

ACTION-ANGLE VARIABLES AND A KAM THEOREM FOR b -POISSON MANIFOLDS

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ABSTRACT. In this article we prove an action-angle theorem for b -integrable systems on b -Poisson manifolds improving the action-angle theorem contained in [LMV11] for general Poisson manifolds in this setting. As an application, we prove a KAM-type theorem for b -Poisson manifolds.

1. INTRODUCTION

The classic Hamilton's equations are the equations of the flow of a Hamiltonian vector field X_H determined by a Hamiltonian function H and a Darboux symplectic form in position and momenta $\omega_0 = \sum_i dp_i \wedge dq_i$ via the correspondence $\iota_{X_H}\omega_0 = -dH$. Symplectic geometry generalizes these equations to the general scenario of Hamiltonian systems associated to a closed non-degenerate 2-form ω (general symplectic form).

Among the class of Hamiltonian systems, the sub-class of integrable systems plays a central role. An integrable system on a $2n$ -dimensional symplectic manifold is given by $n - 1$ additional first integrals f_i with the property that each integral (including H) is preserved by the Hamiltonian flow of the other integrals. This condition is classically known as *involutivity* of the first integrals and can be written in terms of the Poisson bracket as $\{f_i, f_j\} = 0$.

Given an integrable system on a symplectic manifold there exist privileged coordinates in a neighbourhood of an invariant compact submanifold called action-angle coordinates ([M36], [LL76] and [A74]¹). These coordinates not only lead to a direct integration of the Hamiltonian system [L1855] but also they completely describe the geometry of the set of invariant submanifolds. The invariant manifolds are indeed tori named after Liouville. Action-angle variables describe the set of invariant submanifolds as a fibration by tori in a neighbourhood of a prescribed Liouville torus. The Liouville-Mineur-Arnold theorem also proves that the symplectic structure in this fibration

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¹ The action integrals already appear in a paper by Einstein [E17]. We refer to [F13] for a historical review of the contributions to the action-angle theorem.

has a Darboux-type form and that the Hamiltonian motion is linear when restricted to the invariant tori.

A generalization of the Liouville-Mineur-Arnold theorem for general Poisson manifolds is contained in [LMV11] where a set of action-angle coordinates is obtained for regular points. Poisson manifolds [P1809] constitute a natural generalization of symplectic manifolds that takes singularities (which may correspond to equilibria) into account. Poisson geometry has constituted a field of research on its own in the last decades. Many Hamiltonian systems (including integrable systems) are naturally formulated in this Poisson setting (see [LMV11] and [Mi14]).

In this paper we consider a particular class of Poisson manifolds which have been called in the literature b -symplectic manifolds, b -Poisson manifolds, and log-symplectic manifolds (see [NT96], [M93], [GMP11], [GMP12], [GMPS13], [G14L], [GLPR14], [MO1]). The study of b -symplectic manifolds is motivated by the study of deformation quantization and pseudodifferential operators on symplectic manifolds with boundary (see for instance [NT96] and [M93]). A natural set of examples of integrable systems on b -symplectic manifolds comes from this source.

In this article we improve the action-angle theorem in [LMV11] for b -Poisson manifolds by extending the construction in [LMV11] to the singular set determined by the b -Poisson structure. Our result is optimal in the sense that the number of action coordinates is half the dimension of the manifold. Our proof strongly uses the main ideas of Duistermaat in [D80] and [LMV11].

One of the sources of examples of Poisson manifolds with dense symplectic leaves comes from celestial mechanics where a *singular* change of coordinates takes a Darboux symplectic form to a symplectic form which “blows up” on a meagre singular set (see example 3.1.2 in this paper, other examples are contained in [Mc80] and [D82]). This includes b -symplectic manifolds but also other more degenerate examples as b^k -symplectic manifolds (see [S13]).

In these examples coming from celestial mechanics it is crucial to understand which geometry and dynamics of the integrable system persists under *small* perturbations of the system. In particular to determine whether there are invariant tori and if quasiperiodic trajectories are preserved. Classical KAM theory studies these phenomena in the symplectic zone of the manifold but does not take care of the behaviour at points on the singular set. In this paper, we address this problem by proving a KAM theorem for Liouville tori in the singular set of a b -Poisson manifold².

Organization of this paper: Section 2 is devoted to preliminaries on action-angle coordinates, b -Poisson manifolds and KAM theory. Section 3

²KAM theorems in the Poisson context have been an object of investigation by Fortunati and Wiggins (see [FW14]). Their result applied to the b -case only considers regular points.

concentrates on integrable systems on b -symplectic manifolds and a catalogue of examples of Hamiltonian systems in b -Poisson manifolds. Section 4 contains the statement and the proof for the action-angle theorem for b -Poisson manifolds. Section 5 contains a KAM theorem for b -Poisson manifolds.

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2. PRELIMINARIES

2.1. Action-angle coordinates for Poisson manifolds. We start by recalling the definition of an integrable system on a Poisson manifold.

Definition 1 (Integrable system on a Poisson manifold). *Let (M, Π) be a Poisson manifold of (maximal) rank $2r$ and of dimension N . An s -tuple of functions $F = (f_1, \dots, f_s)$ on M is a **(Liouville) integrable system** on (M, Π) if*

- (1) f_1, \dots, f_s are independent (i.e., their differentials are independent on a dense open subset of M);
- (2) f_1, \dots, f_s are in involution (pairwise);
- (3) $r + s = N$.

Viewed as a map, $F : M \rightarrow \mathbb{R}^s$ is called the **momentum map** of (M, Π, F) .

Consider a point $m \in M$ where the df_i are independent and the rank of Π is $2r$. By the Frobenius theorem, the Hamiltonian vector fields X_{f_1}, \dots, X_{f_s} which span an involutive distribution define an integrable distribution at m . Denote by \mathcal{F}_m the integral manifold of this distribution passing through m . When the r -dimensional manifold \mathcal{F}_m is compact, the action-angle coordinate theorem proved in [LMV11] (Theorem 1.1 in the paper) says the following:

Theorem 2. *Let (M, Π) be a Poisson manifold of dimension N and (maximal) rank $2r$. Suppose that $F = (f_1, \dots, f_s)$ is an integrable system on (M, Π) and assume m and \mathcal{F}_m satisfy the conditions above.*

Then there exist \mathbb{R} -valued smooth functions $(\sigma_1, \dots, \sigma_s)$ and \mathbb{R}/\mathbb{Z} -valued smooth functions $(\theta_1, \dots, \theta_r)$ defined in a neighborhood U of \mathcal{F}_m such that

- (1) *the functions $(\theta_1, \dots, \theta_r, \sigma_1, \dots, \sigma_s)$ define an isomorphism $U \simeq \mathbb{T}^r \times B^s$, where B^s is an open ball in \mathbb{R}^s ;*

(2) the Poisson structure can be written in terms of these coordinates as

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i}.$$

In particular, the functions $\sigma_{r+1}, \dots, \sigma_s$ are Casimirs of Π on U ;

(3) the leaves of the surjective submersion $F = (f_1, \dots, f_s)$ are the level sets of the projection of $\mathbb{T}^r \times B^s$ onto B^s ; in particular, the functions $\sigma_1, \dots, \sigma_s$ depend on the functions f_1, \dots, f_s only.

Remark 3. Note that in the case where (M, Π) is symplectic, i.e. $r = \dim(M)/2 = N/2$ the result above is the classical Liouville-Arnold-Mineur theorem: For a point m as above, the integral manifold \mathcal{F}_m is an r -torus and on a neighborhood around \mathcal{F}_m we have a locally trivial fibration by r -tori, such that the flow of the integrable system is linear on the tori.

2.2. b -Poisson manifolds. A particular class of Poisson manifolds, which are very close to being symplectic, are the so called **b -Poisson manifolds**.

Definition 4. Let (M^{2n}, Π) be a Poisson manifold. If the map

$$p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(TM)$$

is transverse to the zero section, then Π is called a **b -Poisson structure** on M . The pair (M, Π) is called a **b -Poisson manifold**. The vanishing set of Π^n is a hypersurface, which we denote by Z and call the **exceptional hypersurface** of (M, Π) .

Example 5. On the 2-sphere \mathbb{S}^2 with the usual coordinates (h, θ) , the Poisson structure $\Pi = h \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ vanishes transversally along the equator $Z = \{h = 0\}$ and thus it defines a b -Poisson structure on (\mathbb{S}^2, Z) .

Example 6. For higher dimensions we may consider the following product structures: let (\mathbb{S}^2, Z) be the sphere in the example above and (S^{2n}, π_S) be a symplectic manifold, then $(\mathbb{S}^2 \times S, \pi_{\mathbb{S}^2} + \pi_S)$ is a b -Poisson manifold of dimension $2n + 2$. We may replace (\mathbb{S}^2, Z) by a (R, π_R) Radko compact surface (see [GMP12]).

If t is a local defining function³ of Z , then locally around a point of Z a b -Poisson structure has the form

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + t \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial z}.$$

One aim of this paper is to show that for b -Poisson manifolds (M^{2n}, Π) for which the exceptional hypersurface has a trivial normal bundle, we can

³There always exists a global defining function if M is oriented, since we can divide $\bigwedge^n \Pi$ by a global nowhere-vanishing section of $\bigwedge^{2n} TM$.

obtain a stricter version of Theorem 2, for which the compact integral manifolds are n -dimensional Liouville tori, even on the exceptional hypersurface where the rank of Π is only $2n - 2$. In this sense, the result is similar to the symplectic case. Around a Liouville torus lying inside the exceptional hypersurface, there is a neighborhood U with action-angle coordinates $(\theta_1, \dots, \theta_n, \sigma_1, \dots, \sigma_n) : U \rightarrow \mathbb{T}^n \times B^n$ such that

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i} + \frac{1}{c} \sigma_n \frac{\partial}{\partial \theta_n} \wedge \frac{\partial}{\partial \sigma_n}, \tag{1}$$

and the “action coordinates” $\sigma_1, \dots, \sigma_n$ depend only on the integrals of the system. The number $c \in \mathbb{R}$ is the modular period of the connected component of Z in which the Liouville torus lies (see Definition 10 below).

To see why we must assume that the exceptional hypersurface has a non-trivial normal bundle, consider the Möbius band $\mathbb{R}^2 / (x, y) \sim (x + 1, -y)$ with the b -Poisson structure $y \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and the function $f = -\log|y|$. Here, the orbits of $X_f = \frac{\partial}{\partial x}$ do not define a trivial fibration semilocally around the orbit $\{y = 0\}$, so there cannot be action-angle coordinates for this example. For the remainder of this paper, we will assume that the exceptional hypersurface has a trivial normal bundle. That is, it is cut out by a defining function on a tubular neighborhood of Z .

Remark 7. In contrast to the general action-angle coordinate theorem for Poisson manifolds, where we only have action-angle coordinates on the points where Π has full rank, our result specifically concerns the degeneracy locus of Π . Away from the degeneracy locus we can apply the standard action-angle coordinate theorem for symplectic manifolds.

In the next section we will introduce the definitions needed to formulate the result above in a more precise way.

2.3. Basics on b -symplectic geometry. Recall from [GMP12] that a b -manifold (M, Z) is a manifold M together with an embedded hypersurface Z ; a b -vector field is a vector field on M which is tangent to Z , and that the space of b -vector fields are naturally sections of a vector bundle on M called the b -tangent bundle bTM . Sections of the exterior algebra of the b -cotangent bundle ${}^bT^*M := ({}^bTM)^*$ are called b -forms, and there is a natural differential on the space of b -forms which defines a differential complex whose cohomology (called b -cohomology, written ${}^bH^*(M)$) satisfies the following relationship to the cohomology of M and Z ,

Theorem 8 (Mazzeo-Melrose). *The b -cohomology groups of M^{2n} satisfy*

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

Under the Mazzeo-Melrose isomorphism, a b -form of degree p has two parts: its first summand, the *smooth* part, is determined (by Poincaré duality) by integrating the form along any p -dimensional cycle transverse to Z

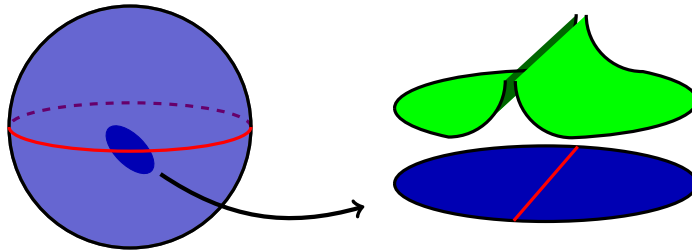


FIGURE 1. A b -function on a b -manifold with a logarithmic singularity near the exceptional hypersurface.

(such an integral is improper due to the singularity along Z , but the principal value of this integral is well-defined). The second summand, the *singular* part, is the residue of the form along Z .

Definition 9. (b -functions) A b -function on a b -manifold (M, Z) is a function which is smooth away from Z , and near Z has the form

$$c \log |t| + g,$$

where $c \in \mathbb{R}, g \in C^\infty$, and t is a local defining function. The sheaf of b -functions is denoted ${}^b\mathcal{C}^\infty$.

In other words, the space of b -functions corresponds to the functions on $M \setminus Z$ whose differential extends to a globally-defined closed b -form on M whose b -cohomology class has zero smooth part.

A closed b -form of degree two which is nondegenerate as a section of $\Lambda^2({}^bT^*M)$ is called a **b -symplectic form**. Dualizing a b -symplectic form gives a b -Poisson bivector, and vice versa; the structures are equivalent ([GMP12]) and we will use the terms b -symplectic manifold and b -Poisson manifold interchangeably. For any b -function on a b -symplectic manifold (M, ω) there is a (smooth) vector field X_{f_n} defined by $\iota_{X_{f_n}} \omega = -df_n$ which we call the **b -Hamiltonian vector field** of f_n . A \mathbb{T}^k action on a b -symplectic manifold (M^{2n}, ω) is called **b -Hamiltonian** if the fundamental vector fields are the b -Hamiltonian vector fields of functions which Poisson commute. Such an action is called **toric** if $k = n$.

The topology of Z . For a given volume form Ω on a Poisson manifold M the associated **modular vector field** u_{mod}^Ω is defined as the following derivation:

$$C^\infty(M) \rightarrow \mathbb{R} : f \mapsto \frac{\mathcal{L}_{X_f} \Omega}{\Omega}.$$

It can be shown (see for instance [We97]) that this is indeed a derivation and, moreover, a Poisson vector field. Furthermore, for different choices of volume form Ω , the resulting vector fields only differ by a Hamiltonian vector field.

The topology of the exceptional hypersurface Z of a b -symplectic structure has been studied in [GMP11] and [GMP12]. In [GMP11] it was shown that

if Z is compact and connected, then it is the mapping torus of any of its symplectic leaves \mathcal{L} by the flow of the any choice of modular vector field u :

$$Z = (\mathcal{L} \times [0, k]) / (x, 0) \sim (\phi(x), k),$$

where k is a certain positive real number and ϕ is the time- k flow of u . In particular, all the symplectic leaves inside Z are symplectomorphic.

In the transverse direction to the symplectic leaves, all the modular vector fields flow with the same speed. This allows the following definition:

Definition 10 (Modular period). *Taking any modular vector field u_{mod}^Ω , the modular period of Z is the number k such that Z is the mapping torus*

$$Z = (\mathcal{L} \times [0, k]) / (x, 0) \sim (\phi(x), k),$$

and the time- t flow of u_{mod}^Ω is translation by t in the $[0, k]$ factor above.

2.4. Basics on KAM theory. The classical KAM theorem – named after Kolmogorov, Arnold and Moser – is a statement about the stability of integrable Hamiltonian systems on symplectic manifolds: Roughly speaking,

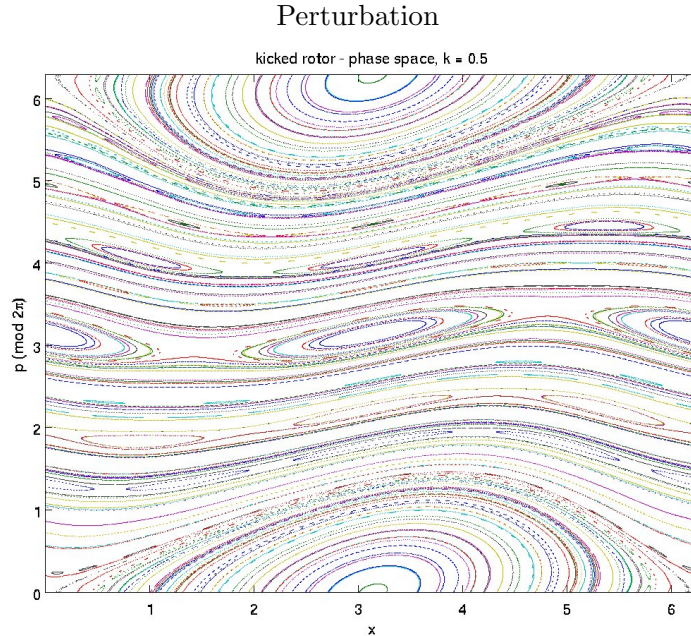


FIGURE 2. The phase space of the kicked rotor: This system is a discrete approximation of the perturbed system arising from the mathematical pendulum (phase space $\mathbb{S}^1 \times \mathbb{R}$) to which a small periodic t -dependent vertical force is added. The variable $x \in \mathbb{S}^1$ is the angle coordinate, the variable $p \in \mathbb{R}$ is the conjugated momentum. The parameter k measures the size of the perturbation.

it implies that “most” Liouville tori of an integrable system persist under sufficiently small perturbations of the Hamiltonian function of the system. In physics, the Hamiltonian function H (or “Hamiltonian” for short) is a function on a symplectic manifold M which determines the evolution of any other function g on M via the equation

$$\dot{g} = \{g, H\}.$$

The pair of a symplectic manifold and a Hamiltonian function is called a **Hamiltonian (dynamical) system**.

An integrable system on a symplectic manifold M together with a Hamiltonian function is called an **integrable Hamiltonian system** if $\{f_i, H\} = 0$ for all integrals f_i . Physically, the last condition corresponds to the fact that the integrals represent constants of motion. From the action-angle coordinate theorem we know that the manifold is semilocally around a Liouville torus symplectomorphic to the product $\mathbb{T}^n \times B^n$ with the standard symplectic structure, with coordinates (φ, y) , and with the integrals $\{y_1, \dots, y_n\}$ being the components of the projection to the B^n component. Restricting to such a neighborhood, we can write their equations of motion explicitly:

$$\begin{aligned} \dot{\varphi} &= \frac{\partial}{\partial y} H(\varphi, y), \\ \dot{y} &= -\frac{\partial}{\partial \varphi} H(\varphi, y). \end{aligned} \tag{2}$$

Here, the expressions in both lines are maps from M to \mathbb{R}^n .

In particular, if we consider the requirement $\{y_i, H\} = 0$ means that H is independent of the angle coordinates φ . Therefore, we can write

$$H(\varphi, y) = h(y)$$

for some function $h : B^n \rightarrow \mathbb{R}$. This system evolves in a very simple way:

$$\begin{aligned} \dot{\varphi} &= \frac{\partial}{\partial y} H(\varphi, y) =: \omega(y) \\ \dot{y} &= 0. \end{aligned} \tag{3}$$

and has solutions of the form

$$y(t) = y_0, \quad \varphi(t) = \varphi_0 + \omega(y_0)t, \tag{4}$$

i.e. the angle coordinates wrap around the torus $\mathbb{T}^n \times \{y_0\}$ in a linear fashion. The components of $\omega(y_0)$ are *rationally dependent* if

$$\omega \cdot k := \sum_{i=1}^n \omega_i k_i = 0 \quad \text{for some } k \in \mathbb{Z}^n \setminus \{0\}.$$

Otherwise, if the components of $\omega(y_0)$ are rationally independent, the motion is called *quasi-periodic* and the resulting trajectory fills the torus densely. In

KAM theory a particular kind of rationally independent frequency vectors are of interest — so-called *strongly non-resonant* or *Diophantine* frequency vectors:

Definition 11 (Diophantine condition). *An n -tuple $\omega \in \mathbb{R}^n$ is called Diophantine if there exist $L, \gamma > 0$ such that*

$$|\omega \cdot k| \geq \frac{L}{|k|^\gamma} \quad \text{for all } k \in \mathbb{Z}^n \setminus \{0\}, \quad (5)$$

where $|k| := \sum_{i=1}^n |k_i|$.

Not only do rationally dependent resp. independent frequency vectors result in very different kinds of motion along the tori, they also behave very differently under *perturbations* of the Hamiltonian system. Indeed, it can be shown that the tori corresponding to rationally dependent frequency vectors – so-called resonant tori – are generically destroyed by arbitrarily small perturbations. In contrast, strongly non-resonant tori survive under sufficiently small perturbations. That is, there is a symplectomorphism on a neighborhood of the torus taking the perturbed trajectory to the linear flow on a torus with unchanged frequency vector. The precise conditions are stated in the following theorem:

Theorem 12 (KAM). *Let $H(\varphi, y) = h(y)$ be an analytic function on $\mathbb{T}^n \times B^n$ with frequency map $\omega(y) := \frac{\partial}{\partial y} h(y)$. If $y_0 \in B^n$ has Diophantine frequency vector $\omega := \omega(y_0)$ and if the non-degeneracy condition holds:*

$$\det \frac{\partial}{\partial y} \omega(y_0) \neq 0, \quad (6)$$

then the torus $\mathbb{T}^n \times \{y_0\}$ persists under sufficiently small perturbations of H . That is, if P is any analytic function on $\mathbb{T}^n \times B^n$ and $\epsilon > 0$ sufficiently small, the perturbed system

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus \mathcal{T} close to $\mathbb{T}^n \times \{y_0\}$.

Moreover, the flow γ^t of the perturbed system on \mathcal{T} is conjugated via a diffeomorphism $\psi : \mathbb{T}^n \rightarrow \mathcal{T}$ to the linear flow with frequency vector ω on \mathbb{T}^n , i.e.

$$\psi^{-1} \circ \gamma^t \circ \psi(\varphi_0) = \varphi_0 + \omega t.$$

The basis is Kolmogorov’s theorem, which we state below and whose proof is the heart of KAM theory. It tells us that we can “correct” a sufficiently small perturbation of a certain type of Hamiltonian via a symplectomorphism close to the identity. The KAM Theorem follows as an easy corollary. Before stating Kolmogorov’s Theorem, we introduce the following

Definition 13 (Kolmogorov normal form). *Let $H : \mathbb{T}^n \times B^n \rightarrow \mathbb{R}$ be an analytic function. We say that H is in **Kolmogorov normal form** (with frequency vector ω) if it is of the form*

$$H(\varphi, y) = E + \omega \cdot y + Q(\varphi, y) \quad (7)$$

for some $E \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ and a function $Q : \mathbb{T}^n \times B^n \rightarrow \mathbb{R}$ which is quadratic in y , meaning that

$$Q(\varphi, 0) = 0, \quad \frac{\partial}{\partial y} Q(\varphi, 0) = 0.$$

The Kolmogorov normal form is called **non-degenerate** if

$$\det \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \partial_y^2 Q(\varphi, 0) d\varphi =: \det \langle \partial_y^2 Q(\cdot, 0) \rangle \neq 0.$$

Theorem 14 (Kolmogorov). *Let $H : \mathbb{T}^n \times B_r^n \rightarrow \mathbb{R}$ be in non-degenerate Kolmogorov normal form (7) with Diophantine frequency vector ω . Consider any (analytic) perturbation $P : \mathbb{T}^n \times B_r^n \rightarrow \mathbb{R}$. Then there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$ there exists a symplectomorphism*

$$\psi : \mathbb{T}^n \times B_{r_*}^n \rightarrow \mathbb{T}^n \times B_r^n$$

for some $0 < r_* < r$ which transforms the perturbed Hamiltonian $H_\epsilon := H + \epsilon P$ into Kolmogorov normal form:

$$(H_\epsilon \circ \psi)(\varphi, y) = E_* + \omega \cdot y + Q_*(\varphi, y)$$

where⁴ $\|\psi - \text{id}\|$, $|E_* - E|$ and $\|Q - Q_*\|$ are of order ϵ . In particular ψ is ϵ -close to the identity, i.e. there exists a constant C such that $\|\psi - \text{id}\| < C\epsilon$.

Remark 15. A concrete choice of ϵ_0 in Theorem 14 can be given, but it is not important for the purpose of this paper.

Remark 16. Note that, in the new Kolmogorov normal form K_ϵ , the constant and quadratic parts E_* resp. Q_* are slightly different from the original E resp. Q , but the frequency vector ω is the same.

In Section 5 we will present a KAM result for the b -symplectic case.

3. INTEGRABLE SYSTEMS ON b -SYMPLECTIC MANIFOLDS

An integrable system on a b -symplectic manifold (M^{2n}, ω) , defined according to the standard definition 1 for Poisson manifolds, will only define a distribution of rank at most $2n - 2$ on the exceptional hypersurface Z . The vector field $\frac{\partial}{\partial \theta_n}$ in the expression (1) is actually not a Hamiltonian vector field of the last angle coordinate σ_n (which is a defining function of Z), but it is a b -Hamiltonian vector field of $\log |\sigma_n|$. To obtain an action-angle coordinate theorem for integrable systems on a b -symplectic manifold as in Equation (1), we allow our integrals of motion to be b -functions.

⁴For functions on $\mathbb{T}^n \times B^n$ the norm $\|\cdot\|$ denotes the supremum norm; for maps $\mathbb{T}^n \times B^n \rightarrow \mathbb{T}^n \times B^n$ we regard the target space as a subset of $\mathbb{R}^{2n} \times \mathbb{R}^n$ and use the supremum norm with respect to the Euclidean norm.

Definition 17 (b -integrable system). A b -integrable system on a $2n$ -dimensional b -symplectic manifold (M^{2n}, ω) is a set of n pairwise Poisson commuting b -functions $F = (f_1, \dots, f_{n-1}, f_n)$ (i.e. $\{f_i, f_j\} = 0$), satisfying, $df_1 \wedge \dots \wedge df_n$ is nonzero as a section of $\wedge^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z . We say that a point in M is **regular** if the vector fields X_{f_1}, \dots, X_{f_n} are linearly independent (as smooth vector fields) at it.

Notice that if a point on Z is regular, then at least one of the f_i must be non-smooth there.

On the set of regular points, the distribution given by X_{f_1}, \dots, X_{f_n} defines a foliation \mathcal{F} . We denote the integral manifold through a regular point $m \in M$ by \mathcal{F}_m . If \mathcal{F}_m is compact, then it is an n -dimensional torus (also referred to as “(standard) **Liouville torus**”). Because the X_{f_i} are b -vector fields and are therefore tangent to Z , any Liouville torus that intersects Z actually lies inside Z . Two (b -)integrable systems F and F' are called **equivalent** if there is a map $\mu : \mathbb{R}^n \supset F(M) \rightarrow \mathbb{R}^n$ taking one system to the other: $F' = \mu \circ F$. We will not distinguish between equivalent integrable systems.

Remark 18. Near a regular point of Z , a b -integrable system on a b -symplectic manifold is equivalent to one of the type $F = (f_1, \dots, f_{n-1}, f_n)$, where f_1, \dots, f_{n-1} are C^∞ functions and f_n is a b -function. In fact, we may always assume that $f_n = c \log |t|$, where $c \in \mathbb{R}$ and t is a global defining function for Z .

3.1. Examples of b -integrable systems on b -Poisson manifolds.

3.1.1. *Integrable systems on manifolds with boundary.* Given a b -symplectic manifold (M, ω) and a component W of $M \setminus Z$, $\omega|_W$ is a classic symplectic form on W . The closure of W is a manifold with boundary, and the asymptotics of $\omega|_W$ near this boundary can be described in the following way, following [NT96]. For each $p \in \partial W$, there is a neighborhood diffeomorphic to $\{(x_1, y_1, \dots, x_n, y_n) \mid x_1 \geq 0\}$ on which $\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i>1} dx_i \wedge dy_i$.

This observation enables us to study manifolds with boundary equipped with a symplectic forms in its interior, as long as these symplectic forms have the kind of asymptotics described above. By taking the double of such a manifold with boundary, the boundary becomes a hypersurface, and the symplectic form extends to a b -symplectic form on the double, and we may then apply the results of this paper.

For example, consider the upper hemisphere (including the equator) H_+ of S^2 . The symplectic form $\frac{1}{h} dh \wedge d\theta$ defined on the interior of H_+ extends to a b -symplectic form on the double of H_+ , which is S^2 . This example can be generalized to $H_+ \times M$, where M is any symplectic manifold, endowed with the product symplectic structure. Moreover, any integrable system on the interior of $H_+ \times M$ which has asymptotics compatible with those of ω near the boundary extends to a b -integrable system on the double $S^2 \times M$. For

example, if (f_1, \dots, f_n) is an integrable system on M , then the integrable system $(\log |h|, f_1, \dots, f_n)$ on $H_+ \times M$ extends to the b -integrable system $(\log |h|, f_1, \dots, f_n)$ on $S^2 \times M$.

3.1.2. Restricted 3-body problem. The restricted 3-body problem⁵ is a simplified version of the general 3-body problem where one of the bodies is assumed to have negligible mass (an “asteroid” or “comet” or “small planet”) and hence the two other bodies, the primaries (“star and big planet”), move independently of it according to Kepler’s laws for the 2-body problem. Here we consider the planar restricted 3-body problem, where the massless body lies in the same plane as the other two. The time-dependent self-potential of the small body is then given by

$$U(q, t) = \frac{1 - \mu}{|q - q_1|} + \frac{\mu}{|q - q_2|},$$

where we assume that the masses of the star and big planet are normalized, $q_1 = q_1(t)$ is the position of the star (“Sun”) with mass $1 - \mu$ at time t and $q_2 = q_2(t)$ the position of the planet (“Jupiter”) with mass μ . Typically it is assumed that the primaries revolve along circles centered in their center of masses (the Circular Restricted Three Body Problem), and the dependence with respect to time of the potential can be removed in a rotating frame of coordinates. In general, the primaries move around their center of mass on elliptic orbits with some non-zero eccentricity (the Elliptic Restricted Three Body Problem).

The Hamiltonian of the system is given by

$$H(q, p, t) = p^2/2 - U(q, t), \quad (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where $p = \dot{q}$ is the momentum of the planet.

Because of the rotational symmetry of the problem, it is natural to introduce polar coordinates (r, α) to describe the problem. For $q = (X, Y) \in \mathbb{R}^2 \setminus \{0\}$ we consider polar coordinates the $(r, \alpha) \in \mathbb{R}^+ \times \mathbb{T}$ ($X = r \cos \alpha, Y = r \sin \alpha$). We transform the momentum variables $p = (P_X, P_Y)$ in such a way that the change of coordinates

$$(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$$

is canonical, i.e. the symplectic structure remains the same.

To study the behaviour at infinity, it is standard to introduce the so-called **McGehee coordinates** (x, α, y, G) , where

$$r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+,$$

which can be then extended to the infinity $x = 0$.

This transformation is non-canonical (and because of this in the classical literature in celestial mechanics the symplectic structure is then totally

⁵Special thanks to Amadeu Delshams for explaining us the Elliptic Restricted Three Body Problem and revealing its connection to our Poisson structures.

abandoned; an exception is [DKRS14]), i.e. the symplectic structure changes and one can check that it is given for $x > 0$ by

$$-\frac{4}{x^3}dx \wedge dy + d\alpha \wedge dG.$$

This extends naturally to a b^3 -symplectic structure in the sense of [S13] on $\mathbb{R} \times \mathbb{T} \times \mathbb{R}^2$. The integrable two body problem for $\mu = 0$ (‘star plus asteroid’) on the initial system of coordinates, where action-angle variables are given by the so-called Delaunay coordinates (which give rise to a degenerate Hamiltonian which only depends essentially on one action), gets transformed to a b^3 -integrable system with the new structure.

4. ACTION-ANGLE VARIABLES FOR b -POISSON MANIFOLDS

The goal of this section is to prove the existence of action-angle variables for b -Poisson manifolds (Theorem 21 below). In the following subsections, let (M, ω, F) be a b -integrable system on the b -symplectic manifold (M, ω) with critical hypersurface Z . We start by establishing a local normal form result for integrable systems.

4.1. Darboux-Carathéodory Theorem. The following result *locally* extends a set of n Poisson commuting functionally independent b -functions to a “ b -Darboux” coordinate system:

Theorem 19. *Let (M^{2n}, ω) be a b -symplectic manifold, m be a point on the exceptional hypersurface Z , and f_1, \dots, f_n be b -functions, defined on a neighborhood of m , which Poisson commute and have the property that $X_{f_1} \wedge \dots \wedge X_{f_n}$ is a nonzero section of $\Lambda^n({}^bTM)$ at m . Then there exist b -functions (q_1, \dots, q_n) around m such that*

$$\omega = \sum_{i=1}^n df_i \wedge dq_i.$$

and the vector fields $\{X_{f_i}, X_{q_j}\}_{i,j}$ commute.

Moreover, if f_n is not a smooth function, i.e. $f_n = c \log |t|$ for some $c \neq 0$ and some local defining function t of Z , then the q_i functions can be chosen to be smooth functions for which $(f_1, \dots, f_{n-1}, t, q_1, \dots, q_n)$ are local coordinates around m .

Proof. For this proof, we will adopt the notation that for a 1-form μ , the vector field X_μ is the vector field satisfying $\iota_{X_\mu} \omega = -\mu$. We begin by inductively constructing a collection $\{\mu_1, \dots, \mu_n\}$ of 1-forms with the property that

$$\begin{aligned} \mu_i(X_{f_j}) &= \delta_i^j \\ \mu_i(X_{\mu_j}) &= 0 \quad \text{for } j < i. \end{aligned}$$

Assume that we have successfully constructed μ_j for $j < i$, and moreover that this construction satisfies

$$0 \neq X_{f_1} \wedge \dots \wedge X_{f_n} \wedge X_{\mu_1} \wedge \dots \wedge X_{\mu_{i-1}}$$

(as a section of $\Lambda^*({}^bTM)$) locally near m , and consider the problem of constructing μ_i . Let P_i and K_i be the subbundles (of bTM and ${}^bT^*M$ respectively) defined by

$$\begin{aligned} P_i &= \text{span}(\{X_{f_j}, X_{\mu_k}\}_{j \neq i, k < i}) \\ K_i &= \ker(P_i) \end{aligned}$$

The bundle K_i is a codimension- $(n + i - 2)$ subbundle of ${}^bT^*M$. By the inductive hypothesis, X_{f_i} is not contained in P_i , so there is a section μ_i of K_i for which $\mu_i(X_{f_i}) = 1$. The fact that P_i and X_{f_i} are in the kernel of df_i , but X_{μ_i} is not, reveals that $X_{\mu_i} \notin \text{span}(P_i \cup X_{f_i})$, so

$$0 \neq X_{f_1} \wedge \dots \wedge X_{f_n} \wedge X_{\mu_1} \wedge \dots \wedge X_{\mu_i},$$

completing the induction.

Because the df_i and μ_i are b -forms, the X_{f_i} and X_{μ_i} are b -vector fields. Using the fact that the f_i functions Poisson commute, and the properties of the μ_i constructed above,

$$\begin{pmatrix} \omega(X_{f_i}, X_{f_j}) & \omega(X_{\mu_i}, X_{f_j}) \\ \omega(X_{f_i}, X_{\mu_j}) & \omega(X_{\mu_i}, X_{\mu_j}) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so $\omega = \sum_{i=1}^n df_i \wedge \mu_i$. To check that the μ_i are closed (and therefore exact) in a neighborhood of m , we apply the Cartan formula for the exterior derivative to calculate that

$$\begin{aligned} d\mu_i(X_{f_j}, X_{f_k}) &= X_{f_j}(\mu_i(X_{f_k})) - X_{f_k}\mu_i(X_{f_j}) - \mu_i([X_{f_j}, X_{f_k}]) \\ &= X_{f_j}\delta_i^k - X_{f_k}\delta_i^j - \mu_i(0) = 0 \\ d\mu_i(X_{f_j}, X_{\mu_k}) &= X_{f_j}\mu_i(X_{\mu_k}) - X_{\mu_k}\mu_i(X_{f_j}) - \mu_i([X_{f_j}, X_{\mu_k}]) \\ &= X_{f_j}(0) - X_{\mu_k}\delta_i^j - \mu_i(0) = 0 \\ d\mu_i(X_{\mu_j}, X_{\mu_k}) &= X_{\mu_j}\mu_i(X_{\mu_k}) - X_{\mu_k}\mu_i(X_{\mu_j}) - \mu_i([X_{\mu_j}, X_{\mu_k}]) \\ &= X_{\mu_j}(0) - X_{\mu_k}(0) - \mu_i(0) = 0. \end{aligned}$$

Letting q_i be any local primitive of μ_i yields the first part of the result.

In the case when $f_n = c \log |t|$ is non-smooth, we can modify our inductive construction of the μ_i so that in addition to requiring that the μ_i be in K_i , we also insist that they be in $T^*M \subseteq {}^bT^*M$. To check that this restriction is consistent with the requirement that $\mu_i(X_{f_i}) = 1$, we must check that the kernel of X_{f_i} is not identically equal to T^*M , i.e. X_{f_i} does not vanish at m when viewed as a section of TM . But this is clear from the fact that X_{f_i} does not vanish at m when viewed as a section of bTM , and

$$0 = \{f_n, f_i\} = \frac{cdt}{t}(X_{f_i}).$$

This proves that the q_i can be chosen to be smooth functions. The fact that $\{X_{f_1}, \dots, X_{f_{n-1}}, X_t, X_{q_1}, \dots, X_{q_n}\}$ pairwise commute follows from the fact that $\{X_{f_i}, X_{q_j}\}_{i,j}$ do, so $(f_1, \dots, f_{n-1}, t, q_1, \dots, q_n)$ indeed are local coordinates.

□

4.2. Foliation by Liouville tori. Suppose m is a regular point of a b -integrable system and the Liouville torus is contained in the exceptional hypersurface. The following theorem shows that the b -integrable system is isomorphic to one of the form

$$(\mathbb{T}^n \times B^n, \omega, F := (f_1, \dots, f_{n-1}, f_n = \log |t|)), \quad (8)$$

Here, “isomorphic” means that there is a b -symplectomorphism of a neighborhood of the torus with the b -manifold above that takes the integrals of motion to F .

Proposition 20. *Let $m \in Z$ be a regular point of a b -integrable system (M, ω, F) . Assume that the integral manifold \mathcal{F}_m through m is compact (i.e. a torus \mathbb{T}^n). Then there exists a neighborhood U of \mathcal{F}_m and a diffeomorphism*

$$\phi : U \simeq \mathbb{T}^n \times B^n,$$

which takes the foliation \mathcal{F} to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.

Proof. The proof is the same as in [LMV11] (Prop. 3.2), the only difference being that the last integral f_n in our system $F = (f_1, \dots, f_n)$ is a b -function, $f_n = \log |t|$ for some t . However, the foliation given by the (b -)Hamiltonian vector fields of F is the same as the one given by the level sets of the smooth function $\tilde{F} := (f_1, \dots, f_{n-1}, t)$. Then, as in [LMV11], choosing an arbitrary Riemannian metric on M defines a canonical projection $\psi : U \rightarrow \mathcal{F}_m$. Setting $\phi := \psi \times \tilde{F}$ we have a commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{T}^n \times B^n \\ & \searrow \tilde{F} & \downarrow p \\ & & B^n \end{array} \quad (9)$$

which provides the necessary isomorphism of b -integrable systems. □

4.3. The statement and the proof of the action-angle theorem.

Theorem 21. *Let $(M, \omega, F = (f_1, \dots, f_{n-1}, f_n = \log |t|))$ be a b -integrable system, and let $m \in Z$ be a regular point for which the integral manifold containing m is compact, i.e. a Liouville torus \mathcal{F}_m . Then there exists an open*

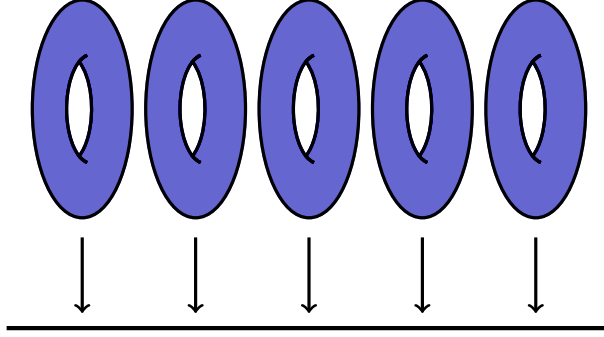


FIGURE 3. Fibration by Liouville tori

neighborhood U of the torus \mathcal{F}_m and coordinates $(\theta_1, \dots, \theta_n, \sigma_1, \dots, \sigma_n) : U \rightarrow \mathbb{T}^n \times B^n$ such that

$$\omega|_U = \sum_{i=1}^{n-1} d\sigma_i \wedge d\theta_i + \frac{c}{\sigma_n} d\sigma_n \wedge d\theta_n, \quad (10)$$

where the coordinates $\sigma_1, \dots, \sigma_n$ depend only on F and the number c is the modular period of the component of Z containing m .

By Proposition (20), we may assume our b -integrable system is in the form in equation (8), where f_1, \dots, f_{n-1} are smooth. Although the vector fields X_{f_1}, \dots, X_{f_n} define a torus action on each of the Liouville tori, $\mathbb{T}^n \times \{b\}_{b \in B^n}$, individually, it is not guaranteed that their flow defines a torus action on all of $\mathbb{T}^n \times B^n$. Our next goal is to construct an equivalent b -integrable system whose fundamental vector fields *do* define a \mathbb{T}^n action on a neighborhood of $\mathbb{T}^n \times \{0\}$.

Step 1 – uniformization of periods (constructing a \mathbb{T}^n action). Denote by $\Phi_{X_f}^s$ the time- s flow of the (b -)Hamiltonian vector fields X_f and consider the combined flow of the (b -)Hamiltonian vector fields X_{f_1}, \dots, X_{f_n} ,

$$\begin{aligned} \Phi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) &\rightarrow (\mathbb{T}^n \times B^n) \\ ((s_1, \dots, s_n), (x, b)) &\mapsto \Phi_{X_{f_1}}^{s_1} \circ \dots \circ \Phi_{X_{f_n}}^{s_n} ((x, b)). \end{aligned}$$

Because the X_{f_i} are complete and commute with one another, this defines an \mathbb{R}^n -action on $\mathbb{T}^n \times B^n$. When restricted to a single orbit $\mathbb{T}^n \times \{b\}$ for some $b \in B^n$, the kernel of this action is a discrete subgroup of \mathbb{R}^n , hence a lattice Λ_b , called the **period lattice** of the orbit $\mathbb{T}^n \times \{b\}$. Since the orbit is compact, the rank of Λ_b is n .

The lattice Λ_b will in general depend on b . The process of *uniformization* entails modifying the action so that $\Lambda_b = \mathbb{Z}^n$ for all b . For any $b \in B^{n-1} \times \{0\}$ and any $a_i \in \mathbb{R}$, the vector field $\sum a_i X_{f_i}$ on $\mathbb{T}^n \times \{b\}$ is the b -hamiltonian vector field of $a_n \log |t| + \sum_{i=1}^{n-1} a_i f_i$, where f_i are smooth for $i < n$. However, by Proposition 4 of [GMPS13], if such a vector field is 1-periodic, then

$a_n = \pm c$, where c is the modular period of the component of Z containing m . Therefore, for all $b \in B^{n-1} \times \{0\}$, the lattice Λ_b is contained in $\mathbb{R}^{n-1} \times c\mathbb{Z} \subseteq \mathbb{R}^n$. To perform the uniformization, pick smooth functions

$$(\lambda_1, \lambda_2, \dots, \lambda_n) : B^n \rightarrow \mathbb{R}^n$$

such that

- $(\lambda_1(b), \lambda_2(b), \dots, \lambda_n(b))$ is a basis for the period lattice Λ_b for all $b \in B^n$
- λ_i^n vanishes along $B^{n-1} \times \{0\}$ for $i < n$, and λ_i^n equals the modular period c along $B^{n-1} \times \{0\}$. Here, λ_i^j denotes the j^{th} component of λ_i .

Such functions λ_i exist that satisfy the first condition (perhaps after shrinking B^n) by the implicit function theorem, using the fact that the Jacobian of the equation $\Phi(\lambda, m) = m$ is regular with respect to the s variables. The fact that they can be chosen to satisfy the second condition is due to the discussion above.

Using these functions λ_i we define the “uniformized” flow

$$\begin{aligned} \tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) &\rightarrow (\mathbb{T}^n \times B^n) \\ ((s_1, \dots, s_n), (x, b)) &\mapsto \Phi\left(\sum_{i=1}^n s_i \lambda_i(c), (x, b)\right). \end{aligned}$$

The period lattice of this \mathbb{R}^n action is constant now (namely \mathbb{Z}^n) and hence the action naturally defines a \mathbb{T}^n action.

Step 2 – the \mathbb{T}^n action is b -Hamiltonian. We want to find b -functions $\sigma_1, \dots, \sigma_n$ such that the b -hamiltonian vector fields X_{σ_i} are precisely the fundamental vector fields of the \mathbb{T}^n action we constructed above, $Y_i = \sum_{j=1}^n \lambda_i^j X_{f_j}$.

The Cartan formula for b -symplectic forms (where the differential is the one of the complex of b -forms [GMP12]⁶) gives

$$\mathcal{L}_{Y_i} \mathcal{L}_{Y_i} \omega = \mathcal{L}_{Y_i} (d(\iota_{Y_i} \omega) + \iota_{Y_i} d\omega) \quad (11)$$

$$= \mathcal{L}_{Y_i} \left(d\left(-\sum_{j=1}^n \lambda_i^j df_j\right) \right) \quad (12)$$

$$= -\mathcal{L}_{Y_i} \left(\sum_{j=1}^n d\lambda_i^j \wedge df_j \right) = 0 \quad (13)$$

where in the last equality we used the fact that λ_i^j are constant on the level sets of F . Recall from [LMV11] (see Claim 2 in page 1856) that if Y is any complete periodic vector field, and P is a bivector field for which $\mathcal{L}_Y \mathcal{L}_Y P = 0$, then $\mathcal{L}_Y P = 0$. This proves that $\mathcal{L}_{Y_i} \omega = 0$, so the vector fields Y_i are Poisson vector fields.

⁶The decomposition of a k -form in the b -complex as $\omega = \frac{dt}{t} \wedge \alpha + \beta$ for α, β De Rham forms proved in [GMP12] allows to extend the Cartan formula to the setting of b -forms

To show that each $\iota_{Y_i}\omega$ has a ${}^bC^\infty$ primitive, it suffices to show that the smooth part of $[\iota_{Y_i}\omega]$, i.e. the first summand in its image under the Mazzeo-Melrose isomorphism (see Theorem 8) ${}^bH^1(\mathbb{T}^n \times B^n) \cong H^1(\mathbb{T}^n \times B^n) \oplus H^0(\mathbb{T}^n \times B^n)$, is zero. But this follows from the fact that the value of the smooth part $H^1(\mathbb{T}^n \times B^n)$ of $\iota_{Y_i}\omega$ is determined by integrating it along loops which are tangent to the fibers. Since the kernel of $\iota_{Y_i}\omega$ contains the tangent space to the torus fibers, these integrals are zero.

Hence the Y_i are b -Hamiltonian and we denote their Hamiltonian functions by σ_i . Because λ_i^n vanish along $B^{n-1} \times \{0\}$ for $i < n$, the forms $\iota_{Y_i}\omega = -\sum_{j=1}^n \lambda_i^j df_j$ are smooth for $i < n$, so σ_i are smooth for $i < n$. Because λ_n^n equals c along $B^{n-1} \times \{0\}$, σ_n has the form $c \log |t|$ for some defining function t .

Step 3 – apply the Darboux-Carathéodory theorem. The construction above gives us candidates $\sigma_1, \dots, \sigma_n = c \log |t|$ for the the “action coordinates.” By the Darboux-Carathéodory theorem we can *locally* find a coordinate system

$$(\sigma_1, \dots, \sigma_{n-1}, t, q_1, \dots, q_n)$$

such that

$$\omega = \sum_{i=1}^{n-1} d\sigma_i \wedge dq_i + \frac{c}{t} dt \wedge dq_n. \quad (14)$$

Since the vector fields $X_{\sigma_i} = \frac{\partial}{\partial q_i}$ ($i = 1, \dots, n$) are the fundamental vector fields of the \mathbb{T}^n -action, in the local chart introduced above the flow of the vector fields gives a linear action on the q_i coordinates.

Therefore, if the functions $\sigma_1, \dots, \sigma_{n-1}, t$ were initially defined on an open set $U \subset M$, we can extend them to the whole set $U' := p^{-1}(p(U))$ (i.e. the union of all tori that intersect non-trivially with U). We denote the extensions of these functions by the same symbols.

The vector fields $\frac{\partial}{\partial q_i}$ ($i = 1, \dots, n$) have period 1, so we can view q_i ($i = 1, \dots, n$) as S^1 valued coordinates. We denote them by the “angle” variable θ_i for this reason.

It remains to check that the extended functions $(\sigma_1, \dots, \sigma_{n-1}, t, \theta_1, \dots, \theta_n)$ define a coordinate system on U' and that ω still has the form

$$\omega = \sum_{i=1}^{n-1} d\sigma_i \wedge d\theta_i + \frac{c}{t} dt \wedge d\theta_n. \quad (15)$$

It is clear that $\{\sigma_i, \theta_j\} = \delta_{ij}$ on U' . To show that $\{\theta_i, \theta_j\} = 0$, we note that this relation holds on U and flowing with the vector fields X_{σ_k} we see that it holds on the whole set U' :

$$X_{\sigma_k}(\{\theta_i, \theta_j\}) = \{\{\theta_i, \theta_j\}, \sigma_k\} = \{\theta_i, \delta_{ij}\} - \{\theta_j, \delta_{ik}\} = 0.$$

This verifies that ω has the form (15) above and in particular, we conclude that the derivatives of the functions $\sigma_1, \theta_1, \dots, \sigma_{n-1}, \theta_{n-1}, t, \theta_n$ are independent on U , hence these functions define a coordinate system. This completes the proof of the action-angle coordinate theorem.

5. KAM THEOREM FOR b -INTEGRABLE SYSTEMS

We will prove a stability result of KAM type for Hamiltonian systems on b -Poisson manifolds, allowing Hamiltonian functions which are b -functions. For a given Hamiltonian function $H \in {}^b C^\infty(M)$ the Hamiltonian equations are completely analogous to the symplectic case: Any function $g \in C^\infty(M)$ evolves according to the equation $\dot{g} = \{g, H\} = X_H(g)$.

Assume that we are given a b -integrable system on M . From the action-angle coordinate theorem for b -symplectic manifolds, Theorem 21, it follows that we can semilocally (around a Liouville torus) replace the given b -integrable system by the functions y_1, \dots, y_n on $\mathbb{T}^n \times B^n$, where the y_i are the projections to the i -th component of B^n and the b -symplectic structure is

$$\frac{c}{y_1} d\varphi_1 \wedge dy_1 + \sum_{i=2}^{n-1} d\varphi_i \wedge dy_i.$$

We want to study the stability of quasi-periodic motion inside the exceptional hypersurface $Z = \{y_1 = 0\}$ of $(\mathbb{T}^n \times B^n, \omega)$. As in the symplectic case, we consider a Hamiltonian “compatible” with the integrable system in the sense that it only depends on the action coordinates y ,

$$H(\varphi, y) = k \log |y_1| + h(y). \quad (16)$$

The resulting equations of motion are

$$\begin{aligned} \dot{\varphi}_1 &= \frac{k}{c} + \frac{y_1}{c} \frac{\partial h}{\partial y_1}(y) \\ \dot{\varphi}_i &= \frac{\partial h}{\partial y_i}(y) \quad i = 2, \dots, n \\ \dot{y}_1 &= 0 \\ \dot{y}_i &= 0 \quad i = 2, \dots, n. \end{aligned} \quad (17)$$

In terms of notation, if $x \in \mathbb{R}^n$, we will write \tilde{x} for the \mathbb{R}^{n-1} vector obtained by omitting the first component $\tilde{x} := (x_2, \dots, x_n)$. We see that on Z the angle coordinates $(\varphi_1, \dots, \varphi_n)$ evolve with frequency

$$\left(\frac{k}{c}, \frac{\partial h}{\partial y_2}(y), \dots, \frac{\partial h}{\partial y_n}(y) \right) =: \left(\frac{k}{c}, \tilde{\omega} \right)$$

on an n -torus $\mathbb{T}^n \times \{\text{const}\} \subset Z$.

We want to study the effect of adding a perturbation $\epsilon P \in {}^b C^\infty(M)$. Writing

$$P(\varphi, y) = k' \log |y_1| + \epsilon f(\varphi, y) \quad (18)$$

the equations of motion of the perturbed system are:

$$\begin{aligned}
\dot{\varphi}_1 &= \frac{k + \epsilon k'}{c} + \frac{y_1}{c} \frac{\partial}{\partial y_1} (h(y) + \epsilon f(\varphi, y)) \\
\dot{\varphi}_i &= \frac{\partial}{\partial y_i} (g(y) + f(\varphi, y)) = \tilde{\omega}_i(y) + \epsilon \frac{\partial f}{\partial y_i}(\varphi, y), \quad i = 2, \dots, n \\
\dot{y}_1 &= -\epsilon \frac{y_1}{c} \frac{\partial f}{\partial \varphi_1}(\varphi, y) \\
\dot{y}_i &= -\epsilon \frac{\partial f}{\partial \varphi_i}(\varphi, y).
\end{aligned} \tag{19}$$

Notice that the hypersurface $Z = \{y_1 = 0\}$ is preserved by the perturbation. In the following discussion of stability of the orbits we restrict ourselves to the case where the motion starts inside Z (and necessarily remains there); the other case is the classical KAM theorem for symplectic manifolds.

We want to consider the case where the function f in the expression (18) for the perturbation P has the form

$$f(\varphi, y) = f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1), \tag{20}$$

where f_1 is an analytic function and f_2, f_3 are smooth functions. In particular, this case occurs if f does not depend on φ_1 .

Theorem 22 (KAM Theorem for b -Poisson manifolds). *Let $\mathbb{T}^n \times B_r^n$ be endowed with the standard b -symplectic structure. Consider a b -function $H = k \log |y_1| + h(y)$ on this manifold, where h is analytic. Let y_0 be a point in B_r^n with first component equal to zero, so that the corresponding level set $\mathbb{T}^n \times \{y_0\}$ lies inside the exceptional hypersurface Z .*

Assume that the frequency map $\tilde{\omega}(y) := \frac{\partial h}{\partial y}(y)$ has a Diophantine value $\tilde{\omega} := \tilde{\omega}(y_0)$ at $y_0 \in B^n$ and that it is non-degenerate at y_0 in the sense that the Jacobian $\frac{\partial \tilde{\omega}}{\partial y}(y_0)$ is regular.

Then the torus $\mathbb{T}^n \times \{y_0\}$ persists under sufficiently small perturbations of H which have the form mentioned above, i.e. they are given by ϵP , where $\epsilon \in \mathbb{R}$ and $P \in {}^b C^\infty(\mathbb{T}^n \times B_r^n)$ has the form

$$\begin{aligned}
P(\varphi, y) &= \log |y_1| + f(\varphi, y) \\
f(\varphi, y) &= f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1).
\end{aligned}$$

More precisely, if $|\epsilon|$ is sufficiently small, then the perturbed system

$$H_\epsilon = H + \epsilon P$$

admits an invariant torus \mathcal{T} .

Moreover, there exists a diffeomorphism $\mathbb{T}^n \rightarrow \mathcal{T}$ close⁷ to the identity taking the flow γ^t of the perturbed system on \mathcal{T} to the linear flow on \mathbb{T}^n with frequency vector $(\frac{k+\epsilon k'}{c}, \tilde{\omega})$.

⁷By saying that the diffeomorphism is “ ϵ -close to the identity” we mean that, for given H, P and r , there is a constant C such that $\|\psi - \text{id}\| < C\epsilon$.

Proof. First assume that $y_0 = 0$. We will prove the general case later on. We consider the restrictions of h and f_1 to Z as functions on B_r^{n-1} resp. $\mathbb{T}^{n-1} \times B_r^{n-1}$:

$$\bar{h}(\tilde{y}) := h(0, \tilde{y}), \quad \bar{f}_1(\tilde{\varphi}, \tilde{y}) := f_1(\tilde{\varphi}, 0, \tilde{y}).$$

By the Kolmogorov theorem for symplectic manifolds, Theorem 14, there exists a constant $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ there is a symplectomorphism $\bar{\psi} : \mathbb{T}^{n-1} \times B_{r_*}^{n-1} \rightarrow \mathbb{T}^{n-1} \times B_r^{n-1}$ such that

$$(\bar{h} + \epsilon \bar{f}_1) \circ \bar{\psi} = h_*$$

is a function in Kolmogorov normal form on $\mathbb{T}^{n-1} \times B_{r_*}^{n-1}$ with frequency vector $\tilde{\omega}$. Denoting $\bar{\psi}(\tilde{\varphi}, \tilde{y}) =: (\tilde{\varphi}', \tilde{y}')$ we define a Poisson diffeomorphism

$$\psi : \mathbb{T}^n \times B_{r_*}^n \rightarrow \mathbb{T}^n \times B_r^n, \quad \psi(\varphi, y) := (\varphi_1, \tilde{\varphi}', y_1, \tilde{y}').$$

Then for $(\varphi, y) \in Z \subset \mathbb{T}^n \times B_{r_*}^n$:

$$\begin{aligned} (H + \epsilon P) \circ \psi(\varphi, y) &= (H + \epsilon P)(\varphi_1, \tilde{\varphi}', y_1, \tilde{y}') = \\ &= (k + \epsilon k') \log |y_1| + \epsilon y_1 f_2(\varphi_1, \tilde{\varphi}', y_1, \tilde{y}') + \epsilon f_3(\varphi_1, y_1) + \\ &\quad + h(y_1, \tilde{y}') + \epsilon f_1(\tilde{\varphi}', y_1, \tilde{y}'). \end{aligned}$$

By the argument above, on Z (i.e. $y_1 = 0$) the two last terms can be written as

$$h(y_1, \tilde{y}') + \epsilon f_1(\tilde{\varphi}', y_1, \tilde{y}') = \bar{h}(\tilde{y}') + \epsilon \bar{f}_1(\tilde{\varphi}', \tilde{y}') = h_*(\tilde{\varphi}, \tilde{y}).$$

Keeping in mind that h_* is of Kolmogorov normal form with frequency vector $\tilde{\omega}$,

$$h_*(\tilde{\varphi}, \tilde{y}) = E_* + \tilde{\omega} \times \tilde{y} + Q_*(\tilde{\varphi}, \tilde{y}),$$

where $E_* \in \mathbb{R}$, $Q_* = \mathcal{O}(|y|^2)$, we see by looking at the equations of motion (19) that the trajectories of $H_* := (H + \epsilon P) \circ \psi$ on Z are precisely given by quasi-periodic motion with frequency $(\frac{k + \epsilon k'}{c}, \tilde{\omega})$ on the tori $\mathbb{T}^n \times \{m\}$ for $m \in B_{r_*}$.

The flow of the Hamiltonian vector field associated to $H + \epsilon P$ is conjugated under ψ to the flow of the Hamiltonian vector field associated to H_* :

$$\gamma_{X_{H + \epsilon P}}^t = \psi \circ \gamma_{X_{H_*}}^t \circ \psi^{-1}.$$

Since the flow $\gamma_{X_{H_*}}^t$ leaves the torus $\mathbb{T}^n \times \{0\}$ invariant, the flow of the perturbed system $\gamma_{X_{H + \epsilon P}}^t$ leaves $\mathcal{T} := \psi(\mathbb{T}^n \times \{0\}) \cong \mathbb{T}^n$ invariant.

In conclusion, the motion induced by $H + \epsilon P$ on \mathcal{T} is conjugated via ψ to the quasi-periodic motion on $\mathbb{T}^n \times \{0\}$ with frequency $(\frac{k + \epsilon k'}{c}, \tilde{\omega})$.

Since the diffeomorphism $\bar{\psi}$ obtained from the Kolmogorov theorem for symplectic manifolds is ϵ -close to the identity, the transformation ψ we construct is also ϵ -close to the identity.

Now consider the case where $y_0 \neq 0$. Let τ be the translation which takes y_0 to 0. Note that τ only changes the last $n - 1$ components since we assume

that the first component of y_0 is already 0, so in particular τ is a Poisson diffeomorphism

$$\mathbb{T}^n \times B_{r'}(y_0) \rightarrow \mathbb{T}^n \times B_{r'}(0),$$

where $B_{r'}(y_0)$ is a ball around y_0 of some radius $r' > 0$ contained in the original ball B_r^n and we endow the sets with the b -symplectic structure inherited from $\mathbb{T}^n \times B_r^n$. Now we apply the argument above (case $y_0 = 0$) to the Hamiltonian $H \circ \tau$ and the perturbation $P \circ \tau$. Denote the diffeomorphism obtained there by ψ_0 ,

$$\psi_0 : \mathbb{T}^n \times B_{r'_*}^n \rightarrow \mathbb{T}^n \times B_{r'_*}^n.$$

Setting $\psi := \tau \circ \psi_0 \circ \tau^{-1}$ we obtain a Poisson diffeomorphism $\mathbb{T}^n \times B_{r'_*}(y_0) \rightarrow \mathbb{T}^n \times B_{r'_*}^n(y_0)$ which is ϵ -close to the identity and conjugates the motion on $\mathcal{T} := \psi(\mathbb{T}^n \times \{y_0\})$ to quasi-periodic motion on $\mathbb{T}^n \times \{y_0\}$ with frequency $(\frac{k+\epsilon k'}{c}, \tilde{\omega})$. \square

Remark 23. Note that the first component of the frequency vector changes; only the last $n - 1$ components of ω are preserved. Moreover, since we only assume the Diophantine condition for the last $n - 1$ components, the orbit through p_0 might not fill the whole torus $\mathbb{T}^n \times \{y_0\}$ densely. However, even in these cases the torus $\mathbb{T}^n \times \{y_0\}$ is invariant.

Remark 24. A special case is where the functions H and P are smooth, i.e. the log-component is zero:

$$H = h \in C^\infty(B_r^n), \quad P = f \in C^\infty(\mathbb{T}^n \times B_r^n).$$

In this case we do not have to make the assumption that f has the form given in Equation (20) to obtain stability of the orbits inside Z . From the equations of motion it is clear that the trajectory starting at a point inside a symplectic leaf $\mathcal{L} \subset Z$ will stay inside the leaf. This is true also after adding the perturbation. Hence the stability of the orbit follows directly from the symplectic KAM theorem: If H is in b -Kolmogorov normal form with vanishing log component (i.e. a C^∞ function) and with Diophantine frequency vector $\tilde{\omega}$ and if P is any C^∞ perturbation, then there is a symplectomorphism on a neighborhood of the orbit *inside* \mathcal{L} which is close to the identity and takes the perturbed orbit to a nearby $n - 1$ torus $\{\varphi_1\} \times \mathbb{T}^{n-1} \times \{y\}$. The perturbed motion is conjugated to linear motion in the $\tilde{\varphi} := (\varphi_2, \dots, \varphi_n)$ coordinates with frequency $\tilde{\omega}$. Note that we only transform inside the leaf \mathcal{L} here – for showing stability this is sufficient.

Remark 25. In view of example 3.1.1, this KAM theorem can be useful to study perturbations of integrable systems on manifolds with boundary.

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