Characterization of a Class of Differential Equations

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Abstract

This paper deals with a characterization of nonlinear systems of the form $\dot{x}_\gamma(t) = f\left(x_\gamma(t), u(t/\gamma)\right)$ when the parameter $\gamma \to \infty$. In particular, we are interested in the uniform convergence of the sequence of functions $x_\gamma\left(\gamma t\right)$. Necessary conditions and sufficient ones are derived for this uniform convergence to happen.

Keywords: nonlinear systems, consistent operator, uniform convergence

1 Introduction

Hysteresis is a nonlinear behavior encountered in a wide variety of processes including biology, optics, electronics, ferroelectricity, magnetism, mechanics, structures, among other areas. The detailed modeling of hysteresis systems using the laws of Physics is an arduous task, and the obtained models are often too complex to be used in applications. For this reason, alternative models of these complex systems have been proposed [15, 1, 8, 6, 9]. These models do not come, in general, from the detailed analysis of the physical behavior of the systems with hysteresis. Instead, they combine some physical understanding of the system along with some kind of black-box modeling.

This way of describing hysteresis systems led to the proliferation of hysteresis models in the last two decades. A search in the Web of Knowledge database gives more than 2000 publications. The question that arises naturally is: do these research works describe really hysteresis phenomena? In other words, does the researcher who proposes a new hysteresis model have a mathematical rule to decide whether the model they propose is indeed a hysteresis one?

Surprisingly enough, such a rule exists only for a limited number of hysteresis processes: those that possess the so-called rate-independence property. This property states that, under a time-scale change, the relationship output versus input is unchanged. Hysteresis systems that are rate-independent are listed in the survey paper [10]. However, in the last two decades, researchers have acknowledged the importance of rate-dependent processes in applications [4, 3, 2]. For this reason, a recent effort [5] proposed a mathematical framework that proposes a rule to decide whether or not a system may be hysteretic. The rule proposed in [5] shows that, for an input/output system with input $u(t/\gamma)$ and output $x_\gamma(t)$, the convergence of the sequence of functions $t \to x_\gamma\left(\gamma t\right)$ as $\gamma \to \infty$ is a necessary condition for the hysteresis. The previous formulation is used to study the hysteresis behavior of the generalized Duhem model [11] and the LuGre friction model [12].

In the present paper, we consider the differential equation $\dot{x} = f(x, u)$. Our objective is to derive necessary conditions and also sufficient ones for the uniform convergence of the sequence of functions $t \to x_\gamma\left(\gamma t\right)$.

This paper is organized as follows. Section 2 presents the system of study and the assumptions under which the study is performed. Sections 3 and 4 present; respectively, necessary conditions and sufficient ones for the uniform convergence of the sequence of functions $x_\gamma\left(\gamma t\right)$ as $\gamma \to \infty$. Conclusions are given in Section 5.

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2 Problem Statement

The class of systems under study is

$$\dot{x}(t) = f\big(x(t), u(t)\big),\ t \geq 0, \tag{1}$$
$$x(0) = x_0, \tag{2}$$

where initial condition x_0 and state $x(t)$ take value in $\mathbb{R}^m$, and input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for some strictly positive integers n and m. The mapping $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a well-defined continuous function. Because of the continuity of the right-hand side of (1), the system (1)-(2) has a maximal solution which is defined on an interval of the form $[0, \omega)$, $\omega > 0$ [14, p. 67–70]. In this paper, we assume that the system (1)-(2) has a unique Carathéodory solution for all $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$.

Consider the time scale change $s_\gamma(t) = t/\gamma, \forall\gamma > 0, \forall t \geq 0$. When the input $u \circ s_\gamma$ is used instead of u, system (1)-(2) becomes

$$\dot{x}_\gamma(t) = f\big(x_\gamma(t), u \circ s_\gamma(t)\big),\ t \geq 0, \tag{3}$$
$$x_\gamma(0) = x_0, \tag{4}$$

which can be written for all $\gamma > 0$ as

$$\sigma_\gamma(t) = x_0 + \gamma \int_0^t f\big(\sigma_\gamma(\tau), u(\tau)\big) d\tau,\ \forall t \in [0, \omega_\gamma), \tag{5}$$

where $\sigma_\gamma = x_\gamma \circ s_{1/\gamma}$ and $[0, \omega_\gamma)$ is the maximal interval for the existence of solutions σ_γ.

We seek necessary conditions and also sufficient conditions for the uniform convergence of the sequence of functions σ_γ.

3 Necessary Conditions

Our aim in this section is to derive necessary conditions for the uniform covergence of the sequence of functions σ_γ.

Lemma 3.1. *Assume that the maximal solution of system (1)-(2) is defined on $\mathbb{R}_+$ for all $(u, x_0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m$. Suppose that there exists a function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|x(t)| \leq h\big(|x_0|, \|u\|_\infty\big), \forall t \geq 0, \tag{6}$$

for each initial state $x_0 \in \mathbb{R}^n$ and each input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. Assume that there exists a function $q_u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$ such that $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$. Then, we have $f\big(x_0, u(0)\big) = 0$, $q_u(0) = x_0$, and $f\big(q_u(t), u(t)\big) = 0$, $\forall t \geq 0$.

Proof. From the fact that $\|u\|_\infty = \|u \circ s_\gamma\|, \forall\gamma > 0$ and Inequality (6) it comes that

$$\|x_\gamma\|_\infty \leq h(|x_0|, \|u\|_\infty) = a. \forall\gamma > 0,$$

Thus, we get from the continuity of σ_γ that

$$|\sigma_\gamma(t)| \leq a,\ \forall t \geq 0,\ \forall\gamma > 0. \tag{7}$$

Inequality (7) along with the continuity of function f and the boundedness of the input u imply that there exists a constant $r > 0$ independent of γ, such that $|f(\sigma_\gamma(\tau), u(\tau))| \leq r$, $\forall\tau \geq 0$, $\forall\gamma > 0$. This means that we can apply the Dominated Lebesgue Theorem in Equation (5) and get

$$\lim_{\gamma\to\infty} \int_0^t f(\sigma_\gamma(\tau), u(\tau))\, d\tau = \int_0^t f(q_u(\tau), u(\tau))\, d\tau,\ \forall t \geq 0, \tag{8}$$

where the continuity of f and the fact that $\lim_{\gamma\to\infty}\|\sigma_\gamma - q_u\|_\infty = 0$ are used. By Equation (7) we have $\|\sigma_\gamma - x_0\|_\infty/\gamma \to 0$ as $\gamma\to\infty$. Thus, we obtain from (5) and (8) that

$$\int_0^t f\left(q_u(\tau), u(\tau)\right) d\tau = 0,\ \forall t \geq 0,$$

which gives $f\big(q_u(t), u(t)\big) = 0$ for almost all $t \geq 0$. From the continuity of functions f, q_u, and u it comes that

$$f\big(q_u(t), u(t)\big) = 0, \quad \text{for all } t \geq 0. \tag{9}$$

Since $\sigma_\gamma(0) = x_0$, $\forall\gamma > 0$ it comes that

$$q_u(0) = x_0. \tag{10}$$

Finally, taking $t = 0$ in (9) and using (10) provides the necessary condition

$$f\big(x_0, u(0)\big) = 0, \tag{11}$$

which completes the proof. □

Remark 1. Once chosen an input u, the term $u(0)$ is given so that any initial condition x_0 for which we have $\lim_{\gamma\to\infty}\|\sigma_\gamma - q_u\|_\infty = 0$ should satisfy (11).

4 Sufficient Conditions

In this section, we derive sufficient conditions to ensure that the sequence of functions σ_γ converges uniformly as $\gamma\to\infty$.

Definition 4.1. *[7] A continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing, satisfies $\beta(0) = 0$, and $\lim_{t\to\infty}\beta(t) = \infty$.*

Lemma 4.1. *[11] Consider a function $z : [0,\omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0,\omega)$.*

(ii) *There exist $z_1, z_2 \geq$ such that $z_1, z(0) < z_2$ and $\dot{z}(t) \leq 0$ for almost all $t \in [0,\omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1\big)$, $\forall t \in [0,\omega)$.

Corollary 4.1. *Consider a function $z : [0,\omega) \subset \mathbb{R}_+ \to \mathbb{R}_+$, where ω may be infinite. Assume the following*

(i) *The function z is absolutely continuous on each compact subset of $[0,\omega)$.*

(ii) *There exist a class $\mathcal{K}_\infty$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and $z_1, z_2, z_3 \geq 0$ such that $\max\big(\beta^{-1}(z_3), z_1, z(0)\big) < z_2$, and $\dot{z}(t) \leq -\beta(z(t)) + z_3$ for almost all $t \in [0,\omega)$ that satisfy $z_1 < z(t) < z_2$.*

Then, $z(t) \leq \max\big(z(0), z_1, \beta^{-1}(z_3)\big)$, $\forall t \in [0,\omega)$.

Proof. We have $\dot{z}(t) \leq 0$ for almost all $t \in [0,\omega)$ that satisfy $\max\big(\beta^{-1}(z_3), z_1\big) < z(t) < z_2$, and hence the result follows directly from Lemma 4.1. □

Lemma 4.2. *Assume that there exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ such that*

$$f\big(q_u(t), u(t)\big) = 0,\ \forall t \geq 0, \tag{12}$$

$$q_u(0) = x_0. \tag{13}$$

Define $y_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ as

$$y_\gamma(t) = \sigma_\gamma(t) - q_u(t) = x_\gamma(\gamma t) - q_u(t), \forall\gamma > 0, \tag{14}$$

for all $t \in [0,\omega_\gamma)$. Suppose that we can find a continuously differentiable function $V : \mathbb{R}^m \to \mathbb{R}_+$ that satisfies the following:

(i) *V is positive definite, that is $V(0) = 0$ and $V(\alpha) > 0, \forall 0 \neq \alpha \in \mathbb{R}^m$.*

(ii) *V is proper, that is $V(\alpha) \to \infty$ as $|\alpha| \to 0$.*

(iii) *There exist $\delta > 0$ and $\beta \in \mathcal{K}_\infty$ satisfying:*

$$\begin{cases} \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t) + q_u(t), u(t)\big) \le -\beta\big(|y_\gamma(t)|\big), \\ \text{for all } t \in [0, \omega_\gamma) \text{ and } \forall \gamma > 0 \text{ that satisfy } |y_\gamma(t)| < \delta. \end{cases} \tag{15}$$

Then,

- $\omega_\gamma = +\infty, \forall \gamma > 0$. *Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*
- $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$.

Proof. Since V is positive definite and proper, there exists $\beta_1, \beta_2 \in \mathcal{K}_\infty$ such that (see [7, p. 145])

$$\beta_1(|\alpha|) \le V(\alpha) \le \beta_2(|\alpha|), \forall \alpha \in \mathbb{R}^m. \tag{16}$$

From (5), we get for almost all $t \in [0, \omega_\gamma)$, $\forall \gamma > 0$ that

$$\dot{y}_\gamma(t) = \gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t), \tag{17}$$

$$y_\gamma(0) = 0. \tag{18}$$

For any $\gamma > 0$, define $V_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ as $V_\gamma(t) = V\big(y_\gamma(t)\big), \forall t \in [0, \omega_\gamma)$. Note that the function V_γ is absolutely continuous on each compact subset of $[0, \omega_\gamma)$ as a composition of a continuously differentiable function V and an absolutely continuous function y_γ. Then, we get for almost all $t \in [0, \omega_\gamma)$ and all $\gamma > 0$ that

$$\dot{V}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha=y_\gamma(t)} \cdot \dot{y}_\gamma(t) = \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha=y_\gamma(t)} \cdot \Big[\gamma f\big(y_\gamma(t) + q_u(t), u(t)\big) - \dot{q}_u(t)\Big]. \tag{19}$$

Let $\Omega = \big(0, \beta_1(\delta)\big)$. By (16) we have for any $\gamma > 0$, and for almost all $t \in [0, \omega_\gamma)$ that

$$V_\gamma(t) \in \Omega \Rightarrow |y_\gamma(t)| < \delta. \tag{20}$$

We conclude from (15), (19), and (20) that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + \|\dot{q}_u\|_\infty \left|\frac{dV(\alpha)}{d\alpha}\Big|_{\alpha=y_\gamma(t)}\right|, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, we deduce from the continuity of $\frac{dV(\alpha)}{d\alpha}$, the boundedness of $\dot{q}_u$, and (20) there exists some $b > 0$ independent of γ such that

$$\dot{V}_\gamma(t) \le -\gamma\,\beta(|y_\gamma(t)|) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Hence, (16) implies

$$\dot{V}_\gamma(t) \le -\gamma\,\beta \circ \beta_2^{-1}\big(V_\gamma(t)\big) + b, \text{ for almost all } t \in [0, \omega_\gamma), \forall \gamma > 0 \text{ that satisfy } V_\gamma(t) \in \Omega.$$

Thus, Corollary 4.1 and the fact that $V_\gamma(0) = 0, \forall \gamma > 0$, imply that $V_\gamma(t) \le \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right), \forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma)$ where $\gamma_0 = \frac{b}{\beta \circ \beta_2^{-1} \circ \beta_1(\delta)}$. Therefore, (16) implies that

$$|y_\gamma(t)| \le \beta_1 \circ \beta_2 \circ \beta^{-1}\left(\frac{b}{\gamma}\right), \forall \gamma > \gamma_0, \forall t \in [0, \omega_\gamma). \tag{21}$$

Thus, $\omega_\gamma = +\infty, \forall \gamma > \gamma_1$ for some $\gamma_1 > 0$, and $\lim_{\gamma\to\infty} \|y_\gamma\|_\infty = 0$, which is equivalent to $\lim_{\gamma\to\infty} \|\sigma_\gamma - q_u\|_\infty = 0$. On the other hand, (21) and the fact that $\sigma_\gamma = y_\gamma + q_u$ imply that there exists some $E, \gamma^* > 0$ such that $\|\sigma_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$, and hence $\|x_\gamma\|_\infty \le E, \forall \gamma > \gamma^*$. □

Lemma 4.3. *Consider the nonlinear system [13]*

$$\dot{x} = f(x,u) = Ax + \Phi(x) + R(u), \quad (22)$$
$$x(0) = x_0, \quad (23)$$
$$y = Dx, \quad (24)$$

where $x_0 \in \mathbb{R}^m$, A is an $m \times m$ Hurwitz matrix[2]*, D is an $m \times m$ matrix, input $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, state x, output y take values in $\mathbb{R}^m$, function $R \in C^0(\mathbb{R}^n, \mathbb{R}^m)$, and a locally Lipshitz function $\Phi \in C^0(\mathbb{R}^m, \mathbb{R}^m)$. Assume the following:*

(i) *The exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ such that $q_u(0) = x_0$ and*

$$Aq_u(t) + \Phi\big(q_u(t)\big) + R\big(u(t)\big) = 0, \forall t \geq 0.$$

(ii) *There exist $c_1 > 0$, $c_2 > 0$, $\xi > 0$ and $r > 2$ such that*

$$\left|\alpha \cdot \left[\Phi\big(\alpha + q_u(t)\big) - \Phi\big(q_u(t)\big)\right]\right| \leq c_1 |\alpha|^2 + c_2 |\alpha|^r, \text{ for almost all } t \geq 0, \forall \alpha \in \mathbb{R}^m \text{ that satisfy } |\alpha| < \xi.$$

(iii) *One has $c_1 < \frac{1}{2\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue for the $m \times m$ positive-definite symmetric matrix P that satisfies*[3]

$$PA + A^T P = -I_{m \times m}. \quad (25)$$

Let x_γ, y_γ be respectively the state and the output of (22)-(24) when we use the input $u \circ s_\gamma$ instead of u.

Then,

- *All solutions of (22)-(24) are bounded. Furthermore, there exist $E, \gamma^* > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, for any solution x_γ of the system (3)-(4).*
- $\lim_{\gamma \to \infty} \|F_\gamma - Dq_u\|_\infty = 0$*, where $F_\gamma : \mathbb{R}_+ \to \mathbb{R}^m$ is defined as $F_\gamma(t) = y_\gamma(\gamma t), \forall t \geq 0, \forall \gamma > 0$.*

Proof. Since Φ is locally Lipschitz, the right-hand side of (22) is locally Lipschitz relative to x and hence the system (22) has a unique solution. The function q_u satisfies (12)-(13) in Lemma 4.2 because of (i).

Consider the continuously differentiable quadratic Lyapunov function candidate $V : \mathbb{R}^m \to \mathbb{R}$ such that $V(\alpha) = \alpha^T P \alpha$, $\forall \alpha \in \mathbb{R}^m$. Since P is symmetric, we have $\forall \alpha \in \mathbb{R}^m$ that

$$\lambda_{\min} |\alpha|^2 \leq V(\alpha) = \alpha^T P \alpha \leq \lambda_{\max} |\alpha|^2,$$

where $\lambda_{\min}$ is the smallest eigenvalue of the matrix P. Thus V is positive definite and proper. Since P is symmetric we have

$$\left|\frac{dV(\alpha)}{d\alpha}\right| = 2|P\alpha| \leq 2\lambda_{\max}|\alpha|, \forall \alpha \in \mathbb{R}^m. \quad (26)$$

We have by (25) that

$$\frac{dV(\alpha)}{d\alpha} \cdot A\alpha = 2P\alpha \cdot A\alpha = \alpha^T \left(PA + A^T P\right)\alpha = -|\alpha|^2, \forall \alpha \in \mathbb{R}^m. \quad (27)$$

From Condition (i) we get for all $\gamma > 0$ that

$$\begin{aligned} \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot f(y_\gamma + q_u, u) &= \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + Aq_u + \Phi(y_\gamma + q_u) + R(u)\Big] \\ &= \frac{dV(\alpha)}{d\alpha}\Big|_{\alpha = y_\gamma} \cdot \Big[Ay_\gamma + \Phi(y_\gamma + q_u) - \Phi(q_u)\Big]. \end{aligned} \quad (28)$$

where y_γ is defined in (14).

[2]that is each eigenvalue of A has a strictly negative real part.

[3]the existence of the matrix P in (25) is guaranteed because A is Hurwitz [7, p.136].

We get from (28), (27), (26) and Condition (ii) that

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le (-1+2c_1\,\lambda_{\max})\,|y_\gamma(t)|^2 + 2c_2\,\lambda_{\max}\,|y_\gamma(t)|^r, \tag{29}$$
$$\forall\gamma>0 \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)|<\xi,$$

where $[0,\omega_\gamma)$ is the maximal interval of existence of σ_γ and y_γ. This leads to

$$\left.\frac{dV(\alpha)}{d\alpha}\right|_{\alpha=y_\gamma(t)} \cdot f\big(y_\gamma(t)+q_u(t),u(t)\big) \le -\frac{1-2c_1\,\lambda_{\max}}{2}\,|y_\gamma(t)|^2, \tag{30}$$
$$\forall\gamma>0, \text{ for almost all } t\in[0,\omega_\gamma) \text{ that satisfy } |y_\gamma(t)| < \min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right).$$

Thus, (15) in is satisfied with $\beta(v) = \frac{1-2c_1\,\lambda_{\max}}{2}v^2$, $\forall v\ge 0$ and $\delta = \min\left(\sqrt[r-2]{\frac{1-2c_1\,\lambda_{\max}}{4c_2\,\lambda_{\max}}},\ \xi\right)$. Hence all conditions of Lemma 4.2 are satisfied so that the solution of (22) is bounded. Morover, there exist $E,\gamma^*>0$ such that $\|x_\gamma\|_\infty \le E, \forall\gamma>\gamma^*$. Futhermore, we have $\lim_{\gamma\to\infty}\|\sigma_\gamma - q_u\|_\infty = 0$. Thus, we deduce from (24) that $\lim_{\gamma\to\infty}\|F_\gamma - Dq_u\|_\infty = 0$. □

Example. Consider the system

$$\dot{x} = -x + x^3 - u, \tag{31}$$
$$x(0) = 0. \tag{32}$$

where state x takes values in $\mathbb{R}$ and input $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ is defined as $u(t) = 0.1\sin(t), \forall t\ge 0$. The system (31)-(32) has the form (22)-(24), with $x=y$, $m=n=1$, $A=-1$, $\Phi(\alpha)=\alpha^3$, $R(\alpha)=-\alpha$, $\forall\alpha\in\mathbb{R}$, and $D=1$. Observe that P in (25) equals 1/2 which mean that $\lambda_{\min}=\lambda_{\max}=1/2$. We have $u(0)=0$ and u is bounded with

$$u(\cdot)\in[u_{\min},u_{\max}] = [-0.1, 0.1]. \tag{33}$$

Define the function $\chi:\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]\to\left[-\frac{2}{3\sqrt{3}},\frac{2}{3\sqrt{3}}\right]$ as $\chi(v) = -v+v^3, \forall v\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. The function χ is strictly decreasing, bijective and its inverse function is continuous. Hence, there exists a function $q_u\in C^0(\mathbb{R}_+,\mathbb{R})\cap L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $q_u(\cdot)\in\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, $q_u(0)=0$ and

$$\chi\big(q_u(t)\big) = -q_u(t)+q_u^3(t) = u(t), \forall t\ge 0. \tag{34}$$

It can be checked using (33) that $\|q_u\|_\infty < 0.11$ (see Figure (1b)). Thus $q_u(\cdot)\ne\frac{1}{\sqrt{3}}$. This fact and (34) implies that the function $\dot{q}_u = \dot{u}/\left(1-3q_u^2\right)$ is bounded so that $q_u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$. Hence Condition (i) of Lemma 4.3 is satisfied.

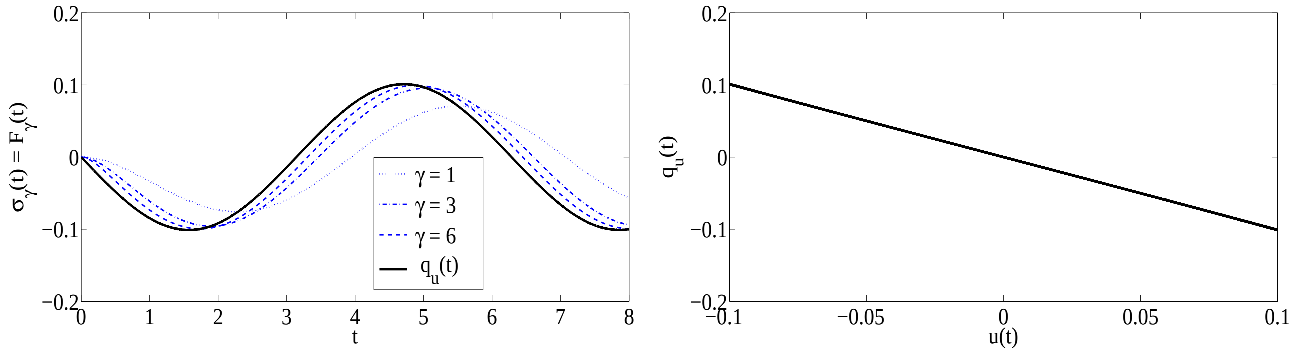
On the other hand, we have for all $\alpha\in\mathbb{R}$ that

$$\alpha\left(\Phi(\alpha+q_u)-\Phi(q_u)\right) = 3q_u^2\alpha^2 + 3q_u\alpha^3+\alpha^4. \tag{35}$$

Since $\|q_u\|_\infty < 0.11$, one has $\left\|3q_u^2\right\|_\infty < 0.0363 = c_1$. Hence it follows from (35) that for any $\xi>0$ we have

$$\alpha\big[\Phi\big(\alpha+q_u(t)\big)-\Phi\big(q_u(t)\big)\big] \le c_1\alpha^2 + (3\|q_u\|_\infty+\xi)\,\alpha^3 \tag{36}$$
$$\forall\alpha\in\mathbb{R}^m \text{ that satisfy } |\alpha|<\xi, \text{ for almost all } t\ge 0.$$

Thus, Condition (ii) in Lemma 4.3 is satisfied with $c_2 = 3\|q_u\|_\infty+\xi$. Moreover, we have $c_1 < 1 = \frac{1}{2\,\lambda_{\max}}$ which implies that Condition (ii) in Lemma 4.3 is also satisfied. Therefore, the solution of (31)-(32) is bounded, that there exist $E,\gamma^*>0$ such that $\|x_\gamma\|_\infty\le E, \forall\gamma>\gamma^*$, and that $\lim_{\gamma\to\infty}\|\sigma_\gamma-q_u\|_\infty = \lim_{\gamma\to\infty}\|F_\gamma-q_u\|_\infty = 0$ (observe that $\sigma_\gamma(\cdot)=F_\gamma(\cdot)$ because $x(\cdot)=y(\cdot)$). This is illustrated in Figure 1a.



(a) $F_\gamma(t)$ versus t for system (31).

(b) $q_u(t)$ versus $u(t)$ for system (31).

Figure 1: Simulations.

5 Conclusion

In [5] a rule for deciding whether a process may or may not be a hysteresis is proposed for causal operators such that a constant input leads to a constant output. That rule involves checking whether the so-called consistency and strong consistency properties hold. In this paper we derived necessary conditions and sufficient ones for the uniform convergence of the shifted solutions $\sigma_\gamma : t \to x_\gamma(\gamma t)$ of the system $\dot{x} = f(x, u \circ s_\gamma)$. This uniform convergence is related to consistency. Does this mean that the concept of consistency can be extended to study operators for which the property that a constant input leads to a constant output, that property does not hold?

This paper explores this issue for systems of the form $\dot{x} = f(x, u)$, however, no clear cut answer may be drawn for the obtained results.

Indeed, the necessary conditions alone cannot guarantee whether the uniform convergence of σ_γ when $\gamma \to \infty$ happens or not. The sufficient conditions do imply that convergence but do not guarantee that the hysteresis loop of the operator is not trivial. In the example, we have seen that q_u is a function of u so that the hysteresis loop is a curve and we cannot acertain from this whether system (31) is a hysteresis or not. This is a future research line.

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