consider the map:

$$\iota_d : S_k(N/d)^2 \to S_k(N)$$

given by $(f,g) \mapsto f(q)+g(q^d)$. Then we define:

$$S_k(N)^{\text{old}} := \sum_{p|N} \iota_p(S_k(N/p)^2)$$

and we denote by $S_k(N)^{\text{new}}$ the orthogonal complement of $S_k(N)^{\text{old}}$ with respect to the Petersson scalar product. Since those spaces are stable under the action of the Hecke operators we have that:

Theorem 1.4. *The spaces $S_k(N)^{\text{old}}$ admits an orthogonal basis of eigenforms for $\mathfrak{h}_{k,N}$. The space $S_k(N)^{\text{new}}$ admits an orthogonal basis of eigenforms for $\mathfrak{h}_k$.*

In particular we can fix a basis $\{f\}$ of normalized eigenforms for the set $S_k(N,\chi)^{\text{new}}$ and we have the following decomposition:

$$S_k(N,\chi)^{\text{new}} = \bigoplus_{f\in\{f\}} S_k(N,\chi)[f],$$

which is a decomposition in one-dimensional spaces, i.e. $S_k(N,\chi)[f] = \mathbb{C}\cdot f$.

For the space of oldforms we have a different situation. In fact if f is new in level $M \mid N$ we have that the space $S_k(N,\chi)[f]$ admits a basis of normalized eigenforms. In order to describe it, we consider the Hecke polynomial of f at a prime p defined as follows:

$$T^2 - a_p(f)T + \chi(p)p^{k-1}.$$

This polynomial admits two roots that we will call $\alpha_p(f)$ and $\beta_p(f)$. It is easy to check that

$$f(q) - \beta_p(f)f(q^p) \in M_k(Mp,\chi)$$

is a normalized eigenform which admits $\alpha_p(f)$ as eigenvalue for U_p. Consider a choice $\mu := \{\mu_p(f)\}_{p|N/M} \in \{\alpha_p(f),\beta_p(f)\}_{p|N/M}$. Then a basis of $S_k(N,\chi)[f]$ is determined by the set of elements of the form

$$f_\mu(q) := f(q) - \sum_{p|N/M} \mu_p(f)\cdot f(q^p), \tag{1.7}$$

for all possible choices of μ. In particular, the space $M_k(N,\chi)[f]$ has dimension $2^{\#\{p|N/M\}}$. Given an element $g \in M_k(N,\chi)[f]$, we say that f is the associated *primitive newform* and that M is the *conductor* of g.

1.4 Eisenstein series

Let $\chi : (\mathbb{Z}/N_\chi\mathbb{Z})^\times \to \mathbb{C}$ be a Dirichlet character of conductor N_χ and let $\mathbb{Q}_\chi$ denote the finite extension of $\mathbb{Q}$ generated by the values of χ. Let

$$\tau(\chi) = \sum_{a=1}^{N_\chi} \chi(a) e^{\frac{2\pi i a}{N_\chi}}$$

be the Gauss sum associated with the Dirichlet character χ.

Consider an integer k such that $\chi(-1) = (-1)^k$. If either $k > 2$ or $k \geq 1$ and χ is non-trivial, then we define the holomorphic Eisenstein series $\tilde{E}_{k,\chi}$ of weight k and nebentype character χ to be the modular form defined by:

$$\tilde{E}_{k,\chi}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{\chi^{-1}(n)}{(mNz+n)^k} \in M_k(N_\chi,\chi).$$

It admits a Fourier expansion, but in this form it is not normalized ($a_1 \neq 1$). We define the normalized holomorphic Eisenstein series $E_{k,\chi}(z)$ of weight k and nebentype character χ as:

$$E_{k,\chi}(z) = \frac{N_\chi^k(k-1)!}{2(-2\pi i)^k\tau(\chi^{-1})} \cdot \tilde{E}_{k,\chi}(z). \tag{1.8}$$

Proposition 1.5. *Let $\sigma_{k-1,\chi}$ denote the function on the positive integers defined by $\sigma_{k-1,\chi}(n) := \sum_{d|n}\chi(d)d^{k-1}$. Then $E_{k,\chi}$ is a newform of level N_χ and its q-expansion is*

$$E_{k,\chi}(z) := \frac{L(\chi, 1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1,\chi}(n)q^n \in M_k(N_\chi, \chi)$$

Proof. It is a classical result, see for instance [Hid93, Prop. 5.1], [Shi76, (3.4)], or [Mia76] □

Consider a multiple N of N_χ and let $\chi_{_N}$ denote the character mod N induced by χ. For every $k \geq 1$ such that $\chi(-1) = (-1)^k$, we define the *non-holomorphic Eisenstein series* of weight k and level N attached to the character $\chi_{_N}$ as the function on $\mathcal{H} \times \mathbb{C}$ given by the rule

$$\tilde{E}_{k,\chi_{_N}}(z,s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{{\chi_{_N}}^{-1}(n)}{(mNz+n)^k} \cdot \frac{y^s}{|mNz+n|^{2s}}. \tag{1.9}$$

This series only converges for $\Re(s) > 1 - k/2$, but we have the following result:

Theorem 1.6. *The Eisenstein series $\tilde{E}_{k,\chi_N}(z,s)$ admits a continuation to a meromorphic function of the variable s and satisfies a functional equation relating values at s and at $1-k-s$.*

Proof. See [Hid93, §9.3, Theorem 1] or [Mia76, §7.2]. □

For any fixed s in the region of convergence, $\tilde{E}_{k,\chi_{_N}}(z,s)$ is a real analytic modular form. For $k > 2$, or $k \geq 1$ but $\chi \neq 1$, the series arising by setting $s = 0$ is holomorphic in z and gives rise to a modular form

$$\tilde{E}_{k,\chi_{_N}}(z) := \tilde{E}_{k,\chi_{_N}}(z,0) \in M_k(N,\chi).$$

In a way similar to that of equation (1.8) we can define the normalization of $\tilde{E}_{k,\chi_{_N}}$ as follows:

$$E_{k,\chi_{_N}}(z) := \frac{N^k(k-1)!}{2(2\pi i)^k\tau(\chi^{-1})} \cdot \tilde{E}_{k,\chi_{_N}}(s) \tag{1.10}$$

Proposition 1.7. *The Eisenstein series $E_{k,\chi_{_N}}(z)$ and $E_{k,\chi}(z)$ have the same eigenvalues for all the good Hecke operators T_n, $n \nmid N$.*

Proof. It can be easily seen from [Shi76, (3.3)] that $E_{k,\chi_{_N}}(z)$ is a linear combination of $E_{k,\chi}(dz)$ for $d \mid N/N_\chi$, hence the result follows. This can also be seen by specializing the formula of [Mia76, Theorem 7.1.3] in our case. Notice that with respect to the latter reference, the normalization we adopt here is non-canonical. In fact $a_1(E_{k,\chi_N}) \neq 1$ in general, but it always lies in $\mathbb{Q}_\chi$. □

We can explicitly describe the action of the Shimura-Maass derivative operator on the Eisenstein series with the following:

Proposition 1.8. *For $t < k/2$ we have that $\tilde{E}_{k,\chi_{_N}}(z,-t) \in M_k^{\mathrm{nh}}(N,\chi)$. In particular:*

$$\tilde{E}_{k,\chi_{_N}}(z,-t) = \frac{(k-2t-1)!}{(k-t-1)!}(-4\pi)^t\delta_{k-2t}^t\tilde{E}_{k-2t,\chi_{_N}}(z). \tag{1.11}$$

Proof. For any value of s, the series $\tilde{E}_{k,\chi_{_N}}(z,s)$ belongs to $M_k^{\mathrm{an}}(N,\chi)$ and one verifies that:

$$\delta_k\tilde{E}_{k,\chi_{_N}}(z,s) = -\frac{s+k}{4\pi}\tilde{E}_{k+2,\chi_{_N}}(z,s-1).$$

Continue by induction on t and put $s = 0$ to get the result. □

1.5 Algebraic modular forms

For this section we follow the approach of [Kat73] as explained in [BDP13]. Let R be a ring in which N is invertible and let A be an elliptic curve over R.

We say that a couple (A, t) is an *elliptic curve with* $\Gamma_1(N)$*-level structure* if t is a closed immersion of the scheme $\mathbb{Z}/N\mathbb{Z}$ in A defined over R. This immersion gives rise to a section $s : \mathrm{Spec}(R) \to A$ of order N which is the image of the section 1 of $\mathbb{Z}/N\mathbb{Z}$. We write $\mathcal{E}(N)$ for the set of isomorphism classes of elliptic curves with $\Gamma_1(N)$-level structure, where we consider isomorphisms of elliptic curves preserving the $\Gamma_1(N)$-level structure.

We say that a triple $(A/R, t, \omega)$ is a *marked elliptic curve* if (A, t) is an elliptic curve with $\Gamma_1(N)$-level structure and ω is a global section of the sheaf or relative differentials $\Omega^1_{A/R}$ over R. We write $\widetilde{\mathcal{E}}(N)$ for the set of isomorphism classes of marked elliptic curves, where the isomorphism is given by the pullback at the level of differentials.

Define the *Tate elliptic curve* $\mathrm{Tate}(q) := \mathbb{G}_m/q^{\mathbb{Z}}$ over $\mathbb{Z}[[q]]$ equipped with some level structure t defined over $\mathbb{Q}((q^{1/d}))$, for some $d \mid N$, and the canonical differential $\omega_{\mathrm{can}} := du/u$. Then the triple $(\mathrm{Tate}(q), t, \omega_{\mathrm{can}})$ is defined over $\mathbb{Q}((q^{1/d}))$.

Definition 1.9. An *algebraic modular form* of weight k and level N defined over a field F is a rule which associates to every marked elliptic curve $(A, t, \omega) \in \widetilde{\mathcal{E}}(N)_R$ defined over an F-algebra R an element $f(A, t, \omega) \in R$ such that:

(1) (base change compatibility) For all F-algebra homomorphisms $\phi : R \to R'$, $f((A, t, \omega) \otimes R') = \phi(f(A, t, \omega))$;

(2) (weight k condition) for all $\lambda \in R^\times$, $f(A, t, \lambda\omega) = \lambda^{-k} f(A, t, \omega)$;

(3) (q-expansion) $f(\mathrm{Tate}(q), t, \omega_{\mathrm{can}}) \in F[[q^{1/d}]]$.

We say that f is a cusp form if $f(\mathrm{Tate}(q), t, \omega_{\mathrm{can}}) \in q^{1/d} F[[q^{1/d}]]$.

Consider now the open modular curve $Y_1(N)$ classifying the elliptic curves with $\Gamma_1(N)$-level structure and its compactification $X_1(N)$ obtained by adding the cusps. For $N \geq 3$, $\Gamma_1(N)$ is torsion free, hence $Y_1(N)$ is a fine moduli scheme admitting a smooth proper model over $\mathbb{Z}[1/N]$, representing the functor $R \mapsto \mathcal{E}(N)_R$, whose set of $\mathbb{C}$-points can be identified with the quotient $\mathcal{H}/\Gamma_1(N)$. An algebraic modular form defines a modular form in the classical sense by mean of the following rule: for each $\tau \in \mathcal{H}$,

$$f(\tau) := f(\mathbb{C}/\langle 1, \tau\rangle, 1/N, 2\pi i dz),$$

where z is the standard coordinate of $\mathcal{H}$.

Let $\pi : \mathcal{E} \to Y_1(N)$ be the universal elliptic curve with level N structure over $Y_1(N)$ and $\underline{\omega} := \pi_* \Omega_{\mathcal{E}/Y_1(N)}$. An algebraic modular form f gives rise to an element of $H^0(Y_1(N), \underline{\omega}^k)$ via the following map:

$$\omega_f(A, t) := f(A, t, \omega)\omega^k,$$

where $(A, t) \in Y_1(N)$ and ω is any generator for $\Omega^1_{E/R}$. Assuming that $N \geq 5$, the cusps of $X_1(N)$ are regular in the sense of [DS05, §3.2], hence the line bundle $\underline{\omega}$ admits an extension to $X_0(N)$ characterized by the property that $H^0(X_0(N)_F, \underline{\omega}^k) = M_k(N)_F$.

1.6 Overconvergent modular forms and Coleman classicality result

Fix a prime number $p \nmid N$ and continue writing $X_1(N)$ for the rigid analytic space associated with the curve $X_1(N)_{\mathbb{Q}_p}$. Take the Eisenstein series E_{p-1} as a lift of the Hasse invariant and consider the *ordinary* *locus*:

$$X_1(N)^{\mathrm{ord}} := \{x \in X_1(N)(\mathbb{C}_p) \mid \mathrm{ord}_p E_{p-1}(x) = 0\}$$

and also, for $\epsilon > 0$:

$$X_1(N)^{<\epsilon} := \{x \in X_1(N)(\mathbb{C}_p) \mid \mathrm{ord}_p E_{p-1}(x) < \epsilon\}.$$

Then for any complete subfield $K \subset \mathbb{C}_p$ we can define:

$$M_k^{(p)}(N)_K := H^0(X_1(N)_K^{\mathrm{ord}}, \underline{\omega}^k), \qquad M_k^{\mathrm{oc}}(N)_K := \varinjlim_{\epsilon>0} H^0(X_1(N)_K^{<\epsilon}, \underline{\omega}^k).$$

In particular, $M_k^{\mathrm{oc}}(N)_K \subset M_k^{(p)}(N)_K$. On these spaces we have an action of the Hecke operators and Diamond operators and we can consider the bad Hecke operator $U = U_p$ and the V operator, acting on q-expansions by the rules:

$$Uf(q) := \sum_{n\geq 0} a_{pn} q^n, \qquad Vf(q) := \sum_{n\geq 0} a_n q^{pn}.$$

We also define the *Serre derivative operator* as follows:

$$d := q\frac{d}{dq} : M_k^{\mathrm{oc}}(N)_K \to M_{k+2}^{\mathrm{oc}}(N)_K.$$

Finally, we define the p-depletion of f:

$$f^{[p]} := (1 - VU)f(q) = \sum_{p\nmid n} a_n q^n.$$

that we can see as the derivative of the modular form F defined by:

$$F(q) := \sum_{p\nmid n} \frac{a_n(f)}{n} q^n,$$

more precisely:

$$dF = f^{[p]}.$$

We call such a F the *overconvergent primitive* of f, it can be defined as $d^{-1}f := \lim_{t\to -1} d^t f$.

The operator U is completely continuous on $M_k^{\mathrm{oc}}(N)_K$ and it induces a slope decomposition. If f is an eigenform, $U(f) = a_p(f) \cdot f$ and its *slope* is the p-order of its eigenvalue, i.e. $\mathrm{ord}_p(a_p(f))$. We say that *f has slope j* if $\mathrm{ord}_p(a_p(f)) = j$ and that f is *ordinary* if $\mathrm{ord}_p(a_p(f)) = 0$, i.e. if $a_p(f)$ is a p-adic unit. Hida's ordinary projector is defined as the following operator:

$$e_{\mathrm{ord}} := \lim_{n\to\infty} U^{n!}. \tag{1.12}$$

This defines a Hecke equivariant projection from $M_k^{\mathrm{oc}}(N)_K$ to its ordinary subspace

$$M_k^{\mathrm{oc,ord}}(N)_K := e_{\mathrm{ord}} M_k^{\mathrm{oc}}(N)_K.$$

Consider now the roots α and β of the Hecke polynomial of f at p. We can order them in such a way that $\mathrm{ord}_p(\alpha) \leq \mathrm{ord}_p(\beta)$. We can define two *$p$-stabilizations* of f as follows:

$$f_\alpha(q) := f(q) - \beta f(q^p) \quad \text{and} \quad f_\beta(q) := f(q) - \alpha f(q^p),$$

in particular we have that $Uf_\alpha = \alpha f_\alpha$ and $Uf_\beta = \beta f_\beta$. If f is ordinary, then $\mathrm{ord}_p(\alpha) = 0$ and we call f_α the *ordinary p-stabilization of f*. For an ordinary modular form we always have an ordinary p-stabilization. In weight $k = 1$ we might have 2 distinct ordinary p-stabilizations.

Theorem 1.10. *For every $k \geq 2$, all overconvergent modular forms of slope strictly less than $k-1$ are classical. In particular, for every $f \in M_k^{\mathrm{oc,ord}}(N)$ there exists a modular form $g \in M_k(N)$ such that f is the ordinary p-stabilization of g.*

Proof. See [Col95]. □

From this theorem we can identify $M_k^{\mathrm{oc,ord}}(N)$ with the space $M_k(\Gamma_1(N) \cap \Gamma_0(p))$. For a Dirichlet character χ modulo N, the above results can be summarized in the following diagram:

$$\begin{array}{ccccc} S_k(Np,\chi)_{\mathbb{C}_p} & \subset & S_k^{\mathrm{oc}}(N,\chi) & \subset & S_k^{(p)}(N,\chi) \\ \cap & & \cap & & \cap \\ M_k(Np,\chi)_{\mathbb{C}_p} & \subset & M_k^{\mathrm{oc}}(N,\chi) & \subset & M_k^{(p)}(N,\chi). \end{array} \tag{1.13}$$

1.7 CM points and elliptic units

Let $K = \mathbb{Q}(\sqrt{-D})$ be a quadratic imaginary field with ring of integers $\mathcal{O}_K$. For every $c \in \mathbb{Z}_{\geq 1}$ we denote the unique order of conductor c in K by:

$$\mathcal{O}_c := \mathbb{Z} + c\mathcal{O}_K.$$

In particular, $\mathcal{O}_1 = \mathcal{O}_K$. We say that an elliptic curve $A/\mathbb{C}$ has complex multiplication by $\mathcal{O}_c$ if $\mathrm{End}(A) = \mathcal{O}_c$. Consider the Picard group $\mathrm{Pic}(\mathcal{O}_c)$ of rank one projective $\mathcal{O}_c$-modules up to isomorphisms, then to every element $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$ we can associate the elliptic curve $\mathbb{C}/\mathfrak{a}$, which has complex multiplication by $\mathcal{O}_c$. Let us write $\mathcal{E}_{\mathrm{CM}}(\mathcal{O}_c)$ for the set of elliptic curves having CM by $\mathcal{O}_c$ up to isomorphism and let:

$$\mathcal{E}_{\mathrm{CM}} := \cup_{c \geq 1} \mathcal{E}_{\mathrm{CM}}(\mathcal{O}_c).$$

Every elliptic curve with complex multiplication by $\mathcal{O}_c$ can be seen as a point $j(\mathfrak{a}) := j(A_{\mathfrak{a}}) \in X(1)$ via the j-invariant, hence $\mathcal{E}_{\mathrm{CM}} \subset X(1)$. We may write $X(1)_{\mathrm{CM}}$ for the set of CM points on $X(1)$ and, similarly, $X_0(N)_{\mathrm{CM}}$ and $X_1(N)_{\mathrm{CM}}$ for the fiber of $X(1)_{\mathrm{CM}}$ via the obvious projection maps.

We have an action of $\mathrm{Pic}(\mathcal{O}_c)$ on the set $\mathcal{E}_{\mathrm{CM}}(\mathcal{O}_c)$ defined as follows: $\mathfrak{a} * A = A/A[\mathfrak{a}]$, for $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$. Since the action is simply transitive, if we fix A to be the elliptic curve with complex multiplication:

$$A := \mathbb{C}/\mathcal{O}_c, \tag{1.14}$$

we can obtain all the other ones via the action of the Picard group, hence we can define $A_{\mathfrak{a}} = \mathfrak{a} * A = \mathbb{C}/\mathfrak{a}^{-1}$. If H is the ring class field of K of conductor c we know that $\mathrm{Pic}(\mathcal{O}_c) \simeq \mathrm{Gal}(H|K)$ via the reciprocity map of global class field theory, arithmetically normalized, i.e. a prime ideal $\wp$ maps to the Frobenius element $\sigma_{\wp}$. The Shimura reciprocity law tells us that:

$$j(A_{\mathfrak{a}}) = \sigma_{\mathfrak{a}} j(A).$$

Assumption 1.11. There exists a cyclic ideal $\mathfrak{N}$ of $\mathcal{O}_K$ of norm N i.e. such that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$. We also assume that N is coprime to c

This is usually called *Heegner hypothesis*. The Shimura reciprocity law and the Heegner hypothesis ensure that A is defined over H, hence $j(A_{\mathfrak{a}}) \in X(1)(H)$ and that the fibers in $X_1(N)$ and $X_0(N)$ are defined over H, too.

CM points on $X_1(N)$: let us consider the map $X_1(N) \to X(1)$. We can consider the fiber of $j(\mathfrak{a})$ in $X_1(N)$. It is the set of couples $x(\mathfrak{a}) = (A_{\mathfrak{a}}, t_{\mathfrak{a}})$ where $t_{\mathfrak{a}}$ is a $\Gamma_1(N)$-level structure induced by the choice of a section $t_{\mathfrak{a}} : \mathrm{Spec}(H) \to A[\mathfrak{N}]$.

Remark 1.12. The choice of the $\Gamma_1(N)$-structure $t_{\mathfrak{a}}$ is not unique, then the map $\mathfrak{a} \mapsto x(\mathfrak{a})$ is not well defined since we might have several choices for the level structure. For this reason we introduce here a choice that will be useful in what follows. Consider $x := x(\mathcal{O}_c) = (A, t)$ where t is a choice of $\Gamma_1(N)$-structure. Then we have a natural map $A \mapsto A_{\mathfrak{a}}$ which induces a $\Gamma_1(N)$-structure on $A_{\mathfrak{a}}$: this determines a point $x(\mathfrak{a}) \in X_1(N)$ uniquely. The set of points:

$$x(\mathfrak{a}) \in X_1(N)(H)$$

defined in this way is in a single orbit for the action of $\mathrm{Pic}(\mathcal{O}_c)$.

CM points and marked elliptic curves: it is useful to point out another choice that will be considered later on. Starting from a CM point $x = x(\mathcal{O}_c) \in X_1(N)$ we can define a marked elliptic curve $\tilde{x}(\mathcal{O}_c) := (A, t, \omega_A)$ by fixing a nonzero $\omega_A \in \Omega^1_{A/H}$. Once such a choice is made, we can define in a coherent way $\tilde{x}(\mathfrak{a}) := (A_{\mathfrak{a}}, t_{\mathfrak{a}}, \omega_{\mathfrak{a}})$ such that $(A_{\mathfrak{a}}, t_{\mathfrak{a}}) = x(\mathfrak{a})$ and $\omega_{\mathfrak{a}}$ is chosen such that ω is its pullback via the map $A \to A_{\mathfrak{a}}$, so that $\tilde{x}(\mathfrak{a})$ are in a single orbit for the action of $\mathrm{Pic}(\mathcal{O}_c)$.

CM points on $X_0(N)$: In a similar way to that of $X_1(N)$, we can consider $X_0(N) \to X(1)$ and the fiber of $j(\mathfrak{a})$ in $X_0(N)$. It is a collection of points $x(\mathfrak{a}) \in X_0(N)(H)$ consisting in couples $(A_{\mathfrak{a}}, A_{\mathfrak{a}}[\mathfrak{N}])$. All of possible CM points of conductor c on $X_0(N)$ are obtained from x with the action of Galois group $\mathrm{Gal}\,(H|K)$ and the Atkin-Lehner involutions as explained for instance in [Gro84]. Such a point is often called in the literature a *Heegner point*. We use here the same notation as for $X_1(N)$ since in what follows it will not create ambiguities.

Elliptic units: As explained in [DD06, §1], we fix a choice of a modular unit $U \in \mathcal{O}^{\times}_{Y_0(N)}$, that is, a holomorphic and nowhere vanishing function on $Y_0(N)$ which extends to a meromorphic function on $X_0(N)$. Then the evaluation of U at CM points is an algebraic number. More precisely, if we consider $x \in X_0(N)$, then the element $u := U(x)$ satisfies the following properties:

$$\begin{aligned} & u \in \mathcal{O}_H[1/N]^{\times} \\ & (\sigma - 1)(u) \in \mathcal{O}_H^{\times}, \qquad \text{for all } \sigma \in \mathrm{Gal}\,(H|K); \end{aligned}$$

In a similar way we can define $u(\mathfrak{a})$ and all of these units are in the same orbit via the action of $\mathrm{Pic}(\mathcal{O}_c)$.

Chapter 2

Complex L-functions

In this chapter we introduce one of the main tools that we are going to use in our dissertation: the L-functions.

Their story started from the Riemman zeta function, the Dirichlet L-functions and the subsequent generalizations: zeta function of number fields, Hecke L-functions, etc. The theory of motives is a unifying framework which allows to undercover what an L-function should be in great generality.

The general philosophy is roughly the following: to a "motive" M we can associate an L-function $L(M,s)$ of a complex variable s which is defined as an Euler product for each prime p. For a prime $\ell \neq p$ one considers the ℓ-adic realization M_ℓ of the motive and defines:

$$L_p(M,T) = \Big(\det(1 - T\sigma_p | M_\ell^{I_p})\Big)$$

where σ_p is the Frobenius element at p in $G_{\mathbb{Q}}$, and I_p is the inertia subgroup at p. The L-function associated to M is then:

$$L(M,s) := \prod_p L_p(M, p^{-s})^{-1}.$$

This L-function (conjecturally) admits a meromorphic continuation and a functional equation. The study of the special values of $L(M,s)$ outside the region of definition contains key information on the motive M and leads to a better understanding of its properties.

We have seen in the introduction the prototypical examples which are the analytic class number formula (where the motive is a number field) and the Birch and Swinnerton-Dyer conjecture (where the motive is an elliptic curve defined over a number field). Our main motivation in this dissertation is the Birch and Swinnerton-Dyer conjecture and its equivariant refinement (see Chap. 4). All these are special cases of the (much) more general equivariant Tamagawa number conjecture (ETNC) for motives.

Although the theory of motives and motivic L-functions is a very active and deep field of study, it will be sufficient for our purposes to introduce L-functions in terms of compatible systems of λ-adic representations. Nevertheless, the compatible systems we are going to treat arise from motives (of an elliptic curve, of a modular form, excetera). For this reason we will sometimes talk about the associated motive as the object from which the compatible system that we are treating arise, without any further explaination about the nature of motives.

Given a compatible system of λ-adic representations V, its L-function is defined as:

$$L(V,s) = \prod_p \Phi_p(p^{-s})$$

where $\Phi_p(p^{-s})$ are the *Euler factors* of $L(V,s)$. They are quite simple polynomials evaluated at p^{-s}. This L-function is a product indexed by the finite places but we can complete it with a *factor at infinity*

$L_\infty(V,s)$, a product of gamma functions which is strictly dependent on the object M that generates the compatible system V.

In particular we will need the L-functions associated with modular forms, Hecke characters and Artin Representations. The other L-functions will arise from those three cases by tensoring the associated compatible systems of λ-adic representations. Most important to our discussion will be the Rankin double product L-function and the Garrett triple product L-function.

2.1 Compatible systems of λ-adic representations

Let G be a profinite group and let E be a topological field. A *representation of G of dimension n with coefficients in E* is a continuous group homomorphism:

$$\rho : G \to \mathrm{GL}_n(E).$$

It is equivalent to ask for an n-dimensional vector space V over E on which G acts continuously and linearly.

Given two representations ρ and ρ' of G whose underlying vector spaces are V and V', and given a group extension $G \subset \widetilde{G}$, we can define new representations using linear algebra operations, for example:

- the *dual representation* (or *contragradient representation*) $\rho^\vee$ whose underlying vector space is $V^\vee = \mathrm{Hom}_E(V,E)$;
- the *direct sum representation* $\rho \oplus \rho'$, whose underlying vector space is $V \oplus V'$;
- the *tensor product representation* $\rho \otimes \rho'$, whose underlying vector space is $V \otimes V'$;
- the *induced representation from G to $\widetilde{G}$* $\mathrm{Ind}_G^{\widetilde{G}}(\rho)$, associated with the vector space $V \otimes_{E[G]} E[\widetilde{G}]$;
- the *symmetric square representation* $\mathrm{Sym}^2(\rho)$ which is the subrepresentation of $V\otimes V$ invariant under the map $(v,w) \mapsto (w,v)$;
- the *adjoint representation* $\mathrm{Ad}(\rho)$, whose underlying vector space is the space of trace zero endomorphisms of V, $\mathrm{End}^0(V)$;
- the *determinant representation* $\det \rho$, whose underlying vector space is $\wedge^n V$.

We will deal with *Galois representations* which are representations of the absolute Galois group $G_K = \mathrm{Gal}\,(K^{\mathrm{sep}}|K)$ of some field K. A Galois representation is said to be a *global representation* if K is a global field and a *local representation* if K is a local field. A representation is said to be an *Artin representation* if $E \subset \mathbb{C}$. It is said to be an *ℓ-adic representation* if $E \subset \overline{\mathbb{Q}}_\ell$. Sometimes we call it λ-adic, if λ is the finite place above ℓ induced by the inclusion.

Let now $K \mid \mathbb{Q}_p$ be a finite extension and $\rho : G_K \to \mathrm{GL}_n(E)$. Let $I \subset G_K$ be the inertia subgroup and define:

$$V^I = \{v \in V \mid v^\sigma = v, \forall \sigma \in I\}.$$

This is a sub vector space of V and it defines a representation ρ^I. We say that ρ is *unramified* if $\rho = \rho^I$ (i.e. if $\rho(I) = \{1\}$), otherwise we say that ρ is *ramified.* Denote by σ_p the *arithmetic Frobenius* of G_K, i.e. a lift of the Frobenius morphism which acts on the residue fields as $x \mapsto x^{\#\mathcal{O}_K/\mathfrak{m}_K}$.

We define the *characteristic polynomial* of ρ as:

$$\Phi(\rho)(T) := \det(1 - T\rho(\sigma)|_{V^I}) \in E[T]$$

We say that ρ is *integral* if $\phi(\rho)(T) \in \mathcal{O}_E[T]$.

If K is a number field, we say that $\rho : G_K \to \mathrm{GL}_n(E)$ is *unramified at* $\wp \subset \mathcal{O}_K$ if $\rho|_{G_{K_\wp}}$ is unramified and that it is *ramified at* $\wp$ otherwise. We define the characteristic polynomial of the Frobenius at $\wp$ as:

$$\Phi_\wp(\rho) := \Phi(\rho|_{G_{K_\wp}}) \in E[T]$$

For unramified primes, the degree of the characteristic polynomial equals the dimension of the representation while for ramified primes it does not exceed it. Assume that E is a number field and λ is a finite place of E whose norm is $\ell = \mathbf{N}\lambda$. A family of λ-adic representations is a collection $V = \{\rho_\lambda : G_K \to \mathrm{GL}_n(E_\lambda)\}$ indexed by the finite places λ of E (to ease the notation we drop the index).

Definition 2.1. A *compatible system of λ-adic representations* of K is a family of λ-adic representations $V = \{\rho_\lambda\}$ such that:

(1) there exists a finite set S of places of K, independent on ℓ, such that each representation ρ_λ is unramified outside $S \cup \{\mathcal{L} \mid \ell\}$;

(2) the characteristic polynomials $\Phi_\wp(\rho_\lambda)$ are in $E[T]$. Moreover they do not depend on λ, if $\wp \nmid \ell$.

The minimal set S for which this holds is the *exceptional set of* V and the primes $\wp \in S$ (resp. $\wp \notin S$) are called *bad primes* (resp. *good primes*) of V.

Since a compatible system of λ-adic representations is essentially a collection of vector spaces, we can consider all of the fundamental operations that we can do with vector spaces, as seen above. In particular we can take the dual, the direct sum, the tensor product, the induced, the symmetric square, the adjoint, etc.

Up to an extension of the scalars in E, there is an obvious notion of isomorphism of representations. It is important to remark that the characteristic polynomials are invariant under isomorphism so that we can consider an *isomorphism class of compatible systems of λ-adic representations* and we still denote them by V.

Let V be an isomorphism class of compatible systems of λ-adic Galois representations of K, with exceptional set S and characteristic polynomials $\Phi_\wp(T)$. Then we can associate to V an L-function $L(V,s)$ defined as an Euler product as follows:

$$L(V,s) := \prod_\wp \Phi_\wp((\mathbf{N}\wp)^{-s})^{-1}.$$

We will refer to the factors $\Phi_\wp((\mathbf{N}\wp^{-s}))$ as the *good Euler factors of* $L(V,s)$ if $\wp \notin S$ and as the *bad Euler factors of* $L(V,s)$ if $\wp \in S$. For this L-function we have the following properties as for the Artin formalism:

$$L(V_1 \oplus V_2, s) = L(V_1,s)L(V_2,s), \qquad L(\mathrm{Ind}_K^L V, s) = L(V,s)$$

In general the L-function associated with a compatible system V is only defined for $\Re(s) \gg 0$, but in all of the cases of interest for us it will admit a meromorphic continuation to the complex plane. In order to describe the functional equation one needs to consider the completed L-function which involves the presence of the Gamma factors:

$$\Gamma_\mathbb{R}(s) := \frac{\Gamma(s/2)}{\pi^{s/2}}, \qquad \Gamma_\mathbb{C}(s) := \frac{2\Gamma(s)}{(2\pi)^s}. \tag{2.1}$$

We define the *completed L-function* as:

$$Ł^*(V,s) = L_\infty(V,s)L(V,s),$$

where $L_\infty(V,s)$ is an appropriate product of the above gamma factors. In the cases we consider, the L-function admits a functional equation which is of the following form:

$$\Lambda(V,s) = \varepsilon(V)\Lambda(V^\vee, w-s).$$

where $\varepsilon(V)$ is a complex number of norm one and $\Lambda(V,s) = A(V)^{s/2}L^*(V,s)$, for a well defined positive integer $A(V)$. We do not describe here the general recipe for $L_\infty(V,s)$ nor we will say more about $\varepsilon(V)$ or $A(V)$. Instead we will give the explicit recipe of these objects in the specific cases we will treat. Notice that neither the functional equation nor the meromorphic continuation are known to hold in general! The proof of these properties is often obtained from an explicit integral representation of the L-function and this integral representation is not known to exist in general.

We now briefly discuss the main sources of examples that will appear during the rest of our dissertation.

1) The compatible system arising from an Artin representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{C})$. Since ρ has finite image, it takes value in a number field. Hence, by localization, it gives rise to a collection of λ-adic representations for every λ. This obviously defines a compatible system $V(\rho)$ and we have:
$$L(V(\rho),s) = L(\rho,s). \tag{2.2}$$

2) The compatible system arising from a character of the ideal class group of a quadratic imaginary number field K (a *finite order Hecke character for* K). Such a character:
$$\psi : \mathrm{Cl}_K \to \mathbb{C}^\times \tag{2.3}$$
can be seen as a Galois character of G_K via the reciprocity law of class field theory, which sends a prime ideal $\wp$ to the arithmetic Frobenius $\sigma_\wp$. By abuse of notation, we still call ψ the associated Galois character so that $\psi(\wp) = \psi(\sigma_\wp)$. In this way ψ is an Artin representation, hence it induces a compatible system $V(\psi)$ and:
$$L(V(\psi),s) = L(\psi,s) := \prod_\wp \Big(1 - \frac{\psi(\wp)}{\mathbf{N}\wp^s}\Big)^{-1}$$
The exceptional set S consists of the primes dividing D_K. In this case $L_\infty(\psi,s) = \Gamma_\mathbb{C}(s)$. We will see the more general case of the Hecke characters of quadratic imaginary field in the next section.

3) The compatible system arising from a modular form $f \in S_k(N_f,\chi_f)$, for $k \geq 1$. Recall that a representation is said to be *odd* if the image of the complex conjugations of $G_\mathbb{Q}$ has determinant -1. The results of Shimura, Deligne and Serre-Deligne associate to any cuspidal modular form f an irreducible, odd and 2-dimensional ℓ-adic Galois representation of $G_\mathbb{Q}$ such that for all primes $p \nmid N\ell$ the characteristic polynomial at p is given by:
$$\Phi_p(\rho)(T) = T^2 - a_p(f)T + \chi(p)p^{k-1}$$
These representations actually define a compatible system of representations $V = \{\rho_{f,\ell}\}$ for which:
$$L(V(f),s) = L(f,s) \tag{2.4}$$
The exceptional set S is composed by the primes dividing the level N_f. In this case, $L_\infty(f,s) = \Gamma_\mathbb{C}(s)$

In the discussions which follow it is going to be important to distinguish very similar L-functions. For this reason we introduce more notation. We will call an L-function *primitive* if it is the L-function associated with a compatible system of Galois representations V. We will denote it by $L(V,s)$. We will call an L-function *imprimitive* if it is defined by the same Euler product of $L(V,s)$ for almost all primes. In particular if we remove the Euler factor corresponding to the set of primes above N, we will denote the corresponding L-function by $L_N(V,s)$. Sometimes imprimitive L-functions are a little bit trickier and they are obtained by substituting the standard bad Euler factors with some other term. Altought we will not use in general the standard notation, it is worth to remind that those imprimitive L-functions are often denoted by $D_S(V,s)$ (or simply $D(V,s)$ when there is no ambiguity arising from the choice of the bad Euler factors). In particular we can write:
$$L_N(V,s) = \mathcal{E}ul_N(V,s)L(V,s), \qquad \text{and} \qquad D_S(V,s) = \mathcal{E}ul_S(V,s)L(V,s)$$
where $\mathcal{E}ul_N(V,s) = \prod_{\wp\in M}\Phi_\wp((\mathbf{N}\wp)^{-s})$ and $\mathcal{E}ul_S(V,s)$ is an adequate elementary product of bad Euler factor depending on the choice of $D_S(V,s)$.

2.2 Hecke characters and L-functions

Let $K/\mathbb{Q}$ be a number field and denote by $\Sigma_K = \mathrm{Hom}(K, \bar{\mathbb{Q}})$ the set of infinite places of K (remember that we see $\bar{\mathbb{Q}} \subset \mathbb{C}$ via our fixed embedding). We say that $\mathbb{Z}[\Sigma_K]$ is the set of *infinity types*. Any element $\gamma \in \mathbb{Z}[\Sigma_K]$ can be written as:

$$\gamma = \sum_{\sigma \in \Sigma_K} \kappa_\sigma \sigma.$$

Let $\mathfrak{c} \subset \mathcal{O}_K$ be an integral ideal. Let $I_\mathfrak{c}$ denote the group of fractional ideals of K that are coprime to $\mathfrak{c}$ and $J_\mathfrak{c} := \{(\alpha) \mid \alpha \gg 0, \alpha - 1 \in \mathfrak{c}\}$, where $\alpha \gg 0$ means that $\sigma(\alpha) > 0$ for every real embedding $\sigma \in \Sigma_K$).

Definition 2.2. A *Hecke character* of infinity type $\gamma \in \mathbb{Z}[\Sigma_K]$ for K is a homomorphism

$$\psi : I_\mathfrak{c} \longrightarrow \mathbb{C}^\times, \quad \text{such that} \quad \psi((\alpha)) = \alpha^\gamma = \prod_\sigma \sigma(\alpha)^{\kappa_\sigma}, \quad \forall \alpha \in J_\mathfrak{c}.$$

The largest ideal $\mathfrak{c}_\psi$ for which ψ is a Hecke character modulo $\mathfrak{c}_\psi$ is called *conductor* of ψ.

Consider the norm characters of $\mathbb{Q}$ and K defned by:

$$\mathbf{N}((a)) = |a|, \qquad \mathbf{N}_K = \mathbf{N} \circ N_\mathbb{Q}^K$$

and the trivial character $\mathbb{1}_K$ for any number field K. Their infinity types are, respectively, 1, $\sum_\sigma \sigma$ and 0. They all have trivial conductor and their image lies in the positive real numbers.

If $\gamma = 0$ we say that ψ is a *finite order* character. Such a character factors through $I_\mathfrak{c}/P_\mathfrak{c}$, where $P_\mathfrak{c}$ is the set of principal ideals coprime with $\mathfrak{c}$. For this reason, it can be seen as a character of G_K via the reciprocity map as explained in the previous section.

Given two Hecke characters $\psi_1, \psi_2 : I_\mathfrak{c} \to \mathbb{C}$ of infinity type γ, their quotient $\psi = \psi_1/\psi_2$ is an Hecke character of infinity type 0. Hence, given an Hecke character of infinity type γ, all the other characters with the same infinity type can be recovered multiplying by a finite order Hecke character.

Definition 2.3. Consider an ideal $\mathfrak{c} \subset \mathfrak{c}_\psi$. We define the L-function associated with the Hecke character ψ to be defined by the Dirichlet series:

$$L_\mathfrak{c}(\psi, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ (\mathfrak{a},\mathfrak{c})=1}} \frac{\psi(\mathfrak{a})}{\mathbf{N}_K(\mathfrak{a})^s} = \prod_{(\wp,\mathfrak{c})=1} \Big(1 - \frac{\psi(\wp)}{\mathbf{N}_K(\wp)^s}\Big)^{-1}.$$

In particular, $L(\psi, s) := L_{\mathfrak{c}_\psi}(\psi, s)$ is the primitive L-function associated with ψ.

It is a classical result due to Hecke that the L-function $L_\mathfrak{c}(\psi, s)$ associated with the Hecke character ψ admits a meromorphic continuation to $\mathbb{C}$ and a functional equation relating values at s and $1 - s$. This result can be found in [Lan70, XIII and XIV] proved both in the classical way and by means of Tate's argument (see also Tate's thesis, [CF67, XV]). From the definitions it is easy to verify that for any $k \in \mathbb{Z}$ we have

$$L(\psi, s) = L(\psi \mathbf{N}_K^k, s + k). \tag{2.5}$$

There exists a number $w(\psi)$ called weight of ψ such that $\kappa_\sigma + \kappa_{\bar{\sigma}} = w(\psi)$, for all σ. We define the *central character of* ψ, denoted by ε_ψ, to be an Hecke character of $\mathbb{Q}$ (which can be seen as a Dirichlet character) such that:

$$\psi|_\mathbb{Q} = \varepsilon_\psi \mathbf{N}_K^{w(\psi)}.$$

We now restrict our attention to the case where $K = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field of discriminant $-D$. For this, we fix an embedding $\sigma : K \to \mathbb{C}$ so that the only other complex embedding is given by $\bar{\sigma}$. In this way, we simply write α instead of $\sigma(\alpha)$. In particular, a Hecke character ψ of K of infinity type $\kappa_1\sigma + \kappa_2\bar{\sigma}$ is given on $J_{\mathfrak{c}}$ by $\psi((\alpha)) = \alpha^{\kappa_1}\bar{\alpha}^{\kappa_1}$. Thanks to our choice of $\sigma : K \to \mathbb{C}$ we will write (κ_1, κ_2) for the infinity type of such a character.

For any Hecke character ψ of infinity type (κ_1, κ_2) define $\psi'(\mathfrak{a}) = \psi(\overline{\mathfrak{a}})$, where $\overline{x}$ denotes complex conjugation. We say that ψ is *self-dual* (or *anticyclotomic*) if $\psi = \psi'$. This forces $\kappa_2 = -\kappa_1$ so that an anticyclotomic character has infinity type $(\kappa, -\kappa)$.

The following lemma is well known.

Lemma 2.4. *Let ψ be a finite order Hecke character of conductor $\mathfrak{c}$. The following are equivalent:*

1. *ψ is self-dual;*
2. *The central character of ψ is trivial;*
3. *ψ is a ring class character of* $\mathrm{Gal}\,(H_c|K)$, *where H_c is the ring class field associated with the order $\mathcal{O}_c$ and c is a positive generator of $\mathfrak{c}_\psi$.*

Proof. It is a representation-theoretical restatement of [Cox89, Theorem 9.18]. □

Given a Hecke character of K of infinity type $(\kappa - 1, 0)$ (or $(0, \kappa - 1)$, we can associate to it a theta series as follows: define the quantities

$$a_n(\psi) = \sum_{\mathfrak{a} \in I^n_{\mathfrak{c}_\psi}} \psi(\mathfrak{a}),$$

where $I^n_{\mathfrak{c}_\psi}$ is the set of the invertible ideals in $I_{\mathfrak{c}_\psi}$ with norm n. Define also $a_0(1) = h_K/w_K$ and $a_0(\psi) = 0$ otherwise. As shown in [Zag08] or [Kan12], the q-expansion

$$\theta_\psi := \sum_{n \geq 0} a_n(\psi) q^n = \sum_{n \geq 0} a_n(\theta_\psi) q^n \in M_\kappa(D_K N^K_{\mathbb{Q}}(\mathfrak{c}_\psi), \chi_K \varepsilon_\psi) \tag{2.6}$$

defines a normalized newform of weight κ, level $D_K N^K_{\mathbb{Q}}(\mathfrak{c}_\psi)$ and nebentype $\chi_K \varepsilon_\psi$. Moreover θ_ψ is Eisenstein if and only if $\psi = \psi'$; otherwise θ_ψ is a cusp form.

Since θ_ψ is a modular form we can associate to it the L-function

$$L(\theta_\psi, s) := \sum_{n \geq 1} \frac{a_n(\theta_\psi)}{n^s}.$$

From the definitions it is easy to see that $L(\theta_\psi, s) = L(\psi, s)$. In fact the compatible systems $\mathrm{Ind}^{\mathbb{Q}}_K(\psi)$ and $V(\theta_\psi)$ are the same. We will consider the completed L-function $L^*(\psi, s) = L_\infty(\psi, s)L(\psi, s)$, where $L_\infty(\psi, s) = \Gamma_{\mathbb{C}}(s)$.

2.3 Double product L-function and Rankin's method

2.3.1 Classical Rankin's L-function

Let $\ell > k > 0$ and consider

$$g = \sum_{n \geq 1} a_n(g) q^n \in S_\ell(N, \chi_g) \quad \text{and} \quad f = \sum_{n \geq 1} a_n(f) q^n \in M_k(N, \chi_f)$$

We do not assume g and f to be newforms, but we do assume them to be eigenforms for all good and bad Hecke operators. Set $\chi := (\chi_g \chi_f)^{-1}$ and let $g^* = \sum_{n \geq 1} \bar{a}_n(g) q^n \in S_\ell(N, \chi_g^{-1})$ denote the modular form whose Fourier coefficients are the complex conjugates of those of g. We have that $a_p(g^*) = a_p(g \otimes \overline{\chi}_g)$ for almost all p.

For a rational prime q we let $(\alpha_q(g_\ell), \beta_q(g_\ell))$ denote the pair of roots of the Hecke polynomial:

$$T^2 - a_q(g)T + \chi_{g,N}(q)q^{\ell-1}$$

that we label in such a way that $\mathrm{ord}_q(\alpha_q(g)) \leq \mathrm{ord}_q(\beta_q(g))$. Note that $(\alpha_q(g), \beta_q(g)) = (a_q(g), 0)$ when $q \mid N$. If the weight is $\ell = 1$ and $q \nmid N$ then both $\alpha_q(g)$ and $\beta_q(g)$ are q-units. In that case we just choose an arbitrary ordering of this pair. Adopt similar notations for f.

The *Rankin L-function* of the convolution of g and f is defined as the Euler product

$$L(g \otimes f, s) = \prod_q L^{(q)}(g \otimes f, s), \tag{2.7}$$

where q ranges over all prime numbers and

$$\begin{aligned} L^{(q)}(g \otimes f, s) =&(1 - \alpha_q(g)\alpha_q(f)q^{-s})^{-1}(1 - \alpha_q(g)\beta_q(f)q^{-s})^{-1} \\ &\times (1 - \beta_q(g)\alpha_q(f)q^{-s})^{-1}(1 - \beta_q(g)\beta_q(f)q^{-s})^{-1}. \end{aligned}$$

Recall that our normalization of the Petersson scalar product on the space of real-analytic modular forms $S_\ell^{\mathrm{an}}(N, \chi) \times M_\ell^{\mathrm{an}}(N, \chi)$ is given by equation (1.5):

$$\langle \phi_1, \phi_2 \rangle_{\ell,N} := \int_{\Gamma_0(N)\backslash\mathcal{H}} \phi_1(z)\overline{\phi_2(z)}y^\ell \frac{dxdy}{y^2}. \tag{2.8}$$

Proposition 2.5 (Shimura). *For all $s \in \mathbb{C}$ with $\Re(s) >> 0$ we have:*

$$L(g \otimes f, s) = \frac{1}{2}\frac{(4\pi)^s}{\Gamma(s)}\langle g^*(z), \tilde{E}_{\ell-k,\chi_N}(z, s - \ell + 1) \cdot f(z)\rangle_{\ell,N} \tag{2.9}$$

Proof. See for instance [Hid93, p. 317, (1)] or [Shi76, (2.4)]. □

This proposition and the functional equation for the Eisenstein series (theorem 1.6) yield the analytic continuation and the functional equation for the $L(f \otimes g, s)$.

Now we want to replace $\tilde{E}_{k-\ell,\chi_N}$ with a rational modular form having coefficients in $\mathbb{Q}_\chi$. To do this we essentially follow the computation of [BDR15]. Choose integers m, t such that

$$\ell = k + m + 2t \quad \text{ and set } \quad j = (\ell + k + m - 2)/2 = \ell - t - 1.$$

For $m \geq 1$ and $t \geq 0$, evaluating equation (2.9) at $s = j$ and using equations (1.11) and (1.10) one finds that

$$\mathfrak{f}_{\mathrm{Ran}}(\ell, k, m) \cdot L(g \otimes f, j) = \langle g^*(z), \delta_m^t E_{m,\chi_N}(z) \cdot f(z)\rangle_{\ell,N}, \tag{2.10}$$

where

$$\mathfrak{f}_{\mathrm{Ran}}(\ell, k, m) = \frac{(-1)^t(m + t - 1)!(j - 1)!(iN)^m}{2^{\ell-1}(2\pi)^{\ell+m-1} \cdot \tau(\chi^{-1})}. \tag{2.11}$$

One could consider the completed L-function $L^*(f \otimes g, s) = L_\infty(f \otimes g, s)L(f \otimes g, s)$, where:

$$L_\infty(f \otimes g, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - k + 1).$$

Since

$$L_\infty(f \otimes g, j) = \frac{4(m + t - 1)!(j - 1)!}{(2\pi)^{\ell+m-1}},$$

we could restate equation (2.10) in the following manner:

$$\frac{(-1)^t(iN)^m}{2^{\ell+1}\tau(\chi^{-1})}L^*(f \otimes g, j) = \langle g^*(z), \delta_m^t E_{m,\chi_N}(z) \cdot f(z)\rangle_{\ell,N}. \tag{2.12}$$

Remark 2.6. The L-function above is actually very close to the L-function associated with the compatible system of Galois representations given by:

$$V(f,g) := V(f) \otimes V(g).$$

Since the eigenvalues of the Frobenius at p are exactly

$$\{\, \alpha_p(f)\alpha_p(g),\ \alpha_p(f)\beta_p(g),\ \beta_p(f)\alpha_p(g),\ \beta_p(f)\beta_p(g) \,\},$$

the good Euler factors of $L(V(f,g),s)$ coincide to those of $L(f\otimes g,s)$. Then $L(f\otimes g,s)$ is an imprimitive L-function associated to $V(f,g)$. This means that the behaviors of $L(f\otimes g,s)$ and $L(V(f,g),s)$ are essentially the same, but there might be a discrepancy of bad Euler factors, so that:

$$L(f \otimes g, s) = \mathcal{E}ul_N(f,g,s)L(V_p(f,g),s)$$

where $\mathcal{E}ul_N(f,g,s)$ is a finite product of Euler factors. The Euler factor can be trivial, for instance in the case when g is a theta series defined by an Hecke character of an imaginary quadratic field.

2.3.2 The Gross-Zagier formula

For more details we suggest to have a look at [Dar04]. Let us now consider an elliptic curve E defined over $\mathbb{Q}$ of conductor N and a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$. Consider the following Heegner hypothesis:

Assumption 2.7 (HH)**.** There exists a cyclic ideal $\mathfrak{N} \in \mathcal{O}_c$ of order N, i.e. $\mathcal{O}_c/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$.

As we have seen in section 1.7, this implies the existence of a point $x \in X_0(N)$ arising from an elliptic curve with complex multiplication by $\mathcal{O}_c$. We fix once and for all a choice of this point.

The theorem of modularity ensures a parametrization $\pi : X_0(N) \to E$ defined over $\mathbb{Q}$ as well as the existence of a modular form $f \in S_2(N, \mathbb{1})$ such that:

$$L(f,s) = L(E,s).$$

Take a finite order anticyclotomic Hecke character ψ. By theorem 2.4 it is a character of $\mathrm{Gal}\,(H|K)$ for H ring class field of conductor c associated with the order $\mathcal{O}_c$ of conductor $c \in \mathbb{Z}_{\geq 1}$. Write $E(H)_{\mathbb{C}} := E(H) \otimes \mathbb{C}$, define $P := \pi(x)$ and consider the point:

$$P_\psi := \sum_{\sigma \in \mathrm{Gal}\,(H|K)} \psi^{-1}(\sigma) P^\sigma \in E(H)_{\mathbb{C}}^{\psi}, \tag{2.13}$$

where:

$$E(H)_{\mathbb{C}}^{\psi} := \{P \in E(H)_{\mathbb{C}} \mid P^\sigma = \psi(\sigma)P\}.$$

Consider also the L-function:

$$L(E/K, \psi, s) := L(V(f) \otimes V(\psi), s),$$

which coincides with $L(f \otimes \theta_\psi, s)$ under our assumption (HH). The Rankin method gives the analytic continuation and the functional equation. The Heegner hypothesis forces the L-function to have odd order at $s = 1$ and the theorem of Gross-Zagier guarantees that:

$$L'(E/K, \psi, 1) \doteq \langle P_\psi, P_\psi \rangle_{\mathrm{NT}},$$

where the dotted equal implies the presence of an unspecified non-zero constant and $\langle -, - \rangle_{\mathrm{NT}}$ is the Néron-Tate height pairing.

Theorem 2.8. *If* $\mathrm{ord}_{s=1} L(E, \psi, s) = 1$*, then* $\dim_{\mathbb{C}} E(H)_{\mathbb{C}}^{\psi} = 1$ *and* P_ψ *is a generator.*

Proof. See the main theorem of [BD90]. □

2.4 Petersson product and L-functions: comparison of various formulae

A classical result due to Petersson (see [Pet49, Satz 6]) relates the special value of L-functions to the Petersson inner product. We have already seen such a result in proposition 2.9. Following the article [Shi76] we want to derive a formula for the Petersson product in the specific case of theta series associated with Hecke characters ψ of imaginary quadratic fields and integer conductor c. More specifically, if ψ has infinity type $(\ell-1,0)$ (or $(0,\ell-1)$), we have:

$$\langle \theta_\psi, \theta_\psi \rangle \doteq L(\psi^2, \ell),$$

where the dotted equal means that we hide a constant. In this section we want to explicitly write down the constant in terms of the fundamental invariants of the quadratic order $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$. This result can be derived directly from [Hid81, Theorem 5.1], but we prefer to give a direct argument here which only relies on the result of Petersson.

For this, given two elements $f \in S_k(N, \chi_f)$ and $g \in M_\ell(N, \chi_g)$ we can consider their *convolution L-function* $D(g, f, s)$ defined by:

$$D(f, g, s) := \sum_{n \geq 1} \frac{a_n(f) a_n(g)}{n^s}$$

whose Euler product is given by factors of the form:

$$\begin{aligned} D_{(p)}(f, g, s) &= \left(1 - \frac{\alpha_p(f)\alpha_p(g)\beta_p(f)\beta_p(g)}{p^{2s}}\right) \qquad (2.14) \\ &= \left(1 - \frac{\alpha_p(f)\alpha_p(g)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p(f)\alpha_p(g)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p(f)\beta_p(g)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p(f)\beta_p(g)}{p^s}\right)^{-1}. \end{aligned}$$

With our notation $\chi = (\chi_g \chi_f)^{-1}$ it is easy to check that:

$$L(f \otimes g, s) = L_N(\chi^{-1}, 2s - k - \ell + 2) \cdot D(f, g, s),$$

where $L_N(\chi, s)$ indicates the Dirichlet L-functions with factors at $p \mid N$ removed. We have already seen in proposition 2.9 that we have a relation between L-functions and Petersson product of real analytic modular forms. The equation (2.5) of [Shi76] can be derived from the above result by computing the residue at $k = \ell$ and it states that:

$$\mathrm{Res}_{s=k} D(f, g, s) = \frac{3}{\pi \Im(N)} \frac{(4\pi)^\ell}{(\ell-1)!} \langle f^*, g \rangle,$$

where $\Im(N) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ is the index of $\Gamma_0(N)$ inside $\mathrm{SL}_2(\mathbb{Z})$. The factor $3/(\pi\Im(N))$ appears because of the different normalization of the Petersson product adopted by Shimura. If we take $f = g^* = \sum \overline{a_n(g)} q^n$ to be the modular form obtained via the complex conjugation of the Fourier coefficients of g, then we find:

$$\mathrm{Res}_{s=k} D(g^*, g, s) = \frac{3}{\pi \Im(N)} \frac{(4\pi)^\ell}{(\ell-1)!} \langle g, g \rangle. \qquad (2.15)$$

Notice that in [Shi76, (2.5)] the complex conjugation doesn't appear. That is a misprint, since the formula is derived from [Shi76, (2.3)], and it was already noticed by Hida in [Hid81, §5].

We introduce the *symmetric square L-function* associated with the modular form g. We define it by the following Euler product:

$$L(\mathrm{Sym}^2(g), s) = \prod_p \left(1 - \frac{\alpha_p(g)^2}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p(g)\beta_p(g)}{p^s}\right)^{-1} \left(1 - \frac{\beta_p(g)^2}{p^s}\right)^{-1}. \qquad (2.16)$$

The result in [Hid81, §5] is expressed in terms of $L(\mathrm{Sym}^2(g) \otimes \chi_g^{-1}, s)$ which can also be seen as an imprimitive adjoint L-function.

Lemma 2.9. *If $g = g^*$ we have that:*

$$\langle g, g\rangle = \frac{\Im(N)\pi}{3} \cdot \frac{(\ell-1)!}{(4\pi)^\ell} \cdot \frac{L_N(\chi,1)}{L_N(\chi^2,2)} \cdot \mathrm{Res}_{s=\ell} L(\mathrm{Sym}^2(g), s)$$

Proof. Starting from equation (2.14), if we put $f = g$ we find:

$$D(g,g,s) = \Big(1 - \frac{\alpha_p(g)^2\beta_p(g)^2}{p^{2s}}\Big) = \Big(1 - \frac{\alpha_p(g)^2}{p^s}\Big)^{-1}\Big(1 - \frac{\alpha_p(f)\beta_p(g)}{p^s}\Big)^{-2}\Big(1 - \frac{\beta_p(g)^2}{p^s}\Big)^{-1}. \tag{2.17}$$

Using the fact that $\alpha_p(g)\beta_p(g) = \chi(p)p^{\ell-1}$ and working on the Euler products one can easily derive the following decomposition of L-function:

$$L_N(\chi^2, 2s - 2\ell + 2) \times D(g,g,s) = L(\mathrm{Sym}^2(g), s) \times L_N(\chi_{_N}, s - \ell + 1). \tag{2.18}$$

We conclude by taking residues and using formula (2.15). □

We now specialize the discussion to the case of a theta series of an imaginary quadratic field associated with an Hecke character of integral conductor c and infinity type $(\ell - 1, 0)$.

Lemma 2.10. *Consider $g = \theta_\psi \in S_\ell(Dc^2, \chi_K)$, where ψ is a Hecke character of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of conductor $c \in \mathbb{Z}$ and infinity type $(\ell-1, 0)$, for $\ell > 1$. Then we have the following decomposition of L-functions:*

$$L(\mathrm{Sym}^2(g), s) = L(\psi^2, s) \cdot \zeta_{Dc^2}(s - \ell + 1)$$

Proof. Remember that $\alpha_p(g)\beta_p(g) = \chi_K(p)p^{\ell-1}$. To prove the decomposition, we rearrange the Euler factors in four categories and we write:

- $p = \wp^2$, for those primes which ramifies, $p \mid D$ and do not divide the conductor c. For these primes, $\psi(p) = \psi^2(\wp) = \alpha_p(g)^2 = p^{\ell-1}$ (see [Hid81, (5.10a)];
- $p = \wp\bar{\wp}$, for those primes which split in K and do not divide c;
- $p = \wp$ for those primes which are inert in K and do not divide c. For these primes we have $\alpha_p(g)^2 = \beta_p(g)^2 = p^{\ell-1}$ and $\psi(p) = p^{\ell-1}$;
- $p \mid c$ for the primes dividing the conductor of ψ, for which $\alpha_p(g) = \beta_p(g) = 0$.

Then we have:

$$\begin{aligned} L(\mathrm{Sym}^2(g), s) = &\prod_{p=\wp^2}\Big(1 - \frac{\alpha_p(g)^2}{p^s}\Big)^{-1} \\ &\times \prod_{p=\wp\bar{\wp}}\Big(1 - \frac{\alpha_p(g)^2}{p^s}\Big)^{-1}\Big(1 - \frac{\beta_p(g)^2}{p^s}\Big)^{-1}\Big(1 - \frac{p^{\ell-1}}{p^s}\Big)^{-1} \\ &\times \prod_{p=\wp}\Big(1 - \frac{p^{\ell-1}}{p^s}\Big)^{-2}\Big(1 + \frac{p^{\ell-1}}{p^s}\Big)^{-1} \end{aligned}$$

and

$$\begin{aligned} L(\psi^2, s) = &\prod_{p=\wp^2}\Big(1 - \frac{\psi^2(p)}{p^s}\Big)^{-1} \\ &\times \prod_{p=\wp\bar{\wp}}\Big(1 - \frac{\psi^2(\wp)}{p^s}\Big)^{-1}\Big(1 - \frac{\psi^2(\bar{\wp})}{p^s}\Big)^{-1} \\ &\times \prod_{p=\wp}\Big(1 - \frac{p^{\ell-1}}{p^s}\Big)^{-1}\Big(1 + \frac{p^{\ell-1}}{p^s}\Big)^{-1}. \end{aligned}$$

An easy comparison of the factors shows that the discrepancy is exactly:

$$\prod_{p\nmid Dc^2}(1-p^{\ell-1-s})=\zeta_{Dc^2}(s-\ell+1).$$

$\square$

Theorem 2.11. *Consider the theta series $\theta_\psi\in S_k(Dc^2,\chi_K)$ as in the previous lemma and assume that $(D,c)=1$. Then we have:*

$$\langle\theta_\psi,\theta_\psi\rangle=\mathfrak{a}_{\mathrm{Pet}}(\ell)\cdot\mathfrak{f}_{\mathrm{Pet}}(\ell)\cdot L(\psi^2,\ell),\tag{2.19}$$

where:

$$\mathfrak{a}_{\mathrm{Pet}}(\ell)=\frac{(\ell-1)!}{\pi^\ell}\quad and\quad \mathfrak{f}_{\mathrm{Pet}}(\ell)=\frac{h_c\sqrt{Dc^2}}{\omega_c}\cdot 2^{2-2\ell}.$$

Here $h_c=\#\mathrm{Pic}(\mathcal{O}_c)$ and ω_c is the number of roots of unity of $\mathcal{O}_c=\mathbb{Z}+c\mathcal{O}_K$.

Proof. Thanks to lemma 2.10 we have that:

$$\mathrm{Res}_{s=\ell}L(\mathrm{Sym}^2(g),s)=L(\psi^2,k)\prod_{p|Dc^2}\Big(1-\frac{1}{p}\Big).\tag{2.20}$$

Since θ_ψ has complex multiplication by χ_K, i.e. $a_p(g)=a_p(g\otimes\chi_K)$ for all $p\nmid Dc^2$, we have that $g=g^*$. Hence we can apply lemma 2.9 that, with equation (2.20), gives us:

$$\langle\theta_\psi,\theta_\psi\rangle=\frac{\Im(Dc^2)\pi}{3}\cdot\frac{(\ell-1)!}{(4\pi)^\ell}\cdot\frac{L_{Dc^2}(\chi_K,1)}{\zeta_{Dc^2}(2)}\prod_{p|Dc^2}\Big(1-\frac{1}{p}\Big)L(\psi^2,k).$$

Combining the Dirichlet class number formula with the classical formula:

$$h_c=\frac{\omega_c}{\omega_K}h_k c\prod_{p|c}\Big(1-\frac{\chi_k(p)}{p}\Big)$$

we can deduce that $L_N(\chi_K,1)=2\pi h_c/\omega_c\sqrt{Dc^2}$. Since moreover $\zeta_N(2)=\frac{\pi^2}{6}\prod_{p|Dc^2}(1-1/p^2)$ we find:

$$\langle\theta_\psi,\theta_\psi\rangle=\frac{\Im(Dc^2)\pi}{3}\cdot\frac{(\ell-1)!}{(4\pi)^\ell}\cdot\frac{12\pi h_c\sqrt{Dc^2}}{\omega_c\pi^2 Dc^2\prod_{p|Dc^2}\Big(1+\frac{1}{p}\Big)}L(\psi^2,k).$$

We get the result noticing that $\Im(N)=N\prod_{p|N}(1+1/p)$. $\square$

The argument given above is in fact slightly different to that of [Hid81, §5]. Hida uses the L-function $L(\mathrm{Sym}^2(g)\otimes\chi_K,s)$ so that the same result reads as follows:

$$\langle\theta_\psi,\theta_\psi\rangle=\frac{(\ell-1)!}{2^{2\ell-2}\pi^\ell}\cdot\frac{h_c\sqrt{Dc^2}}{\omega_c}\cdot\frac{D}{\phi(D)}\cdot L(\psi^2\chi_K,\ell),\tag{2.21}$$

where ϕ is the Euler function. Since $D/\phi(D)=\prod_{p|D}(1-1/p)^{-1}$, it coincides with the bad factors of $L(\psi^2,\ell)$ hence it is equivalent to our result.

For a general theta function one can also find a closed formula, but it involves special values of Dirichlet L-function that do not enjoy a straightforward interpretation in terms of the above invariants. Hence, for a more general theta series we can always write:

$$\langle\theta_\psi,\theta_\psi\rangle=\mathfrak{a}_{\mathrm{Pet}}(\ell)\cdot\mathfrak{f}_{\mathrm{Pet}}(\ell)\cdot L(\psi^2\chi_g,\ell),\tag{2.22}$$

where $\mathfrak{f}_{\mathrm{Pet}}(\ell)=\frac{(\ell-1)!}{\pi^\ell}$ and $\mathfrak{a}_{\mathrm{Pet}}(\ell)=A_\psi\cdot 2^{2-2\ell}$ for some constant A_ψ.

We want to conclude the section by recalling the theorem of Hida that holds for any modular form:

Theorem 2.12. *Let $g \in S_\ell(N,\chi)$ be a normalized newform, then:*

$$L(\mathrm{Sym}^2(g)\otimes\overline{\chi},\ell)=\frac{2^{2\ell}\pi^{\ell+1}}{(\ell-1)!}\cdot A_g\cdot\langle g,g\rangle$$

where $A_g=(\phi(N))/(N_\chi N\cdot\phi(N/N_\chi))$ (here ϕ is the Euler phi).

Proof. See [Hid81, Theorem 5.1]. Pay attention to the different normalization of the Petersson product. □

We might aswell restate the above theorems using the completed L-function instead. It is obtained by adding $L_\infty(\mathrm{Sym}^2(g),s)=\Gamma_{\mathbb{R}}(s-\ell+1)\Gamma_{\mathbb{C}}(s)$ and $L_\infty(\psi^2,s)=\Gamma_{\mathbb{C}}(s)$, so that:

$$L^*(\mathrm{Sym}^2(g)\otimes\overline{\chi},\ell)=2^{\ell+1}A_g\langle g,g\rangle \tag{2.23}$$

$$L^*(\psi^2,\ell)=\frac{2^{\ell-1}\omega_c}{h_c\sqrt{Dc^2}}\langle\theta_\psi,\theta_\psi\rangle. \tag{2.24}$$

Sometimes in the literature, the result of Hida is described using the adjoint L-function associated with g. In particular, since:

$$\mathrm{Ad}(g)\simeq\mathrm{Sym}^2(g)\otimes\overline{\chi}\,\chi_{\mathrm{cyc}}^{1-\ell}$$

we have that:

$$L(\mathrm{Ad}(g),s)=L(\mathrm{Sym}^2(g)\otimes\overline{\chi},s+\ell-1),$$

hence the formula can be written as:

$$L^*(\mathrm{Ad}(g),1)=2^{\ell+1}A_g\langle g,g\rangle. \tag{2.25}$$

2.5 Triple product L-function and Garrett's method

Let us consider three normalized and primitive cuspidal eigenforms

$$f=\sum_{n\geq1}a_n(f)q^n\in S_k(N,\chi_f),\quad g=\sum_{n\geq1}a_n(g)q^n\in S_\ell(N,\chi_g)\quad\text{and}\quad h=\sum_{n\geq1}a_n(h)q^n\in S_m(N_h,\chi_h)$$

of weights $k,\ell,m\geq2$ such that $\chi_g\chi_f\chi_h=\mathbb{1}$. Let $N:=\mathrm{lcm}(N_f,N_g,N_h)$ and let $\mathbb{Q}_{fgh}:=\mathbb{Q}_f\mathbb{Q}_g\mathbb{Q}_h$ be the field generated by the Fourier coefficients of the three modular forms.

We can consider the compatible system of Galois representations associated with the tensor product of the three representations of f, g and h, i.e.:

$$V_p(f,g,h)=V_p(f)\otimes V_p(g)\otimes V_p(h).$$

Definition 2.13 (Garrett-Rankin triple product)**.** We define the *Garrett-Rankin triple product* L-function to be the L-function associated with the compatible system of Galois representations $V_p(f,g,h)$ and we denote it by $L(f\otimes g\otimes h,s)$.

Remark 2.14. Notice that we consider here the primitive L-function associated with $V(f,g,h)$. In particular, the Euler factors at $q\nmid N$ are given by the degree-8 polynomial:

$$\begin{aligned}L^{(q)}(g\otimes f,s)=&(1-\alpha_q(g)\alpha_q(f)\alpha_q(h)T)^{-1}(1-\alpha_q(g)\beta_q(f)\alpha_q(h)T)^{-1}\\&\times(1-\beta_q(g)\alpha_q(f)\alpha_q(h)T)^{-1}(1-\beta_q(g)\beta_q(f)\alpha_q(h)T)^{-1}\\&\times(1-\alpha_q(g)\alpha_q(f)\beta_q(h)T)^{-1}(1-\alpha_q(g)\beta_q(f)\beta_q(h)T)^{-1}\\&\times(1-\beta_q(g)\alpha_q(f)\beta_q(h)T)^{-1}(1-\beta_q(g)\beta_q(f)\beta_q(h)T)^{-1}\end{aligned}$$

evaluated at $T=q^{-s}$. On the contrary, if $q\mid N$ the naive Euler factors at q defined by the above formula do not need to coincide with those of the L-function we are considering. For a precise recipe of the bad Euler factors, see [PSR87].

The result of Garrett (see [PSR87]) ensures that the L-function $L(f\otimes g\otimes h,s)$, completed with the adequate factor at infinity $L_\infty(f\otimes g\otimes h)$, admits a holomorphic continuation and functional equation of the following type:

$$\Lambda(f\otimes g\otimes h,s)=\varepsilon(f,g,h)\cdot\Lambda(f\otimes g\otimes h,k+\ell+m-2-s)$$

The study of the global sign $\varepsilon(f,g,h)$ of the functional equation is always an important part in the study of the L-function and its properties. In our specific case it gives us information about the vanishing of the function $L(f\otimes g\otimes h,s)$ at the central point:

$$c=\frac{k+\ell+m-2}{2}.$$

The global sign for the functional equation of $L(f\otimes g\otimes h,s)$ can be decomposed as a product of local signs:

$$\varepsilon(f,g,h)=\prod_{q|N\infty}\varepsilon_q(f,g,h).$$

The triple (k,ℓ,m) is said to be *unbalanced* if there is a *dominant weight*, i.e. if $k\geq\ell+m$ (resp. $\ell\geq k+m$, resp. $m\geq k+\ell$). In the case when none of the weights is dominant, we say that the triple (k,ℓ,m) is *balanced*. In [Pra90] it is proven that

$$\varepsilon_\infty(f,g,h)=\begin{cases}-1, & \text{if } (k,\ell,m) \text{ is balanced;}\\ +1, & \text{if } (k,\ell,m) \text{ is unbalanced.}\end{cases}$$

From now on we will assume that the local signs $\varepsilon_v(f,g,h)$ are all equal to $+1$. In particular we shall assume that the triple of weights (k,ℓ,m) is unbalanced and that the dominant weight is ℓ, so that:

$$L_\infty(f\otimes g\otimes h,s)=\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+1)\Gamma_{\mathbb{C}}(s-m+1)\Gamma_{\mathbb{C}}(s+2-k-m).$$

This implies of course that the global sign is $+1$ so that the order of vanishing at $s=c$ must be even thanks to the functional equation.

Notice that the condition $\chi_f\chi_g\chi_h=\mathbb{1}$ implies that $k+\ell+m$ must be even, hence we can write:

$$\ell=k+m+2t,\qquad t\geq 0.$$

We define the trilinear period associated to the triple $(\breve{f},\breve{g},\breve{h})\in M_k(N,\chi_f)[f]\times M_\ell(N,\chi_g)[g]\times M_m(N,\chi_h)[h]$ as follows:

$$\mathrm{I}(\breve{f},\breve{g},\breve{h}):=\langle\breve{g}^*,\delta_k^t\breve{f}\cdot\breve{h}\rangle_N,$$

The main result of Ichino, Watson and Woodbury says that there exists a choice of test vector $(\breve{f},\breve{g},\breve{h})$ such that:

$$\frac{|I(\breve{f},\breve{g},\breve{h})|^2}{\langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}=C(f,g,h)\cdot\frac{L^*(f\otimes g\otimes h,c)}{L^*(\mathrm{Ad}(f),1)L^*(\mathrm{Ad}(g),1)L^*(\mathrm{Ad}(h),1)}\tag{2.26}$$

where:

$$C(f,g,h)=C_{\mathrm{Pet}}\cdot\prod_{v|N\infty}C_v(\breve{f},\breve{g},\breve{h})\in\mathbb{Q}^\times.$$

In fact the result of Ichino guarantees the above equality for all of the possible choices of test vectors while the refinements of Watson and Woodbury assure the rationality and non-triviality of the local factor involved in the product. Here C_{Pet} is an absolute constant only depending on the choice of the normalization of the Petersson product while the $C_v(\breve{f},\breve{g},\breve{h})$ are encoding data on the admissible representations of $\mathrm{GL}_2(\mathbb{Q}_v)$ and on the local components at v of $\breve{f},\breve{g}$ and $\breve{h}$. In particular, the constant is independent on the weights.

From this discussion we can derive the following:

Theorem 2.15 (Harris-Kudla, Ichino, Watson). *For an unbalanced triple of modular forms (f, g, h) such that $\ell = k + m + 2t$, $t \geq 0$, there exists a triple of modular forms:*

$$(\breve{f}, \breve{g}, \breve{h}) \in M_k(N, \chi_f)[f] \times M_\ell(N, \chi_g)[g] \times M_m(N, \chi_h)[h]$$

such that

$$\mathfrak{f}_{\mathrm{Gar}}(k, \ell, m) \cdot L(f \otimes g \otimes h, c) = |\mathrm{I}(\breve{f}, \breve{g}, \breve{h})|^2$$

where

$$\mathfrak{f}_{\mathrm{Gar}}(k, \ell, m) = (c-1)!(c-k)!(c-m)!(c-k-m+1)!\pi^{-2\ell} \cdot 2^{1-3\ell-k-m} C_{\mathrm{Gar}}$$

and with $C_{\mathrm{Gar}} = C(\breve{f}, \breve{g}, \breve{h})/(A_f A_g A_h)$ which is independent on the triple of weights (k, ℓ, m);

Proof. Starting from equation (2.26), substitute the adjoint L-function using equation (2.25) and make explicit the factor at infinity of $L^*(f \otimes g \otimes h, c)$. □

Chapter 3

p-adic interpolation of classical L-functions

The basic setting for the p-adic interpolation of a complex L-function $L(M,s)$ is the following:

(1) one considers a set of special values for $s \in \mathcal{U} \subset \mathbb{C}$, which is *large enough*, and can see $\mathcal{U}$ as a subspace of a p-adically complete space $\widehat{\mathcal{U}}$,

(2) the complex values $L(V,s)$ for $s \in \mathcal{U}$ can be naturally seen as values in a p-adic Banach algebra A,

(3) those values enjoy good p-adic properties of continuity (or analyticity),

then one can create a p-adic L-function $\mathscr{L}_p(M) : \widehat{\mathcal{U}} \to A$ via interpolation. This function satisfies an interpolation and defining property of the form:

$$\mathscr{L}_p(M)(s) = c(M,s) \cdot L(M,s), \qquad \forall s \in \mathcal{U} \tag{3.1}$$

where $c(M,s)$ is a fudge factor that we discuss in more detail below. This new L-function is supposed to encode data about the motive M which is the object of our study, even though it is not defined directly in terms of M. In particular, evaluating the function $\mathscr{L}_p(M)$ at a special point outside the region of interpolation $\mathcal{U}$, the resulting value does not correspond anymore to a complex counterpart and might reveal new information about M. It is important to stress that in some cases $\mathscr{L}_p(M)$ strongly depends on the region of interpolation that one choses. Therefore, different choices for $\mathcal{U}$ could lead to the construction of very different p-adic L-functions arising from the same complex L-function.

There is another way to interpret this interpolation by p-adically varying the motive M, instead of the variable s. More precisely:

(1') fix a variable s, for instance $s = 0$, and p-adically *deform* the motive i.e. consider a family $\mathbb{M} := \{M_k\}_k$ indexed by a p-adically continuous *weight variable* $k \in \mathcal{W}$ such that $M_{k_0} = M$ for some $k_0 \in \mathcal{W}$,

(2') prove that there exists a dense subset $\mathcal{W}^{\mathrm{cl}} \subset \mathcal{W}$ containing k_0 such that the family of complex L-values $L(M_k, 0)$ lie in a p-adic Banach algebra A,

(3') show that those values enjoy good p-adic interpolation properties.

The result of this process is a p-adic L-function $\mathscr{L}_p(\mathbb{M}) : \mathcal{W} \to A$ which interpolates the motive in the following sense:

$$\mathscr{L}_p(\mathbb{M})(k) = c(M_k) \cdot L(M_k, 0), \qquad \forall k \in \mathcal{W}^{\mathrm{cl}}. \tag{3.2}$$

In particular, $\mathscr{L}_p(\mathbb{M})(k_0) = c(M) \cdot L(M, 0)$. This kind of L-functions can be interpreted as a sort of p-adic analogous of the derivative of the classical L-function.

In the formulae (3.1) and (3.2) we did not explain in detail the fudge factors which are fundamental for the creation of the p-adic L-function. Some of those might come from the fact that one uses the completed or the uncompleted classical L-function, some might arise from a non-standard normalization of the L-function and some other are needed to actually make the p-adic interpolation possible. The latter are the most important in order to be able to realize the points (2), (2') and (3), (3') of the above discussion. In particular:

(2),(2') In order to perform a p-adic interpolation in an algebraic way one needs to remove the transcendental part of the complex L-function, without removing too much information with it. The periods are the *canonical transcendental numbers* generating an algebraic value out of the (a priori transcendental) special value $L(M_k, 0)$ in such a way that the resulting function still contains interesting data (we might pick the function itself as period, but the resulting p-adic analogue would not be so interesting...).

Deligne defines a critical motive M as a motive such that $L(M, 0)$ is a critical value, i.e. such that the factor at infinity $L_\infty(M, 0)$ is a non-zero number. For a critical motive, Deligne conjectures that the motivic period $\mathrm{Per}(M)$ is the good candidate to canonically algebrize the special value $L(M, 0)$, so that:

$$\mathrm{Per}(M) \cdot L(M, 0) \in \bar{\mathbb{Q}}.$$

(3),(3') The Euler-like factors arise because in the Euler product definition of L there is $\Phi_p(p^{-s})$, the Euler factor at p. Since the function $k \mapsto p^k$ is p-adically ugly, we need to remove it in order to ensure continuity and perform the p-adic interpolation.

Putting everything together, the general (imprecise) form of a p-adic interpolation formula is the following:

$$\mathscr{L}_p(\mathbb{M})(k) = \mathfrak{e}(M_k) \cdot \mathfrak{a}(k) \cdot \mathfrak{f}(k) \cdot \mathrm{Per}(M_k) \cdot L(M_k, 0), \qquad \forall k \in \mathcal{W}^{\mathrm{cl}} \tag{3.3}$$

and this formula uniquely determines the p-adic L-function $\mathscr{L}_p(\mathbb{M})$. Here

- $\mathfrak{e}(M_k)$ denotes the Euler factors at p that we remove to perform the interpolation. Sometimes we will simply write $\mathfrak{e}(k)$ to lighten the notation, when no confusion is created.
- $\mathfrak{a}(k)$ are the terms which do not behave nicely p-adically arising from the usage of the complete L-function. In particular it will contain powers of π and factorials depending on the weight variable;
- $\mathfrak{f}(k)$ are factors which contain the information about the normalization taken and they behave nicely p-adically (possibly after a wise choice of p);

Apart from the general philosophy, in practice one first needs to construct those p-adic L-functions and then prove the interpolation property. In this sense, a p-adic L-function is a power series in $\mathbb{Z}_p[[T]]$, which can be seen as the Amice transform of a bounded p-adic measure. It is not always an easy goal to achieve the construction and we will not enter in the details of these constructions, although we might give some brief explanations. In this chapter we want to introduce the p-adic L-functions that we need, explaining in details the domain, the region of interpolation and stressing the importance of the constants that we did not show here, expecially the periods. The outcome will be a list of p-adic L-function determined by their respective interpolation properties.

3.1 Complex multiplication and p-adic L-functions

Let us fix a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$. In this section we will introduce:

- Katz two variable p-adic L-function;
- Bertolini-Darmon-Prasanna p-adic L-function (BDP p-adic L-function from now on).

They interpolate special values of complex L-functions along an appropriate subspace of Σ_K, the set of Hecke characters of K. In order to make the p-adic interpolation meaningful we need to endow the space Σ_K with a p-adic topology that we describe briefly.

We first recall the adelic interpretation of the Hecke characters. A Hecke character $\psi : I_{\mathfrak{c}} \to \mathbb{C}^\times$ of infinity type (κ_1, κ_2) can be seen as a character $\psi_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathbb{C}^\times$ on the idèle group of K, which is trivial on $K^\times$. The action of the character is described by:

$$\psi_A(\lambda \cdot x_f \cdot x_\infty) := \psi_{\mathbb{A}}(x_f) \cdot x_\infty^{-\kappa_1} \overline{x}_\infty^{-\kappa_2}, \qquad \forall (\lambda, x_f, x_\infty) \in K^\times \times \widehat{K} \times K_\infty^\times$$

where $K_\infty = K \otimes \mathbb{R}$ and $\widehat{K}^\times$ are the finite idèle. In particular, the data of $\psi_{\mathbb{A}}$ is equivalent to the collection of local characters $\{\psi_{\mathbb{A},v}\}_v$, for all places v of K. For $\psi \in \Sigma_K$, the correspondence $\psi \mapsto \psi_{\mathbb{A}}$ is determined on ideals $\mathfrak{a}$ coprime to the conductor $\mathfrak{c}$ by the following formula:

$$\psi(\mathfrak{a}) = \prod_{v \mid \mathfrak{a}} \psi_{\mathbb{A},v}(\pi_v)^{v(\mathfrak{a})},$$

where we denote by v both the place and the corresponding prime ideal by abuse of notation. In particular, since $\psi_{\mathbb{A}}(\alpha) = 1$ for every $\alpha \in J_{\mathfrak{c}}$ (in fact for every $\alpha \in K^\times$), we recover that $\psi((\alpha)) = \alpha^{\kappa_1} \overline{\alpha}^{\kappa_2}$.

From now on we will denote by ψ both the classical Hecke character and its idèpic interpretation. Let us assume that $\psi \in \Sigma_K$. From our definitions it follows that:

- the image of ψ is an algebraic number;
- on the set $\mathbb{A}_K^{(p)}$ of idèles prime to p, the image of the character lies in $\overline{\mathbb{Z}}_p$.

This implies that we can embed Σ_K inside the set of continuous functions $\mathcal{C}(\mathbb{A}_K^{(p)}, \overline{\mathbb{Z}}_p)$. The latter is naturally endowed with the compact open topology which is equivalent to the topology of uniform convergence on $\overline{\mathbb{Z}}_p$. Then we denote by $\widehat{\Sigma}_K$ the completion of Σ_K with respect to this topology.

In the definition of both Katz and BDP p-adic L-functions, the space of interpolation is some set $\Sigma_0 \subset \Sigma_K$ of characters whose image is bounded, i.e. there exists a finite extension F of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, such that the image of any $\psi \in \Sigma_0$ lies in $\mathcal{O}$.

3.1.1 Periods and complex multiplication

Periods need some more explanations than the other constants described in equation (3.3). Hence we devote this subsection to the introduction of:

- the complex period $\Omega(A)$ and the p-adic period $\Omega_p(A)$ associated to an elliptic curve A with complex multiplication by $\mathcal{O}_K$;
- the complex period $\Omega(\psi)$ and the p-adic period $\Omega_p(\psi)$ associated to an Hecke character ψ of infinity type (κ_1, κ_2).

Since the theory of complex multiplication ensures us that every CM elliptic curve defines an Hecke character of infinity type $(1, 0)$, it is not surprising that we have some relations among those periods.

The $\Omega(\psi)$ are the *motivic periods* associated to an Hecke character ψ of a quadratic imaginary field. While we do not treat the definition here, which is essentially due to Deligne and it can be found in [Sch88], we focus on their properties. In particular, the following proposition uncovers their importance:

Proposition 3.1. *If ψ is a Hecke character of K of infinity type (κ_1, κ_2), $\kappa_1 > \kappa_2$ and m critical for $L(\psi^{-1}, s)$. Then the ratio:*

$$\frac{L(\psi^{-1}, m)}{(2\pi i)^m \Omega(\psi')}$$

lies in the field $\mathbb{Q}_\psi$ and the assignement $\psi \mapsto \frac{L(\psi^{-1}, m)}{(2\pi i)^m \Omega(\psi')}$ is G_K-equivariant.

Proof. See [GS81] and [Bla86]. □

In other terms, the motivic periods $\Omega(\psi)$ describe in the best way the transcendental part of the Hecke L-function. The reason for which we introduce at the same time the periods associated with A is the following proposition which explains the link between them:

Proposition 3.2. *If ψ is a Hecke character of K of infinity type (κ_1, κ_2), then the ratio:*

$$\frac{\Omega(\psi')}{(2\pi i)^{\kappa_2} \Omega(A)^{\kappa_1 - \kappa_2}}$$

is algebraic.

Proof. See [Sch88, II.1.8]. □

Moreover since we define

$$\Omega_p(\psi') := \Omega_p(A)^{\kappa_1 - \kappa_2} \frac{\Omega(\psi')}{(2\pi i)^{\kappa_2} \Omega(A)^{\kappa_1 - \kappa_2}},$$

the quotient $\Omega_p(\psi')/\Omega_p(A)$ is also algebraic. Since we are only concerned with algebraicity properties, this allows us to only define the periods $\Omega(A)$ and $\Omega_p(A)$ in order to proceed in the discussion.

We fix now an elliptic curve A with complex multiplication by $\mathcal{O}_K$, say $\mathbb{C}/\mathcal{O}_K$, and define the complex period $\Omega(A)$. The theory of complex multiplication implies that A is defined over H, the Hilbert class field of K. We chose a regular differential $\omega_A \in \Omega^1(A/H)$ and a non-zero element $\gamma \in H_1(A(\mathbb{C}), \mathbb{Q})$. The complex period is then defined as:

$$\omega_A = \Omega(A) \cdot 2\pi i dz,$$

where z denotes the standard coordinate on $\mathbb{C}/\mathcal{O}_K$. Notice that a different choice of ω_A has the effect of multiplying the period by a scalar in $H^\times$, so that $\Omega(A)$ is only well defined in $\mathbb{C}^\times/H^\times$. In particular, the transcendental part is independent on this choice.

To define the p-adic period $\Omega_p(A)$ we consider the base change of A to $\mathbb{C}_p$ via our fixed embedding $F \subset \mathbb{C}_p$. If we assume that A has good reduction at the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, then we can extend $A_{\mathbb{C}_p}$ to a smooth and proper model $A_{\mathcal{O}_{\mathbb{C}_p}}$ and we can complete along the special fiber to get the formal scheme $\widehat{A}_{\mathcal{O}_{\mathbb{C}_p}}$. We have a non-canonical isomorphism of formal schemes:

$$\iota_p : \widehat{A} \to \widehat{\mathbb{G}}_{\mathrm{m}}.$$

The canonical regular differential on $\widehat{\mathbb{G}}_{\mathrm{m}}$ is defined as du/u, where u is the standard coordinate $\widehat{\mathbb{G}}_{\mathrm{m}}$. We can define the p-adic period $\Omega_p(A)$ as follows:

$$\omega_A = \Omega_p(A) \cdot \iota_p^*(du/u).$$

Having fixed ω_A, the choice ι_p only affects the period $\Omega_p(A)$ by a quantity in ${\mathbb{Z}_p}^\times$. Both the complex and the p-adic period depend H-linearly on the choice of the regular differential ω_A. Since they depend upon it in the same way, their ratio is independent on this choice.

3.1.2 Katz p-adic L-function

Assume that $D \geq 7$ and let $\mathfrak{c} \subseteq \mathcal{O}_K$ be an integral ideal. Fix a prime p, coprime with $\mathfrak{c}$, that splits in K as $p = \wp\bar{\wp}$ (we chose $\wp$ as the prime defining the embedding $K \subset \mathbb{Q}_p$).

Denote by $\Sigma_K(\mathfrak{c})$ the set of Hecke characters of K of conductor dividing $\mathfrak{c}$. We say that a character $\psi \in \Sigma(\mathfrak{c})$ is a *critical character* if $L(\psi^{-1}, 0)$ is critical in the sense of Deligne, i.e. $L_\infty(\psi^{-1}, s)$ has no zeroes nor poles at $s = 0$ (see also the discussion at the beginning of the chapter). We define $\Sigma_{\text{crit}}(\mathfrak{c})$ to be the set of critical characters. This set is naturally the disjoint union of the two subsets

$$\Sigma^{(1)}_{\text{crit}}(\mathfrak{c}) = \{\psi \in \Sigma_{\text{crit}}(\mathfrak{c}) \text{ of infinity type } (\kappa_1, \kappa_2), \kappa_1 \leq 0, \kappa_2 \geq 1\},$$

$$\Sigma^{(2)}_{\text{crit}}(\mathfrak{c}) = \{\psi \in \Sigma_{\text{crit}}(\mathfrak{c}) \text{ of infinity type } (\kappa_1, \kappa_2), \kappa_1 \geq 1, \kappa_2 \leq 0\}.$$

These sets are conjugates with respect to the involution $\psi \mapsto \psi'$ if and only if $\bar{\mathfrak{c}} = \mathfrak{c}$. We denote by $\widehat{\Sigma}_{\text{crit}}(\mathfrak{c})$ the completion of $\Sigma_{\text{crit}}(\mathfrak{c})$ with respect to the compact open topology discussed at the beginning of the section. Since characters in $\Sigma^{(1)}_{\text{crit}}(\mathfrak{c})$ can be p-adically approximated by characters in $\Sigma^{(2)}_{\text{crit}}(\mathfrak{c})$ and viceversa, we have that:

$$\widehat{\Sigma}_{\text{crit}}(\mathfrak{c}) = \widehat{\Sigma}^{(2)}_{\text{crit}}(\mathfrak{c}) = \widehat{\Sigma}^{(1)}_{\text{crit}}(\mathfrak{c})$$

Katz constructs in [Kat76] a p-adic L-function by interpolating the suitably normalized values $L(\psi^{-1}, 0)$ as ψ ranges over $\Sigma^{(2)}_K$. More precisely, there exists a p-adic analytic function

$$\mathscr{L}_p(K) : \widehat{\Sigma}_{\text{crit}}(\mathfrak{c}) \longrightarrow \mathbb{C}_p$$

which is uniquely characterized by the following interpolation property:

$$\frac{\mathscr{L}_p(K)(\psi)}{\Omega_p(A)^{\kappa_1 - \kappa_2}} = \mathfrak{e}_{\mathrm{K}}(\psi) \cdot \mathfrak{a}_{\mathrm{K}}(\psi) \cdot \mathfrak{f}_{\mathrm{K}}(\psi) \cdot \frac{L_{\mathfrak{c}}(\psi^{-1}, 0)}{\Omega(A)^{\kappa_1 - \kappa_2}}, \tag{3.4}$$

for all $\psi \in \Sigma^{(2)}_{\text{crit}}(\mathfrak{c})$ of infinity type (κ_1, κ_2), where

$$\mathfrak{e}_{\mathrm{K}}(\psi) = (1 - \frac{\psi(\wp)}{p})(1 - \psi^{-1}(\bar{\wp}));$$
$$\mathfrak{a}_{\mathrm{K}}(\psi) = \frac{\omega_K}{2} \cdot \Big(\frac{\sqrt{D}}{2}\Big)^{\kappa_2};$$
$$\mathfrak{f}_{\mathrm{K}}(\psi) = \frac{(\kappa_1 - 1)!}{\pi^{\kappa_2}}.$$

This p-adic L-function satisfies the functional equation:

$$\mathscr{L}_p(K)(\psi) = \mathscr{L}_p(K)((\psi')^{-1}\mathbf{N}_K) \tag{3.5}$$

as shown in [Gro80, pp. 90-91].

The set $\widehat{\Sigma}_{\text{crit}}(\mathfrak{c})$ also contains finite order anticyclotomic characters of conductor c dividing $\mathfrak{c}$, hence one can approximate them using characters within the region of interpolation. Recall the choice of an elliptic unit u done in section 1.7 and the Hilbert class field H of conductor c. Define:

$$u_\psi := \begin{cases} \sum_{\sigma \in G_{H|K}} \psi^{-1}(\sigma) u^\sigma \in (\mathcal{O}_H^\times)_{\mathbb{Q}_\psi}, & \text{if } \psi \neq \mathbb{1} \\ \text{any } p\text{-unit } u_\wp \in \mathcal{O}_H[\frac{1}{p}]^\times \text{ s.t. } (u_\wp) = \wp^{h_K}, & \text{if } \psi = \mathbb{1} \end{cases} \tag{3.6}$$

The following result is commonly known as Katz's Kronecker p-adic limit formula. It is an explicit formula for the value of $\mathscr{L}_p(K)$ at a finite order character ψ of G_K (cf. [Kat76, §10.4, 10.5], [Gro80, p. 90], [deS87, Ch. II, §5.2]):

$$\mathscr{L}_p(K)(\psi) = \mathfrak{f}_p(\psi) \cdot \log_p(u_{\bar{\psi}}), \tag{3.7}$$

where

$$\mathfrak{f}_p(\psi) = \begin{cases} \frac{1}{2}(\frac{1}{p}-1) & \text{if } \psi = 1 \\ \frac{-1}{24c}(1-\psi(\bar{\wp}))(1-\frac{\psi(\bar{\wp})}{p}) & \text{if } \psi \neq 1. \end{cases} \tag{3.8}$$

Here $c > 0$ is the smallest positive integer in the conductor ideal of ψ.

3.1.3 Bertolini-Darmon-Prasanna p-adic L-function

Let $f \in S_k(N_f, \chi_f)$ be an eigenform and let K be an imaginary quadratic field of discriminant $-D \leq -7$ fulfilling the Heegner hypothesis relative to f, so that there exists a cyclic ideal $\mathfrak{N} \subset \mathcal{O}_K$ of order N_E.

For any Hecke character ψ of infinity type (κ_1, κ_2), let $L(f, \psi, s)$ denote the L-function associated to the compatible system of Galois representations afforded by the tensor product $\varrho_{f|G_K} \otimes \psi$ of the (restriction to G_K of) the Galois representations attached to f and the character ψ.

As usual, $L(f, \psi, s) = \prod_q L^{(q)}(q^{-s})$ is defined as a product of Euler factors ranging over the set of prime numbers. The Euler factors at the primes q such that $q \nmid N$ are exactly the same as that of the Rankin L-series $L(\theta_\psi \otimes f, s)$ introduced above, but may differ at the primes q such that $q \mid N$ (details can be found in [Gro84] for f modular form of weight 2).

The L-factor at infinity is given, for an hecke character of type (κ_1, κ_2), by the following formula:

$$L_\infty(f, \psi, s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - \min(k-1, m) - \kappa_0)$$

where $m = |\kappa_1 - \kappa_2|$ and $\kappa_0 := \min(\kappa_1, \kappa_2)$. Following our notation, the completed L-function $\Lambda(f, \chi, s)$ satisfies a functional equation of the form:

$$\Lambda(f, \psi, s) = \varepsilon(f, \chi, s)\Lambda(f^*, \overline{\psi}, k + \kappa_1 + \kappa_2 - s)$$

For our convenience we now switch the convention and deal with the L-function $L(f, \psi^{-1}, s)$ rather than $L(f, \psi, s)$. In particular we say that a Hecke character ψ is *critical* if $s = 0$ is a critical point for $L(f, \psi^{-1}, s)$. Critical values for this Rankin L-function were predicted by Deligne and proved by Shimura in [Shi76]. An appropriate discussion of those results which serves our purposes is carried out in [BDP13, §4.1]. The discussion in loc. cit. states that the set of critical character $\Sigma_{f,K}$ in the sense of Deligne can be naturally seen as the disjoint union of the three subsets:

$$\Sigma^{(1)}_{f,K} = \{\psi \in \Sigma_{f,K} \text{ such that } 1 \leq \kappa_1, \kappa_2 \leq k-1\},$$

$$\Sigma^{(2)}_{f,K} = \{\psi \in \Sigma_{f,K} \text{ such that } \kappa_1 \geq k, \kappa_2 \leq 0\},$$

$$\Sigma^{(2')}_{f,K} = \{\psi \in \Sigma_{f,K} \text{ such that } \kappa_2 \geq k, \kappa_1 \leq 0\}.$$

The regions $\Sigma^{(2)}_{f,K}$ and $\Sigma^{(2')}_{f,K}$ are interchanged by the involution $\psi \mapsto \psi'$ and the associated complex period is a power of the CM period $\Omega(A)$. On the region $\Sigma^{(1)}_{f,K}$ the period is the Petersson inner product $\langle f, f \rangle$. The so called Bertolini-Darmon-Prasanna p-adic L-function is obtained by an adequate interpolation of the special values of $L(f, \psi^{-1}, s)$ for an appropriate subset of characters in $\Sigma^{(2)}_{f,K}$.

In particular, we say that a character $\psi \in \Sigma_{f,K}$ is *central critical* if $\kappa_1 + \kappa_2 = k$ and $\varepsilon_\psi = \chi_f$. In this way the point $s = 0$ is critical for $L(f, \psi^{-1}, s)$ and we write $\Sigma^{(i)}_{\text{cc}}$ for the characters in $\Sigma^{(i)}_{f,K}$ which are central critical. In particular we can express them as follows:

$$\Sigma^{(1)}_{\text{cc}} = \{\psi \in \Sigma_{f,K} \text{ of infinity type } (k+j, -j), \text{ where } 1-k \leq \lambda \leq -1\},$$

$$\Sigma^{(2)}_{\text{cc}} = \{\psi \in \Sigma_{f,K} \text{ of infinity type } (k+j, -j), \lambda \geq 0\},$$

$$\Sigma^{(2')}_{\text{cc}} = \{\psi \in \Sigma_{f,K} \text{ of infinity type } (-j, k+j), \lambda \geq 0\}.$$

Let $c \in \mathbb{Z}_{\geq 1}$ be an integer such that $(c, N_f Dp) = 1$. We define the set of characters $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$ to be the subset of central critical Hecke character ψ *of finite type* $(c, \mathfrak{N}, \chi_f)$, i.e. such that $\mathfrak{c}_\psi \mid c\,\mathfrak{N}$ and $\varepsilon_v(f, \psi^{-1}) = +1$ for all finite places v. This set is naturally the disjoint union of two subsets:

$$\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f) = \{\psi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f) \text{ of infinity type } (k+j, -j), \text{ where } 1-k \leq -j \leq -1\},$$

$$\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f) = \{\psi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f) \text{ of infinity type } (k+j, -j), j \geq 0\}.$$

If we take the completion of $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$ with respect to the compact open topology, the resulting space $\widehat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$ contains $\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$ as a dense subset. On the set $\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$ the local sign $\varepsilon_\infty(f, \psi^{\cdot -1})$ is $+1$. Thus the global sign is also positive, hence the central critical value is non-zero most of the times. The Bertolini-Darmon-Prasanna p-adic (Rankin) L-function attached to the pair (f, K) is the function:

$$\mathscr{L}_p(f/K) : \widehat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f) \to \mathbb{C}_p$$

defined by the following interpolation formula:

$$\mathscr{L}_p(f/K)(\psi) = \mathfrak{e}_{\mathrm{BDP}}(\psi) \cdot \mathfrak{a}_{\mathrm{BDP}}(\psi) \cdot \mathfrak{f}_{\mathrm{BDP}}(\psi) \cdot \frac{\Omega_p(A)^{2k+4j}}{\Omega(A)^{2k+4j}} \cdot L(f, \psi^{-1}, 0), \tag{3.9}$$

for all characters $\Psi \in \Sigma^{(2)}_{f,K}$ of type $(k+j, -j)$, for $j \geq 0$, where:

- $\mathfrak{e}_{\mathrm{BDP}}(\psi) = (1 - \alpha_p(f)\psi^{-1}(\bar{\wp}))(1 - \beta_p(f)\psi^{-1}(\bar{\wp}))$,
- $\mathfrak{a}_{\mathrm{BDP}}(\psi) = j!(k+j-1)!\pi^{k+2j-1}$
- $\mathfrak{f}_{\mathrm{BDP}}(\psi) = \left(\frac{2}{c\sqrt{D_K}}\right)^{k+2j-1} \cdot 2^{\sharp q|(D_K, N_E)} \cdot \prod_{q|c} \frac{q - \chi_K(q)}{q-1} \cdot \omega(f, \psi)^{-1}$.

The factor $\omega(f, \psi)$ is defined in [BDP13, (5.1.11)], but we recall here its construction. For ψ of infinity type $(k+j, -j)$, we define $\psi_j := \psi \mathbf{N}_K^j$. Because of our hypothesis, $N = \mathfrak{N}\bar{\mathfrak{N}}$ in $\mathcal{O}_c$. We can choose an integral $\mathcal{O}_c$-ideal $\mathfrak{b}$ and a nonzero element $b \in \mathcal{O}_c$ such that:

$$(\mathfrak{b}, Nc) = 1, \qquad \mathfrak{b}\mathfrak{N} = (b). \tag{3.10}$$

If we call ω_f the generalized eigenvalue of the Atkin-Lehner operator W_N (see [BDP13, Lemma 5.2] for more details), then:

$$\omega(f, \psi) := \omega_f \cdot \chi_f(N^K_{\mathbb{Q}}(\mathfrak{b})^{-1} \cdot \psi_j(\mathfrak{b})(-N)^{k/2+j} b^{-k-2j}. \tag{3.11}$$

It is a root of unity in the field $\mathbb{Q}_\psi(f, \sqrt{-N})$ and it does not depend on the choice $(\mathfrak{b}, b)$ made above (see [BDP13, lemma 5.3]).

Remark 3.3. We would like to remark that the period appearing here is not the same used for the construction of the p-adic L-function in [BDP13, (5.1.15)], where an auxiliary elliptic curve A_0 having complex multiplication by $\mathcal{O}_c$ is used. Nevertheless, thanks to the theory of complex multiplication and proposition 3.2 both the complex and the p-adic periods arising from A and A_0 are compatible in the sense that

$$\frac{\Omega_p(A)}{\Omega(A)} = \frac{\Omega_p(A_0)}{\Omega(A_0)},$$

hence the above formula holds. In particular one may chose an isomorphism $\widehat{A}_0 \to \widehat{\mathbb{G}}_m$ such that the associated canonical differential ω_{can} on $\widehat{A}_0$ is compatible with that of $\widehat{A}$.

Remark 3.4. We do not go into the details of the construction, but it is interesting to remark how the values of the interpolation property are extended to a p-adic L-function. We have that:

$$\mathfrak{a}_{\mathrm{BDP}}(\psi) \cdot \mathfrak{f}_{\mathrm{BDP}}(\psi) \cdot \omega(f,\psi)^{-1} \cdot L(f,\psi^{-1},0)/\Omega(A)^{2k+4j} = \left(\sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O}_c)} \psi_j^{-1}(\mathfrak{a}) \cdot \delta_k^j f(\tilde{x}(\mathfrak{a})) \right)^2, \tag{3.12}$$

where $\tilde{x}(\mathfrak{a})$ is defined in section 1.7. On the p-adic side we have the following formula:

$$\mathscr{L}_p(f/K)(\psi) = \left(\sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O}_c)} \psi^{-1}(\mathfrak{a}) \mathbf{N}_K(\mathfrak{a})^{-j} d^j f^{[p]}(\tilde{x}_p(\mathfrak{a})) \right)^2 \tag{3.13}$$

where $\tilde{x}_p := (x, \omega_{\mathrm{can}})$, $\tilde{x}_p(\mathfrak{a}) = (x(\mathfrak{a}), \omega_{\mathrm{can},\mathfrak{a}})$, with $\omega_{\mathrm{can},\mathfrak{a}}$ is the pullback of ω_{can} via the map $A \to A_{\mathfrak{a}}$. The two results are, respectively, theorems 5.4 with theorem 5.5 and theorem 5.9 of [BDP13]. The second equation is obtained from the first with a comparison of the Shimura-Maass derivative and the Serre derivative of f at CM points, i.e.

$$\delta_k f(\tilde{x}(\mathfrak{a})) = df(\tilde{x}(\mathfrak{a})).$$

which is essentially the content of [BDP13, Proposition 1.12 (3)].

If ψ is a finite order anticyclotomic character of conductor $c \mid \mathfrak{c}$, then $\psi\mathbf{N}_K$ lies outside the region of interpolation but it can be approximated using characters of $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \chi_f)$. The main theorem of [BDP12] asserts that

$$\mathscr{L}_p(f,K)(\psi^{-1}\mathbf{N}_K) = \mathfrak{f}_p(f,\psi) \times \log_{E,p}(P_\psi)^2 \tag{3.14}$$

where $\mathfrak{f}_p(f,\psi) = (1-\psi(\bar{\wp})p^{-1}a_p(f) + \psi^2(\bar{\wp})p^{-1})^2$.

3.2 Hida's work and p-adic L-functions

We now want to detail what we informally described in the introduction of the chapter as *deformation of a motive*. Roughly speaking, given a motive M we expect to find a collection $\mathrm{M} = \{M_k\}_k$ of motives indexed by a p-adic variable which gives a deformation of M, i.e. there exists k_0 such that $\mathrm{M}_{k_0} = M$ and two motives M_k and $M_{k'}$ are in some sense *close enough* whenever k and k' are close enough p-adically.

We treat here a very specific instance of this: *Hida families*. Informally, a Hida family is a collection of modular forms $\{f_k\}_k$ such that the induced residual representations are all equal. In particular we have congruences between two members of the family which tell us that f_k and $f_{k'}$ are p-adically close whenever k and k' are p-adically close. In this case, the concept of being close for two modular forms can be made precise by working on the coefficients of the q-expansions.

In order to describe Hida's result and explain its usefulness for the creation of p-adic L-functions we first need to discuss the set $\mathcal{W}$ in which the *weight variable* k lives and define precisely what an Hida family is for us. We closely follow the definitions given in [DR14, §2.6], but it might be useful for the reader to also have a look to [Hid93], [Wil88], [How07, §2], [Ca1, §2.1] and [Was80].

3.2.1 Iwasawa algebrae and the weight space

We fix a rational prime p and we assume for simplicity that $p \geq 3$. We will use the notation $\Delta_N = (\mathbb{Z}/N\mathbb{Z})^\times$. Let $\mathcal{O}$ be the ring of integers of a finite extension $F \mid \mathbb{Q}_p$ and fix a uniformizer $\pi \in \mathcal{O}$. Write q for the residual characteristic of $\mathcal{O}/\pi\mathcal{O}$. A basic structure result, which can be found for instance in [Neu91, II, prop. 5.7], tells us that:

$$\mathcal{O}^\times \simeq \mu_{q-1} \times (1 + \pi\mathcal{O}),$$

so that we also have $F^\times = \pi^{\mathbb{Z}} \times \mathcal{O}$. The second component $1+\pi\mathcal{O}$ can be further decomposed into $\mu_{q^a} \times W_F$, where $W_F = {\mathbb{Z}_p}^{[F:\mathbb{Q}_p]}$ is the $\mathbb{Z}_p$-free part. In the case of $\mathcal{O} = \mathbb{Z}_p$ we have:

$$ {\mathbb{Z}_p}^\times \simeq \mu_{p-1} \times \Gamma, \qquad \text{where} \qquad \Gamma := 1 + p\mathbb{Z}_p. $$

The projection $\omega : {\mathbb{Z}_p}^\times \twoheadrightarrow \mu_{p-1}$ induces a Dirichlet character modulo p that we denote by the same letter $\omega : \Delta_p \simeq \mu_{p-1}$ (notice that $\mu_{p-1} \subset \bar{\mathbb{Q}}$, hence it can be seen both as complex and p-adic number via our fixed embeddings). The latter is called the *Teichmuller character*. It associates to each class $d \in \Delta_p$ the unique root of unity of $\mathbb{Z}_p$ congruent to d modulo $p-1$. This allows us to identify ${\mathbb{Z}_p}^\times \simeq \Delta_p \times \Gamma$. The second projection $\langle - \rangle : \mathbb{Z}_p^\times \twoheadrightarrow \Gamma$ is none other then $\langle x \rangle = \omega^{-1}(x) \cdot x$.

The Iwasawa logarithm induces the isomorphism $\log_p : \Gamma \simeq p\mathbb{Z}_p$ which implies that Γ is a free $\mathbb{Z}_p$-module of rank one. Remember that $\log_p$ is the unique morphism defined by the usual power series on Γ which extends trivially outside Γ, i.e. it sends p and the roots of unity to 0.

We fix now a topological generator u for Γ once and for all. This choice induces an isomorphism that we will denote as the *base u-logarithm*:

$$ \begin{aligned} \log_u : \Gamma &\simeq \mathbb{Z}_p \\ x &\mapsto \log_u(x) \end{aligned} \tag{3.15} $$

where $\log_u(x) = \log_p(x)/\log_p(u)$. In contrast to the Iwasawa logarithm, this is a non-canonical isomorphism, since it depends on the choice of u. The inverse is given by $z \mapsto u^z := \exp(z \log(u))$, which gives a parametrization of Γ.

Recall that if we consider a profinite group $G = \varprojlim_i G_i$, then we define the Iwasawa algebra associated to G as $\Lambda_G := \mathcal{O}[[G]] := \varprojlim_i \mathcal{O}[G/G_i]$. Given $d \in G$ we write $[d]$ for the corresponding element in $\widetilde{\Lambda}_G$. Fix a positive integer N coprime to p and define the three Iwasawa algebrae:

$$ \Lambda = \mathcal{O}[[\Gamma]], \qquad \widetilde{\Lambda}_N := \mathcal{O}[\Delta_N][[{\mathbb{Z}_p}^\times]] \simeq \mathcal{O}[\Delta_{Np}][[\Gamma]] \quad \text{and} \quad \widetilde{\Lambda} = \widetilde{\Lambda}_1 = \mathcal{O}[[{\mathbb{Z}_p}^\times]]. $$

It is well-known that $\Lambda \simeq \mathcal{O}[[T]]$ via the non-canonical isomorphism $u \mapsto T+1$.

For $\chi \in \widehat{\Delta}_{Np}$, consider the projectors:

$$ e_\chi := \frac{1}{|\Delta_{Np}|} \sum_{d \in \widehat{\Delta}_{Np}} \chi^{-1}(d)[d] \in \widetilde{\Lambda}_N, $$

using the fact that $e_\chi \widetilde{\Lambda}_N = \Lambda e_\chi \simeq \Lambda$, we obtain the decomposition:

$$ \widetilde{\Lambda}_N = \bigoplus_{\chi \in \widehat{\Delta}_{Np}} e_\chi \widetilde{\Lambda}_N \simeq \bigoplus_{\chi \in \widehat{\Delta}_{Np}} \Lambda \tag{3.16} $$

so that we can identify every element $\gamma \in \widetilde{\Lambda}_N$ with a collection of power series $g_\chi(T) \in \mathcal{O}[[T]]$ indexed by $\chi \in \widehat{\Delta}_{Np}$ (see [Was80, p. 243]).

Lemma 3.5. *For every $d \in \Delta_{Np}$ we have: $e_\chi[d] = \chi(d)e_\chi$.*

Remark 3.6. When considering $\widetilde{\Lambda}$ ($N=1$), every character $\chi \in \widehat{\Delta}_p$ is a power of the Teichmüller character, so that $\chi = \omega^i$ for some $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ and we can write $e_i := e_\chi$.

Since $\widetilde{\Lambda}_N$ is a topological $\mathcal{O}$-algebra endowed with the p-adic topology defined by the ideal $(\pi, [u]-1)$, we can consider the formal scheme:

$$ \widetilde{\mathcal{W}}_N := \mathrm{Spf}(\widetilde{\Lambda}_N) = \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\widetilde{\Lambda}_N, -) \simeq \mathrm{Hom}(\Gamma, \mathbb{G}_m(-)). $$

Similarly we define $\mathcal{W} := \mathrm{Spf}(\Lambda)$ and $\widetilde{\mathcal{W}} := \mathrm{Spf}(\widetilde{\Lambda})$. For any topological $\mathcal{O}$-algebra B, the B-points of the above scheme are:

$$\widetilde{\mathcal{W}}_N(B) := \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\widetilde{\Lambda}_N, B) = \mathrm{Hom}_{\mathrm{cont}}(\Delta_{Np} \times \Gamma, B^\times) \simeq \mathrm{Hom}(\Delta_{Np}, B^\times) \times \mathrm{Hom}_{\mathrm{cont}}(\Gamma, B^\times).$$

This determines a decomposition:

$$\widetilde{\mathcal{W}}_N(B) \simeq \coprod_{\chi \in \widehat{\Delta}_{Np}} \mathrm{Hom}(\Lambda, B),$$

hence we have a decomposition of formal schemes:

$$\widetilde{\mathcal{W}}_N = \bigoplus_{\chi \in \widehat{\Delta}_{Np}} \mathcal{W}_\chi, \tag{3.17}$$

where $\mathcal{W}_\chi \simeq \mathcal{W}$, for all χ. Explicitly, for any $\mathcal{O}$-algebra B an element $\nu \in \mathcal{W}_\chi(B)$ is characterized by the fact that if $d \in \Delta_{Np}$, then $\nu([d]) = \chi(d)$ so that, in particular, $\nu(e_\chi) = 1$ and $\nu(e_\psi) = 0$ for $\psi \neq \chi$.

We embed $\mathbb{Z}$ into the set of $\mathcal{O}$-points of $\mathcal{W}$ as follows:

$$\begin{aligned} \mathbb{Z} &\hookrightarrow \mathcal{W}(\mathcal{O}) = \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\Lambda, \mathcal{O}) \\ k &\mapsto \{\nu_k : [u] \mapsto u^k\} \end{aligned} \tag{3.18}$$

We write $\mathcal{W}^{\mathrm{cl}}$ for the set $\mathbb{Z}_{\geq 2}$ inside $\mathcal{W}(\mathcal{O})$ via the map defined in equation (3.18). We call $\mathcal{W}^{\mathrm{cl}}$ the set of *classical points* or *arithmetic primes* of Λ. The evaluation map $\nu_k \in \mathcal{W}(\mathcal{O})$ corresponds to the unique prime ideal $P_k = ([u] - u^k) \subset \Lambda$ such that $\nu_k : \Lambda \twoheadrightarrow \Lambda/P_k \hookrightarrow \mathcal{O}$. The map ν_k can be represented on $\mathcal{O}[[T]]$ under the isomorphism defined by $T \mapsto [u] - 1$ and it is given by $\nu_k(T) = u^k - 1$ i.e. the prime ideal P_k is the one generated by $(T+1) - u^k$.

We explicitly translate the action of the evaluation map for the χ-component of $\widetilde{\mathcal{W}}_N$, i.e. we write the corresponding embeddings $\mathbb{Z} \hookrightarrow \widetilde{\mathcal{W}}_N(\mathcal{O})$ under the decomposition (3.17) which is given, for each $k \in \mathbb{Z}$, by:

$$\nu_k^\chi : (\delta, \xi) \mapsto \chi(\delta)\xi^k, \qquad \forall (\delta, \xi) \in \Delta_{Np} \times \Gamma.$$

It can be useful to think $\Delta_{Np} \times \Gamma$ as $\Delta_N \times {\mathbb{Z}_p}^\times$ under the correspondence $(\delta, \xi) \mapsto (d, x)$. In this sense, since any character of Δ_p is a power of the Teichmuller character for $i \in \{0, \dots p-2\}$ and since $p \nmid N$, we can write $\chi = \chi_t \omega^i$, where χ_t is the *tame character*. In this way, ν_k^χ acts on (d, x) as follows:

$$\nu_k^\chi(d, x) = \chi_t(d) \cdot \omega^{i-k}(x) \cdot x^k$$

Thanks to the rigid GAGA theory we can see $\widetilde{\mathcal{W}}_N$, $\widetilde{\mathcal{W}}$ and $\mathcal{W}$ as rigid analytic spaces. Under the same morphism of equation (3.18) we can see $\mathbb{Z}_p$ as a rigid analytic subspace of the weight space $\mathcal{W}(\mathcal{O})$. In this sense, elements of the Iwasawa algebra Λ can be interpreted as analytic functions on $\mathcal{W}$ while passing to the fraction field we deal with analytic meromorphic functions.

3.2.2 Λ-adic modular forms and Hida main result

The first attempts made by Serre to define p-adic modular forms led him to the study of congruences between modular forms via the study of congruences between the Fourier coefficients. These congruences give rise to the following naive definition of compatible family of modular forms as a collection $\{f_k = \sum a_n(k) q^n\}_k$, for almost all positive integers k (i.e. all but finitely many), such that the following congruence relation is respected:

$$k \equiv k' \pmod{\phi(p^\alpha)} \Longrightarrow a_n(k) \equiv a_n(k') \pmod{p^\alpha}. \tag{3.19}$$

where ϕ is the Euler totient function. In particular, k and k' lie in the same residue class modulo $p-1$. Let us recall that by the work of Deligne we can associate to each f_k a compatible system of λ-adic representations V_k. These congruences imply that if we reduce modulo p, the residual representations of each member of the family are equal. This is the basic concept behind the theory of p-adic deformation of Galois representations.

In a sense that we now make precise, the Iwasawa algebra gives a better description of those congruences and the concept of p-adic family of modular forms, allowing them to enjoy a geometric interpretation. Let us consider at first an element $\mathbf{a} \in \Lambda$, which can be thought of as a power series in $\mathcal{O}[[T]]$. It is easy to check that:

$$k \equiv k' \pmod{\phi(p^\alpha)} \Longrightarrow \nu_k(\mathbf{a}) \equiv \nu_{k'}(\mathbf{a}) \pmod{p^\alpha}. \tag{3.20}$$

This is quite natural if we consider $\mathbf{a} \in \Lambda$ being a rigid analytic function on $\mathcal{W}(\mathcal{O})$. Hence, given $\nu \in \mathcal{W}(\mathcal{O})$, we also denote $\nu(\mathbf{a})$ by $\mathbf{a}(\nu)$ or $\mathbf{a}_\nu$. In particular, the function $\nu_k(\mathbf{a})$ must behave nicely for the p-adic topology, so that it is an analytic function of the variable k. Being an element in the Iwasawa algebra we refer to it as an *Iwasawa function*, as does the literature, too (cf. [Was80, §12.2]).

If $\Lambda_f \mid \Lambda$ is a finite flat extension, then we have a natural map $\mathcal{W}_f \to \mathcal{W}$, called *weight map*, and we can define $\mathcal{W}_f^{\mathrm{cl}} \subset \mathcal{W}_f(\mathcal{O})$ as the pull-back of $\mathcal{W}^{\mathrm{cl}}$ along this map. The set $\mathcal{W}_f^{\mathrm{cl}}$ is called set of *arithmetic primes* of Λ_f. We say that a point $\nu \in \mathcal{W}_f^{\mathrm{cl}}$ has *weight* $k \in \mathbb{Z}_{\geq 2}$ if $\nu\mid_\Lambda = \nu_k$. In terms of prime ideals, ν and ν_k correspond to $P \subset \Lambda_f$ and $P_k \subset \Lambda$ and the weight map is just $P \mapsto P_k := P \cap \Lambda$.

If we consider a formal power series $\mathbf{f} = \sum_{n\geq 1} \mathbf{a}_n q^n \in \Lambda_f[[q]]$ it is natural to define the action of $\nu \in \mathcal{W}(\mathbb{C}_p)$ on $\mathbf{g}$ as follows:

$$\nu(\mathbf{f}) = \sum_{n\geq 1} \nu(\mathbf{a})q^n \in \mathbb{C}_p[[q]].$$

In light of the above discussion we know that if two classical points have weights congruent modulo $\phi(p^\alpha)$ then the q-expansions of the corresponding specializations of $\mathbf{f}$ are congruent modulo p^α. In other terms, a formal q-expansion as above is a p-adic analytic family of q-expansions in $\mathbb{C}_p[[q]]$. To mimic the naive definition of p-adic family of modular form, we need to impose that the realizations $\nu(\mathbf{f})$ for $\nu \in \mathcal{W}_f^{\mathrm{cl}}$ are classical modular forms.

Remark 3.7. In order to lighten the notation we will often write $\mathbf{f}_\nu$ to mean $\nu(\mathbf{f})$.

To properly define the concept of compatible family of modular forms, that we call Λ-*adic modular forms*, we fix now a component of the weight space $\mathcal{W}_\chi \simeq \mathcal{W}$ of $\widetilde{\mathcal{W}}_N$ and the decomposition $\chi = \chi_f\omega^i$ into tame and wild character. Recall that a modular form f is called *p-ordinary* (or ordinary at p) if its p-th Fourier coefficient is a p-adic unit, i.e. $|a_p(f)|_p = 1$.

Definition 3.8. A Λ-adic modular form (resp. cusp form) of tame level N and tame character χ_f is a quadruple $(\Lambda_f, \mathcal{U}_f, \mathcal{U}_f^{\mathrm{cl}}, \mathbf{f})$ where:

(i) Λ_f is a complete, finitely generated and flat extension of Λ;

(ii) $\mathcal{U}_f$ is a non-empty open subset of $\mathcal{W}_f(\mathbb{C}_p)$ and $\mathcal{U}_f^{\mathrm{cl}} \subset \mathcal{W}_f^{\mathrm{cl}}$ is dense in $\mathcal{U}_f$;

(iii) $\mathbf{f} \in \Lambda_f[[q]]$ is a formal q-expansion such that, for all $\nu \in \mathcal{U}_f^{\mathrm{cl}}$ of weight k, $\mathbf{f}_\nu \in \mathbb{C}_p[[q]]$ is the q-expansion of a classical ordinary modular form form in $M_k(Np, \chi_f\omega^{i-k})_{\mathbb{C}_p}$ (resp. $S_k(Np, \chi_f\omega^{i-k})_{\mathbb{C}_p}$).

We denote $\mathbb{M}(N, \chi_f)_{\Lambda_f}$ (reap. $\mathbb{S}(N, \chi_f)_{\Lambda_f}$) the space of Λ-adic modular forms (resp. cusp forms) of tame level N and tame character χ_f having coefficients in Λ_f.

On the spaces of Λ-adic modular forms $\mathbb{M}(N, \chi_f)_{\Lambda_f}$ and $\mathbb{S}(N, \chi_f)_{\Lambda_f}$ we have the Λ-adic analogue of the usual Hecke operators T_n. They are compatible with the weight k specializations, i.e. given a Λ-adic modular form $\mathbf{f}$ and a classical weight k point $\nu \in \mathcal{U}_f^{\mathrm{cl}}$, we have that:

$$T_n(\mathbf{f})_\nu = T_n(\mathbf{f}_\nu).$$

This property can be used to define the Hecke operators on Λ-adic modular forms (see [Hid93, §7.3] for more details). In this sense, the multiplication by $[d] \in \widetilde{\Lambda}_N$, whose specialization in weight k is given by $\chi_f(d)\omega^{i-k}(d)d^k$, is the Λ-adic version of the Diamond operator.

Remark 3.9. In the literature people ofter refer to classical weights (or *arithmetic primes*) in a more general way. They are points of the form $\nu_{k,\varepsilon} : [u] \mapsto \varepsilon(u)u^k$, for a finite order character ε on Γ. Since we only restrict to the case in which ε is trivial, we shall often (but not always!) denote without creating ambiguity:

$$\mathbf{f}_\nu = \mathbf{f}_k$$

for a weight-k classical point ν. Moreover, since the set of integers $k \in \mathbb{Z}_{\geq 2}$ such that $k \equiv i \pmod{p-1}$ is p-adically dense in $\mathbb{Z}_p$, in what follows we will often shrink $\mathcal{U}_f^{\mathrm{cl}}$ to classical points of such weight. With this convention, definition 3.8(iii) will simply becomes:

$$\mathbf{f}_k \in S_k(Np, \chi_f)$$

for all $k \in \mathcal{U}_f^{\mathrm{cl}}$.

Consider now an ordinary eigenform f. Since $|a_p(f)|_p = 1$, the two roots $\alpha_p(f)$ and $\beta_p(f)$ can be ordered in such a way that $\alpha_p(f)$ is a p-adic unit. We define the *ordinary p-stabilization of f* to be:

$$f_\alpha(q) := f(q) - \beta_p(f)f(q^p).$$

The modular form f_α is the only normalized eigenform of level Np such that:

$$T_\ell(f_\alpha) = a_\ell(f)f_\alpha, \quad \forall \ell \neq p \qquad \text{and} \qquad U_p(f_\alpha) = \alpha_p(f)f_\alpha$$

A form that satisfies these properties is said to be ordinary and p-stabilized. If $\beta_p(f) = 0$, then $f = f_\alpha$ is already stabilized.

Definition 3.10. A *Hida family* of tame level N and tame character χ_f is a Λ-adic modular form $(\Lambda_f, \mathcal{U}_f, \mathcal{U}_f^{\mathrm{cl}}, \mathbf{f})$, such that Λ_f is finite flat over Λ and for all $\nu \in \mathcal{W}_f^{\mathrm{cl}}$ of weight k, $\mathbf{f}_\nu$ is an ordinary p-stabilized N-new modular form, i.e. there exists f_k of new level N or Np such that $\mathbf{f}_\nu = f_{k,\alpha}$.

Theorem 3.11 (Hida, 1986). *Let $f_\alpha \in S_k(N_f p, \chi_f \omega^i)$ be an ordinary newform of tame level N_f and tame character χ_f (not necessarily new at p). Consider $F := \mathbb{Q}_p[f_\alpha]$, let $\mathcal{O}$ be its ring of integers and $\Lambda := \mathcal{O}[[\Gamma]]$. Then there exists a unique Hida family $(\Lambda_f, \mathcal{U}_f, \mathcal{U}_f^{\mathrm{cl}}, \mathbf{f})$ of tame level N_f and tame character χ_f such that $\mathbf{f}_\nu = f_\alpha$ for a unique point $\nu \in \mathcal{W}_f^{\mathrm{cl}}$ of weight k.*

Proof. See [Hid86]. □

From now on, we will denote a Hida family by $\mathbf{f}$, its specializations by $\mathbf{f}_\nu$ and the modular form whose p-satabilization equals $\mathbf{f}_\nu$ will be denoted by f_ν (in the simplified settings in which we will identify ν with its weight k, we will write respectively $\mathbf{f}_k$ and f_k). A priori it is unclear wether the weight k specialization of an Hida family $\mathbf{f}$ is always old at p or not. The following lemma clarifies this situation:

Lemma 3.12. *Given a Hida family $(\Lambda_f, \mathcal{U}_f, \mathcal{U}_f^{\mathrm{cl}}, \mathbf{f})$, and a weight $k > 2$ classical point $\nu \in \mathcal{U}_f^{\mathrm{cl}}$, the specialization $\mathbf{f}_\nu$ of the Hida family at ν is always old at p (i.e. $\mathbf{f}_\nu \neq f_k$). On the contrary, we may have $\mathbf{f}_2 = f_2$.*

Proof. See [How07, Lemma 2.1.1] for $k > 2$. The fact that $\mathbf{f}_2 = f_2$ consider, for instance, the modular form associated with an elliptic curve over $\mathbb{Q}$ having multiplicative reduction at p). □

The theorem of Hida is a machine to create Λ-adic modular form. Since we now know they exist, we can combine them to create more. Given Λ-adic modular forms $(\Lambda_g, \mathcal{U}_g, \mathcal{U}_g^{\mathrm{cl}}, \mathbf{g})$, $(\Lambda_f, \mathcal{U}_f, \mathcal{U}_f^{\mathrm{cl}}, \mathbf{f})$ and $(\Lambda_h, \mathcal{U}_h, \mathcal{U}_h^{\mathrm{cl}}, \mathbf{h})$ it is possible to construct new ones, in particular:

- given N such that $N_g \mid N$ and an extension $\Lambda'_g \mid \Lambda_g$, we can consider for each $d \mid N/N_g$ an Iwasawa function $\boldsymbol{\lambda}_d \in \Lambda'_g$ and define:

$$\breve{\mathbf{g}} := \sum_{d|N/N_g} \boldsymbol{\lambda}_d \cdot \mathbf{g}(q^d) \in \mathbb{M}(N, \chi_g)_{\Lambda'_g}.$$

For a weight ℓ specialization map $\nu \in \mathcal{U}_g^{\mathrm{cl}}$ we have that:

$$\breve{\mathbf{g}}_\nu \sum_{d|N/N_g} \nu(\boldsymbol{\lambda}_d) \cdot \mathbf{g}_\nu(q^d) \in M_\ell(N, \chi_g)$$

is a oldform in weight ℓ and level N. When we consider such a $\breve{\mathbf{g}}$ we write $\breve{\mathbf{g}} \in \mathbb{M}(N, \chi_g)_{\Lambda'_g}[\mathbf{g}]$.

- the expression $\mathbf{g}^* \in \Lambda_g$ is the Λ-adic modular form determined on $\nu \in \mathcal{U}_g^{\mathrm{cl}}$ by:

$$(\mathbf{g}^*)_\nu = \mathbf{g}^*_\nu$$

- the *ordinary product of* Λ*-adic modular forms* $e_{\mathrm{ord}}(\mathbf{f} \times \mathbf{g}) \in \Lambda_{fh} := \Lambda_f \otimes_{\mathcal{O}} \Lambda_h$, which is uniquely determined on $(\nu, \mu) \in \mathcal{U}_f^{\mathrm{cl}} \times \mathcal{U}_h^{\mathrm{cl}}$ by:

$$(e_{\mathrm{ord}}(\mathbf{f} \times \mathbf{g}))_{\nu,\mu} = e_{\mathrm{ord}}(\mathbf{f}_\nu \times \mathbf{h}_\mu) = e_{\mathrm{ord}}(f_{\nu,\alpha} \times h_{\mu,\alpha}).$$

The Λ-algebra structure on Λ_{fh} is given by the diagonal embedding on group-like elements so that if ν has weight k and μ has weight m, (ν, μ) has weight $k + m$;

- by [DR14, Prop. 2.18], the expression of the form $e_{\mathrm{ord}}(d^\bullet \mathbf{f}^{[p]} \times \mathbf{h}) \in \Lambda'_{fh} := \Lambda \otimes \Lambda_f \otimes \Lambda_h$ is a Λ-adic modular form, uniquely determined on $(t, \nu, \mu) \in \mathcal{W}^{\mathrm{cl}} \times \mathcal{U}_f^{\mathrm{cl}} \times \mathcal{U}_h^{\mathrm{cl}}$ by:

$$\nu_{t,\nu,\mu}(e_{\mathrm{ord}}(d^\bullet \mathbf{f}^{[p]} \times \mathbf{h})) = e_{\mathrm{ord}}(d^t \mathbf{f}_\nu^{[p]} \times \mathbf{h}_\mu) = e_{\mathrm{ord}}(d^t f_{\nu,\alpha}^{[p]} \times h_{\mu,\alpha}). \tag{3.21}$$

The Λ-algebra structure on Λ'_{fh} is given by the embedding $[u] \mapsto [u]^2 \otimes [u] \otimes [u]$, so that if ν has weight k and μ has weight m, then (t, ν, μ) has weight $2t + k + m$ (here we are using on purpose the letter t both for the specialization and for the weight).

3.2.3 Castellà's p-adic L-function

We now present a two variables extension of BDP's p-adic L-function $\mathscr{L}_p(f/K)$ by allowing the variation of f in Hida family. The resulting two variables p-adic L-function is a slight variation of the one first studied by Castellà in [Ca1]. We propose here a very down to earth construction which better serves our scopes and also provide a slight extension of Castella's p-adic L-function.

Let us consider a Hida family $\mathbf{f} \in \mathbb{S}(N_f, \mathbb{1})$ and let assume that $\mathcal{U}_f^{\mathrm{cl}}$ is fibered over a single residue class modulo $p-1$ for simplicity. Fix for now a weight k specialization $\nu \in \mathcal{U}_f^{\mathrm{cl}}$ and we identify it with its weight since it shall mean no harm. Denote by $\mathbf{f}_k$ the specialization of the Hida family at ν and by f_k the associated newform. Take a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$ where $p = \wp\bar{\wp}$ splits and satisfying the Heegner hypothesis. Consider the choice of the ideal $\mathfrak{N}$ above N_f from this assumption. We write K_p for the completion of K at $\wp$.

As seen in section 3.1.3, BDP p-adic L-function satisfies the equation (3.13) which we restate here for a classical weight k specialization of an Hida family $\mathbf{f}$ as follows: for any character ψ in $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \mathbb{1})$ of infinity type $(k + j, -j)$ (with $j \geq 0$) and conductor c we have that:

$$\mathscr{L}_p(f/K)(\psi) = \left(\sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \psi^{-1}(\mathfrak{a}) \mathbf{N}_K(\mathfrak{a})^{-j} d^j \mathbf{f}_k^{[p]}(\tilde{x}_p(\mathfrak{a})) \right)^2 \tag{3.22}$$

where $\tilde{x}_p(\mathfrak{a}) = \mathfrak{a} * (A, t, \omega_{\text{can}})$ as seen in remark 3.4. To ease the exposition we restrict to a single residue class modulo $p-1$ for j, too.

The idea is to let those characters appearing in equation (3.22) vary in p-adic family, i.e. define some Λ-adic characters which interpolate them. Let us consider a p-adic Hecke character λ of infinity type $(0, 1)$ and conductor $\bar{\wp}$, having image in $\mathcal{O}^\times$, where $\mathcal{O}$ is the ring of integer of a $F|K_p$ finite extension containing the values of λ. We can consider the character $\langle\lambda\rangle$ whose image is in the torsion free subgroup of $\mathcal{O}^\times$. Define the character:

$$\psi_\infty := \langle\lambda\rangle^{-k-2j}\mathbf{N}_K^{k+j}.$$

Since both ψ_∞ and ψ are of infinity type $(k + j, -j)$ there exists a finite order character ψ_b of conductor c such that:

$$\psi = \psi_b\psi_\infty.$$

We call ψ_b the *branch character* associated with ψ and ψ_∞ the infinity part.

We consider now the two variables Iwasawa algebra $R := \Lambda_f \otimes_{\mathcal{O}} \mathcal{O}[[\widetilde{\Gamma}]]$ where we write $\widetilde{\Gamma} = \Gamma$ in order to distinguish the variable to which we are referring. We also write $d \mapsto \widetilde{[d]}$ for the second variable embedding. We can define the R-adic version of ψ_∞ as follows:

$$\Psi_\infty := \left[\frac{\mathbf{N}_K}{\langle\lambda\rangle}\right] \cdot \widetilde{\left[\frac{\mathbf{N}_K}{\langle\lambda\rangle^2}\right]}.$$

This R-adic character satisfy the following interpolation property:

$$\Psi_\infty(k, j) = \langle\lambda\rangle^{-k-2j}\mathbf{N}_K^{k+j}.$$

Similarly $\Psi := \psi_b\Psi_\infty$ so that $\Psi(k, j) = \psi$. In this way, for every fixed $\mathfrak{a} \in \text{Pic}(\mathcal{O}_c)$, $\Psi(\mathfrak{a}) \in R$ is an element in $\Lambda_f[[\widetilde{\Gamma}]]$.

In [Ca1] the author constructs a Λ_f-adic measure μ associated to $\mathbf{f}^{[p]}$ and $\mathfrak{a} \in \text{Pic}(\mathcal{O}_c)$ such that the weight-k-specialization is given by:

$$\int_{\mathbb{Z}_p} z^\lambda d\mu_{\mathfrak{a},k} := \Big(\int_{\mathbb{Z}_p} z^\lambda d\mu_{\mathfrak{a}}\Big)(k) = d^\lambda\mathbf{f}_k^{[p]}(x(\mathbf{a})).$$

We can then define, in a similar way to that of [Ca1], a p-adic L-function in the following way:

$$\mathscr{L}_p(\mathbf{f}/K)(\psi_b) := \sum_{\mathfrak{a}\in\text{Pic}(\mathcal{O}_c)} \Psi(\mathfrak{a})^{-1}\widetilde{[\mathbf{N}_K(\mathfrak{a})]}^{-1}\Big(\int_{\mathbb{Z}_p}\widetilde{[z]}d\mu_{\mathfrak{a}}\Big)$$

such that the specialization in weights k and j is given by:

$$\begin{aligned}
\mathscr{L}_p(\mathbf{f}/K)(\psi_b, k, j) :&= \sum_{\mathfrak{a}\in\text{Pic}(\mathcal{O}_c)} \Psi(\mathfrak{a})(k, j)^{-1} \cdot \widetilde{[\mathbf{N}_K(\mathfrak{a})]}^{-1}(j) \cdot \Big(\int_{\mathbb{Z}_p}\widetilde{[z]}d\mu_{\mathfrak{a}}\Big)(k, j) = \\
&= \sum_{\mathfrak{a}\in\text{Pic}(\mathcal{O}_c)} \psi_b(\mathfrak{a})\langle\lambda(\mathfrak{a})\rangle^{k+2j}\mathbf{N}_K(\mathfrak{a})^{-k}\int_{\mathbb{Z}_p} z^j d\mu_{\mathfrak{a},k} = \\
&= \sum_{\mathfrak{a}\in\text{Pic}(\mathcal{O}_c)} (\langle\lambda(\mathfrak{a})\rangle^{-k-2j}\mathbf{N}_K(\mathfrak{a})^{k+j})^{-1} \cdot \mathbf{N}_K(\mathfrak{a})^j \cdot d^j\mathbf{f}_k^{[p]}(x(\mathbf{a})) = \\
&= \sum_{\mathfrak{a}\in\text{Pic}(\mathcal{O}_c)} \psi(\mathfrak{a})^{-1}\mathbf{N}_K(\mathfrak{a})^j \cdot d^j\mathbf{f}_k^{[p]}(x(\mathbf{a})).
\end{aligned}$$

Hence, by equation (3.22) we have that:

$$\mathscr{L}_p(\mathbf{f}/K)(\psi_b, k, j)^2 = \mathscr{L}_p(f/K)(\Psi(k, j)). \tag{3.23}$$

Remark 3.13. Notice that the choice of the branch character is fundamental. In fact, a different choice gives a totally different L-function since the space of characters naturally splits in branches. In this sense we might see $\mathscr{L}_p(\mathbf{f}/K)$ as a three variables p-adic L-function

$$\mathscr{L}_p(\mathbf{f}/K) : \mathrm{Gal}\,(H_c|K) \times \mathcal{U}_f \times \mathcal{W} \to \mathbb{C}_p$$

but this terminology would be improper since ψ_b is not actually a p-adic variable. This is the reason for which we recast it as branch variable.

Remember that $\Psi(k,j) = \psi$, $\Psi(2,-1) = \psi_b \mathbf{N}_K$. Thanks to formula (3.23), the interpolation formula for this p-adic L-function reads exactly as the formula for BDP p-adic L-function:

$$\mathscr{L}_p(\mathbf{f}/K)(\psi_b,k,j)^2 = \mathfrak{e}_{\mathrm{BDP}}(\psi) \cdot \mathfrak{a}_{\mathrm{BDP}}(\psi) \cdot \mathfrak{f}_{\mathrm{BDP}}(\psi) \cdot \frac{\Omega_p(A)^{2k+4j}}{\Omega(A)^{2k+4j}} \cdot L(f,\psi^{-1},0), \tag{3.24}$$

for all characters $\psi \in \Sigma^{(2)}_{f,K}$ of type $(k+j,-j)$, for $k \geq 0$ and $j \geq 0$, where:

- $\mathfrak{e}_{\mathrm{BDP}}(\psi) = (1-\alpha_p(f)\psi^{-1}(\bar{\wp}))(1-\beta_p(f)\psi^{-1}(\bar{\wp}))$,
- $\mathfrak{a}_{\mathrm{BDP}}(\psi) = j!(k+j-1)!\pi^{k+2j-1}$
- $\mathfrak{f}_{\mathrm{BDP}}(\psi) = \left(\frac{2}{c\sqrt{D_K}}\right)^{k+2j-1} \cdot 2^{\sharp q|(D_K,N_E)} \cdot \prod_{q|c} \frac{q-\chi_K(q)}{q-1} \cdot \omega(f,\psi)^{-1}$

with $\omega(f,\psi)$ as defined in section 3.1.3.

If ψ_b is a finite order anticyclotomic character of conductor dividing c, then $\Psi(2,-1) = \psi_b \mathbf{N}_K$ lies outside the region of interpolation but it can be approximated by letting $(k,j) \mapsto (2,-1)$. The main theorem of [BDP12] asserts that

$$\mathscr{L}_p(\mathbf{f}/K)(\psi_b,2,-1) = \mathfrak{f}_p(f,\psi_b) \times \log_{\omega_E}(P_{\psi_b}) \tag{3.25}$$

where $\mathfrak{f}_p(f,\psi_b) = (1-\alpha_p(f)\psi_b(\bar{\wp})p^{-s})(1-\beta_p(f)\psi_b(\bar{\wp})p^{-s})$.

3.2.4 p-adic interpolation of trilinear periods

Let now f,g,h be three modular forms of weights k,ℓ,m such that $\ell \geq k+m$ and let

$$(\breve{g},\breve{f},\breve{h}) \in M_\ell(N,\chi_g)[g] \times M_k(N,\chi_f)[f] \times M_m(N,\chi_h)[h]$$

be a test vector. In sections 2.3.1 and 2.5 we have seen that trilinear periods of the form:

$$I^g(\breve{g},\breve{f},\breve{h}) = \langle \breve{g}^*, \delta^t \breve{f} \times \breve{h} \rangle$$

play a crucial role because they create a link between a complex L-function and a Petersson product, which provides an integral representation of the function. The problem is that, in general, those periods are far from being algebraic numbers. Luckily we have the following:

Lemma 3.14. *Given an eigenform $g \in S_\ell(N,\chi_g)_K$, for all $\breve{g} \in S_\ell(N)_K[g]$ and $F \in S_k(N)_K$ we have that the product $\langle \breve{g}, F\rangle$ is a K-rational multiple of the period $\langle f,f\rangle$.*

Proof. See [DR14, Lemma 2.12]. □

Hence if we want to interpolate p-adically those values we have to consider the complex period $\langle g,g\rangle$. We define:

$$J^g(\breve{g},\breve{f},\breve{h}) = \frac{\langle \breve{g}^*, \delta^t \breve{f} \times \breve{h} \rangle}{\langle g^*, g^*\rangle}$$

If we can interpolate those periods p-adically we can use the results of sections 2.3.1 and 2.5 to interpolate p-adically the Rankin and the Garrett L-functions.

The idea can be simplified in this way: consider a number field K, a modular form $g \in S_\ell(N_g, \chi_g)_K$ (i.e. assume that $\mathbb{Q}_{\chi_g} \subset K$) and two modular forms $\breve{g} \in S_\ell(N, \chi_g)_K[g]$, $\phi \in S_\ell(N)_K$. Then the expression of the form

$$J^g(\breve{g}, \phi) := \frac{\langle \breve{g}, \phi \rangle}{\langle g, g \rangle}$$

is algebraic in light of the above lemma. Hence we want to interpolate p-adically this kind of expression, which is to say that we want to give a meaning to an expression of the form:

$$J^g_p(\breve{\mathbf{g}}, \boldsymbol{\phi}) := \text{"}\frac{\langle \breve{\mathbf{g}}, \boldsymbol{\phi} \rangle}{\langle \mathbf{g}, \mathbf{g} \rangle}\text{"}$$

for Λ-adic modular forms $(\Lambda_g, \mathcal{U}_g, \mathcal{U}_g^{\mathrm{cl}}, \mathbf{g})$, $(\Lambda_{\breve{g}}, \mathcal{U}_{\breve{g}}, \mathcal{U}_{\breve{g}}^{\mathrm{cl}}, \breve{\mathbf{g}})$ and $(\Lambda_\phi, \mathcal{U}_\phi, \mathcal{U}_\phi^{\mathrm{cl}}, \boldsymbol{\phi})$ such that $\breve{\mathbf{g}} \in \mathbb{S}(N)_{\Lambda_g}[\mathbf{g}]$. For this we have the following:

Proposition 3.15. *Let $\mathcal{K}_g$ be the fraction field of Λ_g. For all $\breve{\mathbf{g}} \in \mathbb{S}(N)_{\Lambda_g}[\mathbf{g}]$ and all $\boldsymbol{\phi} \in \mathbb{S}(N)_{\Lambda_\phi}$ there exists an element $J^g(\breve{\mathbf{g}}, \boldsymbol{\phi}) \in \mathcal{K}_g \otimes_\Lambda \Lambda_\phi$ such that for all classical points $(\nu, \mu) \in \mathcal{U}_g^{\mathrm{cl}} \times_{\mathcal{W}^{\mathrm{cl}}} \mathcal{U}_\phi^{\mathrm{cl}}$ we have:*

$$J^g_p(\breve{\mathbf{g}}, \boldsymbol{\phi})_{\nu,\mu} = \frac{\langle \breve{\mathbf{g}}_\nu, e_{\mathbf{g}_\nu} \boldsymbol{\phi}_\mu \rangle}{\langle \mathbf{g}_\nu, \mathbf{g}_\nu \rangle} = \frac{\langle \breve{g}_\nu, e_{g_\nu} \phi_\mu \rangle}{\langle g_\nu, g_\nu \rangle}$$

Proof. It is [DR14, Lemma 2.19]. □

Notice that having $(\nu, \mu) \in \mathcal{U}_g^{\mathrm{cl}} \times_{\mathcal{W}^{\mathrm{cl}}} \mathcal{U}_\phi^{\mathrm{cl}}$ implies that the weights of ν and μ are the same, so that the Petersson product is well defined. The projection e_{g_ν} appears because although ϕ_μ is an ordinary p-adic modular form, there is no need for it to be classical, hence the Petersson product might be not well defined. Theorem 1.10 implies that its projection onto the g_ν-isotypical subspace is a classical modular form.

3.2.5 Hida-Rankin p-adic L-function

In [Hid93, §7.4] Hida constructed a three-variable p-adic L-function interpolating central critical values of the Rankin L-function associated to the convolution of two Hida families of modular forms. We describe here this p-adic L-function following the notations and normalizations adopted in [BDR15].

Consider two Hida families of tame level N:

$$\mathbf{g} \in \mathbb{S}(N, \chi_g) \quad \text{and} \quad \mathbf{f} \in \mathbb{S}(N, \chi_f).$$

As seen in the previous section, $\mathbf{g}$ and $\mathbf{f}$ are parametrized by the rigid analytic covers $\mathcal{U}_g$ and $\mathcal{U}_f$ of the weight space $\mathcal{W}$. It will be harmless for our purposes to assume that they are fibered over a single residue class modulo $p-1$ and to identify the classical specializations with their weights. With this in mind, for each $(\ell, k) \in \mathcal{U}_g^{\mathrm{cl}} \times \mathcal{U}_f^{\mathrm{cl}}$ we can consider the newforms $g_\ell \in S_\ell(N, \chi_g)$ and $f_k \in S_k(N, \chi_f)$ whose ordinary p-stabilizations are $\mathbf{g}_\ell$ and $\mathbf{f}_k$. We associate to them the Rankin product L-function $L(g_\ell \otimes f_k, s)$ that we have studied in §2.3.1. Assume from now on that $\ell \geq k+1$. By results of Deligne and Shimura we know that the integer j is critical for $L(g_\ell \otimes f_k, s)$ if and only if $j \in [k, \ell-1]$, but we shall restrict to integers in the range

$$j \in \Big[\frac{\ell+k-1}{2}, \ell-1\Big]. \tag{3.26}$$

For a given such j we set

$$t := \ell - j - 1 \quad \text{and} \quad m := \ell - k - 2t. \tag{3.27}$$

We define the range of interpolation as the set

$$\mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}} := \{(\ell, k, j) \in \mathbb{Z}^3_{\geq 2} \mid \ell \geq k+1 \text{ and } j \in [(\ell+k-1)/2, \ell-1]\}$$

From equation (2.10) we can derive that for all $(\ell, k, j) \in \mathcal{U}_{HR}^{\mathrm{cl}}$ we have

$$\mathfrak{f}_{\mathrm{Ran}}(\ell, k, j) \cdot L(g_\ell \otimes f_k, j) = \langle g_\ell^*(z), \delta_m^t E_{m,\chi_N}(z) \cdot f_k(z) \rangle_{\ell,N}. \tag{3.28}$$

Define the algebraic part of $L(g_l \otimes f_k, j)$ as in [BDR15, (9)]:

$$L^{\mathrm{alg}}(g_\ell \otimes f_k, j) := \mathfrak{f}_{\mathrm{Ran}}(\ell, k, j) \frac{L(g_\ell \otimes f_k, j)}{\langle g_\ell^*, g_\ell^* \rangle_{\ell,N}} = \frac{\langle g_\ell^*(z), \delta_m^t E_{m,\chi_N}(z) \cdot f_k(z) \rangle_{\ell,N}}{\langle g_\ell^*, g_\ell^* \rangle_{l,N}}. \tag{3.29}$$

The notation L^{alg} is justified since the ratio of Petersson products, the one of the right hand side of the equation (3.29), is an algebraic number by lemma 3.14. More precisely, it represents the coefficient of the holomorphic projection of $\delta_m^t E_{m,\chi_N}(z) \cdot f_k(z)$, as shown for instance [DR14, Lemma 2.12]). Since g_ℓ and f_k are specializations of a Hida family, we expect to p-adically interpolate those values as (ℓ, k) varies.

In order to interpolate the expression on the right-hand side of (3.29), we use the theorem 3.15. Define the Λ-adic modular form $\boldsymbol{E}_{\chi_N}$ such that for $m \in \mathcal{U}_E^{\mathrm{cl}}$ we have:

$$\boldsymbol{E}_{\chi,\xi} = (E_{m,\chi})_\alpha.$$

By the theorem 3.15 we know that there exists an Iwasawa function $J^g(\mathbf{g}, e_{\mathrm{ord}}(d^\bullet \boldsymbol{E}_\chi^{[p]} \times \mathbf{f}))$ such that:

$$J^g(\mathbf{g}, e_{\mathrm{ord}}(d^\bullet \boldsymbol{E}_\chi^{[p]} \times \mathbf{f}))_{(\ell,k,m,t)} = \frac{\langle \mathbf{g}_\ell^*, e_{\mathrm{ord}}(d^t \boldsymbol{E}_{\chi,m}^{[p]} \times \mathbf{f}_k) \rangle}{\langle \mathbf{g}_\ell^*, \mathbf{g}_\ell^* \rangle}$$

for every quadruple of $(\ell, k, m, t) \in \mathcal{U}_g^{\mathrm{cl}} \times_{\mathcal{W}^{\mathrm{cl}}} (\mathcal{U}_f^{\mathrm{cl}} \times \mathcal{U}_E^{\mathrm{cl}} \times \mathcal{W})$.

Remark 3.16. The latter implies that $\ell = k + m + 2t$. In this way, the choice of m determines t and, according to equations (3.26) and (3.27) we know that it is equivalent to choose m or to chose the central critical value j. In particular, we have that:

$$m \in [1, \ell - k].$$

For this reason, we will use the variable j as third variable instead of the variable m and remember the definitions of m and t in equations (3.26) and (3.27).

Definition 3.17. We define the Hida-Rankin (three variables) p-adic L-function as the element:

$$\mathscr{L}_p{}^g(\mathbf{g}, \mathbf{f}) := J_p^g(\mathbf{g}, e_{\mathrm{ord}}(d^\bullet \boldsymbol{E}_\chi^{[p]} \times \mathbf{f})) \in \mathcal{K}_g \otimes_\Lambda (\Lambda_g \otimes \Lambda_E \otimes \Lambda)$$

When evaluating the p-adic L-function at a classical point $(\ell, k, j) \in \mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}}$ we get:

$$\mathscr{L}_p{}^g(\mathbf{g}, \mathbf{f})(\ell, k, j) = \frac{\langle \mathbf{g}_\ell^*, e_{\mathrm{ord}}(d^t \boldsymbol{E}_{m,\chi}^{[p]} \times \mathbf{f}_k) \rangle}{\langle \mathbf{g}_\ell^*, \mathbf{g}_\ell^* \rangle} \tag{3.30}$$

Define

$$\mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j) := \frac{\mathcal{E}(g_\ell, f_k, j)}{\mathcal{E}_1(g_\ell)\mathcal{E}_0(g_\ell)} \tag{3.31}$$

where:

$$\begin{aligned} \mathcal{E}(g_\ell, f_k, j) &= (1 - \beta_p(g_\ell)\alpha_p(f_k)p^{t-\ell+1})(1 - \beta_p(g_\ell)\beta_p(f_k)p^{t-\ell+1}) \\ &\times (1 - \beta_p(g_\ell)\alpha_p(f_k)\chi(p)p^{t-\ell+1})(1 - \beta_p(g_\ell)\beta_p(f_k)\chi(p)p^{t-\ell+1}), \\ \mathcal{E}_1(g_\ell) &= 1 - \beta_p(g_\ell)^2 p^{-\ell}, \\ \mathcal{E}_0(g_\ell) &= 1 - \beta_p(g_\ell)^2 p^{1-\ell}. \end{aligned}$$

The result of [DR14, Theorem 4.7] implies that we have:

$$\mathscr{L}_p{}^g(\mathbf{g},\mathbf{f})(\ell,k,j) = \mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j) \cdot \frac{\langle g_\ell^*(z), \delta_m^t E_{m,\chi_N}(z) \cdot f_k(z)\rangle_{\ell,N}}{\langle g_\ell^*, g_\ell^*\rangle_{l,N}},$$

so that we can find the interpolation formula which describes the three variable Hida-Rankin p-adic L-function using equation (3.29):

$$\begin{aligned}\mathscr{L}_p{}^g(\mathbf{g},\mathbf{f})(\ell,k,j) &= \mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j) \cdot L^{\mathrm{alg}}(g_\ell \otimes f_k, j) = \\ &= \mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j) \cdot \mathfrak{a}_{\mathrm{HR}}(\ell,k,j) \cdot \mathfrak{f}_{\mathrm{HR}}(\ell,k,j) \frac{L(g_\ell \otimes f_k, j)}{\langle g_\ell^*, g_\ell^*\rangle_{\ell,N}}. \end{aligned} \tag{3.32}$$

for all $(\ell,k,j) \in \mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}}$, where:

- $\mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j)$ as in equation 3.31;
- $\mathfrak{a}_{\mathrm{HR}}(\ell,k,m) = (m+t-1)!(j-1)!\pi^{1-\ell-m}$
- $\mathfrak{f}_{\mathrm{HR}}(\ell,k,m) = \frac{(-1)^t (iN)^m}{2^{2\ell+m-2}\cdot\tau(\chi^{-1})}$.

Notice that the point $(1,2,1)$ is not in the region of interpolation $\mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}}$. The following result provides a formula for the value of $\mathscr{L}_p{}^g(\mathbf{g},f)(1,2,1)$. Recall the p-adic iterated integrals from equation (1) of the introduction.

Proposition 3.18. *Suppose that* $\mathbf{g}_1$ *is a classical modular form. Then* $\mathscr{L}_p{}^g(\mathbf{g},\mathbf{f})$ *has no pole at* $(1,2,1)$ *and*

$$\mathscr{L}_p(\mathbf{g},\mathbf{f})(1,2,1) = \int_{\gamma_{g_1}} f_2 \cdot E_{1,\chi_N}. \tag{3.33}$$

Proof. See [CR1, prop. 3.2] □

3.2.6 Garrett-Hida p-adic L-function

We now want to interpolate the triple product L-function. We essentially follow [DR14] and the computations are totally similar to what we have done in the previous section. In particular, in virtue of theorem 2.15 we are reduced to the interpolation of the trilinear periods

$$\mathrm{I}^g(\breve{g}, \breve{f}, \breve{h}) = \langle \breve{g}^*, \delta^t \breve{f} \times \breve{h}\rangle$$

for a choice of test vector $(\breve{f}, \breve{g}, \breve{h}) \in S_k(N,\chi_f)[f] \times M_\ell(N,\chi_g)[g] \times M_m(N,\chi_h)[h]$. Let us recall that we consider an unbalanced triple of weights (ℓ,k,m) with dominant weight ℓ and $t = (\ell-k-m)/2$. Hence:

$$\mathcal{U}_{\mathrm{GH}}^{\mathrm{cl}} := \{(\ell,k,m) \in \mathbb{Z}^3_{\geq 2} \mid \ell \geq k+m\}$$

By the lemma 3.14 the quantity:

$$J^g(\breve{f}, \breve{g}, \breve{h}) := \frac{\langle \breve{g}, \delta^t \breve{f} \times \breve{h}\rangle}{\langle g, g\rangle}$$

is an algebraic number, so that we can define the algebraic part of the triple product L-function as:

$$L^{\mathrm{alg}}(f \otimes g \otimes h, c) := J^g(\breve{f}, \breve{g}, \breve{h})^2 \tag{3.34}$$

In order to perform a p-adic interpolation in a similar way as we did for the Hida-Rankin p-adic L-function, we consider three Hida families $\mathbf{f}, \mathbf{g}, \mathbf{h}$ of tame levels N_f, N_g, N_h and describe the associated Λ-adic test vector $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$.

Definition 3.19. We define the Garrett-Hida (three variables) p-adic L-function to be:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}}) := J_p^g(\mathbf{g}, e_{\mathrm{ord}}(d^{\bullet}\mathbf{f}_{\chi}^{[p]} \times \mathbf{h})) \in \Lambda_g \otimes_{\Lambda} (\Lambda_f \otimes \Lambda_h \otimes \Lambda) \tag{3.35}$$

In this way we have that:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(\ell,k,m) = \frac{\langle \breve{\mathbf{g}}_\ell^*, e_{\breve{\mathbf{g}}_\ell} e_{\mathrm{ord}}(d^t \breve{\mathbf{f}}_k^{[p]} \times \breve{\mathbf{h}}_m)\rangle}{\langle \mathbf{g}_\ell^*, \mathbf{g}_\ell^*\rangle}$$

Define

$$\mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m) := \frac{\mathcal{E}(g_\ell, f_k, h_m)}{\mathcal{E}_1(g_\ell)\mathcal{E}_0(g_\ell)} \tag{3.36}$$

where:

$$\begin{aligned}
\mathcal{E}(g_\ell, f_k, h_m) &= (1-\beta_p(g_\ell)\alpha_p(f_k)\alpha_p(h_m)p^{-c})(1-\beta_p(g_\ell)\beta_p(f_k)\alpha_p(h_m)p^{-c}) \\
&\times (1-\beta_p(g_\ell)\alpha_p(f_k)\beta_p(h_m)p^{-c})(1-\beta_p(g_\ell)\beta_p(f_k)\beta_p(h_m)p^{-c}), \\
\mathcal{E}_1(g_\ell) &= 1-\beta_p(g_\ell)^2 p^{-\ell}, \\
\mathcal{E}_0(g_\ell) &= 1-\beta_p(g_\ell)^2 p^{1-\ell}.
\end{aligned}$$

A simple computation shows that $\mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m)$ equals $\mathfrak{e}_{\mathrm{HR}}(g_\ell, f_k, j)$ as defined in (3.31) when $h_m = E_{m,\chi}$. Because of [DR14, Theorem 4.7] we have that:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(\ell,k,m) = \mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m)\cdot \frac{\mathrm{I}(\breve{g}_\ell, \breve{f}_k, \breve{h}_m)}{\langle g_\ell^*, g_\ell^*\rangle}.$$

Since

$$|\mathrm{I}(\breve{f},\breve{g},\breve{h})|^2 = \mathrm{I}(\breve{f},\breve{g},\breve{h})\cdot\overline{\mathrm{I}(\breve{f},\breve{g},\breve{h})} = \mathrm{I}(\breve{f},\breve{g},\breve{h})\cdot \mathrm{I}(\breve{f}^*,\breve{g}^*,\breve{h}^*) \quad \text{and} \quad \langle g_\ell, g_\ell\rangle = \langle g_\ell^*, g_\ell^*\rangle$$

using theorem 2.15 we find the interpolation formula which relates the p-adic L-function to the values $L(\breve{g}_\ell \otimes \breve{f}_k \otimes \breve{h}_m, c)$, where $c = (k+\ell+m-2)/2$, for all $(\ell,k,m)\in\mathcal{U}_{\mathrm{GH}}^{\mathrm{cl}}$. It is given by:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(\ell,k,m)\mathscr{L}_p{}^g(\breve{\mathbf{g}}^*,\breve{\mathbf{f}}^*,\breve{\mathbf{h}}^*)(\ell,k,m) = \mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m)^2\cdot \mathfrak{f}_{\mathrm{Gar}}(k,\ell,m)\cdot\frac{L(\breve{f}_k\otimes\breve{g}_\ell\otimes\breve{h}_m, c)}{\langle g_\ell^*, g_\ell^*\rangle^2}. \tag{3.37}$$

Being both $\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})$ and $\mathscr{L}_p{}^g(\breve{\mathbf{g}}^*,\breve{\mathbf{f}}^*,\breve{\mathbf{h}}^*)$ Iwasawa meromorphic functions, we have that their quotient:

$$\omega(\ell,k,m) := \frac{\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(k,\ell,m)}{\mathscr{L}_p{}^g(\breve{\mathbf{g}}^*,\breve{\mathbf{f}}^*,\breve{\mathbf{h}}^*)(k,\ell,m)} \tag{3.38}$$

is a meromorphic Iwasawa function, too. Hence we can rewrite the interpolation formula (3.37) as follows:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(\ell,k,m)^2 = \mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m)^2\cdot \mathfrak{a}_{\mathrm{GH}}(k,\ell,m)\cdot \mathfrak{f}_{\mathrm{GH}}(k,\ell,m)\cdot\frac{L(\breve{f}\otimes\breve{g}\otimes\breve{h}, c)}{\langle g_\ell^*, g_\ell^*\rangle^2} \tag{3.39}$$

where:

- $\mathfrak{e}_{\mathrm{GH}}(g_\ell, f_k, h_m)$ as in equation (3.36);
- $\mathfrak{a}_{\mathrm{GH}}(k,\ell,m) = (c-1)!(c-k)!(c-m)!(c-k-m+1)!\pi^{-2\ell}$;
- $\mathfrak{f}_{\mathrm{GH}}(k,\ell,m) = C_{\mathrm{Ran}}\cdot\omega(\ell,k,m)\cdot 2^{1-3\ell-k-m}$.

The point $(k,\ell,m) = (1,2,1)$ lies outside the region of interpolation, hence its special value is no more related with its complex counterpart. Proposition 2.6 of [DLR15] describes it as a p-adic iterated integral in the following way:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{\mathbf{h}})(1,2,1) = \int_{\breve{\gamma}_{g_\alpha}} \breve{f}\cdot\breve{h}. \tag{3.40}$$

Chapter 4

Proof of the main results

In this chapter, we describe and proof theorem 0.6 and theorem 0.7, which constitute the main results of [CR1] and [CR2]. We begin by describing the general setting of the elliptic Stark conjectures, with special emphasis on our case, then we move to the proofs of the theorems.

4.1 Elliptic Stark conjecture and p-adic Gross-Zagier formula

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E and let $f \in S_2(N_E, \mathbb{1})$ be the modular form associated to E by modularity. Fix once and for all a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$ and denote its associated quadratic Dirichlet character by χ_K. Consider the two finite order Hecke characters

$$\psi_g, \psi_h : G_K \to \mathbb{C}^\times$$

of conductors $\mathfrak{c}_g$ and $\mathfrak{c}_h$ which define theta series:

$$g = \theta_{\psi_g} \in M_1(D\mathbf{N}_K(\mathfrak{c}_g), \overline{\chi}), \quad \text{and} \quad h := \theta_{\psi_h} \in M_1(D\mathbf{N}_K(\mathfrak{c}_h), \chi).$$

Consider the associated representations $V_g := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_g)$ and $V_h := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_h)$. We define the four dimensional representation $\rho := \rho_g \otimes \rho_h : G_{\mathbb{Q}} \to \mathrm{SL}_4(L)$ whose underlying vector space is $V_{gh} := V_g \otimes V_h$. If we let $\psi_1 := \psi_g\psi_h$ and $\psi_2 := \psi_g\psi_h'$, the representation V_{gh} splits as the direct sum $V_1 \oplus V_2$, where $V_i := \mathrm{Ind}_K^{\mathbb{Q}}(\psi_i)$. This automatically implies a decomposition of L-functions:

$$L(E, \rho, s) = L(E/K, \psi_1, s)L(E/K, \psi_2, s). \tag{4.1}$$

Notice that the characters ψ_1 and ψ_2 are ring class character of conductors c_1 and c_2 because of lemma 2.4, since they have trivial central character. Let us define $c := \mathrm{lcm}(c_1, c_2)$. Let p be a prime number such that $\mathrm{ord}_p(N_E) \leq 1$ and define N_f to be the tame part of N_E, i.e. $N_f = N_E$ if $p \nmid N_E$ and $N_f = N_E/p$ if $p||N_E$.

Assumption 4.1. We will assume from now on the following hypothesis:

(AR) the *analytic rank* hypotesis: $r_{\mathrm{an}}(E, \rho, s) = 2$;

(HH) The *Heegner hypothesis*: there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N_f\mathbb{Z}$. Fix such an ideal $\mathfrak{N}$ once and for all and assume also that $(c, N_f D) = 1$;

(PP) the prime p splits in K as $p\mathcal{O}_K = \wp\bar{\wp}$, where $\wp = \mathfrak{m}_{\mathbb{C}_p} \cap K$.

The field H cut by ρ is the ring class field H_c where $c = \mathrm{lcm}(c_1, c_2)$. Following the discussion of sections 1.7 and 2.3.2, we now fix a Heegner point $P \in E(H)$ and an elliptic unit $u \in \mathcal{O}_H^\times$ and define P_ψ as in equation (2.13) and u_ψ as in equation (3.6). Assumption (HH), in particular, implies that the order of vanishing of $L(E/K, \psi_i, s)$ and $L(E/K, \bar{\psi}_i, 2)$ is odd at $s = 1$. Since we also assume (AR), we have that

$$\mathrm{ord}_{s=1} L(E/K, \psi_i, s) = 1, \qquad \mathrm{ord}_{s=1} L(E/K, \bar{\psi}_i, s) = 1$$

which implies, by theorem 2.8, that $\dim_L E(H)_L^{\psi_i} = 1 \dim_L E(H)_L^{\bar{\psi}_i}$ and that P_{ψ_i} and $P_{\bar{\psi}_i}$ are generators of these spaces.

Assumptions (HH) and (PP) ensure us that there is a cyclic ideal above the number N_E also when $p || N_E$ and it is $\mathfrak{N}\wp$.

Consider now assumption (PP). The choice of the ideal $\wp$ determines a Frobenius element $\sigma_p \in \mathrm{Gal}\,(H_c|K)$ and we set the eigenvalues of σ_p acting on V_{ψ_g} as follows:

$$\alpha := \psi_g(\sigma_p), \qquad \beta := \psi_g'(\sigma_p) = \alpha^{-1}.$$

Since we are going to use the cuspidal Hida family passing through g explicitly, we recall here the construction which is described in [Hid93, p.235] with a little modification. Recall the Λ-adic character $[\langle\lambda\rangle]$ that we defined in section 3.2.3. We define $\mathbf{g}$ as follows:

$$\mathbf{g}(q) := \sum_{\mathfrak{a} \in I_{c\bar{\wp}}} \psi_g(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{-1} [\langle \lambda(\mathfrak{a}) \rangle] q^{\mathbf{N}_{\mathbb{Q}}^K(\mathfrak{a})} \in \mathbb{S}(N_g, \overline{\chi}) \tag{4.2}$$

and consider $\mathcal{U}_g$ fibered over the residue class of 1 modulo $p - 1$. Then, following the notation introduced in equation (2.6), we have that:

$$\mathbf{g}_\ell(q) = \sum_{\mathfrak{a} \in I_{c\bar{\wp}}} \psi_g(\mathfrak{a}) \lambda(\mathfrak{a})^{\ell-1} q^{\mathbf{N}_{\mathbb{Q}}^K(\mathfrak{a})} = \sum_{n \geq 1} a_n(\psi_g \langle \lambda \rangle^{\ell-1}) q^n \in M_\ell(N_g p, \overline{\chi}), \quad \forall \ell \in \mathcal{U}_g^{\mathrm{cl}}.$$

It is then natural to define, for all $\ell \in \mathcal{U}_g^{\mathrm{cl}}$, the Hecke character $\psi_{\ell-1}$ as follows:

$$\psi_{\ell-1}(\mathfrak{a}) = \psi_g(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{\ell-1}, \quad \text{if } \bar{\wp} \nmid \mathfrak{a}; \tag{4.3}$$

$$\psi_{\ell-1}(\bar{\wp}) = \frac{\overline{\chi}(p) \cdot p^{\ell-1}}{\psi_{\ell-1}(\wp)}, \tag{4.4}$$

so that if we define the theta series $g_\ell := \theta_{\psi_{\ell-1}} \in M_1(N_g, \overline{\chi})$ and if m is a prime that spits in $\mathcal{O}_K$ as $m\mathcal{O}_K = \mathfrak{m}\bar{\mathfrak{m}}$, then we have that:

$$\alpha_m(g_\ell) = \psi_{\ell-1}(\mathfrak{m}), \qquad \beta_m(g_\ell) = \psi_{\ell-1}(\bar{\mathfrak{m}}) \tag{4.5}$$

In particular $\mathbf{g}_\ell = g_{\ell,\alpha}$ and for $\ell = 1$ we have that $\psi_0 = \psi_g$ (hence $g_1 = g$), so that also the weight one specialization is a classical modular form.

We end this section with the following technical:

Definition 4.2. Let L be a number field. A function

$$\mathfrak{f} : \mathcal{U}_g^{\mathrm{cl}} \to \overline{\mathbb{Q}}$$

is L-admissible if it extends to a meromorphic Iwasawa function on $\mathcal{U}_g$ having no pole at 1 and satisfying $\mathfrak{f}(1) \in L^\times$.

Similarly, a two variables function

$$\mathfrak{f} : \mathcal{U}_{fg}^{\mathrm{cl}} \to \overline{\mathbb{Q}}$$

is L-admissible if it extends to a meromorphic Iwasawa function on $\mathcal{U}_{fg}$ having no pole at $(2, 1)$ and satisfying $\mathfrak{f}(2, 1) \in L^\times$.

4.2 Discussion on hypothesis C and C'

Theorem 1.10 is very important but it only works for $k \geq 2$. This is the reason for which we need hypothesis C and C' (see the discussion in the introduction). In the case $k = 1$ it might happen that $S_k^{\mathrm{ord}}(Np, \chi) \subsetneq S_k^{\mathrm{oc,ord}}(N, \chi)$ and the p-adic iterated integrals might lead to more exotic results. The elliptic Stark conjecture in this setting is discussed in [DLR2] and it involves significant new features.

For this reason we need to have some control on the space of overconvergent modular forms of weight one. More precisely, let $g \in M_1(N_g, \chi_g)$ be a normalized eigenform of weight one. We say that g is regular at p if $\alpha_p(f) \neq \beta_p(f)$, otherwise we say that g is irregular at p. Take g_α to be a p-stabilization and consider the generalized eigenspace:

$$S_1^{\mathrm{oc}}(N, \chi_g)[[g_\alpha^*]] := \bigcup_{n \geq 1} \ker(I_{g_\alpha^*}^n).$$

Then we consider the following:

Assumption 4.3 (Hypothesis C)**.** The generalized eigenspace $S_1^{\mathrm{oc}}(N, \chi_g)[[g_\alpha^*]]$ is non-trivial and it only consists in classical modular forms.

Assumption 4.4 (Hypothesis C')**.** The following holds:

- if g is a cusp form then it is regular at p and is not the theta series of a character of a real quadratic field in which p splits;
- if g is an Eisenstein series then it is irregular at p.

The following proposition explains the relations between the two items in hypothesis C' and hypothesis C:

Proposition 4.5. *(1) If g is a cusp form and it is regular at p, then the natural inclusion:*

$$S_1(Np, \chi_g)_{\mathbb{C}_p}[g_\alpha] \subset S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]]$$

is an isomorphism if and only if g is not induced from a character of a real quadratic field in which p splits.

(2) If g is an Eisenstein series then the space $S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]]$ is non-trivial if and only if g is irregular.

Proof. This is proven in [DLR15, prop. 1.1, prop. 1.2] building on the results of [BeDi16]. □

As explained in [DLR15, §1], while hypothesis C and C' are equivalent when g is a cusp form, hypothesis C' could be weaker than hypothesis C when g is Eisenstein.

In the special case of theta series induced by ring class characters of an imaginary quadratic field K, hypothesis C' for Eisenstein series is automatic, since $g = \theta_\psi$ is an Eisenstein series if and only if $\psi^2 = 1$, hence $\alpha_p(g) = \beta_p(g)$. When $g = \theta_\psi$ is a cusp forms, the regularity assumption is sufficient for hypothesis C' (hence hypothesis C) to hold, because $\rho_g = \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$ and if $\rho_g = \mathrm{Ind}_F^{\mathbb{Q}}(\xi)$ for a character ξ of a real quadratic field F, then $\mathrm{Gal}\,(H|K) = C_4$ and F is the single real quadratic subfield of F. In this setting, p cannot split in F.

4.3 Proof of theorem 0.6

In this section we will treat the proof of theorem 0.6. Its hypothesis are slightly more restrictive than the ones from the general setting. More precisely:

Assumption 4.6. The following are assumed through all this section:

(ES) $\psi_h = \mathbb{1}$ is the trivial character, so that $h = E_{1,\chi_K}$. We write $\psi := \psi_g$.

(GR) the prime p in assumption 4.1(PP) is such that $p \nmid N := \mathrm{lcm}(N_E, Dc^2)$

Assumption (GR) implies that all the modular forms involved have good reduction at p. Under assumption (ES) and the self-duality condition for ρ we have some consequences that we list for the comfort of the reader:

- ψ is a ring class character cutting $H = H_c$, the ring class field of K of conductor $c \in \mathbb{Z}_{\geq 1}$;
- $V := V_1 = V_2 = \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$;
- $L(E, \rho, s) = L(E/K, \psi, s)^2$.

We want to apply Rankin's method as described in section 2.3.1 to $L(E/K, \psi, s) = L(f \otimes g, s)$. In order to do this, we chose

$$\breve{f} \in S_2(N)[f], \quad \text{and} \quad \breve{g} \in M_1(N, \chi_K)[g]$$

two normalized eigenform for all Hecke operators. To be explicit, we can write:

$$\breve{g}(q) = g(q) - \sum_{d|N/N_g} \mu_d(g) g(q^d).$$

as explained in section 1.3 and we define the Λ-adic modular form $\breve{\mathbf{g}}$ as follows:

$$\breve{\mathbf{g}}(q) := \mathbf{g}(q) - \sum_{d|N/N_g} \mu_d(g) \mathbf{g}(q^d),$$

Lemma 4.7. *There exists a meromorphic function $\mathcal{E}ul_N(\breve{f}, \breve{g}, s)$ such that:*

$$L(\breve{g} \otimes \breve{f}, s) = \mathcal{E}ul_N(\breve{f}, \breve{g}, s) \cdot L(f, \psi, s)$$

and $\mathcal{E}ul_N(\breve{f}, \breve{g}, 1) \in \mathbb{Q}_\psi(\breve{f})^\times$.

Proof. Comparing the naive Euler factors of $L(\breve{f} \otimes \breve{g}, s)$ with those of $L(f, \psi, s)$ using [Gro84, equation (20.2)], we find that:

$$\mathcal{E}ul_N(\breve{f}, \breve{g}, s) := \frac{L(\breve{f} \otimes \breve{g}, s)}{L(f, \psi, s)} = \frac{\prod_{q||N_E, q\nmid D}(1 - a_q(f)q^{-s})^2 \prod_{q|D, q\nmid N_E}(1 - a_q(f)a_q(g)q^{-s} + q^{1-2s})}{\prod_{q|N/(N_E,D)}(1 - a_q(\breve{f})a_q(\breve{g})q^{-s})},$$

which is of course a meromorphic function whose values at integes lie in $\mathbb{Q}_\psi(\breve{f})$. An easy verification of the factors shows that $\mathcal{E}ul_N(\breve{f}, \breve{g}, 1)$ is well defined and nonzero. □

Lemma 4.8. *For every $\ell \in \mathcal{U}_g^{\mathrm{cl}} \cup \{1\}$ we define the Hecke character Ψ_ℓ of conductor c as follows:*

$$\Psi_\ell := \psi_{\ell-1}^{-1} \mathbf{N}_K^{\frac{\ell+1}{2}}, \text{ of infinity type } \Big(2 + \frac{\ell - 3}{2}, -\frac{\ell - 3}{2}\Big).$$

Then there exists a number $\mathcal{E}ul_N^{\mathrm{HR}}(\breve{g}_\ell, \breve{f}) \in \mathbb{Q}_\psi(\breve{f})$ such that we have the following equality of critical L-values:

$$L(\breve{g}_\ell \otimes \breve{f}, (\ell + 1)/2) = \mathcal{E}ul_N^{\mathrm{HR}}(\breve{g}_\ell, \breve{f}) \cdot L(f, \Psi_\ell^{-1}, 0)$$

and $\mathcal{E}ul_N^{\mathrm{HR}}((\breve{g}_1, \breve{f}) \neq 0$.

Proof. The Euler factors defining $L(\breve{g}_\ell \otimes \breve{f}, s)$ and $L(f, \psi_{\ell-1}^{-1}, s)$ at good primes coincide, so that the only discrepancies arise for primes $q \mid N$. Those factors arise from Hecke polynomials with coefficients in $\mathbb{Q}_\psi(\breve{f})$ and lemma 4.7 guarantees that $\mathcal{E}ul_N^{\mathrm{HR}}(\breve{g}_1, \breve{f}) \neq 0$. □

Lemma 4.9. *Consider the factor $\omega(f, \Psi_\ell)$ appearing in BDP interpolation formula 3.9. Then:*

$$\omega(f, \Psi_\ell) = (-1)^{\frac{\ell-1}{2}} \frac{\psi_{\ell-1}(\mathfrak{N})}{N_E^{\frac{\ell-1}{2}}} \quad \textit{and} \quad \omega(f, \Psi_1) = \psi(\mathfrak{N}).$$

Proof. For simplicity call $\Psi := \Psi_\ell$ the Hecke character of infinity type $(\kappa + \lambda, -\lambda)$, where $\kappa = 2$ and $\lambda = (\ell - 3)/2$. Define

$$\Psi_\lambda := \Psi \mathbf{N}_K^\lambda.$$

Following the definition of equation (3.11), choose an ideal $\mathfrak{b} \subset \mathcal{O}_c$ relatively prime to pcN_E and an element $b_N \in \mathcal{O}_c$ such that $\mathfrak{b}\mathfrak{N} = (b_N)$. Since in our case $\omega_f = 1$ and $\varepsilon_f = \mathbb{1}$, equation (3.11) gives

$$\omega(f, \Psi) = \Psi_\lambda(\mathfrak{b}) \cdot (-1)^{\kappa/2+\lambda} N_E^{\kappa/2+\lambda} b_N^{-\kappa-2\lambda}. \tag{4.6}$$

Since

$$\Psi_\lambda(\mathfrak{b}) \Psi_\lambda(\mathfrak{N}) = \Psi_\lambda((b_N)) = b_N^{\kappa+2\lambda}.$$

We can substitute into (4.6) and find:

$$\omega(f, \Psi) = (-1)^{\kappa/2+\lambda} \frac{N_E^{\kappa/2+\lambda}}{\Psi_\lambda(\mathfrak{N})} = (-1)^{\kappa/2+\lambda} \frac{N_E^{\kappa/2}}{\Psi(\mathfrak{N})}.$$

Using now the definition of Ψ and the fact that $\kappa + 2\lambda = \ell - 1$ and $\kappa = 2$, we find

$$\omega(f, \Psi) = (-1)^{\frac{\ell-1}{2}} \cdot \frac{\psi_{\ell-1}(\mathfrak{N}) N_f}{N_E^{\frac{\ell+1}{2}}} = (-1)^{\frac{\ell-1}{2}} \cdot \frac{\psi_{\ell-1}(\mathfrak{N})}{N_E^{\frac{\ell-1}{2}}},$$

as claimed.

□

Lemma 4.10. *For every $\ell \in \mathcal{U}_g^{\mathrm{cl}} \cup \{1\}$ we define the Hecke character Φ_ℓ of conductor c as follows:*

$$\Phi_\ell := \psi_{\ell-1}^{-2} \mathbf{N}_K^\ell, \text{ of infinity type } (\ell, 2 - \ell).$$

Then there exists a non-zero number $\mathcal{E}ul_N^{\mathrm{Pet}}(\breve{g}_\ell) \in \mathbb{Q}_\psi^\times$ such that we have the following equality:

$$\langle \breve{g}_\ell, \breve{g}_\ell \rangle_N = \frac{\Im(Dc^2)}{\Im(N)} \cdot \mathcal{E}ul_N^{\mathrm{Pet}}(\breve{g}_\ell) \cdot \mathfrak{a}_{\mathrm{Pet}}(\ell) \cdot \mathfrak{f}_{\mathrm{Pet}}(\ell) \cdot L(\Phi_\ell^{-1}, 0).$$

where $\mathfrak{a}_{\mathrm{Pet}}(\ell)$ and $\mathfrak{f}_{\mathrm{Pet}}(\ell)$ are as defined in theorem 2.11.

Proof. By proposition 3.15 we know that the function:

$$\mathcal{E}ul_N^{\mathrm{Pet}}(\breve{g}_\ell) := \frac{\langle \breve{g}_\ell, \breve{g}_\ell \rangle_N}{\langle g_\ell, g_\ell \rangle_N} \tag{4.7}$$

interpolates well p-adically and we may compute it explicitly using equation (2.15). We get the result using theorem 2.11. □

Lemma 4.11. *The following formulae hold true:*

$$\Phi_\ell(\wp) = \frac{\beta_p(g_\ell)^2}{p^{\ell-2}}, \qquad \Phi_\ell(\bar\wp) = \frac{p^\ell}{\beta_p(g_\ell)^2}, \qquad \Psi_\ell(\bar\wp) = \frac{p^{\frac{\ell+1}{2}}}{\beta_p(g_\ell)}.$$

Proof. It is a straightforward computation using equation (4.5). □

We now define:

$$\mathcal{E}ul_N(\ell) := \frac{\mathcal{E}ul_N^{\mathrm{HR}}(\breve g_\ell, \breve f)}{\mathcal{E}ul_N^{\mathrm{Pet}}(\breve g_\ell)}$$
$$\mathfrak{f}_\infty(\ell) := \frac{\mathfrak{f}_{\mathrm{HR}}(\ell)\cdot \mathfrak{f}_{\mathrm{K}}(\Phi_\ell)}{\mathfrak{f}_{\mathrm{BDP}}(\Psi_\ell)\cdot \mathfrak{f}_{\mathrm{Pet}}(\ell)}$$

consider the function:

$$\mathfrak{f} : \mathcal{U}_g^{\mathrm{cl}} \to \mathbb{C}_p, \quad \mathfrak{f}(\ell) := \mathcal{E}ul_N(\ell)\cdot \mathfrak{f}_\infty(\ell).$$

From section 3.2.5 we can restrict Hida-Rankin p-adic L-function to the line $(\ell, k, j) = (\ell, 2, (\ell+1)/2) \in \mathcal{U}_{\mathrm{HR}}$. In particular, $(\ell, 2, (\ell+1)/2) \in \mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}}$ for every $\ell \in \mathcal{U}_g^{\mathrm{cl}}$ and the interpolation formula (3.32) reads as follows

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}}, \breve f)(\ell) = \mathfrak{e}_{\mathrm{HR}}(\ell)\cdot \mathfrak{a}_{\mathrm{HR}}(\ell)\cdot \mathfrak{f}_{\mathrm{HR}}(\ell)\cdot \frac{L(\breve g_\ell \otimes \breve f, (\ell+1)/2)}{\langle \breve g_\ell^*, \breve g_\ell^*\rangle_N}, \tag{4.8}$$

where we write $\mathscr{L}_p{}^g(\breve{\mathbf{g}}, \breve f)(\ell) := \mathscr{L}_p{}^g(\breve{\mathbf{g}}, \breve f)(\ell, 2, (\ell+1)/2)$, $\mathfrak{e}_{\mathrm{HR}}(\ell) = \mathfrak{e}_{\mathrm{HR}}(g_\ell, f, (\ell+1)/2)$ and similarly for the other terms. Before the main theorem we need a last:

Lemma 4.12. *The following equalities are true:*

$$\mathfrak{e}_{\mathrm{K}}(\Phi_\ell)\cdot \mathfrak{e}_{HR}(\ell) = \mathfrak{e}_{\mathrm{BDP}}(\Psi_\ell),$$
$$\mathfrak{a}_{\mathrm{K}}(\Phi_\ell)\cdot \mathfrak{a}_{HR}(\ell) = \mathfrak{a}_{\mathrm{BDP}}(\Psi_\ell)\cdot \mathfrak{a}_{\mathrm{Pet}}(\ell).$$

Proof. The first part follows from lemma 4.11, the second part is a straightforward comparison. □

Then we have the following:

Theorem 4.13. *The function $\mathfrak{f}$ is $\mathbb{Q}_\psi(\breve f)$-admissible in the sense of definition 4.2, i.e. it interpolates to a meromorphic Iwasawa function on $\mathcal{U}_g$ and $\mathfrak{f}(1) \in \mathbb{Q}_\psi(\breve f)^\times$. For all $\ell \in \mathcal{U}_g$ we have the following factorization of p-adic L-functions:*

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}}, \breve f)(\ell)\cdot \mathscr{L}_p(K)(\Phi_\ell) = \mathfrak{f}(\ell)\cdot \mathscr{L}_p(f/K)(\Psi_\ell).$$

Proof. First of all notice that for $\ell \in \mathcal{U}_g^{\mathrm{cl}} \cap \mathbb{Z}_{\geq 3}$, we have that:

- $(\ell, 2, (\ell+1)/2) \in \mathcal{U}_{\mathrm{HR}}^{\mathrm{cl}}$, following section 3.2.5,
- $\Psi_\ell \in \Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \mathbb{1})$, following section 3.1.3,
- $\Phi_\ell \in \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$, following section 3.1.2,

hence it is meaningful to compare the values of the three L-functions. Using Lemmas 4.8 and 4.10 we can restate formula (4.8) as follows:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}}, \breve f)(\ell) = \mathcal{E}ul_N(\ell)\cdot \mathfrak{e}_{\mathrm{HR}}(\ell)\cdot \mathfrak{a}_{\mathrm{HR}}(\ell)\cdot \frac{\mathfrak{f}_{\mathrm{HR}}(\ell)}{\mathfrak{f}_{\mathrm{Pet}}(\ell)}\cdot \frac{L(f, \Psi_\ell^{-1}, 0)}{L(\Phi_\ell^{-1}, 0)}.$$

Substituting Katz interpolation formula (3.4) and BDP interpolation formula (3.9), the complex and the p-adic periods defined in section 3.1.1 are simplified and we find that:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{f})(\ell)=\mathcal{E}ul_N(\ell)\cdot\frac{\mathfrak{e}_{\mathrm{HR}}(\ell)\mathfrak{e}_{\mathrm{K}}(\Phi_\ell)}{\mathfrak{e}_{\mathrm{BDP}}(\Psi_\ell)}\cdot\frac{\mathfrak{a}_{\mathrm{HR}}(\ell)\cdot\mathfrak{a}_{\mathrm{K}}(\Phi_\ell)}{\mathfrak{a}_{\mathrm{BDP}}(\Psi_\ell)\cdot\mathfrak{a}_{\mathrm{Pet}}(\ell)}\cdot\frac{\mathfrak{f}_{\mathrm{HR}}(\ell)\cdot\mathfrak{f}_{\mathrm{K}}(\Phi_\ell)}{\mathfrak{f}_{\mathrm{BDP}}(\Psi_\ell)\cdot\mathfrak{f}_{\mathrm{Pet}}(\ell)}\cdot\frac{\mathscr{L}_p(f/K)(\Psi_\ell)}{\mathscr{L}_p(K)(\Phi_\ell)}.$$

Using lemma 4.12 we find that for all $\ell\in\mathcal{U}_g^{\mathrm{cl}}\cap\mathbb{Z}_{\geq 3}$

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{f})(\ell)\cdot\mathscr{L}_p(K)(\Phi_\ell)=\mathfrak{f}(\ell)\cdot\mathscr{L}_p(f/K)(\Psi_\ell).$$

It remains to prove that $\mathfrak{f}_\infty(\ell)$ extends to an analytic function. Using the explicit expressions for the various factors involved and lemma 4.4, we derive that:

$$\mathfrak{f}_\infty(\ell)=-\frac{\Im(Dc^2)}{\Im(N)}\cdot\prod_{q|c}\frac{q-\chi_K(q)}{q-1}\cdot\frac{N\cdot 2^{-q|(N_E,D)}}{h_c\cdot D}\cdot\frac{\psi_{\ell-1}(\mathfrak{N})}{c^{3-\ell}N_E^{\frac{\ell-1}{2}}}. \tag{4.9}$$

Since $p\nmid N$, the factor $\mathfrak{f}_\infty(\ell)$ extends to an meromorphic Iwasawa function on $\mathcal{U}_g$. The value at 1 is non-zero and the same holds for $\mathcal{E}ul_N(1)$ as seen in lemmas 4.8 and 4.10. This proves the result. □

Since we have a factorization of p-adic L-functions on the whole space $\mathcal{U}_g$, which is fibered over 1 modulo $p-1$, we may ask ourselves what happens at $\ell=1$. In particular, because of lemmas 4.8 and 4.10, the number $\mathcal{E}ul(1)$ is well-defined and non-zero. Using the explicit equation (4.9) we find:

$$\mathfrak{f}_\infty(1)=-\frac{\Im(Dc^2)}{\Im(N)}\cdot\prod_{q|c}\frac{q-\chi_K(q)}{q-1}\cdot\frac{N\cdot 2^{-q|(N_E,D)}}{h_c\cdot Dc^2}\cdot\psi(\mathfrak{N})\neq 0$$

hence $\mathfrak{f}_\infty(1)\neq 0$ and $\mathfrak{f}(\ell)$ is $\mathbb{Q}_\psi(\breve{f})$-admissible. With the help of Katz functional equation (3.5) we find:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{f})(1)\cdot\mathscr{L}_p(K)(\psi^{-2})=\mathfrak{f}(1)\cdot\mathscr{L}_p(f/K)(\psi^{-1}\mathbf{N}_K).$$

The formulae for the special values of BDP and Katz p-adic L-functions (equations (3.7) and (3.14)) allow us to conclude that:

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{f})(1)=\mathcal{E}ul_N(1)\cdot\frac{\mathfrak{f}_\infty(1)\mathfrak{f}_p(f,\psi)}{\mathfrak{f}_p(\psi)}\cdot\frac{\log_{E,p}(P_\psi)^2}{\log_p(u_{g_\alpha})}$$

The proof of theorem 0.6 then follows from proposition 3.18. To be specific, the factor $\lambda(\breve{f},\breve{g})$ appearing in theorem 0.6 is given by:

$$\lambda(\breve{f},\breve{g})=\mathcal{E}ul_N(1)\cdot\frac{\mathfrak{f}_\infty(1)\mathfrak{f}_p(f,\psi)}{\mathfrak{f}_p(\psi)}.$$

4.4 Proof of theorem 0.7 and elliptic Stark conjecture when $p||N_E$

Remember assumption (PP) and consider now the setting in which $p||N_E$, i.e. we have multiplicative reduction at p. We denote by N_f the tame level of f, by which we mean that $N_f=N_E/p$, so that $f\in S_2(pN_f)$, and p does not divide cN_fD.

In a setting of bad reduction for f, the omega factor contained in BDP's interpolation formula and the local factors C_v of Garrett-Hida p-adic L-function are difficult to analyze in family. Moreover, to get the explicit formula for the constant appearing in the elliptic Stark conjecture in the good reduction case, we needed to assume that the levels were all equal. In order to avoid these problems we let f vary in Hida family, too. In this way, the higher weight specializations are the p-stabilizations of modular forms of level N_f not divisible by p. In this way all of the constants are easy to treat. For this reason we need to consider Castella's extension of BDP's p-adic L-function.

Let $\mathbf{f}$ be a cuspidal Hida familiy passing through f and let $\mathbf{g}$ be the Hida family described in equation (4.2) passing through g_α. In particular we have that

$$\mathbf{f}_2 = f \quad \text{and} \quad \mathbf{g}_1 = g_\alpha.$$

Recall the three ring class characters ψ_1, ψ_2 defined in section 4.1 and set $\psi_0 := \psi_g/\psi'_g$. We define the following families of characters:

$$\Psi_{gh}(k,\ell) := (\psi_{g,\ell-1}\psi_h)^{-1}\mathbf{N}_K^{\frac{k+\ell-1}{2}}, \qquad \text{s.t. } \Psi_{gh}(2,1) = \psi'_1\mathbf{N}_K \tag{4.10}$$

$$\Psi_{gh'}(k,\ell) := (\psi_{g,\ell-1}\psi'{}_h)^{-1}\mathbf{N}_K^{\frac{k+\ell-1}{2}}, \qquad \text{s.t. } \Psi_{gh}(2,1) = \psi'_2\mathbf{N}_K, \tag{4.11}$$

$$\Phi_g(\ell) := (\psi_{g,\ell-1}^2\chi)^{-1}\mathbf{N}_K^{\ell}, \qquad \text{s.t. } \Phi_g(1) = \psi'_0\mathbf{N}_K. \tag{4.12}$$

Then, we consider the space of classical interpolation $\mathcal{U}_{gf}^{\mathrm{cl}}$ as the set:

$$\mathcal{U}_{gf}^{\mathrm{cl}} := \{(\ell,k) \in \mathbb{Z}^2 \mid k > 2, \ell \geq k+1\}$$

and notice that, for $(\ell,k) \in \mathcal{U}_{fg}^{\mathrm{cl}}$, the following holds:

- $(\ell,k,1) \in \mathcal{U}_{\mathrm{GH}}^{\mathrm{cl}}$, following section 3.2.6,
- $\Psi_{gh}(k,\ell), \Psi_{gh'}(k,\ell) \in \Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\mathbb{1})$, following section 3.1.3, since they have infinity type $(\kappa+\lambda,-\lambda)$, where $\kappa = k$ and $\lambda = (\ell-k-1)/2 \geq 0$. Moreover, following the notation of section 3.2.3, they have branch characters ψ_1 and ψ_2 respectively.
- $\Phi_g(\ell) \in \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$, following section 3.1.2, since it has infinity type $(\kappa_1,\kappa_2) = (\ell, 2-\ell)$.

Since we restrict to the plane $\mathcal{U}_{gf} \subset \mathcal{U}_{\mathrm{HR}}$ via the above map, we shall denote

$$\mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{h})(\ell,k) := \mathscr{L}_p{}^g(\breve{\mathbf{g}},\breve{\mathbf{f}},\breve{h})(\ell,k,1), \quad \mathfrak{e}_{\mathrm{GH}}(k,\ell) := \mathfrak{e}_{\mathrm{GH}}(g_\ell,f_k,h)$$

and use a similar notation for all the other constants appearing in the various interpolation formulae.

Lemma 4.14. *The following formulae hold true:*

$$\Phi_g(\ell)(\wp) = \frac{\beta_{g_\ell}^2\chi(p)}{p^{\ell-2}}, \qquad \Phi_g(\ell)(\bar{\wp}) = \frac{p^\ell}{\beta_{g_\ell}^2\chi(p)},$$

$$\Psi_{gh}(k,\ell)(\bar{\wp}) = \frac{p^{\frac{\ell+k-1}{2}}}{\beta_{g_\ell}\beta_h}, \qquad \Psi_{gh'}(k,\ell)(\bar{\wp}) = \frac{p^{\frac{\ell+k-1}{2}}}{\beta_{g_\ell}\alpha_h}.$$

Proof. This is similar to lemma 4.11 and is obtained using equation (4.5). □

Lemma 4.15. *The following equalities hold true:*

$$\mathfrak{e}_K(\Phi_g(\ell))^2 \cdot \mathfrak{e}_{\mathrm{GH}}(k,\ell)^2 = \mathfrak{e}_{\mathrm{BDP}}(k,\Psi_{gh}(k,\ell))^2\mathfrak{e}_{\mathrm{BDP}}(k,\Psi_{gh'}(k,\ell))^2,$$
$$\mathfrak{a}_{\mathrm{GH}}(k,\ell) \cdot \mathfrak{a}_{\mathrm{K}}(\Phi_g(\ell))^2 = \mathfrak{a}_{\mathrm{BDP}}(\Psi_{gh}(k,\ell)) \cdot \mathfrak{a}_{\mathrm{BDP}}(\Psi_{gh'}(k,\ell)) \cdot \mathfrak{a}_{\mathrm{Pet}}(\ell)^2.$$

Proof. This again follows from a straight-forward computation using Lemma 4.14 for the first equality. □

Lemma 4.16. *The following identities hold true:*

$$\omega(f,\Psi_{gh}(k,\ell)) = (-1)^{\frac{\ell-1}{2}}\frac{\psi_{g,\ell-1}(\mathfrak{N})\psi_h(\mathfrak{N})}{N_f^{\frac{\ell-1}{2}}}, \qquad \omega(f,\Psi_{gh}(k,\ell)) = (-1)^{\frac{\ell-1}{2}}\frac{\psi_{g,\ell-1}(\mathfrak{N})\psi'_h(\mathfrak{N})}{N_f^{\frac{\ell-1}{2}}}.$$

In particular we have

$$\omega(f,\Psi_{gh}(2,1)) = \psi_1(\mathfrak{N}) \quad \text{and} \quad \omega(f,\Psi_{gh}(2,1)) = \psi_2(\mathfrak{N}).$$

Proof. The proof is the same as for lemma with N_E substituted with N_f. □

Let

$$L_0 = K(\psi_g, \psi_h, \tau(\chi), \sqrt{N_g}, \sqrt{N_h})$$

denote the extension of $\mathbb{Q}$ generated by the values of the Hecke characters ψ_g and ψ_h, the Gauss sum associated to the Dirichlet character χ and the square-roots of N_g and N_h. Fix test vectors $\breve{\mathbf{f}}$, $\breve{\mathbf{g}}$ and $\breve{h}$ as in §3.2.6, giving rise to the p-adic L-function $\mathscr{L}_p{}^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{h})$.

Theorem 4.17. *There exists a quadratic extension L/L_0 and a L-admissible function $\mathfrak{f}$ on $\mathcal{U}_{fg}$ (in the sense of definition 4.2) such that the following factorization of two-variable p-adic L-functions holds:*

$$\mathscr{L}_p{}^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{h})(k,\ell) \cdot \mathscr{L}_p(K)(\Phi_{\mathbf{g}}(\ell)) = \mathfrak{f}(k,l) \cdot \mathscr{L}_p(\mathbf{f}/K)(\psi_1, k, \ell) \cdot \mathscr{L}_p(\mathbf{f}/K)(\psi_2, k, \ell).$$

Proof. We have a decomposition of classical L-functions given by

$$L\Big(f_k \otimes g_\ell \otimes h, \frac{k+\ell-1}{2}\Big) = \mathcal{E}ul_N(k,\ell) \cdot L(f_k, \Psi_{gh}(k,\ell)^{-1}, 0) \cdot L(f_k, \Psi_{gh'}(k,\ell)^{-1}, 0). \tag{4.13}$$

Since $p \nmid N$, the Euler factor in $\mathcal{E}ul_N(k,\ell)$ interpolates p-adically and gives rise to an L_0-admissible function. Combine theorem 2.11 and equation (4.13) with the interpolation formula (3.39). Then use equations (3.9) and (3.4) to replace the classical L-functions with their respective p-adic avatar. An easy check shows that the periods simplify.

Thanks to Lemma 4.15 one obtains the following equality, true for every $(k,\ell) \in \mathcal{U}_{fg}^{\mathrm{cl}}$:

$$\mathscr{L}_p{}^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{h})(k,l)^2 \cdot \mathscr{L}_p(K)(\Phi_g(\ell))^2 = \mathfrak{f}_0(k,\ell) \cdot \mathscr{L}_p(\mathbf{f}/K)(\psi_1, k, \ell)^2 \cdot \mathscr{L}_p(\mathbf{f}/K)(\psi_2, k, \ell)^2 \tag{4.14}$$

where

$$\mathfrak{f}_0(k,\ell) = \frac{\mathcal{E}ul_N(k,\ell)}{\mathfrak{f}_{\mathrm{Pet}}(\ell)^2} \cdot \frac{\mathfrak{f}_{\mathrm{GH}}(\ell,k,1) \cdot \mathfrak{f}_{\mathrm{K}}(\Phi_g(\ell))^2}{\mathfrak{f}_{\mathrm{BDP}}(\Psi_{gh}(k,\ell)) \cdot \mathfrak{f}_{\mathrm{BDP}}(\Psi_{gh'}(k,\ell))} \cdot \frac{\prod_{v|N\infty} W_v}{\omega(k,\ell)}.$$

Let us show that $\mathfrak{f}_0$ is L_0-admissible. Notice that the L_0-admissibility of almost all terms appearing in the numerator and denominator of $\mathfrak{f}_0(k,\ell)$ follows directly from the definitions. Lemma 4.16 determines the L_0-admissibility of $\omega(f_k, \Psi_{gh})\omega(f_k, \Psi_{gh'})$, since $p \nmid N_f$, and the constant C_{Ran} appearing in $\mathfrak{f}_{\mathrm{GH}}(\ell, k, 1)$ is independent on the weights k and ℓ.

The L_0-admissibility of the function $\omega(k,\ell)$ appearing in $\mathfrak{f}_{\mathrm{GH}}(\ell,k,1)$ follows by the same argument as in the last part of the proof of [DLR15, Theorem 3.9]. Hence we have proved that $\mathfrak{f}_0$ is L_0-admissible and the theorem follows after taking the square-roots on both sides of (4.14). □

We now make in force the assumption 4.6(ES), i.e. $\psi = \psi_g$ is a ring class character of conductor $c \in \mathbb{Z}_{\geq 1}$ and $\psi_h = 1$ is the trivial character. In this setting the characters defined in equations (4.10) and (4.11) coincide, i.e. $\Psi_{gh} = \Psi_{gh'}$, thus we simply denote this character by Ψ_g, which have branch character $\psi = \psi_g$.

Assumption 4.18. (BR) the prime p of assumption 4.1(PP) is such that $p \nmid N := \mathrm{lcm}(N_f, Dc^2)$.

Theorem 4.19. *There exists a two-variable L-admissible function $\mathfrak{f}(k,\ell)$ such that the following factorization of p-adic L-functions holds:*

$$\mathscr{L}_p{}^g(\mathbf{f}, \mathbf{g})(k,\ell) \cdot \mathscr{L}_p(K)(\Phi_g(\ell)) \;=\; \mathfrak{f}(k,\ell) \cdot \mathscr{L}_p(\mathbf{f}/K)(\psi_g, k, \ell)^2.$$

Proof. After setting $\Psi := \Psi_g(k, \ell)$, it is easy to verify that

$$L(\breve{g}_\ell \otimes \breve{f}_k, (\ell + k - 1)/2) = \mathcal{E}ul_N(k, \ell) \cdot L(f_k, \Psi^{-1}, 0)$$

where $\mathcal{E}ul_N(k, \ell)$ stands for a product of Euler factors at primes dividing N. The function $\mathcal{E}ul_N(k, \ell)$ gives rise to an admissible function as shown in lemma 4.7. From this, the proof proceeds along similar lines as in the proof of Theorem 4.13, replacing the BDP p-adic L-function with Castella's two variable extension described in §3.2.3.

The explicit expression for the admissible function appearing in the above statement is very similar to that of the case $p \nmid N_E$, namely

$$\mathfrak{f}(k, \ell) = \mathcal{E}ul_N(k, \ell) \cdot (-1)^{\frac{k}{2}} \frac{\Im(D_K c^2)}{\Im(N)} \frac{N \cdot 2^{-q|(D_K, N_f)}}{h_c \cdot D_K} \frac{\psi_{\ell-1}(\mathfrak{N})}{c^{3-\ell} N_f^{\frac{\ell-1}{2}}}. \tag{4.15}$$

□

Theorem 0.7 relative to the extension of theorem 0.4 and 0.6 in the case of bad reduction is then achieved by evaluating the factorizations of theorems 4.17 and 4.19 at the point $(k, \ell) = (2, 1)$ and applying the various formulae for the special values (3.40), (3.18), (3.25) and (3.7)

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