From expanded digraphs to voltage and line digraphs

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Abstract

In this note we present a general approach to construct large digraphs from small ones. These are called expanded digraphs, and, as particular cases, we show their close relationship between voltage digraphs and line digraphs, which are two known approaches to obtain dense digraphs. In the same context, we show the equivalence between the vertex-splitting and partial line digraph techniques. Then, we give a sufficient condition for a lifted digraph of a base line digraph to be again a line digraph. Some of the results are illustrated with two well-known families of digraphs. Namely, De Bruijn and Kautz digraphs.

Keyword: Digraph, adjacency matrix, regular partition, quotient digraph, voltage digraphs, lifted digraph, partial line digraphs, vertex-split digraphs.

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1 Introduction

In the study of interconnection and communication networks, the theory of digraphs plays a key role because, in many cases, the links between nodes are unidirectional. In this theory, there are three concepts that have shown to be very fruitful to construct good and efficient networks. Namely, those of quotient digraphs, voltage digraphs and (partial) line digraphs. Roughly speaking, quotient digraphs allow us to obtain a simplified or 'condensed' version of a bigger digraph, while the voltage and line digraph techniques do the converse by 'expanding' a smaller digraph. From this point of view, it is natural that the three techniques have close relationships. In this paper we explore some of such interrelations by introducing a general construction that we call expanded digraphs. These digraphs are obtained from a base graph whose vertices become vertex sets in the new graph, and the adjacencies are defined from a set of mappings. A special case is obtained when such a mappings are defined within a group, so obtaining the lifted graphs of base graphs with assigned voltages (elements of the group) on its arcs. In this context, we show that De Bruijn and Kautz digraphs can be defined as lifted digraphs of smaller De Bruijn digraphs. Moreover, it is proved that, under some sufficient conditions, the lifted digraph of a base graph that is a line digraph is again a line digraph. In the more general case of nonrestricted maps, we consider the quotient graphs, and the equivalent constructions of vertex-split digraphs and partial line digraphs. Here, it turns out that the line digraph and quotient operations commute. Finally, it is proved that the techniques of vertex-splitting and partial line digraph are equivalent, and some consequences are derived.

1.1 Background

Let us first recall some basic terminology and notation concerning digraphs. For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks on the subject; for instance Chartrand and Lesniak [3] or Diestel [4].

Through this note, $\Gamma = (V, E)$ denotes a digraph, with vertex set V and arc set E. An arc from vertex u to vertex v is denoted by either (u, v), uv, or $u \to v$. We allow *loops* (that is, arcs from a vertex to itself), and *multiple arcs.* The set of vertices adjacent to and from $v \in V$ is denoted by $\Gamma^-(v)$ and $\Gamma^+(v)$, respectively. Such vertices are referred to as *in-neighbors* and *out-neighbors* of v, respectively. Moreover, $\delta^-(v) = |\Gamma^-(v)|$ and $\delta^+(v) = |\Gamma^+(v)|$ are the *in-degree* and *out-degree* of vertex v, and Γ is *d-regular* when $\delta^+(v) = \delta^-(v) = d$ for any $v \in V$.

2 Expanded digraphs

Expanded digraphs are, in fact, a type of compounding that consists in connecting together several copies of a (di)graph by setting some (directed) edges between any two copies.

Indeed, let $\Gamma = (V, E)$ be a (base) digraph on n vertices, etc. (We allow loops and multiple arcs). Assume that each vertex $v \in V$ has assigned a vetex set U_v, and each arc $e = (u, v) \in E$ has assigned a mapping $\phi_{uv} : U_u \to U_v$.

Definition 2.1. *Let $\Phi = \{\phi_{uv} : (u, v) \in E\}$. The* expanded digraph Γ^Φ *of* Γ *with respect to Φ has vertex set $V(\Gamma^\Phi) = \{U_v : v \in V\}$, and there is an arc from $x \in U_u$ to $y \in U_v$ whenever $(u, v) \in E$ and $\phi_{uv}(x) = y$.*

Some important particular cases of this construction are obtained when the mappings in Φ are defined from a group:

- **Cayley digraphs**. Let G be a group with generating set Δ having δ elements. If Γ is a singleton with assigned vertex set G, δ loops, and each loop e has the mapping $\phi_e : h \to hg$, with $g \in \Delta$, then the expanded digraph Γ^Φ is the *Cayley digraph* $\mathrm{Cay}(G, \Delta)$.
- **Coset digraphs**. Let G be a group with generating set Δ having δ elements and with subgroup H. If Γ is a singleton with assigned vertex set $\{Hh : h \in G\}$, and δ loops, and each loop e has the mapping $\phi_e : Hh \to Hhg$, with $g \in \Delta$, then the expanded digraph Γ^Φ corresponds to the *coset digraph* $\mathrm{Coset}(G, H, \Delta)$.

Two natural generalizations of these concepts are the following (as far as we know, the second one is a new proposal):

- **Lifted (of voltage) digraphs** or **expanded Cayley digraphs**. Let G be a group with generating set Δ having δ elements. If each vertex of Γ is assigned to the vertex set G, and each arc e has the mapping $\alpha(e) = \phi_e : h \to hg$, with $g \in \Delta$, then the expanded digraph Γ^Φ is the so-called *lifted digraph* Γ^α (see Section 3).
- **Expanded coset digraphs**. Let G be a group with generating set Δ having δ elements and with subgroup H. If each vertex of Γ is assigned to the vertex set $\{Hh : h \in G\}$, and each arc e has the mapping $\phi_e : Hh \to Hhg$, with $g \in \Delta$, then we refer to the corresponding expanded digraph Γ^Φ as the *expanded coset digraph* Γ^α.

3 Voltage and lifted digraphs

When a group is involved in the setting of the mappings, the symmetry of the obtained constructions yield digraphs with large automorphism groups. To our knowledge, one of the first papers where voltage (undirected) graphs were used for construction of dense graphs was that of Alegre, Fiol and Yebra [1], but without using the name of 'voltage graphs'. This name was coined previously by Gross [9]. For more information, see Gross and Tucker [10], Baskoro, Branković, Miller, Plesník, Ryan and Siráň [2], and Miller and Siráň [12].

Let Γ be a digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$. Then, given a group G with generating set Δ, a voltage assignment of Γ is a mapping $\alpha : E \to \Delta$. The lift Γ^α is the digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times G$, where there is an arc from vertex (u, g) to vertex (v, h) if and only if $uv \in E$ and $h = g\alpha(uv)$. Such an arc is denoted by (uv, g).

3.1 De Bruijn and Kautz digraphs

Here we show that both De Bruijn and Kautz digraphs can be seen as lifted digraphs of the former. First, recall that a *De Bruijn digraph* $B(d, \ell)$ has vertices $x_1x_2 \dots x_\ell$, where $x_i \in \mathbb{Z}_d$ for $i = 1, \dots, \ell$, and adjacencies

$$x_1x_2 \dots x_\ell \quad \rightsquigarrow \quad x_2x_3 \dots x_\ell y, \qquad y \in \mathbb{Z}_d.$$

Similarly, a *Kautz digraph* $K(d, \ell)$ has vertices $x_1x_2 \dots x_\ell$, where $x_i \in \mathbb{Z}_{d+1}$, $x_i \neq x_{i+1}$ for $i = 1, \dots, \ell - 1$, and adjacencies

$$x_1x_2 \dots x_\ell \quad \rightsquigarrow \quad x_2x_3 \dots x_\ell y, \qquad y \neq x_\ell.$$

Lemma 3.1. *The equality $B(d, k + 1) = B(d, k)^\alpha$ holds with*

$$\begin{aligned} \alpha : E(B(d, k)) &\to \mathbb{Z}_d \\ x_1 \dots x_k x_{k+1} &\mapsto x_{k+1}. \end{aligned}$$

Similarly, Kautz digraphs are lifted digraphs of De Bruijn digraphs.

Lemma 3.2. *The equality $K(d, k + 1) = B(d, k)^\beta$ holds with*

$$\begin{aligned} \beta : E(B(d, k)) &\to \mathbb{Z}_{d+1} \\ x_1 \dots x_k x_{k+1} &\mapsto x_{k+1}, \end{aligned}$$

where $x_i \in \mathbb{Z}_{d+1} \setminus \{0\}$.

4 Quotient digraphs and line digraphs

In the more general context of nonrestricted maps, we now consider the quotient graphs, and the equivalent constructions of vertex-split digraphs and partial line digraphs.

4.1 Regular partitions and quotient digraphs

Let $\Gamma = (V, E)$ be a digraph with n vertices. A partition π of its vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_m$, for $m \leq n$, is called *regular* if the number c_{ij} of arcs from a vertex $u \in U_i$ to vertices in U_j only depends on i and j. The *quotient digraph* of Γ with respect to π, denoted by $\pi(\Gamma)$, has vertices the subsets U_i, $i = 1, \ldots, m$, and c_{ij} parallel arcs from vertex U_i to vertex U_j.

4.2 Line digraphs

In the *line digraph* $L(\Gamma)$ of a digraph Γ, each vertex represents an arc of Γ, $V(L(\Gamma)) = \{uv : (u, v) \in E(G)\}$, and a vertex uv is adjacent to a vertex wz when the arc (u, v) is adjacent to the arc (w, z): $u \to v(= w) \to z$. Line digraphs have shown to be very interesting structures in the study of dense digraphs (that is, digraphs with a large number of vertices for given degree and diameter). Moreover, it is know that the iteration of the line digraph technique yields digraphs with maximum connectivity. For more details, see, for instance the papers by Fiol, Yebra, and Alegre [7, 8], and Fàbrega and Fiol [5]. Furthermore, by the Heuchenne's condition [11], a digraph Γ is a line digraph if and only if, for every pair of vertices u and v, either $\Gamma^+(u) = \Gamma^+(v)$ or $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$.

4.3 Regular partitions vesus line digraphs

The following result shows that the quotient and line digraph operations commute.

Proposition 4.1. *Every regular partition π of a digraph Γ induces a regular partition π' in its line digraph $L(\Gamma)$ and*

$$L(\pi(\Gamma)) = \pi'(L(G)).$$

4.4 From line digraphs to line digraphs

Now we show a sufficient condition for being a line digraph be kept when applying the voltage digraph technique.

Proposition 4.2. *Let $\Gamma = (V, E)$ be a digraph endowed with a voltage assignment α. If Γ is a line digraph, and for every pair of vertices u and v with common out-neighbor sets $\Gamma^+(u) = \Gamma^+(v) = \{x_1, \ldots, x_\delta\}$, we have*

$$\alpha(ux_i)\alpha(vx_i)^{-1} = \alpha(ux_j)\alpha(vx_j)^{-1}, \qquad i, j = 1, \ldots, \delta, \tag{1}$$

then, the lifted digraph Γ^α is again a line digraph.

Proof. It suffices to prove that Γ^α satisfies Heuchenne's condition. With this aim, let $ux_i, ux_j \in E$, so that in Γ^α the vertex (u,g) is adjacent to the vertices $(x_i, g\alpha(ux_i))$ and $(x_j, g\alpha(ux_j))$. Now, if there is a vertex $v \in V$ such that $vx_i \in E$, we also have $vx_j \in E$ (because Γ is a line digraph). But the former implies that vertex (v,h), where $h = g\alpha(ux_i)\alpha(vx_i)^{-1}$ is adjacent to vertex

$$(x_i, h\alpha(vx_i)) = (x_i, g\alpha(ux_i)).$$

Moreover, from (1), we get that $\alpha(xv_j) = \alpha(vx_i)\alpha(ux_i)^{-1}\alpha(ux_j)$. Thus, the vertex (v,h) is also adjacent to the vertex

$$(x_j, h\alpha(vx_j)) = (x_j, g\alpha(ux_j)).$$

Consequently, the vertices (u,g) and (v,h) satisfy Heuchenne's condition and Γ^α is a line digraph. □

4.5 Vertex-splitting

Let us now consider what we call *the vertex-splitting method* to "blow up" a digraph. Given a digraph $\Gamma = (V,E)$ on n vertices and m arcs, the vertex-split digraph $S_\mu(\Gamma)$, where $n \le \mu \le m$ is constructed as follows. Every vertex $v \in V$ is split into $\iota(v)$ vertices $v_1, \ldots, v_{\iota(v)}$, where $\iota(v) \le \delta^-(v)$. Thus the order of $S_\mu(\Gamma)$ is

$$\mu = \sum_{v\in V} \iota(v) \le \sum_{v \in V} \delta^-(v),$$

satisfying $n \le \mu \le m$. Moreover, for each arc $vw \in E$, we choose any vertex, say w_j, with $1 \le j \le \iota(w)$, and set the arcs $v_i w_j$ for every $i = 1, \ldots, \iota(v)$. We proceed in this way with all the arcs of Γ, with the condition that, in the end, all vertices of $S_\mu(\Gamma)$ must have nonzero indegree. (Notice that this is always possible, as $\iota(u) \le \delta^-(u)$ for every $u \in V$.)

4.6 Partial line digraphs

The above method is shown to be equivalent to the partial line digraph technique first proposed by Fiol and Lladó in [6], which is as follows. Given the digraph $\Gamma(V,E)$ as above, let $E' \subseteq E$ a subset of μ arcs satisfying $\{v : uv \in E'\} = V$, so that $n \le \mu \le m$. Then in the *partial line digraph* of Γ, denoted by $L_\mu(G)$, each vertex represent and arc of E', and a vertex uv is adjacent to the vertices $v'w$ if $vw \in E$ and, if $v' = v$ when $vw \in E'$, or v' is any vertex such that $v'w \in E'$, otherwise.

Lemma 4.3. *By appropriately chosen the above vertices v' in the construction of the vertex-split and partial line digraphs of the digraph Γ, we have the isomorphism*

$$S_\mu(\Gamma) \cong L_\mu(\Gamma).$$

Proof. Let the partial line digraph $L_\mu(\Gamma)$ be constructed from the arc set E'. Then, in constructing $S_\mu(\Gamma)$, every vertex v of Γ is split into the vertices $v_1 \ldots, v_{\iota(v)}$ if and only if $v_j v \in E'$ for every $j = 1, \ldots, \iota(\Gamma)$. Now, for every $v \in V$, assume that $uv \in E'$. Then, in $S_\mu(\Gamma)$ we have $v_i = u$ for some $j = 1 \ldots, \iota(v)$. Assuming that $vw \in E$, we have to consider two cases:

- If $vw \in E'$, then in $S_\mu(\Gamma)$ we choose every vertex v_i, $i = 1 \ldots, \iota(v)$, to be adjacent to the vertex $w_j = v$.
- Otherwise, in $L_\mu(\Gamma)$ we choose the vertex uv to be adjacent to the vertex $v'w$, where $v' = w_j$ (for the chosen vertex w_j in $S_\mu(\Gamma)$).

Then, it is clear that this gives the desired equivalence between the two digraphs. □

In particular, when $\iota(u) = \delta^-(u)$ for every u, we have $S_m(\Gamma) \cong L(\Gamma)$.

The above equivalence can be used to prove the following:

Lemma 4.4. *Every partial line digraph $L_\mu\Gamma$ of a digraph Γ can be seen as an expanded digraph of Γ with appropriate set Φ of mappings.*

Moreover, the diameter and mean distance of the vertex-split digraph $S_\mu(\Gamma)$ is only increased by at most one. This result was proved for the case of partial line digraphs in [6]. Thus, its proof is also a consequence of the above equivalence.

Proposition 4.5. *Let Γ be a digraph different from a cycle, with diameter D and mean distance $\overline{D}$. Then, the diameter D^* and mean distance $\overline{D}^*$ of the vertex-split digraph $S_\mu(\Gamma)$, with $\mu > m$, satify:*

$$\begin{aligned} D^* &= D + 1; \\ \overline{D}^* &< \overline{D} + 1. \end{aligned}$$

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