# ANALYSIS OF NONLINEAR AND NON-SMOOTH DYNAMICS OF A SELF-OSCILLATING SERIES RESONANT INVERTER

L. Benadero [1] , E. Ponce[2] and A. El Aroudi[3*]

[1]Departament de Física,
Universitat Politècnica de Catalunya, Barcelona, Spain,
email: luis@fa.upc.edu
[2]Departamento de Matemática Aplicada,
Escuela Técnica Superior de Ingeniería, Sevilla, Spain,
email: eponcem@us.es
[3]Departament d'Enginyeria Electrònica, Elèctrica i Automàtica,
Universitat Rovira i Virgili, Tarragona, Spain,
email: abdelali.elaroudi@urv.cat

## ABSTRACT

*In this paper, the dynamics of a dc-ac resonant self-oscillating LC series inverter is analyzed from the point of view of piecewise smooth dynamical systems. Our system is defined by two symmetric configurations and its bifurcation analysis can be given in a one dimensional parameter space, thus finding a non smooth transition between two strongly different dynamics. The oscillating regime, which is the one useful for applications and involves a repetitive switching action between those configurations, is given whenever their open loop equilibrium is a focus. Otherwise, the only attractors are equilibrium points of node type whose stable manifolds preclude the appearance of oscillations.*
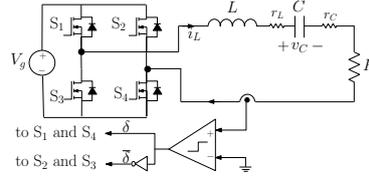
Figure 1. Schematic diagram of the LC series resonant inverter.

## 1   INTRODUCTION

In this paper we deal with an analysis of the LC series resonant inverter, similar to the one developed in [1] for its LC parallel counterpart. We put in evidence some relevant differences between these two implementations from the point of view of dynamics and bifurcations, which are mainly related to the location of the equilibria regarding the switching manifold. The rest of this paper is organized as follows. Section 2 presents the mathematical switched model of the system and its normalization, thus resulting in a unique bifurcation parameter. In section 3, we find that the transition from spiral to node of the open loop equilibrium further implies a non smooth global bifurcation, thus inhibiting the desired oscillatory mode. Finally, concluding remarks are drawn in the last section.

## 2   SYSTEM DESCRIPTION AND MATHEMATICAL MODELING

Figure 1 shows the circuit diagram of the system considered in this study that is an LC series resonant inverter [2]. he switches $S_1$ and $S_4$ are ON when $i_L > 0$ ($\delta = 1$), and they are turned OFF when $i_L < 0$ ($\delta = 0$). The switches $S_2$ and $S_3$ are driven in a complementary way to $S_1$ and $S_4$. Let $v_C$ be the voltage of the output capacitor, $i_L$ the inductor current and $\mathbf{z} = (v_C, i_L)^\intercal$ the vector state. Let also $u$ be the variable determined by the control in the form $u = 2\delta - 1$, that is $u = 1$ if $i_L > 0$ and $u = -1$ if $i_L < 0$. Let us define $\tau$ and $\mathbf{x} = (x_1, x_2)^\intercal$ as follows:

$$\tau = \omega_0 t, \quad x_1 = \frac{v_C}{V_g}, \quad x_2 = \frac{i_L Z_0}{V_g}.$$

The dynamical model of the system is as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \tag{1}$$

$$h(\mathbf{x}) = \mathbf{C}^\intercal \mathbf{x}, \tag{2}$$

where the matrix $\mathbf{A}$ and the vector $\mathbf{B}$ are redefined as

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{1}{Q} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that above parameters are the natural frequency $\omega_0$, the characteristic impedance $Z_0$ and the quality factor $Q$ of the LCR resonant series circuit, which are given by the expressions

$$\omega_0 = \sqrt{\frac{1}{LC}}, \quad Z_0 = \sqrt{\frac{L}{C}}, \quad Q = \frac{Z_0}{R_S},$$

where $R_S = R + r_C + r_L$ is the equivalent series resistance of the circuit. Note also that the open loop system (1), in which the switch variable $u$ remains constant, either $u = 1$ or $u = -1$, has as unique attractor the equilibrium point $\bar{\mathbf{x}} = (u, 0)^\intercal$. The eigenvalues of the matrix $\mathbf{A}$ are

$$p^\pm = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1},$$

and it can be deduced, due the physical restriction $Q > 0$, which implies eigenvalues with negative real part, that the open loop equilibrium is always stable. However, there is a minor transition at $Q = 1/2$, because the eigenvalues change from real to complex values. If $Q > 1/2$, the two eigenvalues are complex conjugated, so that the equilibrium is surrounded by spiraling trajectories. Otherwise, if $Q \leq 1/2$, the equilibrium is a node, and so the orbits tend to the stable manifold corresponding to the eigenvector associated to the highest or to the lowest eigenvalue considering forward or backward time evolution respectively. Unlike in the linear system, we will prove that in our piecewise smooth system (1)-(2), a non trivial non smooth bifurcation is produced at the same value $Q = 1/2$.

## 3  PIECEWISE SMOOTH ANALYSIS

### 3.1  The switching manifold and the sliding subset

Recall that from (2), the switching manifold is defined here as $\Sigma = \{\mathbf{x} : x_2 = 0\}$. According to the Filippov theory, sliding dynamics can occur in a subset $\Sigma_S$ of the switching manifold $\Sigma$, if the vector fields $\mathbf{F}^+$ and $\mathbf{F}^-$ satisfy the condition

$$\Sigma_S = \left\{ \mathbf{x} \in \Sigma : \left( \nabla h(\mathbf{x}) \cdot \mathbf{F}^+(\mathbf{x}) \right) \left( \nabla h(\mathbf{x}) \cdot \mathbf{F}^-(\mathbf{x}) \right) < 0 \right\}, \tag{3}$$

in which $\nabla(\cdot)$ is the gradient operator. This means that in a sliding region, the vector field points inwards or outwards at both sides of $\Sigma_S$. Conversely, in the points not belonging to $\Sigma_S$, the vector field crosses $\Sigma$. Roughly speaking, three different cases of switching dynamics can exist, one of them corresponding to simple crossing associated to Carathéodory solutions. The other two cases are the attracting and the rejecting sliding motions. In our case, the field $\mathbf{F}^+$ and $\mathbf{F}^-$ points outwards $\Sigma_S$, so the sliding is repelling and it is defined in the subset

$$\Sigma_S = \left\{ \mathbf{x} : -1 < x_1 < 1, x_2 = 0 \right\}.$$

### 3.2  The non oscillatory dynamics

We deal first with the non oscillatory dynamics. Actually, this a malfunction of the inverter in real applications, which occurs under the over damping condition, that is in the parameter domain $0 < Q \leq 1/2$. This case is illustrated in Fig. 2 using $Q = 0.4$, where some ad hoc trajectories have been depicted. If $0 < Q \leq 1/2$, the eigenvalues of the matrix $\mathbf{A}$ are real and negative, and so the dynamics evolving around each equilibrium cannot cross their corresponding stable manifolds. The consequence of this fact is that for any arbitrary trajectory, at most only one switching can be produced and therefore, the oscillating regimen cannot be attained. The boundary of attraction between the twin equilibrium points, which is also depicted in Fig. 2 using red color, is made up of three pieces: the sliding subset $\Sigma_S$ and the part of stable manifold corresponding to the lowest (more negative) eigenvalue in the valid side of the state plane for each equilibrium.

### 3.3  The self oscillating dynamics

In the following, we consider the quality factor restricted to the range $Q > 1/2$. Then, system (1)-(2) has an oscillatory dynamics, which is the one useful for inverter applications. Notice that for the linear case (1), with either $u = 1$ or $u = -1$, we have naturally a focus dynamics converging to an equilibrium, so that the self sustained oscillation is only enabled by the switching action introduced in (2). To prove this, let us choose an initial point located in the upper half plane. The dynamical integration forces the trajectory to cross $\Sigma$ at a point $(y_1 > 1, y_2 = 0)$,
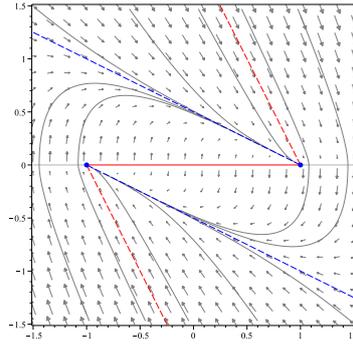
Figure 2: Boundaries of the attraction basin of the twin equilibria (blue points) for the system (1)-(2) with $Q = 0.4 < 1/2$, defined by the rejecting sliding segment and the eigenvectors corresponding to the lowest eigenvalue (red lines). A scheme of the piecewise smooth vector field and some illustrative trajectories have been also plotted. Notice that they tend to the stable manifold corresponding to the highest eigenvalue (blue lines).
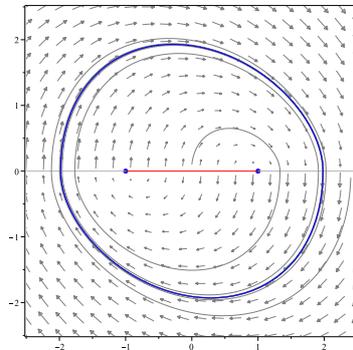


Figure 3: Oscillatory dynamics for system (1)-(2) with $Q = 1.5 > 1/2$, thus converging to a limit cycle, depicted in blue color, which is defined by two half cycles connected each other.

because it evolves clockwise around the right side equilibrium $\overline{\mathbf{x}}^+ = (1, 0)$. Then, the trajectory enters the lower half plane, so evolving clockwise around the left side equilibrium $\overline{\mathbf{x}}^- = (-1, 0)$ to reach and cross $\Sigma$ again at a point $(y_1 < -1, y_2 = 0)$. This process is repeated indefinitely, thus converging the trajectory to a finite limit cycle, that is the oscillatory dynamics, due to the dissipative character of the system. To get an expression of the stable limit cycle, it turns out more convenient to introduce the bifurcation parameter $\gamma$ as the quotient between the real and the imaginary parts of the focus eigenvalue $p^+$ that is

$$p^+ = -\frac{1}{2Q} + i\sqrt{1 - \frac{1}{4Q^2}} = \sigma + i\omega_r = \sigma\left(1 + \frac{i}{\gamma}\right),$$

in which

$$\gamma = \frac{\sigma}{\omega_r} = -\frac{1}{2Q\sqrt{1 - 1/(4Q^2)}} = -\frac{1}{\sqrt{4Q^2 - 1}} < 0.$$

Fig. 4 shows the evolution of the parameter $\gamma$ in terms of the quality factor $Q$. Notice that $\omega_r$ is the ratio between the free running frequency $\omega$ and the natural frequency $\omega_0$ in the real system, that is $\omega = \omega_r\omega_0$. Thus, if we take the new time and variables

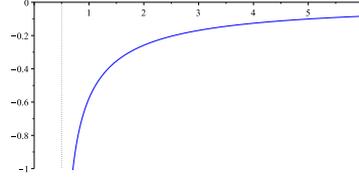$$\theta = \omega_r\tau, \quad y_1 = x_1, \quad y_2 = \omega_r x_2,$$

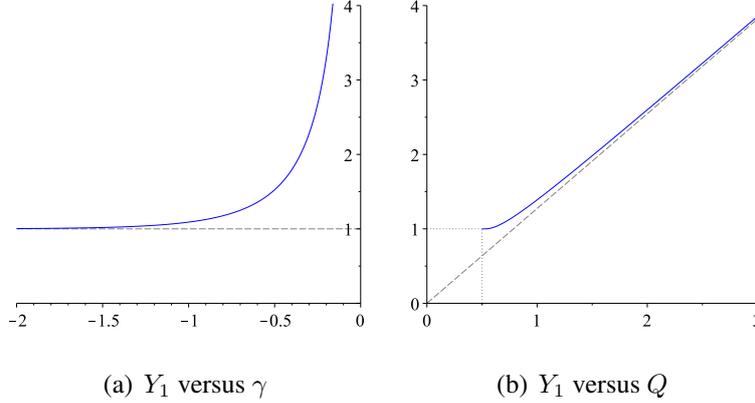Figure 4. The plot of the focus parameter $\gamma$ versus the quality factor $Q$.



(a) $Y_1$ versus $\gamma$        (b) $Y_1$ versus $Q$

Figure 5: Amplitude of the limit cycle versus $\gamma$ and $Q$. The gray dashed lines are the asymptotes to which the amplitude tends for high absolute values of the corresponding parameter.

and take into account that $\omega_r^{-2} = \gamma^2 + 1$, we obtain from (1)-(2) the normalized system

$$\frac{d\mathbf{y}}{d\theta} = \begin{pmatrix} 0 & \gamma^2 + 1 \\ -1 & 2\gamma \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \tag{4}$$

$$h(\mathbf{y}) = \mathbf{C}^\mathsf{T} \mathbf{y}, \tag{5}$$

in which $\mathbf{y} = (y_1, y_2)$. Accordingly, the switching manifold is redefined as $\Sigma = \{\mathbf{y} : y_2 = 0\}$. Taking into account the symmetry of the vector field with respect to the origin, we focus our attention only to the half-plane $y_2 \geq 0$, where $u = 1$ with the focus located at point $(1, 0)$. Thus, solving equation (4) with $u = 1$, we get

$$\begin{pmatrix} y_1(\theta) - 1 \\ y_2(\theta) \end{pmatrix} = \mathbf{\Phi}(\theta) \begin{pmatrix} y_1(0) - 1 \\ y_2(0) \end{pmatrix}, \tag{6}$$

where $\mathbf{\Phi}(\theta)$ is an evolution operator given by

$$\mathbf{\Phi}(\theta) = \mathrm{e}^{\gamma\theta} \begin{pmatrix} \cos\theta - \gamma\sin\theta & (\gamma^2 + 1)\sin\theta \\ -\sin\theta & \cos\theta + \gamma\sin\theta \end{pmatrix}. \tag{7}$$

Since we are dealing with orbits for $y_2 \geq 0$ starting at $\Sigma$ and returning to $\Sigma$ at time $\theta_1$ after surrounding the focus, we can write $y_2(0) = y_2(\theta_1) = 0$ in (6) thus resulting $\theta_1 = \pi$. This simple solution reflects the fact that any orbit running from $\Sigma$ to $\Sigma$ surrounding the focus, will last exactly half time of the cycle because both focus are at $\Sigma$ itself. Imposing also the symmetry condition $y_1(\theta_1) = -y_1(0)$, we obtain after some algebra an expression for the amplitude of the limit cycle, as the crossing point of the limit cycle at $\Sigma$, namely

$$Y_1 = y_1(\theta_1) = \coth\left(-\frac{\gamma\pi}{2}\right).$$

(a) $Q = 5\,(\gamma = -0.3535)$         (b) $\gamma = -1\,(Q = 0.7071)$
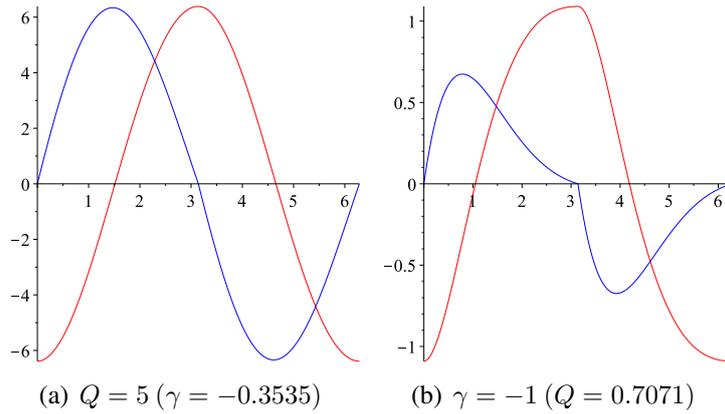
Figure 6. Waveforms of $y_1(\theta)$ in red and of $y_2(\theta)$ in blue, for the limit cycle.

In Fig. 5, the value of the normalized variable $y_1$ at the switching condition, called here $Y_1$, is represented in front of the two parameters $\gamma$ and $Q$. It is worth noting that if the quality factor is high enough, the expression $Y_1 \approx 4Q/\pi$ is a reasonable approximation for the amplitude of the steady oscillation. Also, if $\gamma$ is made negative enough, $Y_1$ converges to its lowest value $Y_1 = 1$. Both asymptotic behaviors can be seen in the same diagrams. In Fig. 6, the normalized values $(y_1, y_2)$ for one cycle of the steady state oscillation have been represented for two different parameters $Q = 5\,(\gamma = -0.3535)$ and $\gamma = -1\,(Q = 0.7071)$. Focusing on applications, let us define a new variable $y_R$ to account for the relative load voltage. Thus, recalling that $y_2 = \omega_r i_L Z_0 / V_g$ and considering the voltage divider relation $\alpha = R/R_S$ between the load $R$ and the series equivalent $R_S$ we deduce that

$$y_R = \frac{i_L R}{V_g} = \frac{\alpha y_2}{\omega_r Q}, \tag{8}$$

and for one of the two symmetrical half cycles in the steady state we deduce from (6)-(7) the expression $y_2(\theta) = (1 + Y_1)e^{\gamma\theta}\sin\theta$, and then $y_R(\theta) = Y_R(\gamma)g(\gamma,\theta)$ follows, in which the constant $Y_R$ is a sort of amplitude and $g$ takes care of the dependence on time. These terms are

$$Y_R = -2\alpha\gamma\left(1 - \coth\left(\frac{\gamma\pi}{2}\right)\right), \quad g(\gamma,\theta) = e^{\gamma\theta}\sin\theta.$$

**ACKNOWLEDGMENTS**

**REFERENCES**

[1] E. Ponce, L. Benadero, A. El Aroudi and L. Martínez-Salamero, "Sliding Bifurcations in Resonant Inverters," *International Multi-Conference on Systems, Signals and Devices,* accepted, SSD'2017.

[2] R. Erickson and D. Maksimovic, *Fundamentals of Power Electronics*, 2nd ed.   Springer, 2001.