# On a Caginalp phase-field system with two temperatures and memory

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**Abstract.** The Caginalp phase-field system has been proposed in [4] as a simple mathematical model for phase transition phenomena. In this paper, we are concerned with a generalization of this system based on the Gurtin-Pipkin law with two temperatures for heat conduction with memory, apt to describe transition phenomena in nonsimple materials. The model consists of a parabolic equation governing the order parameter which is linearly coupled with a nonclassical integrodifferential equation ruling the evolution of the thermodynamic temperature of the material. Our aim is to construct a robust family of exponential attractors for the associated semigroup, showing the stability of the system with respect to the collapse of the memory kernel. We also study the spatial behavior of the solutions in a semi-infinite cylinder, when such solutions exist.

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## 1. Introduction

We are interested in this paper in the study of the following variant of the Caginalp phase-field system[1]:

$$\begin{cases} u_t - \Delta u + f(u) = \varphi - \Delta\varphi, \\ \varphi_t - \Delta\varphi_t - \displaystyle\int_0^\infty k(s)\Delta\varphi(t-s)\,\mathrm{d}s = -u_t, \end{cases} \tag{1.1}$$

subject to homogeneous Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = 0$$

[1]Here and below, all physical parameters are set equal to one.

(here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$) and with initial data

$$u(0) = u_0 \qquad \text{and} \qquad \varphi(-t) = \varphi_0(-t), \quad t \geq 0.$$

The original Caginalp phase-field system reads

$$u_t - \Delta u + f(u) = \theta, \tag{1.2}$$

$$\theta_t - \Delta\theta = -u_t. \tag{1.3}$$

It has been proposed in [4] as a simple mathematical model for phase transition phenomena, such as melting-solidification phenomena (e.g. ice). The two unknown functions $u$ and $\theta$ are, respectively, the order parameter and the relative temperature (relative to the equilibrium melting temperature) of the system occupying the volume $\Omega$, while the nonlinearity $f$ is the derivative of a double-well potential $F$. System (1.2)–(1.3) has been much studied from a mathematical point of view (see e.g. [1, 2, 3, 8, 9, 10, 24, 26, 29, 30, 31, 32, 40, 49, 56]). This system can be derived by introducing the (total Ginzburg–Landau) free energy

$$\Psi = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2\Big)\,\mathrm{d}x.$$

Then, the evolution equation for the order parameter $u$ is given by

$$u_t = -\frac{D\Psi}{Du}, \tag{1.4}$$

where $\frac{D}{Du}$ denotes the variational derivative with respect to $u$. Next, denoting by $\mathbf{q}$ the heat flux, the evolution of the enthalpy

$$H = u + \theta,$$

is ruled out by the energy equation

$$H_t = \nabla \cdot \mathbf{q}. \tag{1.5}$$

Equations (1.2)-(1.3) then follow from (1.4)-(1.5), assuming the classical Fourier law for heat conduction,

$$\mathbf{q} = -\nabla\theta. \tag{1.6}$$

Now, a major drawback of the Fourier heat law is the infinite speed of propagation of thermal disturbances, deemed physically unreasonable and called paradox of heat conduction (see [11]). Thus, several alternatives to (1.6) have been proposed in order to derive more realistic models: in particular, the Caginalp phase-field system, supplemented with either the Maxwell-Cattaneo law or other constitutive laws for the heat flux coming from thermomechanics, has been studied by several authors, see e.g. [13, 38, 39, 41, 42, 43, 44, 46].

Still also the type III heat conduction theory, as well as the classical Fourier one, suffers from some theoretical drawbacks (see [28]) which are overcome by (1.1).

A different approach to heat conduction has been proposed in the Sixties in [5, 6, 7], where it was observed that *two* temperatures are involved in the definition of the entropy, namely the conductive temperature $\theta$, influencing the heat conduction contribution, and the thermodynamic temperature $\varphi$, appearing in the heat supply part. For time-independent models, these two temperatures coincide in absence of heat supply. Nonetheless, they are in general different in the time dependent framework, although no heat is supplied to the system: for instance, this happens in non-simple materials. In that case, the two temperatures are related by the (linearized) law

$$\theta = \varphi - \Delta\varphi.$$

Accordingly, equation (1.2) reads

$$u_t - \Delta u + f(u) = \theta = \varphi - \Delta\varphi. \tag{1.7}$$

The Caginalp phase-field system with two temperatures has been studied in [20] for the classical Fourier law, in [47] for the type III thermomechanics theory (see [33]) with two temperatures recently proposed in [53] (see also [21]), and in [48] for the theory of two-temperature-generalized thermoelasticity proposed in [55] and based on the Maxwell-Cattaneo law

$$\mathbf{q} + \tau\mathbf{q}_t = -\nabla\varphi, \tag{1.8}$$

where $\tau > 0$ is a given relaxation parameter. In fact, our model is a generalization of the one studied in [55] which is obtained for the particular choice of the exponential kernel.

In this paper, we continue the study initiated in [15] of a theory of two temperatures with memory, where the classical Fourier law is substituted by the Coleman-Gurtin (if $a > 0$) or Gurtin-Pipkin law (if $a = 0$) for the heat flux, namely (see [12, 34])

$$\mathbf{q}(t) = -a\nabla\varphi(t) - \int_0^\infty k(s)\nabla\varphi(t-s)\,\mathrm{d}s, \quad a \in [0,1),$$

based on the key assumption that the evolution of $\mathbf{q}$ is influenced by the past history of the temperature gradient, through a suitable (nonnegative) summable memory kernel $k$ of total mass $1-a$ which characterizes the memory properties of the material. In that case, the evolution law (1.5) for the enthalpy

$$H = \theta + u = \varphi - \Delta\varphi + u$$

becomes

$$\varphi_t(t) - \Delta\varphi_t(t) - a\Delta\varphi(t) - \int_0^\infty k(s)\Delta\varphi(t-s)\,\mathrm{d}s = -u_t(t), \quad t > 0. \tag{1.9}$$

Collecting (1.7) and (1.9), we arrive at the following system

$$\begin{cases} u_t - \Delta u + f(u) = \varphi - \Delta\varphi, \\ \varphi_t - \Delta\varphi_t - a\Delta\varphi - \int_0^\infty k(s)\Delta\varphi(t-s)\,\mathrm{d}s = -u_t, \end{cases} \tag{1.10}$$

and, taking $a = 0$, we end up with (1.1).

In the first part of the paper, we consider the family of problems (1.1) in which the memory kernel $k$ is replaced by the rescaled kernel

$$k_\varepsilon(s) = \frac{1}{\varepsilon} k\Big(\frac{s}{\varepsilon}\Big),$$

for every $\varepsilon \in (0,1]$, namely

$$\begin{cases} u_t - \Delta u + f(u) = \varphi - \Delta\varphi, \\ \varphi_t - \Delta\varphi_t - \displaystyle\int_0^\infty k_\varepsilon(s)\Delta\varphi(t-s)\,\mathrm{d}s = -u_t, \end{cases} \tag{1.11}$$

subject to the same initial and boundary conditions.

Well-posedness and regularity results for (1.11) have been obtained in [15]. Furthermore, the existence of the global attractor (see, e.g, [50, 54]), as well as its upper semicontinuity with respect to $\varepsilon$, have also been established. The latter gives results concerning the stability of the system with respect to the "collapse" of the memory kernel as $\varepsilon \to 0$. Indeed, $k_\varepsilon$ converges in the distributional sense to the Dirac mass of weight one at the origin; in turn, (1.11) formally collapses as $\varepsilon \to 0$ into the "limiting" system

$$\begin{cases} u_t - \Delta u + f(u) = \varphi - \Delta\varphi, \\ \varphi_t - \Delta\varphi_t - \Delta\varphi = -u_t. \end{cases} \tag{1.12}$$

Note however that lower semicontinuity results on the global attractor are not known.

Our main result establishes the existence of a family of exponential attractors $\mathcal{E}_\varepsilon$ for the corresponding semigroups, which is *robust* (i.e. both upper and lower semicontinuous) with respect to $\varepsilon$, see Section 5. In particular, we prove that $\mathcal{E}_\varepsilon$ is close (in the sense of the symmetric Hausdorff semidistance) to $\mathcal{E}_0$, the exponential attractor of (1.12). This means that the longtime dynamics of the two models become closer and closer as $\varepsilon \to 0$, so that the formal limit is now rigorously justified.

In the second part of the paper, we are interested in the study of the spatial behavior of the solutions. Spatial decay estimates for partial differential equations are related to the Saint-Venant principle which is both a mathematical and a thermomechanical aspect which has deserved much attention in the last years (see [35] and the references therein). Such studies describe how the influence of the perturbations on a part of the boundary is damped for the points which are far away from the perturbations. Spatial decay estimates for elliptic [22], parabolic [36, 37], hyperbolic [23] and/or combinations of such [52] have been obtained in the last years. However, as far as nonlinear equations are concerned, such a knowledge is limited (see [41, 42, 43, 45, 46]).

What is usual is to consider a semi-infinite cylinder whose finite end is perturbed and see what happens when the spatial variable goes to infinity. However, we do not study the existence of solutions to this problem; in fact, this does not seem to be an easy task (see, e.g., [45]). We thus assume the existence of solutions and then only study the spatial asymptotic behavior in that case. More precisely, we obtain a Phragmén-Lindelöf alternative, i.e., either a growth or a decay estimate.

## 2. Assumptions and Functional Setting

**The nonlinearity**

We assume that $f \in \mathcal{C}^3(\mathbb{R})$ satisfies $f(0) = 0$ along with the dissipation conditions

$$\inf_{u\in\mathbb{R}} f'(u) > -\infty \tag{2.1}$$

and

$$f(u)u \geq c_1 F(u) - c_2 \geq -c_3, \quad \forall u \in \mathbb{R}, \tag{2.2}$$

for some $c_1 > 0$, $c_2, c_3 \geq 0$, having defined $F(u) = \int_0^u f(y)\mathrm{d}y$.

**The memory kernel**

$k$ is a nonnegative summable function of total mass equal to 1, having the explicit form

$$k(s) = \int_s^\infty \mu(y)\,\mathrm{d}y.$$

Here, $\mu \in L^1(\mathbb{R}^+)$ is a nonincreasing, nonnegative, absolutely continuous function satisfying, for some $\delta > 0$,

$$\mu'(s) + \delta\mu(s) \leq 0, \quad \forall s > 0. \tag{2.3}$$

Note that the exponential kernel $k(s) = \mathrm{e}^{-\delta s}$ complies with all the assumptions, nonetheless $\mu$ might be unbounded at the origin. In what follows, we set $\kappa := \int_0^\infty \mu(s)\,\mathrm{d}s$.

**Remark 2.1.** For the rescaled memory kernels $\mu_\varepsilon$ defined by

$$\mu_\varepsilon(s) = -k'_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\Big(\frac{s}{\varepsilon}\Big),$$

condition (2.3) implies

$$\mu'_\varepsilon(s) + \frac{\delta}{\varepsilon}\mu_\varepsilon(s) \leq 0, \quad \forall s > 0. \tag{2.4}$$

We also remark that

$$\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s = \frac{\kappa}{\varepsilon}. \tag{2.5}$$

**Functional spaces.**

We denote by $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ the space $L^2(\Omega)$ endowed with the standard scalar product and norm. Let $-\Delta$ be the Dirichlet operator with domain

$$\mathfrak{D}(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$$

and let us denote

$$A := -\Delta + \mathbb{I}.$$

For $\sigma \in \mathbb{R}$, we introduce the scale of (compactly) nested Hilbert spaces

$$\mathrm{H}^\sigma = \mathfrak{D}(A^{\frac{\sigma}{2}})$$

with inner products and norms

$$\langle w, v\rangle_\sigma = \langle A^{\frac{\sigma}{2}} w, A^{\frac{\sigma}{2}} v\rangle, \quad \|w\|_\sigma = \|A^{\frac{\sigma}{2}} w\|.$$

We omit the superscript $\sigma$ whenever it equals zero. The symbol $\langle\cdot,\cdot\rangle$ also stands for the duality product between $\mathrm{H}^\sigma$ and its dual space $\mathrm{H}^{-\sigma}$. Note that, by the Poincaré inequality, the following equivalence holds

$$\lambda_0\|w\|_1^2 \le \|\nabla w\|^2 \le \|w\|_1^2,$$

for some $\lambda_0 \in (0,1)$ which is independent of $w \in \mathrm{H}^1$.

We also introduce the so-called *memory spaces*

$$\mathcal{M}_\varepsilon^\sigma = L^2_{\mu_\varepsilon}(\mathbb{R}^+;\mathrm{H}^\sigma), \quad \langle \eta,\xi\rangle_{\mathcal{M}_\varepsilon^\sigma} = \int_0^\infty \mu_\varepsilon(s)\langle\eta(s),\xi(s)\rangle_\sigma \mathrm{d}s,$$

and we denote by [2]

$$T_\varepsilon\eta = -\eta', \quad \mathfrak{D}(T_\varepsilon) = \big\{\eta \in \mathcal{M}_\varepsilon^2 : \eta' \in \mathcal{M}_\varepsilon^2,\, \eta(0) = 0\big\},$$

the infinitesimal generator of the strongly continuous semigroup of right translations on the memory space $\mathcal{M}_\varepsilon^2$.

Finally, we denote by

$$\mathcal{H}_\varepsilon^\sigma = \mathrm{H}^{2\sigma+2} \times \mathrm{H}^{2\sigma+2} \times \mathcal{M}_\varepsilon^{2\sigma+2}$$

the hierarchy of the *extended phase spaces*, endowed with the natural scalar product. Again, we omit the superscript $\sigma$ whenever it equals zero. In particular,

$$\mathcal{H}_\varepsilon = \mathrm{H}^2 \times \mathrm{H}^2 \times \mathcal{M}_\varepsilon^2 \qquad \text{and} \qquad \mathcal{H}_\varepsilon^1 = \mathrm{H}^4 \times \mathrm{H}^4 \times \mathcal{M}_\varepsilon^4.$$

**The reformulated problem**

As in [15], we first reformulate the problem in the history framework proposed by Dafermos in [18]. To this aim, we introduce for $t \ge 0$ and $s > 0$ the *integrated past history* $\eta = \eta^t(s)$ of the variable $\varphi$, formally defined as

$$\eta^t(s) = \int_0^s \varphi(t-y)\,\mathrm{d}y.$$

Accordingly, the second equation in (1.11) translates into the system

$$\begin{cases} A\varphi_t - \displaystyle\int_0^\infty \mu_\varepsilon(s)\Delta\eta(s)\,\mathrm{d}s = -u_t, \\ \eta_t = T_\varepsilon\eta + \varphi, \end{cases}$$

in the two unknowns $\varphi = \varphi(t)$ and $\eta = \eta^t(s)$, with corresponding initial data

$$\varphi(0) = \varphi_0 \qquad \text{and} \qquad \eta^0(s) = \eta_0(s) := \int_0^s \varphi_0(-y)\,\mathrm{d}y.$$

We refer the readers to [16, 17] for more details on this approach.

[2]Here $\eta'$ denotes the derivative of $\eta$ with respect to the internal variable $s$.

In summary, the model under investigation reads

$$\begin{cases} u_t - \Delta u + f(u) = A\varphi, \\ A\varphi_t - \displaystyle\int_0^\infty \mu_\varepsilon(s)\Delta\eta(s)\,\mathrm{d}s = -u_t, \\ \eta_t = T_\varepsilon\eta + \varphi, \end{cases} \tag{2.6}$$

for $t > 0$, in the unknown variables $(u(t), \varphi(t), \eta^t)$, supplemented with Dirichlet boundary conditions and initial conditions at $t = 0$

$$(u(0), \varphi(0), \eta^0) = (u_0, \varphi_0, \eta_0). \tag{2.7}$$

**Notation**

Throughout the paper $c, c' > 0$ stand for generic constants and $Q(\cdot)$ for a nonnegative increasing function allowed to vary within a same line and only influenced by the structural data of the problem. In particular, they are independent of $\varepsilon$.

## 3. The dissipative semigroups $S_\varepsilon(t)$

The initial value problem (2.6)-(2.7) has been studied in [15] for any given initial datum $(u_0, \varphi_0, \eta_0) \in \mathcal{H}_\varepsilon$, where it is shown to possess a unique global solution $(u(t), \varphi(t), \eta^t)$ such that

$$(u, \varphi, \eta) \in L^\infty(\mathbb{R}^+; \mathcal{H}_\varepsilon),$$

and

$$u_t \in L^\infty(\mathbb{R}^+; \mathrm{H}) \cap L^2(\mathbb{R}^+; \mathrm{H}^1), \qquad \varphi_t \in L^2(\mathbb{R}^+; \mathrm{H}^2).$$

Besides, the third component $\eta^t$ of the solution fulfills the explicit representation formula

$$\eta^t(s) = \begin{cases} u(t) - u(t-s), & 0 < s \le t, \\ \eta_0(s-t) + u(t) - u_0, & s > t. \end{cases}$$

Accordingly, for every fixed $\varepsilon \in [0,1]$, the map

$$S_\varepsilon(t) : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon, \quad t \ge 0,$$

acting by the formula

$$S_\varepsilon(t)(u_0, \varphi_0, \eta_0) = (u(t), \varphi(t), \eta^t)$$

defines a dynamical system, or semigroup on $\mathcal{H}_\varepsilon$. In this section we recall a number of results proved in [15] on the continuity and energy properties of $S_\varepsilon(t)$.

**Theorem 3.1.** *Let $z_1, z_2 \in \mathcal{H}_\varepsilon$ be given and set $S_\varepsilon(t)z_i = (u_i(t), \varphi_i(t), \eta_i^t)$. Then, there exists $c = c(\|z_i\|_{\mathcal{H}_\varepsilon}) > 0$ independent of $\varepsilon$ and of the particular choice of $z_i$ such that*

$$\|S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2\|_{\mathcal{H}_\varepsilon} \le ce^{ct}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}, \quad \forall t \ge 0. \tag{3.1}$$

*In particular, the semigroup* $S_\varepsilon(t)$ *is strongly continuous. Furthermore, for every* $t > 0$,

$$\|\partial_t u_1(t) - \partial_t u_2(t)\|^2 + \int_0^t (\|\nabla\partial_t u_1(s) - \nabla\partial_t u_2(s)\|^2 \tag{3.2}$$

$$+ \|A\partial_t\varphi_1(s) - A\partial_t\varphi_2(s)\|^2)\mathrm{d}s \tag{3.3}$$

$$\le ce^{ct}\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon}.$$

**Theorem 3.2.** *There exist* $\omega > 0$ *and* $K' \ge 0$, *independent of* $\varepsilon$, *such that*

$$\|S_\varepsilon(t)z\|^2_{\mathcal{H}_\varepsilon} + \|\partial_t u_\varepsilon(t)\|^2 \le Q(\|z\|_{\mathcal{H}_\varepsilon})\mathrm{e}^{-\omega t} + K', \quad \forall t \ge 0, \tag{3.4}$$

*and*

$$\int_0^\infty \big(\|\nabla\partial_t u_\varepsilon(t)\|^2 + \|A\partial_t\varphi_\varepsilon(t)\|^2\big)\mathrm{d}t \le Q(\|z\|_{\mathcal{H}_\varepsilon}), \tag{3.5}$$

*for every* $z \in \mathcal{H}_\varepsilon$. *In particular,* $K' = 0$ *if* $f$ *satisfies* (2.2) *with* $c_2 = c_3 = 0$.

**Remark 3.3.** It is worth noticing that the above theorem prescribes in particular the *exponential decay* of the solutions to the linear model, corresponding to $f \equiv 0$.

The main results from [15] concerning the longterm behavior of $S_\varepsilon(t)$ are collected in the two following theorems. The first says that the semigroup $S_\varepsilon(t)$ is *dissipative*.

**Theorem 3.4.** *For every* $\varepsilon \in [0,1]$, *the semigroup* $S_\varepsilon(t)$ *possesses a bounded absorbing set* $B_\varepsilon \subset \mathcal{H}_\varepsilon$. *Namely, for any bounded set* $B \subset \mathcal{H}_\varepsilon$, $\exists t_0 = t_0(B)$ *such that*

$$S_\varepsilon(t)B \subset B_\varepsilon, \qquad \forall t \ge t_0.$$

*Besides,* $B_\varepsilon$ *is bounded and its size in* $\mathcal{H}_\varepsilon$ *is independent of* $\varepsilon$.

Indeed, this is a straightforward consequence of Theorem 3.2, since it is enough to set

$$B_\varepsilon = \{z \in \mathcal{H}_\varepsilon \, : \, \|z\|_{\mathcal{H}_\varepsilon} \le R_0\},$$

for $R_0$ large enough (independent of $\varepsilon$).

The next result shows that $S_\varepsilon(t)$ has an exponentially attracting set, which is a bounded set of $\mathcal{H}^1_\varepsilon$. This is established in the proof of [15, Theorem 5.1].

**Theorem 3.5.** *There exists* $r_1 > 0$, $\omega_1 > 0$ *and* $M_1 > 0$ *such that the ball*

$$B^1_\varepsilon = \{z \in \mathcal{H}^1_\varepsilon \, : \, \|z\|_{\mathcal{H}^1_\varepsilon} \le r_1\}$$

*is an exponentially attracting set for* $S_\varepsilon(t)$, *i.e.*

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B_\varepsilon, B^1_\varepsilon) \le M_1\mathrm{e}^{-\frac{\omega_1}{2}t}, \qquad \forall t \ge 0,$$

*for every* $\varepsilon \in [0,1]$.

## 4. Further Dissipativity

In this section we deepen the asymptotic study of the semigroup showing the existence of a *compact* family of exponentially attracting sets for $S_\varepsilon(t)$, see Theorem 4.3 below. Indeed, it is weel known since [51] that the embedding $\mathcal{H}^1_\varepsilon \subset \mathcal{H}_\varepsilon$ is not compact, due to the presence of the memory component.

The first step in this direction is proving that the semigroup is dissipative also when acting on $\mathcal{H}^1_\varepsilon$.

**Lemma 4.1.** *There exist $\vartheta > 0$, $R_1 > 0$, and a positive function $Q_1(\cdot)$ such that*

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le Q_1(r)\mathrm{e}^{-\vartheta t} + R_1, \qquad \forall t \ge 0,$$

*whenever $\|z\|_{\mathcal{H}^1_\varepsilon} \le r$, for every $\varepsilon \in [0,1]$.*

*Proof.* We first observe that, since $\|z\|_{\mathcal{H}^1_\varepsilon} \le r$ implies $\|z\|_{\mathcal{H}_\varepsilon} \le r$, owing to (3.4), we have

$$\|f(u(t))\|_2 + \|f'(u(t))\|_2 + \|f''(u(t))\|_{L^\infty} \le Q(\|u(t)\|_2) \le Q(r), \quad \forall t \ge 0. \tag{4.1}$$

Besides, by Theorem 3.4, we find $t_0 = t_0(r)$ such that

$$\|S_\varepsilon(t)z\|_{\mathcal{H}_\varepsilon} \le R_0, \quad t \ge t_0. \tag{4.2}$$

In turn, by (3.5), we also get

$$\int_{t_0}^{\infty} \|\nabla u_t(s)\|^2 \mathrm{d}s \le c, \quad t \ge t_0, \tag{4.3}$$

where here and along the proof, the generic constant $c > 0$ might depend on $R_0$ but is independent on $r$.

To improve our dissipativity estimates we consider a formal argument which can be rigorously justified within a standard Galerkin scheme. The product of system (2.6) by $(\Delta^2 u_t, \Delta^2 A\varphi, -\Delta^3 A\eta)$ in $\mathrm{H} \times \mathrm{H} \times L^2_{\mu_\varepsilon}(\mathbb{R}^+, \mathrm{H})$ yields

$$\begin{aligned}
&\|\Delta u_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 = \langle A\varphi, \Delta^2 u_t\rangle - \langle f(u), \Delta^2 u_t\rangle,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta A\varphi\|^2 - \int_0^\infty \mu_\varepsilon(s)\langle \Delta\eta(s), \Delta^2 A\varphi\rangle \mathrm{d}s = -\langle \Delta^2 A\varphi, u_t\rangle,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\eta\|^2_{\mathcal{M}^*_\varepsilon} - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|\Delta\eta(s)\|^2_* \mathrm{d}s = -\int_0^\infty \mu_\varepsilon(s)\langle \Delta\eta(s), \Delta^2 A\varphi\rangle \mathrm{d}s,
\end{aligned}$$

where

$$\|v\|^2_* = \|\nabla v\|^2 + \|\Delta v\|^2.$$

Note that $\|\cdot\|_*$ is equivalent to $\|\cdot\|_2$ in $\mathrm{H}^2$. Hence, denoting by $\mathcal{M}^*_\varepsilon$ the space $\mathcal{M}^2_\varepsilon$ endowed with the equivalent norm

$$\|\eta\|^2_{\mathcal{M}^*_\varepsilon} = \int_0^\infty \mu_\varepsilon(s)\|\eta(s)\|^2_* \mathrm{d}s,$$

the functional

$$E_1(t) = \frac{1}{2}\big(\|\nabla\Delta u(t)\|^2 + \|\Delta A\varphi(t)\|^2 + \|\Delta\eta^t\|^2_{\mathcal{M}^*_\varepsilon}\big)$$

satisfies the basic energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1 + \frac{1}{2}\|\Delta u_t\|^2 - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|\Delta\eta(s)\|^2_*\mathrm{d}s \le Q(r), \tag{4.4}$$

having estimated

$$|\langle f(u), \Delta^2 u_t\rangle| \le \|\Delta f(u)\|\|\Delta u_t\| \le Q(r) + \frac{1}{2}\|\Delta u_t\|^2$$

in light of (4.1). Next, we formally differentiate in time the first equation in (2.6), getting

$$u_{tt} - \Delta u_t + f'(u)u_t = A\varphi_t$$

which we multiply by $\Delta^2 u_t$, and we consider the product of the second equation in (2.6) by $\Delta^2 A\varphi_t$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_t\|^2 + \|\nabla\Delta u_t\|^2 + \langle f'(u)u_t, \Delta^2 u_t\rangle = \langle A\varphi_t, \Delta^2 u_t\rangle,$$

$$\|\Delta A\varphi_t\|^2 - \int_0^\infty \mu_\varepsilon(s)\langle\Delta^2\eta(s), \Delta A\varphi_t\rangle\mathrm{d}s = -\langle A\varphi_t, \Delta^2 u_t\rangle.$$

Adding the results and owing to (2.5), we easily see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_t\|^2 + \frac{1}{2}\|\nabla\Delta u_t\|^2 + \frac{1}{2}\|\Delta A\varphi_t\|^2 \le \frac{\kappa}{2\varepsilon}\|\Delta\eta\|^2_{\mathcal{M}^*_\varepsilon} + Q(r)\|\nabla u_t\|^2, \tag{4.5}$$

having observed that

$$\langle f'(u)u_t, \Delta^2 u_t\rangle \le Q(r)\|\nabla u_t\|\|\nabla\Delta u_t\|,$$

and

$$\int_0^\infty \mu_\varepsilon(s)\langle\Delta^2\eta(s), \Delta A\varphi_t\rangle\mathrm{d}s \le \frac{1}{2}\|\Delta A\varphi_t\|^2 + \frac{1}{2}\left(\int_0^\infty \mu_\varepsilon(s)\,\mathrm{d}s\right)\|\Delta\eta\|^2_{\mathcal{M}^*_\varepsilon}.$$

Now note that, in view of (2.4), we have

$$\frac{\delta}{\varepsilon}\|\Delta\eta\|^2_{\mathcal{M}^*_\varepsilon} \le -\int_0^\infty \mu'_\varepsilon(s)\|\Delta\eta(s)\|^2_*\mathrm{d}s. \tag{4.6}$$

As a consequence, the energy functional

$$\Theta(t) = E_1(t) + \frac{\gamma}{2}\|\Delta u_t\|^2$$

(here $\gamma > 0$ is to be chosen) satisfies in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta + \frac{1}{2\varepsilon}(\delta - \kappa\gamma)\|\Delta\eta\|^2_{\mathcal{M}^*_\varepsilon} \le Q(r)(1 + \|\nabla u_t\|^2).$$

Thus, provided that $\gamma < \delta/\kappa$, owing to (3.5), we deduce that

$$\Theta(t) \le \Theta(0) + Q(r)(1+t) \le Q(r)(1+t), \quad t \ge 0.$$

Now we note that by comparison in the first equation of (2.6)

$$\|u(t)\|_4 \le \|u_t(t)\|_2 + \|f(u(t))\|_2 + \|\varphi(t)\|_4. \tag{4.7}$$

Therefore, we see that

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le Q(r)(t+1), \quad t \ge 0.$$

In particular, this yields

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le Q(r), \quad t \in [0, t_0]. \tag{4.8}$$

Let us now take any $t \ge t_0$. We consider the functional

$$\Psi_1(t) = -\int_0^\infty \mu_\varepsilon^\star(s)\langle \Delta A\eta^t(s), \Delta A\varphi(t)\rangle \mathrm{d}s,$$

where $\mu_\varepsilon^\star(s) = \mu_\varepsilon(s_\star)\chi_{(0,\varepsilon s_\star]}(s) + \mu_\varepsilon(s)\chi_{(\varepsilon s_\star,\infty]}(s)$, $\mu_\star : \mathbb{R}^+ \to [0,\infty)$ being defined as[3]

$$\mu_\star(s) = \mu(s_\star)\chi_{(0,s_\star]}(s) + \mu(s)\chi_{(s_\star,\infty]}(s),$$

and $s_* > 0$ is fixed in order to satisfy

$$\int_{s_\star}^\infty \mu(s)\mathrm{d}s \ge \frac{3}{4}\kappa.$$

Reasoning as in [15, Theorem 3.1], due to (4.2) for $t \ge t_0$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 + \frac{\nu}{\varepsilon}\|\Delta A\varphi\|^2 \le \frac{1}{2}\|\Delta u_t\|^2 - \frac{c}{\varepsilon}\int_0^\infty \mu_\varepsilon'(s)\|\Delta\eta(s)\|_*^2\mathrm{d}s + \frac{c}{\varepsilon}\|\Delta\eta\|^2_{\mathcal{M}_\varepsilon^*}, \tag{4.9}$$

for some $\nu > 0$. We introduce the further functional

$$\Lambda_1(t) = \Theta(t) + \varrho\varepsilon\Psi_1(t) + \varrho\|\Delta u(t)\|^2,$$

for some given $\varrho \in (0, 1/2)$. Note that, by multiplying the equation for $u$ by $\Delta^2 u$, thanks to (4.2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \|\nabla\Delta u\|^2 = -\langle \Delta f(u), \Delta u\rangle + \langle \Delta A\varphi, \Delta u\rangle \le \frac{\nu}{2}\|\Delta A\varphi\|^2 + c.$$

Taking into account (4.4), (4.5) and (4.9), we are thus led to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \varrho\|\nabla\Delta u\|^2 + \frac{1}{2}\big(1-\varrho\big)\|\Delta u_t\|^2 \\
&\qquad + \frac{\nu\varrho}{2}\|A\Delta\varphi\|^2 - \big(\frac{1}{2} - c\varrho\big)\int_0^\infty \mu_\varepsilon'(s)\|\Delta\eta(s)\|_*^2\mathrm{d}s \\
&\qquad \le c(1 + \|\nabla u_t\|^2) + (c\varrho + \frac{\gamma\kappa}{2\varepsilon})\|\Delta\eta\|^2_{\mathcal{M}_\varepsilon^*}.
\end{aligned}$$

Invoking (4.6), it is apparent that, properly choosing $\varrho$ and $\gamma$, we end up with the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \nu_1\Lambda_1 + \frac{1}{4}\|\Delta u_t\|^2 \le c(1 + \|\nabla u_t\|^2), \quad t \ge t_0,$$

for some $\nu_1 > 0$. An application of the Gronwall Lemma on $[t_0, t]$ provides, in view of (4.3),

$$\Lambda_1(t) \le \Lambda_1(t_0)e^{-\nu_1(t-t_0)} + c, \quad t \ge t_0,$$

[3] Using $\mu_\star$ instead of $\mu$ is needed when $\mu$ is unbounded at the origin (see [51]).

which, owing to (4.8) for $t = t_0$ and (4.7), implies

$$\|S_\varepsilon(t)z\|^2_{\mathcal{H}^1_\varepsilon} \leq Q(r)e^{-\nu_1(t-t_0)} + c, \quad t \geq t_0.$$

Collecting this last inequality and (4.8) the proof is completed. □

### 4.1. Compact exponentially attracting set

Due to the lack of compactness of the embedding $\mathcal{H}^1_\varepsilon \subset \mathcal{H}_\varepsilon$, we need to introduce a Banach space $\mathcal{W}_\varepsilon$ which is compactly embedded in the phase space. This is done following [27, Proposition 5.4], where it is shown that

$$\mathcal{K}_\varepsilon = \{\eta \in \mathcal{M}^4_\varepsilon,\, \partial_s\eta \in \mathcal{M}^2_\varepsilon,\, \eta(0) = 0,$$
$$\sup_{x\geq 1} x \int_{\varepsilon x}^{\infty} \mu_\varepsilon(s)\|\eta(s)\|_2^2\,\mathrm{d}s < \infty\} \subset \mathcal{M}^4_\varepsilon,$$

with norm

$$\|\eta\|^2_{\mathcal{K}_\varepsilon} = \|\eta\|^2_{\mathcal{M}^4_\varepsilon} + \varepsilon^2\|\partial_s\eta\|^2_{\mathcal{M}^2_\varepsilon} + \sup_{x\geq 1} x \int_{\varepsilon x}^{\infty} \mu_\varepsilon(s)\|\eta(s)\|_2^2\,\mathrm{d}s,$$

is continuously embedded in $\mathcal{M}^4_\varepsilon$. Besides, its closed balls are closed in $\mathcal{M}^2_\varepsilon$ and the compact embedding

$$\mathcal{K}_\varepsilon \Subset \mathcal{M}^2_\varepsilon$$

holds. As a consequence, the product space

$$\mathcal{W}_\varepsilon = \mathrm{H}^4 \times \mathrm{H}^4 \times \mathcal{K}_\varepsilon \subset \mathcal{H}^1_\varepsilon$$

is compactly embedded in $\mathcal{H}_\varepsilon$. We also recall a suitable formulation of [25, Lemma 5.2], which is crucial when working in $\mathcal{K}_\varepsilon$.

**Lemma 4.2.** *Let $\eta_0 \in \mathcal{K}_\varepsilon$. Assume that $\eta$ satisfies the Cauchy problem*

$$\begin{cases} \partial_t\eta = T_\varepsilon\eta + \varphi, \\ \eta^0 = \eta_0, \end{cases}$$

*on $(0,T)$, for some $T > 0$. Then, $\eta^t \in \mathcal{K}_\varepsilon$ and*

$$\|\eta^t\|^2_{\mathcal{K}_\varepsilon} \leq 2(t+2)\mathrm{e}^{-\delta t}\|\eta_0\|^2_{\mathcal{K}_\varepsilon} + c\|\varphi(t)\|_4^2, \qquad \forall t \in (0,T).$$

The main result of this section is the following

**Theorem 4.3.** *There exists $\varrho > 0$ such that*

$$\mathcal{B}_\varepsilon = \{z \in \mathcal{W}_\varepsilon \,:\, \|z\|_{\mathcal{W}_\varepsilon} \leq \varrho\}$$

*is exponentially attracting for $S_\varepsilon(t)$, with an attraction rate independent of $\varepsilon$, namely*

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B_\varepsilon, \mathcal{B}_\varepsilon) \leq M\mathrm{e}^{-\varkappa t}, \qquad \forall t \geq 0,$$

*for some $M, \varkappa > 0$ independent of $\varepsilon$. Furthermore, $\mathcal{B}_\varepsilon$ absorbs itself in a finite time $t_\varrho \geq 0$ (independent of $\varepsilon$).*

*Proof.* We need a number of steps.

**Step I.** We first note that the analogous of Lemma 4.1 holds, replacing $\mathcal{H}^1_\varepsilon$ with $\mathcal{W}_\varepsilon$, namely there exist $\vartheta_2 > 0$, $R_2 > 0$, and a positive function $Q_2(\cdot)$ such that

$$\|S_\varepsilon(t)z\|_{\mathcal{W}_\varepsilon} \le Q_2(r)\mathrm{e}^{-\vartheta_2 t} + R_2, \qquad \forall t \ge 0,$$

whenever $\|z\|_{\mathcal{W}_\varepsilon} \le r$, for every $\varepsilon \in [0,1]$.

Indeed, this is a simple application of Lemma 4.2, recalling the inclusion

$$\|z\|_{\mathcal{H}^1_\varepsilon} \le \|z\|_{\mathcal{W}_\varepsilon}.$$

**Step II.** There is $r_2 > 0$ such that the ball of $\mathcal{W}_\varepsilon$ with radius $r_2$ attracts the set $B^1_\varepsilon$ found in Theorem 3.5, with an attraction rate independent of $\varepsilon$.

To this aim let $z = (u_0, \varphi_0, \eta_0) \in B^1_\varepsilon$ be given. We decompose

$$S_\varepsilon(t)z = (u(t), \varphi(t), \eta^t) = (0, 0, \psi^t) + (u(t), \varphi(t), \xi^t),$$

where

$$\begin{cases} \partial_t \psi = T_\varepsilon \psi, \\ \psi^0 = \eta_0, \end{cases} \qquad\qquad \begin{cases} \partial_t \xi = T_\varepsilon \xi + \varphi(t), \\ \xi^0 = 0. \end{cases}$$

We readily get from $\mu' + \delta\mu \le 0$ the exponential decay

$$\|\psi^t\|^2_{\mathcal{M}^4_\varepsilon} \le Q(r_1)\mathrm{e}^{-\delta t}.$$

Now, observe that, from Lemma 4.1,

$$\|S_\varepsilon(t)z\|_{\mathcal{H}^1_\varepsilon} \le Q_1(r_1), \qquad \forall t \ge 0.$$

Then, by comparison,

$$\|\xi^t\|_{\mathcal{M}^4_\varepsilon} \le Q(r_1).$$

An application of Lemma 4.2, noting that here the initial datum is null, yields

$$\|\xi^t\|_{\mathcal{K}_\varepsilon} \le Q(r_1) \quad \Rightarrow \quad \|(u(t), \varphi(t), \xi^t)\|_{\mathcal{W}_\varepsilon} \le Q(r_1).$$

Therefore, the thesis follows setting $r_2 = Q(r_1)$.

**Step III.** We now choose $\varrho$ strictly greater than $r_2$ and $R_2$ and we define

$$\mathcal{B}_\varepsilon = \{z \in \mathcal{W}_\varepsilon \,:\, \|z\|_{\mathcal{W}_\varepsilon} \le \varrho\}.$$

Since $\rho > r_2$, it is clear that $\mathcal{B}_\varepsilon$ exponentially attracts $B^1_\varepsilon$ at a uniform rate with respect to $\varepsilon$. Now we prove that $\mathcal{B}_\varepsilon$ absorbs itself in a finite time. Indeed, having in mind Step I, we choose $t_\varrho > 0$ such that

$$\mathcal{Q}_2(\varrho)\mathrm{e}^{-\vartheta_2 t} + R_2 \le \varrho, \qquad \forall t \ge t_\varrho.$$

This implies that

$$\|S_\varepsilon(t)z\|_{\mathcal{W}_\varepsilon} \le \varrho, \qquad \forall t \ge t_\varrho,$$

whenever $\|z\|_{\mathcal{W}_\varepsilon} \le \varrho$, hence

$$S_\varepsilon(t)\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon, \qquad \forall t \ge t_\varrho,$$

as claimed.

**Step IV.** Owing to the transitivity of the exponential attraction and to the (uniform with respect to $\varepsilon$) continuity of $S_\varepsilon(t)$, we infer from the fact that $\mathcal{B}_\varepsilon$ exponentially attracts the attracting ball $B^1_\varepsilon \subset \mathcal{H}^1_\varepsilon$ that the basin of attraction is the absorbing set $B_\varepsilon$, hence the whole phase space.

□

## 5. Robust exponential attractors

We report here a suitable version of the main abstract result from [16, Theorem A.2] (see also [19]), ensuring the existence of robust exponential attractors for $S_\varepsilon(t)$. In order to make this statement precise, we introduce the lifting map $\mathbb{L}_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ defined as

$$\mathbb{L}_\varepsilon(u,\varphi) = \begin{cases} (u,\varphi,0), & \varepsilon > 0, \\ (u,\varphi), & \varepsilon = 0, \end{cases}$$

and the projection onto the first two components of $\mathcal{H}_\varepsilon$, namely $\mathbb{P} : \mathcal{H}_\varepsilon \to \mathrm{H}^2 \times \mathrm{H}^2$ acting as $\mathbb{P}(a,b,c) = (a,b)$.

**Theorem 5.1.** *Assume that*

(H0) *there exist $R > 0$ and $t^\star > 0$, both independent of $\varepsilon$, and a family of closed sets $\mathbb{B}_\varepsilon \subset B_{\mathcal{H}_\varepsilon}(R)$ such that*

$$S_\varepsilon(t)\mathbb{B}_\varepsilon \subset \mathbb{B}_\varepsilon, \qquad \forall t \geq t^\star,$$

*and $\mathbb{B}_\varepsilon$ is exponentially attracting in $\mathcal{H}_\varepsilon$, with an attraction rate independent of $\varepsilon$.*

*Assume furthermore that there exist $\Lambda_j \geq 0$, $\lambda \in [0,\frac{1}{2})$, $\alpha \in (0,1]$ and a continuous increasing function $\Sigma : [0,1] \to [0,\infty)$ with $\Sigma(0) = 0$ (all independent of $\varepsilon$) such that the following conditions hold.*

(H1) *The map $S_\varepsilon = S_\varepsilon(t^\star)$ satisfies, for every $z_1, z_2 \in \mathbb{B}_\varepsilon$,*

$$S_\varepsilon z_1 - S_\varepsilon z_2 = L_\varepsilon(z_1,z_2) + K_\varepsilon(z_1,z_2),$$

*where*

$$\|L_\varepsilon(z_1,z_2)\|_{\mathcal{H}_\varepsilon} \leq \lambda \|z_1 - z_2\|_{\mathcal{H}_\varepsilon},$$
$$\|K_\varepsilon(z_1,z_2)\|_{\mathcal{W}_\varepsilon} \leq \Lambda_1 \|z_1 - z_2\|_{\mathcal{H}_\varepsilon}.$$

(H2) *There holds*

$$\|S^n_\varepsilon z - \mathbb{L}_\varepsilon S^n_0 \mathbb{P} z\|_{\mathcal{H}_\varepsilon} \leq \Lambda^n_2 \Sigma(\varepsilon), \qquad \forall z \in \mathbb{B}_\varepsilon,\ \forall n \in \mathbb{N}.$$

(H3) *There holds*

$$\|S_\varepsilon(t) z - \mathbb{L}_\varepsilon S_0(t) \mathbb{P} z\|_{\mathcal{H}_\varepsilon} \leq \Lambda_3 \Sigma(\varepsilon), \qquad \forall z \in \mathbb{B}_\varepsilon,\ \forall t \in [t^\star, 2t^\star].$$

(H4) *The map*

$$z \mapsto S_\varepsilon(t)z : \mathbb{B}_\varepsilon \to \mathbb{B}_\varepsilon$$

*is Lipschitz continuous, with a Lipschitz constant independent of* $t \in [t^\star, 2t^\star]$ *and of* $\varepsilon$*. Here,* $\mathbb{B}_\varepsilon$ *is endowed with the metric topology of* $\mathcal{H}_\varepsilon$*.*

(H5) *The map*

$$(t, z) \mapsto S_\varepsilon(t)z : [t^\star, 2t^\star] \times \mathbb{B}_\varepsilon \to \mathbb{B}_\varepsilon$$

*is Hölder continuous of exponent* $\alpha$ *(with a constant that may depend on* $\varepsilon$*). Again,* $\mathbb{B}_\varepsilon$ *is endowed with the metric topology of* $\mathcal{H}_\varepsilon$*.*

*Then there exists a family of compact sets* $\mathcal{E}_\varepsilon \subset \mathbb{B}_\varepsilon$*, called exponential attractors, such that*

$$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon, \qquad \forall t \geq 0,$$

*with the following additional properties.*

- $\mathcal{E}_\varepsilon$ *attracts* $\mathbb{B}_\varepsilon$ *with an exponential rate which is uniform with respect to* $\varepsilon$*, that is,*

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}\big(S_\varepsilon(t)\mathbb{B}_\varepsilon, \mathcal{E}_\varepsilon\big) \leq M_1 e^{-\kappa t}, \qquad \forall t \geq 0,$$

  *for some* $\kappa > 0$*.*
- *The fractal dimension of* $\mathcal{E}_\varepsilon$ *is uniformly bounded with respect to* $\varepsilon$*, that is,*

$$\dim_{\mathcal{H}_\varepsilon}[\mathcal{E}_\varepsilon] \leq M_2.$$

- *There holds*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}_\varepsilon}\big(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon \mathcal{E}_0\big) \leq M_3[\Sigma(\varepsilon)]^\tau,$$

  *for some* $\tau \in (0, 1]$*.*

*The positive constants* $\kappa, \tau$ *and* $M_j$ *are independent of* $\varepsilon$ *and can be explicitly calculated.*

We now proceed with the verification of the assumptions of the theorem.

### 5.1. Proof of (H0)

Set

$$\mathbb{B}_\varepsilon = \mathcal{B}_\varepsilon$$

given by Theorem 4.3. Note that $\mathcal{B}_\varepsilon$ is closed in $\mathcal{H}_\varepsilon$. To start with, take $t^\star > t_\rho$ to be possibly increased so that (H1) holds true.

### 5.2. Proof of (H1)

For any given $z_1, z_2 \in \mathbb{B}_\varepsilon$, we decompose the difference of the solutions as

$$S_\varepsilon(t)z_1 - S_\varepsilon(t)z_2 = L_\varepsilon(t)(z_1 - z_2) + K_\varepsilon(t)(z_1, z_2),$$

where

$$L_\varepsilon(t)z = (v(t), \chi(t), \xi^t) \qquad \text{and} \qquad K_\varepsilon(t)(z_1, z_2) = (w(t), \psi(t), \zeta^t)$$

solve the problems

$$\begin{cases} v_t - \Delta v = A\chi, \\ A\chi_t - \displaystyle\int_0^\infty \mu_\varepsilon(s)\Delta\xi(s)\,\mathrm{d}s = -v_t, \\ \xi_t = T_\varepsilon \xi + \chi, \\ L_\varepsilon(0)z = z, \end{cases} \tag{5.1}$$

and

$$\begin{cases} w_t - \Delta w + f(u_1) - f(u_2) = A\psi, \\ A\psi_t - \displaystyle\int_0^\infty \mu_\varepsilon(s)\Delta\zeta(s)\,\mathrm{d}s = -w_t, \\ \zeta_t = T_\varepsilon \zeta + \psi, \\ K_\varepsilon(0)(z_1, z_2) = 0, \end{cases} \tag{5.2}$$

where $S_\varepsilon(t)z_i = (u_i(t), \varphi_i(t), \eta_i^t)$. Note that $L_\varepsilon(t)z$ is a semigroup, since it is the solving operator of (2.6) corresponding to $f \equiv 0$. Thus, in light of Theorem 3.2 and Remark 3.3, we have the exponential decay of $L_\varepsilon(t)z$,

$$\|L_\varepsilon(t)(z_1 - z_2)\|^2_{\mathcal{H}_\varepsilon} \le c\,\mathrm{e}^{-\omega t}\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon}, \quad \forall t \ge 0, \tag{5.3}$$

where, here and along this section, $c \ge 0$ denotes any constant possibly depending on the size of $\mathbb{B}_\varepsilon$. Owing to the continuous dependence estimate (3.1), this immediately yields

$$\|K_\varepsilon(t)(z_1, z_2)\|^2_{\mathcal{H}_\varepsilon} \le c\,\mathrm{e}^{ct}\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon}, \quad \forall t \ge 0. \tag{5.4}$$

Furthermore, the following higher-order estimates hold true for $K_\varepsilon(t)(z_1, z_2)$:

**Theorem 5.2.** *There exists $c > 0$, depending on the size of $\mathbb{B}_\varepsilon$, such that, for every $t \ge 0$, we have*

$$\|K_\varepsilon(t)(z_1, z_2)\|_{\mathrm{H}^3 \times \mathrm{H}^4 \times L^2_{\mu_\varepsilon}(\mathbb{R}^+;\mathrm{H}^4)} \le c\,\mathrm{e}^{ct}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}.$$

*Besides, setting $K_\varepsilon(t)(z_1, z_2) = (w(t), \psi(t), \zeta^t)$,*

$$\sup_{t \ge 0} \int_t^{t+1} \big(\|w_t(s)\|_2^2 + \frac{1}{\varepsilon}\|\Delta\zeta(s)\|^2_{\mathcal{M}^*_\varepsilon}\big)\mathrm{d}s \le c\,\mathrm{e}^{ct}\|z_1 - z_2\|^2_{\mathcal{H}_\varepsilon}.$$

*Proof.* Along the proof, we perform formal computations which can be rigorously justified within a standard Galerkin scheme. The product of system (5.2) by $(\Delta^2 w_t, \Delta^2 A\psi, -\Delta^3 A\zeta)$ in $\mathrm{H} \times \mathrm{H} \times L^2_{\mu_\varepsilon}(\mathbb{R}^+, \mathrm{H})$ yields

$$\|\Delta w_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta w\|^2 = \langle A\psi, \Delta^2 w_t\rangle - \langle f(u_1) - f(u_2), \Delta^2 w_t\rangle,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta A\psi\|^2 - \int_0^\infty \mu_\varepsilon(s)\langle\Delta\zeta(s), \Delta^2 A\psi\rangle\mathrm{d}s = -\langle\Delta^2 A\psi, w_t\rangle,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\zeta\|^2_{\mathcal{M}^*_\varepsilon} - \frac{1}{2}\int_0^\infty \mu'_\varepsilon(s)\|\Delta\zeta(s)\|^2_*\mathrm{d}s = -\int_0^\infty \mu_\varepsilon(s)\langle\Delta\zeta(s), \Delta^2 A\psi\rangle\mathrm{d}s.$$

Hence, defining

$$E_1(t) = \frac{1}{2}\big(\|\nabla\Delta w(t)\|^2 + \|\Delta A\psi(t)\|^2 + \|\Delta\zeta^t\|^2_{\mathcal{M}^*_\varepsilon}\big),$$

we obtain the basic inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1 + \frac{1}{2}\|\Delta w_t\|^2 - \frac{1}{2}\int_0^\infty \mu_\varepsilon'(s)\|\Delta\zeta(s)\|_*^2 \mathrm{d}s \le c\|u_1 - u_2\|_2^2, \tag{5.5}$$

having used

$$|\langle f(u_1) - f(u_2), \Delta^2 w_t\rangle| \le \|\Delta f(u_1) - \Delta f(u_2)\|\|\Delta w_t\| \le c\|u_1 - u_2\|_2^2 + \frac{1}{2}\|\Delta w_t\|^2,$$

in light of Theorem 3.2. We now introduce the functional

$$\Psi_1(t) = -\int_0^\infty \mu_\varepsilon^\star(s)\langle\Delta A\zeta^t(s), \Delta A\psi(t)\rangle \mathrm{d}s,$$

where $\mu_\varepsilon^\star : \mathbb{R}^+ \to [0,\infty)$ is defined as in the proof of Lemma 4.1. Reasoning as above, it is easy to check that we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 + \frac{\nu}{\varepsilon}\|\Delta A\psi\|^2 \le \frac{1}{2}\|\Delta w_t\|^2 - \frac{c}{\varepsilon}\int_0^\infty \mu_\varepsilon'(s)\|\Delta\zeta(s)\|_*^2 \mathrm{d}s + \frac{c}{\varepsilon}\|\Delta\zeta\|_{\mathcal{M}_\varepsilon^*}^2, \tag{5.6}$$

for some $\nu > 0$. We introduce the further functional

$$\Lambda_1(t) = E_1(t) + \varepsilon\varrho\Psi_1(t) + \varrho\|\Delta w(t)\|^2,$$

for some given $\varrho \in (0, 1/2)$. Note that, by multiplying the equation for $w$ by $\Delta^2 w$, thanks to (5.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|^2 + \|\nabla\Delta w\|^2 &= -\langle\Delta f(u_1) - \Delta f(u_2), \Delta w\rangle + \langle\Delta A\psi, \Delta w\rangle \\ &\le \frac{\nu}{2}\|\Delta A\psi\|^2 + c\|u_1 - u_2\|_2^2.\end{aligned}$$

Taking into account (5.5) and (5.6), we are thus led to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \varrho\|\nabla\Delta w\|^2 + \frac{1}{2}(1-\varrho)\|\Delta w_t\|^2 + \frac{\nu\varrho}{2}\|A\Delta\psi\|^2 \\ &\quad - \Big(\frac{1}{2} - c\varrho\Big)\int_0^\infty \mu_\varepsilon'(s)\|\Delta\zeta(s)\|_*^2 \mathrm{d}s \\ &\quad \le c\|u_1 - u_2\|_2^2 + \frac{c}{\varepsilon}\varrho\|\Delta\zeta\|_{\mathcal{M}_\varepsilon^*}^2.\end{aligned}$$

Invoking (4.6), it is apparent that we can properly choose $\varrho$ to get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_1 + \nu_1\Lambda_1 + \frac{1}{4}\|\Delta w_t\|^2 + \frac{c}{\varepsilon}\|\Delta\zeta\|_{\mathcal{M}_\varepsilon^*}^2 \le c\|u_1 - u_2\|_2^2,$$

for some $\nu_1 > 0$. Since $\Lambda_1(0) = 0$, an application of the Gronwall Lemma and an integration in time complete the proof. □

**Remark 5.3.** Note that

$$\|K_\varepsilon(t)(z_1, z_2)\|_{\mathcal{H}_\varepsilon^1} \le c\,\mathrm{e}^{ct}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}.$$

This is readily seen by differentiating the first equation in (5.2) with respect to time,

$$w_{tt} - \Delta w_t + f'(u_1)u_t + [f'(u_1) - f'(u_2)]u_{2t} = A\psi_t,$$

and multiplying the result by $\Delta^2 w_t$:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w_t\|^2 + \|\nabla\Delta w_t\|^2 \\
&= \langle f''(u_1)\partial_t u_1 \nabla u + f''(u_1)\partial_t u \nabla u_2 + [f''(u_1) - f''(u_2)]\partial_t u_2 \nabla u_2, \nabla\Delta w_t\rangle \\
&\quad + \langle f'(u_1)\nabla u_t + (f'(u_1) - f'(u_2))\nabla\partial_t u_2, \nabla\Delta w_t\rangle + \langle A\psi_t, \Delta^2 w_t\rangle.
\end{aligned}$$

Then, multiplying the second equation in (5.2) by $\Delta^2 w_t$, the last term above reads

$$\langle A\psi_t, \Delta^2 w_t\rangle = \int_0^\infty \mu_\varepsilon(s)\langle \Delta\zeta(s), \Delta^2 w_t\rangle \,\mathrm{d}s - \|\Delta w_t\|^2,$$

leading to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w_t\|^2 + \|\nabla\Delta w_t\|^2 + \|\Delta w_t\|^2 \\
&= \langle f''(u_1)\partial_t u_1 \nabla u + f''(u_1)\partial_t u \nabla u_2 + [f''(u_1) - f''(u_2)]\partial_t u_2 \nabla u_2, \nabla\Delta w_t\rangle \\
&\quad + \langle f'(u_1)\nabla u_t + (f'(u_1) - f'(u_2))\nabla\partial_t u_2, \nabla\Delta w_t\rangle \\
&\quad + \int_0^\infty \mu_\varepsilon(s)\langle \Delta\zeta(s), \Delta^2 w_t\rangle \,\mathrm{d}s \\
&\leq \frac{1}{2}\|\nabla\Delta w_t\|^2 + c(\|u_t\|_1^2 + \frac{1}{\varepsilon}\|\Delta\zeta\|^2_{\mathcal{M}_\varepsilon^*}) \\
&\quad + c(1 + \|\partial_t u_1\|_1^2 + \|\partial_t u_2\|_1^2)\|u_1 - u_2\|_2^2.
\end{aligned}$$

Then Theorem 3.1 and the integral inequality in Theorem 5.2 allow to conclude that

$$\|w_t(t)\|_2 \leq c e^{ct}\|z_1 - z_2\|_{\mathcal{H}_\varepsilon}, \quad t \geq 0,$$

hence the elliptic equation $-\Delta w = -f(u_1) + f(u_2) + A\psi - w_t \in \mathrm{H}^2$ gives the desired bound in $\mathrm{H}^4$.

Owing to (5.3), (5.4), Remark 5.3 and Lemma 4.2, it is readily seen that, for $t^\star$ large enough independent of $\varepsilon$, $L_\varepsilon(z_1, z_2) = L_\varepsilon(t^\star)(z_1 - z_2)$ and $K_\varepsilon(z_1, z_2) = K_\varepsilon(t^\star)(z_1, z_2)$ satisfy (H1).

### 5.3. Proof of (H2) – (H4)

The sufficient conditions (H2)-(H3) with $\Sigma(\varepsilon) = \sqrt[8]{\varepsilon}$ follow directly from [15]. Indeed, Theorem 6.1 therein reads as follows:

**Theorem 5.4.** *Let $\varepsilon > 0$ and $T > 0$. For any $z \in \mathcal{H}_\varepsilon$ such that $\|z\|_{\mathcal{H}_\varepsilon} \leq R$, there holds*

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{\mathrm{H}^2\times\mathrm{H}^2} \leq Q_T(R)\sqrt[8]{\varepsilon}, \quad \forall t \in [0, T], \tag{5.7}$$

*where $Q_T(\cdot)$ depends on $T$. Besides, for any $t \geq 0$,*

$$\|\eta_\varepsilon^t\|_{\mathcal{M}_\varepsilon^2} \leq \|\eta_0\|_{\mathcal{M}_\varepsilon^2}\mathrm{e}^{-\delta t/4\varepsilon} + Q(R)\sqrt{\varepsilon}. \tag{5.8}$$

In turn, (H4) is a straightforward consequence of Theorem 3.1.

**5.4. Proof of** (H5)

Owing to Theorem 3.1, we accomplish our purpose by the Hölder inequality once we prove that, for any $z \in \mathbb{B}_\varepsilon$, the solution $S_\varepsilon(t)z = (u(t), \varphi(t), \eta^t)$ satisfies

$$\int_{t^\star}^{2t^\star} (\|u_t(s)\|_2^2 + \|\varphi_t(s)\|_2^2 + \|\eta_t(s)\|^2_{\mathcal{M}^2_\varepsilon})\mathrm{d}s \le \frac{c}{\varepsilon}. \tag{5.9}$$

Recalling that $S_\varepsilon(t)\mathbb{B}_\varepsilon \subset \mathbb{B}_\varepsilon$ for $t \ge t^\star$, we integrate (4.4) over $[t^\star, 2t^\star]$: taking into account (2.4), we bound the first term in (5.9). In a similar way, multiplying the second equation in (2.6) by $A\varphi_t$ and exploiting (3.4), we obtain

$$\|A\varphi_t\|^2 = -\langle u_t, A\varphi_t\rangle + \int_0^\infty \mu_\varepsilon(s)\langle \Delta\eta(s), A\varphi_t(s)\rangle \mathrm{d}s \le \frac{1}{2}\|A\varphi_t\|^2 + \frac{c}{\varepsilon},$$

so that a further integration in time leads to the control of the second term in (5.9). Finally, the control in the memory variable is obtained differentiating in time the third equation in (2.6) with the usual estimates, relying on the integral estimate on $\|\varphi_t\|_2^2$.

**Remark 5.5.** It is worth recalling that the global attractors $\mathcal{A}_\varepsilon$, whose existence follows from Theorem 4.3 (see also [15, Theorem 5.1]), are invariant sets contained in each compact attracting set. In particular, $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$, so that the fractal dimension of $\mathcal{A}_\varepsilon$ is uniformly bounded with respect to $\varepsilon$. We can note that, in [15], only the upper semicontinuity at $\varepsilon = 0$ of the family of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\ge 0}$ has been established. Namely,

$$\lim_{\varepsilon\to 0} \mathrm{dist}_{\mathcal{H}_\varepsilon}\big(\mathcal{A}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{A}_0\big) = 0.$$

In particular, there is no explicit estimate in terms of $\varepsilon$ here, contrary to the above family of exponential attractors for which we have a full continuity result.

## 6. Spatial behavior of solutions

In this section, we study the spatial behavior of the solutions of system (1.11) in the semi-infinite cylinder $R = (0, \infty) \times D$, where $D$ is a two-dimensional bounded domain which is smooth enough to allow the use of the divergence theorem. We supplement the equations with the following boundary conditions

$$u = \varphi = 0, \quad \text{on } (0,\infty) \times \partial D \times (0,T), \tag{6.1}$$

$$u(0, x_2, x_3, t) = h(x_2, x_3, t), \;\; \varphi(0, x_2, x_3, t) = m(x_2, x_3, t) \quad \text{on } \{0\}\times D\times(0,T), \tag{6.2}$$

where $T > 0$ is a given final time, and null initial conditions

$$u|_{t=0} = \varphi|_{t\le 0} = 0 \quad \text{on } R. \tag{6.3}$$

As far as the nonlinear term $f$ is concerned, we assume that there exists a positive constant $d$ such that

$$f(s)s + ds^2 \ge 0, \text{ and } F(s) + ds^2 \ge 0,$$

where $F$ is as above. In particular, it is clear that the function $f(s) = s^3 - s$ satisfies these conditions. Actually, any function of the form $f(s) = a|s|^k s - bs$, $a, k > 0$, is admissible.

Our aim is to obtain an alternative of Phragmén-Lindelöf type, meaning that the solutions decrease (resp., grow) in a negative (resp., positive) exponential way with respect to the spatial variable.

In view of this, we first introduce the functional

$$F_\omega(z,t) = \int_0^t \int_{D(z)} \exp(-2\omega s)\Big(u_{,1}(s)u_s(s) + \varphi(s)\int_0^\infty \mu(\tau)\eta^s_{,1}(\tau)d\tau\Big)\mathrm{d}a\mathrm{d}s,$$

where $D(z) = \{x \in R, x_1 = z\}$ and $\omega$ is an arbitrary positive constant to be fixed later; here

$$v_s = \partial v/\partial s \qquad \text{and} \qquad v_{,1} = \partial v/\partial x_1.$$

There holds, owing to the boundary and initial conditions and the divergence theorem,

$$F_\omega(z+h,t) - F_\omega(z,t) = \int_0^t \int_{R(z,z+h)} \exp(-2\omega s)W\mathrm{d}x\mathrm{d}s,$$

where $R(z, z+h) = \{x \in R, z < x_1 < z+h\}$ and

$$\begin{aligned} W =& u_i u_{,is} + u_s^2 + f(u)u_s + (\varphi_s - \Delta\varphi_s)(\varphi - \Delta\varphi) \\ &+ \int_0^\infty \mu(\tau)\eta^s_{,i}(\tau)\varphi_{,i}(s)\mathrm{d}\tau + \int_0^\infty \mu(\tau)\Delta\eta^s(\tau)\Delta\varphi(s)\mathrm{d}\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} W =& \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}|\nabla u|^2 + F(u) + \frac{1}{2}(\varphi - \Delta\varphi)^2\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\int_0^\infty \mu(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau\right) \\ &- \frac{1}{2}\int_0^\infty \mu'(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau + |u_s|^2. \end{aligned}$$

We then obtain

$$
\begin{aligned}
&F_\omega(z+h,t) - F_\omega(z,t) \\
&\quad = \frac{\exp(-2\omega t)}{2} \int_{R(z,z+h)} \Big( |\nabla u|^2 + 2F(u) + (\varphi - \Delta\varphi)^2 \\
&\quad + \int_0^\infty \mu(s)(\eta^t_{,i}(s)\eta^t_{,i}(s) + \Delta\eta^t(s)\Delta\eta^t(s))\mathrm{d}s \Big)\mathrm{d}x \\
&\quad + \int_0^t \int_{R(z,z+h)} \exp(-2\omega s) \Big( |u_s|^2 - \frac{1}{2}\int_0^\infty \mu'(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) \\
&\quad + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau \Big)\mathrm{d}x\mathrm{d}s \\
&\quad + \omega \int_0^t \int_{R(z,z+h)} \exp(-2\omega s) \Big( |\nabla u|^2 + 2F(u) + (\varphi - \Delta\varphi)^2 \\
&\quad + \int_0^\infty \mu(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau \Big)\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

and a direct differentiation gives

$$
\begin{aligned}
&\frac{\partial F_\omega(z,t)}{\partial z} = \frac{\exp(-2\omega t)}{2} \int_{D(z)} \Big( |\nabla u|^2 + 2F(u) + (\varphi - \Delta\varphi)^2 \\
&\quad + \int_0^\infty \mu(s)(\eta^t_{,i}(s)\eta^t_{,i}(s) + \Delta\eta^t(s)\Delta\eta^t(s))\mathrm{d}s \Big)\mathrm{d}a \\
&\quad + \int_0^t \int_{D(z)} \exp(-2\omega s) \Big( |u_s|^2 \\
&\quad - \frac{1}{2}\int_0^\infty \mu'(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau \Big)\mathrm{d}a\mathrm{d}s \\
&\quad + \omega \int_0^t \int_{D(z)} \exp(-2\omega s) \Big( |\nabla u|^2 + 2F(u) + (\varphi - \Delta\varphi)^2 \\
&\quad + \int_0^\infty \mu(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau \Big)\mathrm{d}a\mathrm{d}s.
\end{aligned}
$$

Next, we introduce the functional

$$
G_\omega(z,t) = \int_0^t \int_{D(z)} \exp(-2\omega s) u_{,1} u \,\mathrm{d}a\mathrm{d}s.
$$

It is easy to see that

$$
\begin{aligned}
G_\omega(z+h,t) - G_\omega(z,t) &= \frac{\exp(-2\omega t)}{2} \int_{R(z,z+h)} |u|^2 \mathrm{d}x \\
&+ \int_0^t \int_{R(z,z+h)} \exp(-2\omega s)(|\nabla u|^2 + f(u)u - (\varphi - \Delta\varphi)u + \omega u^2)\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Furthermore

$$\frac{\partial G_\omega(z,t)}{\partial z} = \frac{\exp(-2\omega t)}{2}\int_{D(z)} |u|^2 \mathrm{d}a$$
$$+\int_0^t\int_{D(z)} \exp(-2\omega s)(|\nabla u|^2 + f(u)u - (\varphi - \Delta\varphi)u + \omega u^2)\mathrm{d}a\mathrm{d}s.$$

We finally set

$$H_\omega(z,t) = \int_0^t\int_{D(z)} \exp(-2\omega s)\varphi_{,1}\varphi \mathrm{d}a\mathrm{d}s,$$

so that

$$\frac{\partial H_\omega(z,t)}{\partial z} = \int_0^t\int_{D(z)} \exp(-2\omega s)(|\nabla\varphi|^2 + \varphi\Delta\varphi)\mathrm{d}a\mathrm{d}s.$$

Let $\phi$ be a positive constant to be fixed and set

$$J_\omega = F_\omega + \phi G_\omega + 2\omega H_\omega.$$

Noting that

$$\frac{\partial J_\omega(z,t)}{\partial z} = \frac{\exp(-2\omega t)}{2}\int_{D(z)} \Sigma_1 \mathrm{d}a + \int_0^t\int_{D(z)} \exp(-2\omega s)\Sigma_2 \mathrm{d}a\mathrm{d}s,$$

where

$$\Sigma_1 = |\nabla u|^2 + 2F(u) + (\varphi - \Delta\varphi)^2$$
$$+\phi u^2 + \int_0^\infty \mu(s)(\eta^t_{,i}(s)\eta^t_{,i}(s) + \Delta\eta^t(s)\Delta\eta^t(s))\mathrm{d}s$$

and

$$\begin{aligned}\Sigma_2 &= u_s^2 + \phi(|\nabla u|^2 + f(u)u - (\varphi - \Delta\varphi)u)\\ &\quad + \omega(|\nabla u|^2 + 2F(u) + \varphi^2 + 2|\nabla\varphi|^2 + (\Delta\varphi)^2 + \phi u^2)\\ &\quad + \omega\int_0^\infty \mu(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau\\ &\quad - \frac{1}{2}\int_0^\infty \mu'(\tau)(\eta^s_{,i}(\tau)\eta^s_{,i}(\tau) + \Delta\eta^s(\tau)\Delta\eta^s(\tau))\mathrm{d}\tau.\end{aligned}$$

we can choose $\phi$ large enough to guarantee that

$$\phi u^2 + 2F(u) \geq 0$$

and

$$\phi f(u)u + \omega(\varphi^2 + (\Delta\varphi)^2) - \phi(\varphi - \Delta\varphi)u + \phi\omega u^2 + 2\omega F(u) \geq C_0(\varphi^2 + (\Delta\varphi)^2 + u^2),$$

where $C_0$ is a positive constant. Indeed, the first condition is clear and the second one follows from the fact that the determinants of the leading minors of the matrix

$$\begin{pmatrix} \phi\omega - 2\omega d - \phi d & -\frac{\phi}{2} & \frac{\phi}{2} \\ -\frac{\phi}{2} & \omega & 0 \\ \frac{\phi}{2} & 0 & \omega \end{pmatrix}$$

are positive if $\phi$ and $\omega$ are large enough. We thus deduce the existence of a positive constant $C_1$ such that

$$\Sigma_2 \geq C_1\Big(u_s^2 + \varphi^2 + |\nabla\varphi|^2 + (\Delta\varphi)^2 + u^2 + |\nabla u|^2 + \int_0^\infty \mu(\tau)\eta^s_{,i}(\tau)\eta^s_{,i}(\tau)\mathrm{d}\tau\Big)$$

The next step consists in obtaining an estimate on $|J_\omega|$ in terms of the spatial derivative of $J_\omega$. We note that we can find positive constants $C_2$, ..., $C_5$ such that

$$|u_{,1}u_s| \leq C_2\Sigma_2,$$
$$|\varphi \int_0^\infty \mu(\tau)\eta^s_{,1}(\tau)d\tau| \leq C_3\Sigma_2,$$
$$|\phi u_{,1}u| \leq C_4\Sigma_2,$$

and

$$|2\omega\varphi\varphi_{,1}| \leq C_5\Sigma_2.$$

There thus exists a positive constant $C_6 = C_2 + ... + C_5$ such that

$$|J_\omega| \leq C_6\frac{\partial J_\omega}{\partial z},$$

for every $z$ and $t$ positive.

This inequality is classical and yields a Phragmén-Lindelöf alternative. More precisely, if there exists $z_0 \geq 0$ such that $J_\omega(z_0, t) > 0$, then the solution satisfies the estimate

$$J_\omega(z,t) \geq J_\omega(z_0,t)\exp(C_6^{-1}(z - z_0)), \ \ z \geq z_0. \tag{6.4}$$

This estimate gives information in terms of the measure defined in the cylinder. Indeed, it follows that

$$\frac{\exp(-2\omega t)}{2}\int_{R(0,z)} \Sigma_1 \mathrm{d}x + \int_0^t \int_{R(0,z)} \exp(-2\omega s)\Sigma_2 \mathrm{d}x\mathrm{d}s$$

tends to infinity exponentially fast, where $R(0,z) = \{x \in R, x_1 \leq z\}$. On the contrary, when $J_\omega(z,t) \leq 0$, for every $z \geq 0$, it follows that the solution decays and we can obtain an estimate of the form

$$-J_\omega(z,t) \leq -J_\omega(0,t)\exp(-C_6^{-1}z), \ \ z \geq 0.$$

This inequality implies that $J_\omega(z,t)$ tends to zero as $z$ goes to infinity. Furthermore in view of this estimate, it is clear that

$$E_\omega(z,t) \leq E_\omega(0,t)\exp(-C_6^{-1}z), \ \ z \geq 0,$$

where

$$E_\omega(z,t) = \frac{\exp(-2\omega t)}{2}\int_{R(z)} \Sigma_1 \mathrm{d}x + \int_0^t \int_{R(z)} \exp(-2\omega s)\Sigma_2 \mathrm{d}x\mathrm{d}s,$$

and $R(z) = \{x \in R, x_1 > z\}$. Setting finally

$$\mathcal{E}_\omega(z,t) = \frac{1}{2}\int_{R(z)} \Sigma_1 \mathrm{d}x + \int_0^t \int_{R(z)} \Sigma_2 \mathrm{d}x\mathrm{d}s,$$

we can state the

**Theorem 6.1.** *Let* $(u, \varphi)$ *be a smooth solution of the problem defined by system* (1.11)*, the boundary conditions* (6.1)*-*(6.2) *and the initial conditions* (6.3)*. Then, either this solution satisfies the growth estimate* (6.4) *or it satisfies the decay estimate*

$$\mathcal{E}_\omega(z,t) \leq E_\omega(0,t)\exp(2\omega t - C_6^{-1} z), \quad z \geq 0,$$

*where the energies* $\mathcal{E}_\omega$ *and* $E_\omega$ *are defined above.*

It is worth noting that the argument even works in the relaxed case that we assume that $\mu(s) \geq 0$ and $\mu'(s) \leq 0$.

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