

Review Article

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Boundedness of Stable Solutions to Semilinear Elliptic Equations: A Survey

DOI: 10.1515/ans-2017-0008

Received February 3, 2017; accepted February 23, 2017

Abstract: This article is a survey on boundedness results for stable solutions to semilinear elliptic problems. For these solutions, we present the currently known L^∞ estimates that hold for all nonlinearities. Such estimates are known to hold up to dimension 4. They are expected to be true also in dimensions 5 to 9, but this is still an open problem which has only been established in the radial case.

Keywords: Stable Solutions, Semilinear Elliptic Equations, Boundedness of Solutions

MSC 2010: 35K57, 35B65

Communicated by: Antonio Ambrosetti and David Arcoya

Dedicated to Ireneo Peral, with great admiration and friendship

1 Introduction. The Known L^∞ Estimates

This article is a survey on regularity results for *stable* solutions of semilinear elliptic problems. For these solutions, we present the currently known L^∞ estimates that hold for all nonlinearities, or at least for a large class of them. It is Haïm Brezis who stressed, since the mid-nineties, the significance of this problem.

Even though we will briefly comment at the end of this introduction on related results for other nonlinear elliptic problems, the rest of the paper concerns the semilinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $n \geq 1$, and f is a locally Lipschitz nonlinearity.

Following [2], we say that u is a *weak solution* of (1.1) if, with $d_\Omega = \text{dist}(\cdot, \partial\Omega)$, we have $u \in L^1(\Omega)$, $f(u)d_\Omega \in L^1(\Omega)$ and $-\int_\Omega u \Delta \zeta \, dx = \int_\Omega f(u) \zeta \, dx$ for all $\zeta \in C^1(\bar{\Omega})$ with $\zeta = 0$ on $\partial\Omega$.

A weak solution u of (1.1) is said to be *stable* if $f'(u) \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega f'(u) \xi^2 \, dx \leq \int_\Omega |\nabla \xi|^2 \, dx \quad \text{for all } \xi \in C_c^1(\Omega), \quad (1.2)$$

i.e., for all C^1 functions ξ with compact support in Ω . If the solution u is bounded, its stability is equivalent to the nonnegativeness of the first Dirichlet eigenvalue in Ω of the linearized operator $-\Delta - f'(u)$ for (1.1) at u . As a consequence, *local minimizers* of the energy (i.e., minimizers under small perturbations) are stable solutions – whenever the energy and all objects are well defined within a proper functional analytic framework.

In dimensions 10 and higher, there are explicit examples of unbounded (also called singular) stable solutions. The following is one of them.

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Proposition 1.1. For $n \geq 3$, the function $\tilde{u} = -2 \log|x|$ is a weak solution of (1.1) in $\Omega = B_1$, the unit ball, for the nonlinearity $f(u) = 2(n-2)e^u$. For $n \geq 10$, the function \tilde{u} is an $H_0^1(B_1)$ stable weak solution.

This result is easily proved. One verifies that \tilde{u} is a weak solution whenever $n \geq 3$. Next, the linearized operator at \tilde{u} is given by

$$-\Delta - 2(n-2)e^{\tilde{u}} = -\Delta - \frac{2(n-2)}{|x|^2}.$$

Thus, if $n \geq 10$, then its first Dirichlet eigenvalue in B_1 is nonnegative. This is a consequence of *Hardy's inequality*

$$\frac{(n-2)^2}{4} \int_{B_1} \frac{\xi^2}{|x|^2} \leq \int_{B_1} |\nabla \xi|^2 \quad \text{for every } \xi \in H_0^1(B_1)$$

and the fact that $2(n-2) \leq (n-2)^2/4$ if $n \geq 10$.

Thus, in dimensions $n \geq 10$ there exist unbounded H_0^1 stable solutions of (1.1), even in the unit ball and for the exponential nonlinearity. One may wonder if the same could happen in a lower dimension, perhaps for another nonlinearity and in another domain. The regularity results that we present next indicate that dimension 10 should be optimal. However, *a main open question is still to be answered in dimensions $5 \leq n \leq 9$* – see Open Problem 1.6 below. Let us emphasize that here we are talking about regularity results that hold for all nonlinearities f , or at least for a very large class of them. In particular, we will allow nonlinearities with supercritical growth with respect to the Sobolev exponent, as in Proposition 1.1. It is therefore the stability of the solution that will lead to L^∞ estimates.

All this is reminiscent of important results within the theories of minimal surfaces and of harmonic maps (see the survey [6] for these analogies), in which regularity of stable solutions also holds only in certain low dimensions.

There are many nonlinearities for which (1.1) admits a stable weak solution u with $u \neq 0$. Indeed, replace $f(u)$ by $\lambda f(u)$ in (1.1) with $\lambda \geq 0$. That is, consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Assume that f is positive, nondecreasing, and superlinear at $+\infty$, that is,

$$f(0) > 0, \quad f' \geq 0, \quad \lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty. \quad (1.4)$$

Note that in this case we look for positive solutions (when $\lambda > 0$).

Assume (1.4); then there exists an extremal parameter $\lambda^* \in (0, +\infty)$ such that if $0 \leq \lambda < \lambda^*$, then (1.3) admits a minimal classical solution u_λ . Here minimal means smallest. On the other hand, if $\lambda > \lambda^*$, then (1.3) has no classical solution. The family of classical solutions $\{u_\lambda : 0 \leq \lambda < \lambda^*\}$ is increasing in λ , and its limit as $\lambda \uparrow \lambda^*$ is a weak solution $u^* = u_{\lambda^*}$ of (1.3) for $\lambda = \lambda^*$. The function u^* is called *the extremal solution* of (1.3). Moreover, for every $\lambda \in [0, \lambda^*]$ the function u_λ is a *stable* weak solution of (1.3).

The proofs of the previous statements can be found in the book [12, Chapter 3] by Dupaigne. There, among many others, two further interesting results are proved under the hypothesis that f is convex and satisfies (1.4). First, the nonexistence of *weak* solutions of (1.3) for $\lambda > \lambda^*$, a result established in a seminal paper in this topic by Brezis, Cazenave, Martel, and Ramiandrisoa [2]. Second, Martel's result stating that u^* is the unique weak solution of (1.3) for $\lambda = \lambda^*$. The book [12] by Dupaigne also contains proofs of the statements and estimates that we present below. Another nice monograph in this subject is by Dávila [11].

1.1 L^∞ Estimates

We now turn to the core of the paper: L^∞ estimates for stable solutions. As usual, the key point will be to establish *a priori* L^∞ estimates for stable *classical* solutions of (1.1). Once this is done, they lead to the boundedness (and thus regularity) of the extremal solution u^* of (1.3) – simply by approximation of u^* by the stable

classical solutions u_λ with $\lambda \uparrow \lambda^*$ since the a priori estimates will be uniform in λ as $\lambda \uparrow \lambda^*$. In this respect, we could now forget about the extremal problem (1.3), come back to equation (1.1) and ask whether a stable weak solution of (1.1) can be approximated by classical solutions to similar problems. A nice answer was given in [2] (see also [12, Corollary 3.2.1]) and states that this is always possible if $u \in H_0^1(\Omega)$ is a stable weak solution of (1.1) and f is nonnegative and convex. The approximating functions can be taken to be stable classical solutions to problem (1.1) with f replaced by $(1 - \varepsilon)f$, where $\varepsilon \downarrow 0$.

There is extensive literature on a priori estimates for stable solutions, beginning with the seminal paper of Crandall and Rabinowitz [10] from 1975. In more recent years, there have been strong efforts to obtain a priori bounds under minimal assumptions on f , mainly after Brezis and Vázquez [3] raised several open questions – see also the open problems raised by Brezis in [1].

The following three theorems collect all what is currently known in terms of L^∞ estimates. We will give their proofs (with most of details) in Sections 2–4.

We start with a result that suggests the sharpness of dimension 10. However, it requires the nonlinearity f to be convex, satisfy (1.4) and also that

$$f \in C^2(\mathbb{R}) \quad \text{and the limit} \quad \lim_{t \rightarrow +\infty} \frac{f(t)f''(t)}{f'(t)^2} \text{ exists.} \quad (1.5)$$

It is easy to see that if the limit exists, then it must belong to $[0, 1]$, otherwise f would blow up somewhere in $(0, +\infty)$. These hypotheses are satisfied by many nonlinearities, including $f(u) = e^u$, $f(u) = e^{e^u}$ as well as $f(u) = (1 + u)^p$ with $p > 1$.

Theorem 1.2 ([10, 16, 17, 19]). *Let u^* be the extremal solution of (1.3).*

- (i) *Let f be convex and satisfy (1.4) and (1.5). If $n \leq 9$, then $u^* \in L^\infty(\Omega)$.*
- (ii) *If $\Omega = B_1$, $f(u) = e^u$ and $n \geq 10$, then $u^* = -2 \log|x|$ and $\lambda^* = 2(n - 2)$.*

Note that in part (i) we may take $f(u) = e^u$. Together with part (ii) – which contains more information than Proposition 1.1 – this gives a quite detailed answer to the boundedness of the extremal solution, in any dimension, for the exponential nonlinearity. Part (ii) was first established in [16] by using ODEs techniques, while a more flexible PDE proof was given in [3].

Part (i) also applies to $f(u) = (1 + u)^p$ with $p > 1$. For these power nonlinearities, the boundedness of stable solutions holds indeed for $n \leq 10$, and also for $n \geq 11$ and p smaller than the Joseph–Lundgren exponent. For the rest of couples (n, p) , the explicit radial solution is given by $|x|^{-2/(p-1)} - 1$ and coincides with the extremal solution u^* when $\Omega = B_1$. See [3, 5] for more details on this power case.

Theorem 1.2 (i) was proven by Crandall and Rabinowitz [10] and by Mignot and Puel [17], not only for the exponential nonlinearity but also for some others. The improved result under assumption (1.5) is due to Sanchón [19]. The proofs use the equation in (1.3) together with the stability condition (1.2) applied to

$$\xi := (f(u) - f(0))^\alpha \quad \text{or} \quad \xi := f(u)^\alpha - f(0)^\alpha \quad (1.6)$$

for a well chosen positive exponent α . In Section 2, we will present the proof with all details in the case $f(u) = e^u$.

Note that assumption (1.5) on a convex nonlinearity f prevents $f'' \geq 0$ to oscillate at infinity. Think that f could be linear within some intervals approaching infinity, and grow exponentially in the complement of the intervals. We will see that to establish regularity for stable solutions without the additional assumption (1.5) on f is a much harder task.

However, the radial case $\Omega = B_1$ was settled by Cabré and Capella [5] in 2006, as we state next. Note that the estimates apply to every nonlinearity f . This work does not use phase plane analysis, but PDE techniques instead.

Theorem 1.3 ([5]). *Let $n \geq 1$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a locally Lipschitz function. Let $u \in H_0^1(B_1)$ be a stable radial weak solution of (1.1). Then u is either constant, radially increasing or radially decreasing in B_1 . Moreover, the following holds:*

(i) If $n \leq 9$, then $u \in L^\infty(B_1)$ and

$$\|u\|_{L^\infty(B_1)} \leq C_n(\|u\|_{L^1(B_1)} + \|f(u)d_{B_1}\|_{L^1(B_1)}) \tag{1.7}$$

for some constant C_n depending only on n , where $d_{B_1} = \text{dist}(\cdot, \partial B_1)$.

(ii) If $n = 10$, then $|u(|x|)| \leq C|\log|x||$ in B_1 for some constant C .

(iii) If $n \geq 11$, then

$$|u(|x|)| \leq C|x|^{-\frac{n}{2} + \sqrt{n-1} + 2} |\log|x||^{\frac{1}{2}}$$

in B_1 for some constant C .

In this result, the assumption that the weak solution belongs to $H_0^1(B_1)$ is needed; see [12, Example 3.2.1] for this statement. The proof of the theorem will be given in Section 3, but later in this introduction we will comment on it.

From the theorem, and by working with the stable classical solutions $u = u_\lambda$ of (1.3), it is easy to bound the right-hand side of (1.7) uniformly in λ , assuming $\lambda > \lambda^*/2$ and (1.4) on the nonlinearity. Indeed, one first multiplies (1.3) by the first Dirichlet eigenfunction φ_1 of the Laplacian. We obtain

$$\lambda_1 \int_{\Omega} u \varphi_1 \, dx = \lambda \int_{\Omega} f(u) \varphi_1 \, dx.$$

Using now the superlinearity of f at infinity, we deduce a bound for $\int_{\Omega} f(u) \varphi_1 \, dx$ uniform in $\lambda > \lambda^*/2$. From this, multiplying (1.3) by the solution w of $-\Delta w = 1$ in Ω with zero Dirichlet boundary values, and controlling w by φ_1 , we finally bound $\|u\|_{L^1(\Omega)}$ uniformly as $\lambda \uparrow \lambda^*$. Therefore, since the classical solutions $u = u_\lambda$ are radially decreasing by the moving planes method, from the previous theorem we deduce the following result.

Corollary 1.4 ([5]). *Assume that f satisfies (1.4) and that $\Omega = B_1$. Let u^* be the extremal solution of (1.3). If $1 \leq n \leq 9$, then $u^* \in L^\infty(B_1)$.*

We turn now to the nonradial case. We present the currently known estimates that hold for all nonlinearities. These estimates coincide with the available ones for convex nonlinearities satisfying (1.4) – nothing better is known in this case. An a priori L^∞ estimate is known only up to dimension 4.

Theorem 1.5. (i) ([4]) *Let f be any smooth function satisfying $f \geq 0$ in \mathbb{R} and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Assume that $2 \leq n \leq 4$ and let u be a stable classical solution of (1.1). Then, for any compact set $K \subset \Omega$, we have*

$$\|u\|_{L^\infty(\Omega)} \leq C(\Omega, K, \|u\|_{L^1(\Omega)}, \|\nabla u\|_{L^4(\Omega \setminus K)}), \tag{1.8}$$

where C is a constant depending only on the quantities within the parentheses.

(ii) ([4, 18, 20]) *Let f satisfy (1.4), let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $1 \leq n \leq 4$. If $n \in \{3, 4\}$, assume either that f is a convex nonlinearity or that Ω is a convex domain. If u^* is the extremal solution of (1.3), then $u^* \in L^\infty(\Omega)$.*

Note that part (ii) of this theorem does not require assumption (1.5) on f .

In dimensions 2 and 3, and for convex nonlinearities, part (ii) was proved by Nedev [18] in 2000. We will briefly describe its proof in Section 2. Ten years later, Cabré established the first result in dimension 4 in [4]. Note that part (i) holds for *all nonnegative nonlinearities* and that the L^4 norm of ∇u is computed in an arbitrarily small neighborhood of the boundary $\partial\Omega$. As a consequence, we will see that the estimate leads easily to the boundedness of the extremal solution in convex domains. This is part of the statement in (ii).

In part (ii), for $3 \leq n \leq 4$, the paper [4] requires Ω to be convex, while f does not need to be convex. Some years later, Villegas [20] succeeded to use both [4] and [18], when $n = 4$, to remove the requirement that Ω is convex by further assuming that f is convex.

Open Problem 1.6. For convex nonlinearities satisfying (1.4), the boundedness of the extremal solution u^* or the existence of an a priori L^∞ bound for stable classical solutions is still unknown in dimensions $5 \leq n \leq 9$. This is the case even for convex domains which are symmetric with respect to each coordinate hyperplane $\{x_i = 0\}$. It is only known, by Theorem 1.3, in balls.

Let us explain where the convexity of Ω comes into play in Theorem 1.5, and raise an open problem in this direction. As we showed after Theorem 1.3, for a solution u of (1.1) and for nonlinearities f which are superlinear at infinity, one can give a bound for $\|u\|_{L^1(\Omega)}$ depending only on f and Ω . Therefore, estimate (1.8) in Theorem 1.5 reduces the L^∞ bound for a stable solution u to a bound for u in a sufficiently small neighborhood $\bar{\Omega} \setminus K$ of the boundary. In this case, we need an L^4 estimate for ∇u in $\Omega \setminus K$. Thus, the question is now whether there are boundary estimates for stable positive solutions of semilinear elliptic problems. This turns out to be true in convex domains (and here the stability of the solution is not needed) since the moving planes method gives a cone of monotonicity for u at all points close enough to $\partial\Omega$. Such a cone of monotonicity gives automatically an L^∞ control for u near $\partial\Omega$ in terms of $\|u\|_{L^1(\Omega \setminus K)}$, a quantity that, as $\|u\|_{L^1(\Omega)}$, is under control. See [4, 12] for more details.

In dimension 2, Chen and Li [9] extended this boundary estimate to general domains – they removed the convexity assumption on the domain after doing a Kelvin transform at boundary points. One only needs to assume that $f \geq 0$; see [4, 9] for more details. However, in dimensions $n \geq 3$ boundary estimates for stable solutions remain open.

Open Problem 1.7. Are there a priori *boundary estimates* (for instance L^∞ bounds in a neighborhood of the boundary) for stable classical solutions in nonconvex domains Ω of \mathbb{R}^n , $n \geq 3$, for all nonnegative nonlinearities f or at least for a large class of them? It would be desirable that the control is in terms of $\|u\|_{L^1(\Omega)}$.

In this respect, Nedev’s estimate (2.1) presented below is a global integral bound – and thus also a boundary integral bound. In Section 4, it will be used to control, when $n \leq 4$, the “boundary quantity” $\|\nabla u\|_{L^4(\Omega \setminus K)}$ in (1.8).

Let us finally comment on the proofs. Recall (1.6): for Theorem 1.2 the key point is to use a power of f as the test function ξ in the stability condition (1.2). Instead, for Theorems 1.3 and 1.5 we will use other type of test functions. The proofs start by writing the stability condition (1.2) for the test function $\xi = c\eta$, where $\eta|_{\partial\Omega} \equiv 0$. Integrating by parts, we easily deduce that

$$\int_{\Omega} (\Delta c + f'(u)c)\eta^2 \, dx = \int_{\Omega} c^2 |\nabla \eta|^2 \, dx. \tag{1.9}$$

Next, a key point is to choose a function c satisfying an appropriate equation for the linearized operator $\Delta + f'(u)$. In the radial case the choice of c and the final choice of ξ are

$$c = u_r \quad \text{and} \quad \xi = u_r r \left(r^{-\alpha} - \left(\frac{1}{2} \right)_+^{-\alpha} \right),$$

where $r = |x|$, $\alpha > 0$ and ξ is later truncated arbitrarily near the origin to make it Lipschitz. Note that, differentiating $u_{rr} + (n - 1)r^{-1}u_r + f(u) = 0$ with respect to r , we get

$$(\Delta + f'(u))u_r = (n - 1) \frac{u_r}{r^2}. \tag{1.10}$$

This way of proceeding was motivated by the study of stable minimal cones in the theory of minimal surfaces, where one takes $c = |A|$ (the norm of the second fundamental form of the cone), and then later $\xi = cr^{-\alpha}$ (properly truncated), where r is the distance to the vertex of the cone.

For the estimate up to dimension 4 in the nonradial case, as in Theorem 1.5, we will take

$$c = |\nabla u| \quad \text{and} \quad \xi = |\nabla u| \varphi(u), \tag{1.11}$$

where, in dimension $n = 4$, the function φ will be chosen depending on the solution u itself – thus not depending only on f as in Theorem 1.2. A simple computation (see Section 4) shows that

$$(\Delta + f'(u))|\nabla u| = \frac{1}{|\nabla u|} (|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2) \quad \text{in } \Omega \cap \{|\nabla u| > 0\}, \tag{1.12}$$

where $|A|^2 = |A(x)|^2$ is the squared norm of the second fundamental form of the level set of u passing through a given $x \in \Omega \cap \{|\nabla u| > 0\}$, i.e., the sum of the squares of the principal curvatures of the level set. On the other

hand, ∇_T denotes the tangential gradient to the level set. Thus, (1.12) involves geometrical information of the level sets of u . As we will see in Section 4, the choice of the function $\varphi = \varphi(t)$ in (1.11) will depend on certain integrals on the level set $\{u = t\}$.

The next crucial point in dimensions $n \leq 4$ is the use of the *Michael–Simon and Allard Sobolev inequality* (Proposition 4.2 below) in every regular level set of u . This is a remarkable Sobolev inequality on general hypersurfaces of \mathbb{R}^n that involves their mean curvature but has a best constant that is independent of the hypersurface – it depends only on the dimension n .

1.2 Other Nonlinear Elliptic Problems

The previous questions have been studied also for semilinear elliptic problems involving other operators. For the p -Laplace operator and $f(u) = e^u$, minimal and extremal solutions of (1.3) were studied by García-Azorero, Peral and Puel [14, 15]. They established the boundedness of the extremal solution when $n < p + 4p/(p - 1)$, and showed that this condition is optimal. Under hypotheses (1.4) and (1.5), Theorem 1.2 has been extended to the p -Laplace case by Sanchón [19]. The radial case in Theorem 1.3 has been extended by Capella, Sanchón and Cabré [7].

Other works treat the case of the bilaplacian, the mean curvature operator for graphs, the fractional Laplacian, boundary reaction problems (this is related with fractional diffusion), and elliptic systems. Optimal L^∞ bounds are still unknown for many of these problems, in particular for the fractional Laplacian.

Finally, the semilinear equation $-\Delta u = \lambda f(u)$ has also been studied for nonlinearities $f : [0, 1) \rightarrow \mathbb{R}$ which blow up at $u = 1$. This is connected with micro-electro-mechanical devices (MEMS); see the monograph [13].

For more information and references in all these problems, see Dupaigne’s monograph [12] – mainly Chapter 8.

2 The Exponential Nonlinearity. Nedev’s Result

The proof of Theorem 1.2 uses the stability condition with the test function ξ being essentially a power of $f(u)$. Let us give all details of the proof in the case of the exponential nonlinearity.

Proof of Theorem 1.2 (i) when $f(u) = e^u$. Let u be a stable solution of (1.3) with $f(u) = e^u$. Take $\lambda \in (0, \lambda^*)$ so that u is a classical solution, and $\xi = e^{\alpha u} - 1$ in the stability condition (1.2). We deduce

$$\begin{aligned} \int_{\Omega} \lambda e^u (e^{\alpha u} - 1)^2 dx &\leq \int_{\Omega} \alpha^2 e^{2\alpha u} |\nabla u|^2 dx = \frac{\alpha}{2} \int_{\Omega} \nabla(e^{2\alpha u} - 1) \cdot \nabla u dx \\ &= \frac{\alpha}{2} \int_{\Omega} (e^{2\alpha u} - 1)(-\Delta u) dx = \frac{\alpha}{2} \int_{\Omega} \lambda e^u (e^{2\alpha u} - 1) dx, \end{aligned}$$

where we have used that u solves problem (1.3). From the first and last integrals in these inequalities, we deduce that

$$\left(1 - \frac{\alpha}{2}\right) \int_{\Omega} e^{(1+2\alpha)u} dx \leq 2 \int_{\Omega} e^{(1+\alpha)u} dx.$$

Taking any $\alpha < 2$ and using the Hölder inequality, we can bound $\int_{\Omega} e^{(1+2\alpha)u} dx$ by a constant depending only on α and $|\Omega|$. Denoting $p := 1 + 2\alpha$, we conclude that $\lambda e^u \in L^p(\Omega)$ for every $p < 5$, with an a priori bound independent of λ . Thus $-\Delta u$ is in $L^p(\Omega)$, and hence u is controlled in $W^{2,p}(\Omega)$ uniformly in λ . If $n \leq 9$, the exponent p can be chosen such that $\frac{n}{2} < p < 5$, and therefore we deduce a uniform $L^\infty(\Omega)$ bound for u by the Sobolev embedding. \square

Nedev extended the previous argument to give an L^∞ bound in dimensions 2 and 3 for all convex nonlinearities satisfying (1.4). His proof uses the stability condition with $\xi = f(u) - f(0)$, and also tests the equation

in (1.3) by multiplying it by $g(u) := \int_0^u f'(t)^2 dt$. After some clever bounds on several nonlinearities which come up in the argument (see [12, 18]), he deduces the a priori bound

$$\int_{\Omega} f'(u)f(u) dx \leq C. \tag{2.1}$$

Next, since f is convex, we have $-\Delta(f(u) - f(0)) \leq f'(u)(-\Delta u) = \lambda f'(u)f(u)$. Thus $0 \leq f(u) - f(0) \leq v$, where v is the solution of $-\Delta v = \lambda f'(u)f(u)$ with zero Dirichlet boundary values. From (2.1) we deduce that v , and hence also $f(u) - f(0)$, belong to $L^p(\Omega)$ for all $p < n/(n - 2)$. We conclude that u is bounded in $W^{2,p}(\Omega)$ for all $p < n/(n - 2)$. This gives an L^∞ estimate in dimensions 2 and 3.

An alternative proof of Nedev’s bound (2.1) was recently found by Sanchón, Spruck and Cabré [8].

3 The Radial Case

We give here the main ideas in the proof of Theorem 1.3 concerning the radial case. See [5, 12] for the proof with all details. As stated in the theorem, the proof requires $u \in H_0^1(B_1)$, a condition which is needed by [12, Example 3.2.1]. Instead, the equation may be assumed to hold only in $B_1 \setminus \{0\}$ since the test functions in the following argument can be cut off near the origin.

We use the stability in the form (1.9) with $c = u_r$, together with the linearized equation (1.10), to obtain

$$(n - 1) \int_{B_1} u_r^2 \frac{\eta^2}{r^2} dx \leq \int_{B_1} u_r^2 |\nabla \eta|^2 dx. \tag{3.1}$$

Choose now $\eta = r(r^{-\alpha} - (\frac{1}{2})^{-\alpha})_+$, after being cut off near the origin to make it Lipschitz. In the following, however, we do not give all the details of the cut-off argument, which are standard. We deduce

$$(n - 1) \int_{B_1} u_r^2 \left(r^{-\alpha} - \left(\frac{1}{2}\right)^{-\alpha} \right)_+^2 dx \leq \int_{B_1} u_r^2 \left| \nabla \left(r \left(r^{-\alpha} - \left(\frac{1}{2}\right)^{-\alpha} \right)_+ \right) \right|^2 dx.$$

The most singular term near the origin on the right-hand side is $(\alpha - 1)^2 \int_{\Omega} u_r^2 r^{-2\alpha} dx$. The same integral appears as the most singular term on the left-hand side, but with constant $n - 1$ in front. After a simple interpolation argument, we deduce that

$$\int_{B_{1/2}} u_r^2 r^{-2\alpha} dx \leq C_{\alpha,n} \int_{B_{1/2}} u_r^2 r^{n-1} dx \tag{3.2}$$

if $(\alpha - 1)^2 < n - 1$. That is, we need to take α such that

$$1 \leq \alpha < 1 + \sqrt{n - 1}. \tag{3.3}$$

Since u is radial, it is not difficult to see that the integral on the right-hand side of (3.2) can be bounded by the right-hand side of (1.7). With this at hand, the bound in (3.2), valid for the above range of exponents α , leads to an L^∞ estimate for u whenever $n \leq 9$. To see this we just notice that, for $0 < s \leq \frac{1}{2}$, we have

$$\begin{aligned} \left| u(s) - u\left(\frac{1}{2}\right) \right| &= \left| \int_s^{\frac{1}{2}} -u_r dr \right| \leq \int_s^{\frac{1}{2}} |u_r| r^{-\alpha + \frac{n-1}{2}} r^{\alpha - \frac{n-1}{2}} dr \\ &\leq C_n \left(\int_{B_{1/2}} u_r^2 r^{-2\alpha} dx \right)^{\frac{1}{2}} \left(\int_s^{\frac{1}{2}} r^{2\alpha + 1 - n} dr \right)^{\frac{1}{2}} \end{aligned} \tag{3.4}$$

by Cauchy–Schwarz.

The last integral in (3.4) is finite with $s = 0$ if we take $2\alpha + 1 - n > -1$, i.e.,

$$\frac{n-4}{2} < \alpha - 1.$$

Since $n \leq 9$, we have $(n-4)/2 < \sqrt{n-1}$ and we can choose α satisfying this last condition as well as (3.3).

The pointwise estimates in dimensions $n \geq 10$ stated in Theorem 1.3 also follow from (3.2), used together with (3.4). These pointwise bounds give L^q integrability results for H_0^1 stable radial solutions with a range of exponents q depending only on the dimension $n \geq 10$. The range turns out to be optimal, as it is checked with the explicit radial solutions $r^{-2/(p-1)} - 1$ (see [5]).

4 Dimensions $n \leq 4$

In this section, we give most of the details of the proof of Theorem 1.5; see [4, 12] for all details. The key estimate is the following theorem.

Theorem 4.1 ([4]). *Let f be any smooth function and $\Omega \subset \mathbb{R}^n$ any smooth bounded domain. Assume that $2 \leq n \leq 4$. Let u be a positive stable classical solution of (1.1). Then, for every $t > 0$,*

$$\|u\|_{L^\infty(\Omega)} \leq t + \frac{C}{t} |\Omega|^{\frac{4-n}{2n}} \left(\int_{\{0 < u < t\}} |\nabla u|^4 dx \right)^{\frac{1}{2}}, \quad (4.1)$$

where C is a universal constant (in particular, independent of f , Ω and u).

To establish this result it will be essential to apply on every regular level set of u the following remarkable Sobolev inequality.

Proposition 4.2 (Allard; Michael and Simon; see [4, 12]). *Let $M \subset \mathbb{R}^{m+1}$ be an immersed smooth m -dimensional compact hypersurface without boundary. Then, for every $p \in [1, m)$, there exists a constant $C = C(m, p)$ depending only on the dimension m and the exponent p such that, for every C^∞ function $v : M \rightarrow \mathbb{R}$,*

$$\left(\int_M |v|^{p^*} dV \right)^{\frac{1}{p^*}} \leq C(m, p) \left(\int_M (|\nabla v|^p + |Hv|^p) dV \right)^{\frac{1}{p}}, \quad (4.2)$$

where H is the mean curvature of M and $p^* = mp/(m-p)$.

Note that when M is a minimal surface and $p = 1$, inequality (4.2) is equivalent to a universal isoperimetric inequality on subsets of any minimal surface.

Let us start by showing how to use Theorem 4.1 to deduce Theorem 1.5 (i), as well as Theorem 1.5 (ii) in convex domains. After this, we will explain the proof of Theorem 1.5 (ii) for convex nonlinearities. Finally, we will establish Theorem 4.1.

4.1 Proof of Theorem 1.5 (i), and Theorem 1.5 (ii) in Convex Domains

We use a well-known estimate for a positive solution of (1.1) (note that u is positive since we assume $f \geq 0$). We have

$$u \geq c_\Omega \left(\int_\Omega f(u) d_\Omega dx \right) d_\Omega \quad \text{in } \Omega,$$

where $d_\Omega = \text{dist}(\cdot, \partial\Omega)$ and c_Ω is a constant depending only on Ω ; see [12, Proposition A.4.2]. We deduce that in (4.1) we have $\{0 < u < t\} \subset \{0 < d_\Omega < s\}$, where $s = t/(c_\Omega \|f(u) d_\Omega\|_{L^1(\Omega)})$. Now, as explained right after Theorem 1.3, $\|f(u) d_\Omega\|_{L^1(\Omega)}$ can be bounded by below (and by above) by $\|u\|_{L^1(\Omega)}$.

Therefore, given the compact set K in Theorem 1.5 (i), we simply need to take t small enough such that $\{0 < d_\Omega < s\} \subset \Omega \setminus K$. Part (i) is now proved.

From part (i) we easily deduce part (ii) when Ω is convex. First, as we mentioned in the introduction, there are a priori bounds for positive solutions (not necessarily stable) in convex domains, thanks to the moving planes method; see [4, Proposition 3.2] or [12, Theorem 4.5.3]. For $n = 2$, the convexity of Ω is not needed. We apply these boundary bounds to the minimal solutions $u = u_\lambda$ of (1.3) with $\lambda < \lambda^*$. We deduce a bound for $\|u\|_{L^\infty(\{d_\Omega < \varepsilon\})}$ by a constant depending only on Ω and $\|u\|_{L^1(\Omega)}$, where $\varepsilon > 0$ depends only on Ω . As a consequence, $\|f(u)\|_{L^1(\{d_\Omega < \varepsilon\})}$ is also under control by a constant that depends on f in addition. All bounds are, in particular, uniform in λ .

Next, we use standard elliptic regularity theory to control

$$\|\nabla u\|_{L^4(\{d_\Omega < \varepsilon/2\})}$$

by

$$\|u\|_{L^\infty(\{d_\Omega < \varepsilon\})} \quad \text{and} \quad \|f(u)\|_{L^\infty(\{d_\Omega < \varepsilon\})}.$$

We now use part (i) of the theorem applied with the compact set $K := \{d_\Omega \geq \varepsilon/2\}$. Finally, as explained right after Theorem 1.3, the superlinearity of f gives a bound for $\|u\|_{L^1(\Omega)}$ which is uniform in $\lambda > \lambda^*/2$.

4.2 Proof of Theorem 1.5 (ii) for Convex Nonlinearities

We explain here the argument of Villegas [20] to remove the hypothesis that Ω is convex when the dimension $n = 4$, at the price of assuming that f is convex and satisfies (1.4). See [20] for all details.

We consider the minimal solution $u = u_\lambda$ of (1.3) with $\lambda < \lambda^*$. Since f is convex, we have $f(u) - f(0) \leq uf'(u)$. Thus, Nedev's estimate (2.1) gives a control on $\int_\Omega (f(u) - f(0))f(u)/u \, dx$. From this, and by $\lim_{t \rightarrow +\infty} f(t) = +\infty$, it is easy to deduce a uniform bound

$$\int_{\{u>1\}} \frac{f(u)^2}{u} \, dx \leq M. \quad (4.3)$$

Taking $t := \|\nabla u\|_{L^4(\Omega)}$ in Theorem 4.1, we deduce $\|u\|_{L^\infty(\Omega)} \leq C\|\nabla u\|_{L^4(\Omega)}$. Therefore, using that $n = 4$ in the Sobolev embedding, we obtain

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq C\|\nabla u\|_{L^4(\Omega)} \leq C\|u\|_{W^{2,2}(\Omega)} \leq C\|f(u)\|_{L^2(\Omega)} \\ &\leq C\left(f(1)^2|\Omega| + \int_{\{u>1\}} f(u)^2 \, dx\right)^{\frac{1}{2}} \leq C\left(1 + \int_{\{u>1\}} \frac{f(u)^2}{u} \, dx\right)^{\frac{1}{2}} \\ &\leq C(1 + M\|u\|_{L^\infty(\Omega)})^{\frac{1}{2}}, \end{aligned}$$

where we have used (4.3). From this, we conclude a bound for $\|u\|_{L^\infty(\Omega)}$ that is uniform in λ .

4.3 Proof of Theorem 4.1

We make the choice (1.11) and, in particular, we take $c = |\nabla u|$ in (1.9). It is easy to check that, in the set $\{|\nabla u| > 0\}$, we have

$$(\Delta + f'(u))|\nabla u| = \frac{1}{|\nabla u|} \left(\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij} \frac{u_j}{|\nabla u|} \right)^2 \right).$$

Taking an orthonormal basis in which the last vector is the normal $\nabla u/|\nabla u|$ to the level set and the other vectors are the principal directions of the level set, we easily see that the last expression coincides with the right-hand side of (1.12). Therefore, using the stability condition (1.9), we conclude the following result of Sternberg and Zumbrun.

Proposition 4.3 (Sternberg and Zumbrun; see [4, 12]). *Let Ω be a smooth bounded domain of \mathbb{R}^n and let u be a smooth positive stable solution of (1.1). Then, for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} \equiv 0$,*

$$\int_{\{|\nabla u|>0\}} (|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2) \eta^2 \, dx \leq \int_\Omega |\nabla u|^2 |\nabla \eta|^2 \, dx, \quad (4.4)$$

where ∇_T denotes the tangential or Riemannian gradient along a level set of u and

$$|A|^2 = |A(x)|^2 = \sum_{l=1}^{n-1} \kappa_l^2,$$

with κ_l being the principal curvatures of the level set of u passing through x for a given $x \in \Omega \cap \{|\nabla u| > 0\}$.

Note that in the radial case (4.4) reduces exactly to (3.1).

The proof of Theorem 4.1 proceeds as follows. We denote

$$T := \max_{\Omega} u = \|u\|_{L^\infty(\Omega)} \quad \text{and} \quad \Gamma_s := \{x \in \Omega : u(x) = s\}$$

for $s \in (0, T)$. By Sard's theorem, almost every $s \in (0, T)$ is a regular value of u . For these regular values, we thus can apply Proposition 4.2 with $M = \Gamma_s$.

We first use Proposition 4.3 with $\eta = \varphi(u)$, where φ is a Lipschitz function in $[0, T]$ with $\varphi(0) = 0$. The right-hand side of (4.4) becomes

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx &= \int_{\Omega} |\nabla u|^4 \varphi'(u)^2 dx \\ &= \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s \right) \varphi'(s)^2 ds, \end{aligned}$$

by the coarea formula. Thus, (4.4) can be written as

$$\begin{aligned} \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s \right) \varphi'(s)^2 ds &\geq \int_{\{|\nabla u| > 0\}} (|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2) \varphi(u)^2 dx \\ &= \int_0^T \left(\int_{\Gamma_s \cap \{|\nabla u| > 0\}} \frac{1}{|\nabla u|} (|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2) dV_s \right) \varphi(s)^2 ds \\ &= \int_0^T \left(\int_{\Gamma_s \cap \{|\nabla u| > 0\}} (4|\nabla_T |\nabla u|^{\frac{1}{2}}|^2 + (|A| |\nabla u|^{\frac{1}{2}})^2) dV_s \right) \varphi(s)^2 ds. \end{aligned}$$

We conclude that

$$\int_0^T h_1(s) \varphi(s)^2 ds \leq \int_0^T h_2(s) \varphi'(s)^2 ds \quad (4.5)$$

for all Lipschitz functions $\varphi : [0, T] \rightarrow \mathbb{R}$ with $\varphi(0) = 0$, where

$$h_1(s) := \int_{\Gamma_s} (4|\nabla_T |\nabla u|^{\frac{1}{2}}|^2 + (|A| |\nabla u|^{\frac{1}{2}})^2) dV_s, \quad h_2(s) := \int_{\Gamma_s} |\nabla u|^3 dV_s \quad (4.6)$$

for every regular value s of u .

Inequality (4.5), with h_1 and h_2 as defined above, will lead to our L^∞ estimate of Theorem 4.1 after choosing an appropriate test function φ in (4.5).

The rest of the proof differs in every dimension $n = 2, 3, 4$. But in all three cases it will be useful to denote

$$B_t := \frac{1}{t^2} \int_{\{u < t\}} |\nabla u|^4 dx = \frac{1}{t^2} \int_0^t h_2(s) ds, \quad (4.7)$$

where $t > 0$ is a given positive constant as in the statement of the theorem. Note that the quantity B_t is the main part of the right-hand side of our estimate (4.1). Let us start with the following two cases.

Cases $n = 2$ and $n = 3$. In these dimensions we take the simple test function

$$\varphi(s) = \begin{cases} \frac{s}{t} & \text{if } s \leq t, \\ 1 & \text{if } s > t. \end{cases}$$

With this choice of φ and since $h_1(s) \geq \int_{\Gamma_s} |A|^2 |\nabla u| dV_s$ (see definition (4.6)), inequality (4.5) leads to

$$\int_t^T \int_{\Gamma_s} |A|^2 |\nabla u| dV_s ds \leq \int_0^t h_2(s) \frac{1}{t^2} ds = \frac{1}{t^2} \int_{\{u < t\}} |\nabla u|^4 dx =: B_t. \tag{4.8}$$

Next, we use a well-known geometric inequality for the curve Γ_s ($n = 2$) or the surface Γ_s ($n = 3$). It also holds in every dimension $n \geq 2$ and it states

$$|\Gamma_s|^{\frac{n-2}{n-1}} \leq C_n \int_{\Gamma_s} |H| dV_s, \tag{4.9}$$

where H is the mean curvature of Γ_s , C_n is a constant depending only on n and s is a regular value of u . In dimension $n = 2$ this simply follows from the Gauss–Bonnet formula. For $n \geq 3$, inequality (4.9) follows from the Michael–Simon and Allard Sobolev inequality (Proposition 4.2 above) applied with $p = 1$ and $v \equiv 1$.

We also use the classical isoperimetric inequality

$$V(s) := |\{u > s\}| \leq C_n |\Gamma_s|^{\frac{n}{n-1}}. \tag{4.10}$$

Now, (4.9) and (4.10) lead to

$$V(s)^{\frac{n-2}{n}} \leq C_n \int_{\Gamma_s} |H| dV_s \leq C_n \left(\int_{\Gamma_s} |A|^2 |\nabla u| dV_s \right)^{\frac{1}{2}} \left(\int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right)^{\frac{1}{2}}$$

for all regular values s , by Cauchy–Schwarz and since $|H| \leq |A|$. From this, we deduce

$$\begin{aligned} T - t &= \int_t^T ds \leq \int_t^T C_n \left(\int_{\Gamma_s} |A|^2 |\nabla u| dV_s \right)^{\frac{1}{2}} \left(V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right)^{\frac{1}{2}} ds \\ &\leq C_n \left(\int_t^T \int_{\Gamma_s} |A|^2 |\nabla u| dV_s ds \right)^{\frac{1}{2}} \left(\int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds \right)^{\frac{1}{2}} \\ &\leq C_n B_t^{\frac{1}{2}} \left(\int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds \right)^{\frac{1}{2}}, \end{aligned} \tag{4.11}$$

where we have used (4.8) in the last inequality.

Finally, since $V(s) = |\{u > s\}|$ is a nonincreasing function, it is differentiable almost everywhere and, by the coarea formula,

$$-V'(s) = \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \text{ for a.e. } s \in (0, T).$$

In addition, for $n \leq 3$, also $V(s)^{(4-n)/n}$ is nonincreasing in s and thus its total variation satisfies

$$\begin{aligned} |\Omega|^{\frac{4-n}{n}} &\geq V(t)^{\frac{4-n}{n}} = [V(s)^{\frac{4-n}{n}}]_{s=T}^{s=t} \\ &\geq \int_t^T \frac{4-n}{n} V(s)^{\frac{2(2-n)}{n}} (-V'(s)) ds \\ &= \frac{4-n}{n} \int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds. \end{aligned}$$

This, together with the first and the last inequality in (4.11) leads to the desired inequality

$$T - t \leq C_n B_t^{\frac{1}{2}} |\Omega|^{\frac{4-n}{2n}}$$

for $n \leq 3$.

Note that this argument gives nothing for $n \geq 4$ since the integral

$$\int_t^T V(s)^{\frac{2(2-n)}{n}} (-V'(s)) ds = \int_0^{V(t)} \frac{dr}{r^{\frac{2(n-2)}{n}}}$$

in (4.11) is not convergent at $s = T$ (i.e., $r = 0$) because $2(n - 2)/n \geq 1$.

We now turn to the other case.

Case $n = 4$. For $n \geq 4$, we apply the Michael–Simon and Allard Sobolev inequality (4.2) with $M = \Gamma_s$, $p = 2 < m = n - 1$ and $v = |\nabla u|^{1/2}$. Note that we have $|H| \leq |A|$. Recalling (4.6), we obtain

$$\left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} dV_s \right)^{\frac{n-3}{n-1}} \leq C_n h_1(s) \tag{4.12}$$

for all regular values s of u . This estimate combined with (4.5) leads to

$$\int_0^T \left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} dV_s \right)^{\frac{n-3}{n-1}} \varphi(s)^2 ds \leq C_n \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 dV_s \right) \varphi'(s)^2 ds \tag{4.13}$$

for all Lipschitz functions φ in $[0, T]$ with $\varphi(0) = 0$. We only know how to derive an L^∞ estimate for u (i.e., a bound on $T = \max_\Omega u$) from (4.13) when the exponent $(n - 1)/(n - 3)$ on its left-hand side is larger than or equal to the one on the right-hand side, i.e., 3. That is, we need $(n - 1)/(n - 3) \geq 3$, which means $n \leq 4$.

Therefore, by taking $n = 4$, then $(n - 1)/(n - 3) = 3$ and (4.12) becomes

$$h_2^{\frac{1}{3}} \leq C h_1 \quad \text{a.e. in } (0, T) \text{ (when } n = 4), \tag{4.14}$$

where C is a universal constant.

To proceed, the idea is to choose a function φ that tries to violate the inequality (4.5), by imposing $h_1 \varphi^2 = 2h_2 \varphi'^2$. This ODE will lead to the choice (4.16) below. The ODE will not be satisfied in all $[0, T]$ since we must have $\varphi(0) = 0$. For this reason, (4.5) will not be violated, but instead will lead to an estimate.

For every regular value s of u , we have $0 < h_2(s)$ and $h_1(s) < \infty$ (simply by their definition). This together with (4.14) gives $h_1/h_2 \in (0, +\infty)$ a.e. in $(0, T)$. Thus, defining

$$g_k(s) := \min\left(k, \frac{h_1(s)}{h_2(s)}\right)$$

for regular values s and for a positive integer k , we have that $g_k \in L^\infty(0, T)$ and

$$g_k(s) \nearrow \frac{h_1(s)}{h_2(s)} \in (0, +\infty) \quad \text{as } k \uparrow \infty, \text{ for a.e. } s \in (0, T). \tag{4.15}$$

Since $g_k \in L^\infty(0, T)$, the function

$$\varphi_k(s) := \begin{cases} \frac{s}{t} & \text{if } s \leq t, \\ \exp\left(\frac{1}{\sqrt{2}} \int_t^s \sqrt{g_k(\tau)} d\tau\right) & \text{if } t < s \leq T, \end{cases} \tag{4.16}$$

is well defined, Lipschitz in $[0, T]$ and satisfies $\varphi_k(0) = 0$.

Since

$$h_2(\varphi_k')^2 = h_2 \frac{1}{2} g_k \varphi_k^2 \leq \frac{1}{2} h_1 \varphi_k^2 \quad \text{in } (t, T),$$

inequality (4.5) used with $\varphi = \varphi_k$ leads to

$$\int_t^T h_1 \varphi_k^2 ds \leq \frac{2}{t^2} \int_0^t h_2 ds = \frac{2}{t^2} \int_{\{u < t\}} |\nabla u|^4 dx = 2B_t. \quad (4.17)$$

Recall that B_t was defined in (4.7) and that we need to establish $T - t \leq CB_t^{1/2}$.

By (4.15) we have

$$T - t = \int_t^T ds = \sup_{k \geq 1} \int_t^T \sqrt[4]{\frac{h_2 g_k}{h_1}} ds. \quad (4.18)$$

Using (4.17) and Cauchy–Schwarz, we have that

$$\begin{aligned} \int_t^T \sqrt[4]{\frac{h_2 g_k}{h_1}} ds &= \int_t^T (\sqrt{h_1} \varphi_k) \left(\sqrt[4]{\frac{h_2 g_k}{h_1^3}} \frac{1}{\varphi_k} \right) ds \\ &\leq (2B_t)^{\frac{1}{2}} \left(\int_t^T \sqrt{\frac{h_2 g_k}{h_1^3}} \frac{1}{\varphi_k^2} ds \right)^{\frac{1}{2}} \\ &\leq (2B_t)^{\frac{1}{2}} \left(C \int_t^T \sqrt{g_k} \frac{1}{\varphi_k^2} ds \right)^{\frac{1}{2}}. \end{aligned} \quad (4.19)$$

In the last inequality we have used our crucial estimate (4.14).

Finally, we bound the integral in (4.19), using the definition (4.16) of φ_k , as follows:

$$\begin{aligned} \int_t^T \sqrt{g_k} \frac{1}{\varphi_k^2} ds &= \int_t^T \sqrt{g_k} \frac{1}{\varphi_k^2} \frac{\varphi_k'}{\frac{1}{\sqrt{2}} \sqrt{g_k} \varphi_k} ds \\ &= \sqrt{2} \int_t^T \frac{\varphi_k'}{\varphi_k^3} ds = \frac{\sqrt{2}}{2} [\varphi_k^{-2}(s)]_{s=T}^{s=t} \\ &\leq \frac{\sqrt{2}}{2} \varphi_k^{-2}(t) = \frac{\sqrt{2}}{2}. \end{aligned}$$

This bound, together with (4.18) and (4.19) finishes the proof in dimension 4.

Funding: The author is supported by MINECO grant MTM2014-52402-C3-1-P and is part of the Catalan research group 2014 SGR 1083.

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