Nodes of directed graphs ranked by solutions defined on cooperative games

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Abstract

Hierarchical structures, transportation systems, communication networks and even sports competitions can be modeled by means of directed graphs. Since digraphs without a predefined game are considered, the main part of the work is devoted to establish conditions on cooperative games so that they can be used to measure accessibility to the nodes. Games that satisfy desirable properties are called test games. Each ranking on the nodes is then obtained according to a pair formed by a test game and a solution defined on cooperative games whose utilities are given for every ordered coalition. Solutions here proposed are extensions of the wide family of semivalues to games in generalized characteristic function form.

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1. Introduction

Cooperative games with transferable utility have been proved to be appropriate mathematical tools to model many situations coming from different areas of science like economy, social sciences or political sciences. Very close to cooperative games there must always be their solutions because, by means of these solutions, the position of each agent in the scenario described by the game is highlighted. The information needed in any concrete situation to define the corresponding cooperative game is the utility that each group of agents or coalition is able to obtain. All these data are gathered in the *characteristic function*.

The Shapley [19] and the Banzhaf [3, 16] values have a prominent position among the set of solutions for cooperative games. Both of them offer to every player a unique numerical assignment that allow to draw its importance in the game, and in both of them the *marginal contribution* of the players, that is, the difference between the utility that a coalition can obtain with or without a fixed player, are essential to determine the assignment. Nevertheless, there is an outstanding difference between them. The Shapley value is an efficient solution, in the sense that the sum of all the players' assignments coincides with the utility that the grand coalition can get, whereas the Banzhaf value is not.

In an economical context, or even better, in a monetary context, the efficiency is a desirable property that enables a convenient distribution of the total utility, and it gives to the solutions an intuitive idea easy to understand. But in other circumstances, where efficiency is not essential, other non-efficient solution concepts can be considered. In a broad sense, we can think in solutions that assign to each player an expected value of their marginal contributions to the coalitions, according to probability distributions that give a specific weight depending on the cardinality of each coalition. These solutions form a wide family known as semivalues [6], and it contains the Banzhaf and Shapley values, being the latter the only efficient one.

The information gathered by the characteristic function of a classical cooperative game is the utility of all the subsets of players without considering the order in which the coalition is formed. If the situation you want to describe depends on the order of presentation of its agents, the concept of characteristic function must be extended to ordered coalitions. In the literature, games defined on ordered coalitions are called *games in generalized characteristic function form*. In turn, solutions for this class of games must be considered, as Novak and Radzik [15] or Sánchez and Bergantiños [18] did, modifying in a convenient way the Shapley value. Here, following a parallel process, we consider the extension of semivalues to the set of cooperative games defined over ordered coalitions.

All the above mentioned concepts of Game Theory are the ones that we will use to achieve the main purpose of this work. We want to establish mechanisms to obtain measures of the accessibility of each node in oriented networks modelled by directed graphs or, simply, digraphs. The interest in giving a rank among the digraphs' nodes comes from the great amount of structures that digraphs allow to describe: hierarchy structures, comparison between pairs, sports competitions, strategic assessment of nodes in communication or transport networks, etc. There are several works in the literature that have studied this subject. In general, the results they give depend on the fixed criteria they use to obtain the solution. An important group of solutions are based in iterative methods, following Wei [20] or Kendall [11]. These kind of methods were intensively analyzed by Laslier [14] for the particular case of *tournaments*, that is, digraphs where for each pair of nodes there is only one of the two possible oriented edges between them. More recently, Hering et al. [10] provided a procedure to measure the *power* of the nodes in a digraph using a mixture between the iterative and the axiomatic methods.

Concepts and methods of Game Theory have been also used to establish a rank on the nodes of a digraph. Among others, we can remind the papers of Laffond et al. [13], where a symmetric game of zero sum is associated to situations described as tournaments, Gilles et al. [7] or Derks and Gilles [5] who consider games modified by structures of hierarchy which are modelled as digraphs. In general, in these types of works, a game is defined from the oriented structure, or the given game is modified taking into account this structure. Our approach in this paper is different: without giving an axiomatic and inflexible system of properties, we want to determine the features that a cooperative game should satisfy in order to offer an acceptable ordering of the nodes of a digraph with respect their accessibility, without any *a priori* game defined on the nodes of the oriented network. The desirable features give rise to properties of the games and these are obtained taking into account the marginal contributions of each node to the set of ordered coalitions.

Hence, cooperative games in generalized characteristic function form are an essential tool in our analysis.

Near cooperative games there are always their solution concepts. Desirable properties to measure the accessibility of nodes lead to extend semivalues to games in generalized characteristic function form. We will determine the accessibility of nodes selecting a pair given by a game satisfying the required assumptions and a semivalue chosen following some criteria about its coefficients, which are interpreted as a probability distribution among the coalitions. We call *test games* the cooperative games that could be selected to measure the accessibility. This terminology was already considered in Amer et al. [2] to give a measure of the nodes in a digraph using the Shapley value. This work generalizes the previous case and, also, widens the class of test games.

The allocations to nodes have been determined in two stages: (i) dealing with the marginal contributions of each node, which depends on the digraph geometry and on the utilities given by the proposed game and (ii) considering the weighting coefficients for the ordered coalitions, which come from the chosen semivalue. In a digraph with n nodes, stage (i) reduces to obtain marginal contributions over $n-1$ oriented paths (the elementary digraphs). It is there where the properties that a game should fulfill to be considered a test game are established. Stage (ii), strongly bond with probability distributions, allows constructing semivalues in such a way that the weights over ordered coalitions correspond to desired criteria.

Every pair of test game and semivalue provides a measure of the accessibility giving an allocations vector over the nodes. In general, this allocation is not efficient. In fact, the grand coalition's utility does not have any special meaning in the digraph, taking into account that the game is external to it. In order to make comparison among allocations, in each example we use percentages vectors, in a similar way as van der Laan and van den Brink [12] considered the so-called *share functions*: the allocation of each player is divided by the sum of all the allocations (we multiply them by one hundred).

According to these considerations, the paper is organized as follows. In Section 2, basic concepts about digraphs, about cooperative games, cooperative games defined over ordered coalitions and their solutions are described. Section 3 is devoted to extending semivalues to games in characteristic function form. The definition of accessibility of nodes and some first properties are also given. In Section 4 the accessibility is determined over oriented paths, giving expressions to calculate it explicitly. Section 5 relates the concept of accessibility over oriented paths with a type of cooperative games, in such a way that the characteristics of one game to be test game are stated. In Section 6, the semivalue that gives appropriate weightings over ordered coalitions is chosen. A detailed example is presented in Section 7. Finally, in Section 8, a summary of the process is presented.

2. Cooperative games modified by directed graphs

A *digraph* is a pair (N, D) where N is a finite set of *nodes* and D is a binary relation defined on N. Each pair $(i, j) \in N \times N$ corresponds to an oriented edge that links node i to node j. Since we consider digraphs without loops, the *complete digraph* is (N, D_N)

with $D_N = N \times N \setminus \{(i,i) \,/\, i \in N\}$. Fixed N, we identify each digraph (N,D) with the binary relation D. In this way, all digraphs on N are the subsets $D \subseteq D_N$.

A *cooperative game* with transferable utility or TU game is a pair (N,v), where N is a finite set of *players* and $v : 2^N \to \mathbb{R}$ is the so-called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *worth* of coalition S, and satisfies the natural condition $v(\emptyset) = 0$.

By G_N we denote the set of all TU games on N. For a given set of players N, we identify each game (N,v) with its characteristic function v. A TU game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$; a TU game v is called *symmetric* if the utilities only depend on the coaliton size, i.e., $v(S) = f(s)$, where $s = |S|$, and a TU game v is *zero-normalized* when $v(\{i\}) = 0$ for all $i \in N$.

When the utilities also depend on the orders of the players within each coalition, we need to consider TU games in *generalized characteristic function form* as they were introduced by Nowak and Radzik in [15] and by Sanchez and Bergantiños in [18]. Formally, for each nonempty subset $S \subseteq N$, we denote by $H(S)$ the set of all orders of the elements in S. The elements $T \in H(S)$, $\emptyset \neq S \subseteq N$, will be called *ordered coalitions*. A TU game in *generalized characteristic function form* or, for short, a *generalized TU game* is a pair (N,v) where N is a finite set of players and v is a function that assigns to every $T \in H(S)$, $\emptyset \neq S \subseteq N$, a real number $v(T)$. By definition $H(\emptyset) = \{\emptyset\}$ and $v(\emptyset) = 0$ is imposed. We denote the set of all generalized TU games on N by Γ_N.

For a nonempty ordered coalition $T = (i_1, i_2, \ldots, i_s) \in H(S)$, we say that i_{j+1} is the *consecutive* element of i_j in T, for $1 \leq j \leq s-1$ (or i_j is the *previous* element of i_{j+1}). A subset of consecutive elements in T, $Q = (i_p, i_{p+1}, \ldots, i_{p+u})$ with $1 \leq p \leq p+u \leq s$, is called a *consecutive subcoalition* of T.

Definition 2.1. *Given a digraph D defined on N, a consecutive subcoalition $Q = (i_p, i_{p+1}, \ldots, i_{p+u})$ of $T = (i_1, i_2, \ldots, i_s)$ is a connected consecutive subcoalition according to the digraph D if, and only if, $u = 0$ or $(i_j, i_{j+1}) \in D$ for $j = p, \ldots, p+u-1$.*
If, in addition, (i) $p = 1$ or $(i_{p-1}, i_p) \notin D$ and (ii) $p+u = s$ or $(i_{p+u}, i_{p+u+1}) \notin D$, we say that Q is a maximal connected consecutive subcoalition according to D.

Definition 2.2. *Let v and D be a TU game and a digraph respectively defined on N. The game v modified by digraph D is the generalized TU game defined by*

$$v^D(T) = \sum_{Q \in T/D} v(Q') \quad \forall T \in H(S),\ \forall S \subseteq N,\ S \neq \emptyset,$$

where T/D denotes the set of maximal connected consecutive subcoalitions of T according to digraph D, and Q' denotes the (non-ordered) coalition in N formed with the elements of the ordered subcoalition Q.

Remark 2.3. *If v is a symmetric TU game, for every modified game v^D, the utilities of all ordered coalitions only depend on the size of their respective maximal connected consecutive subcoalitions according to digraph D: $v(Q') = f(q)$, where $q = |Q'|$. From now on, we only consider symmetric TU games.*

Example 2.4. *We introduce three symmetric TU games whose utilities are obtained from specific properties of the coalitions related with their respective sizes.*

*(i) The conferences game. For every coalition $S \subseteq N$, the conferences game assigns the number of subcoalitions in S with two or more players (*conferences*): $v_1(S) = f_1(s) = 2^s - s - 1$, $\forall S \subseteq N$.*

(ii) The pairs game. For every coalition $S \subseteq N$, the pairs game assigns the number of subcoalitions in S with two players: $v_2(S) = f_2(s) = s(s-1)/2$, $\forall S \subseteq N$.

(iii) The lengths game. For every coalition $S \subseteq N$, the lengths game assigns the length of a minimal path involving all players as nodes: $v_3(S) = f_3(s) = s - 1$, $\forall S \subseteq N$.

Game v_1 and game $2v_2$ –under the name of messages game– were considered in [9], where social networks without direction have been studied. All three games are zero-normalized.

A *solution* or a *value* on the set of TU games G_N is a function $\Psi : G_N \to \mathbb{R}^N$ which assigns to every game v a vector $\Psi[v]$ with components $\Psi_i[v]$ for all $i \in N$. It represents a method to measure the negotiation strength of the players in the game. The vector space $\mathbb{R}^N$ is called the allocation space. *Semivalues* [6] as solution concepts were introduced by Dubey, Neyman and Weber by means of four axioms. A solution $\psi : G_N \to \mathbb{R}^N$ is a semivalue iff it satisfies the following properties:

A1. *Linearity.* $\psi[\lambda u + \mu v] = \lambda\psi[u] + \mu\psi[v]$ for all $u, v \in G_N$ and $\lambda, \mu \in \mathbb{R}$.

A2. *Anonymity.* $\psi_{\pi i}[\pi v] = \psi_i[v]$ for all $v \in G_N$, $i \in N$ and π permutation of N, where game πv is defined by $(\pi v)(\pi S) = v(S)$ for all $S \subseteq N$.

A3. *Positivity.* If game v is monotonic, then $\psi_i[v] \geq 0$ for all $i \in N$.

A4. *Projection.* $\psi_i[v] = v(\{i\})$ for all $i \in N$ and $v \in A_N$, where A_N denotes the set of additive games in G_N, i.e., games v such that $v(S \cup T) = v(S) + v(T)$ if $S \cap T = \emptyset$ and $S, T \subseteq N$.

In the same paper, another characterization of semivalues by using *weighting coefficients* and *marginal contributions* is provided.

Theorem 2.5. *(Dubey et al., [6]) (a) For every weighting vector $(p_s)_{s=1}^n$ such that*

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1 \quad \text{and} \quad p_s \geq 0 \quad \text{for } s = 1, \ldots, n, \tag{1}$$

the expression

$$\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_{s+1}[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in G_N, \tag{2}$$

where $s = |S|$, defines a semivalue ψ on G_N.
(b) Conversely, every semivalue on G_N is of this form, i.e., there exists a one-to-one map between the semivalues on G_N and the vectors $(p_s)_{s=1}^n$ verifying conditions (1).

The marginal contribution of a player $i \in N$ to a coalition $S \cup \{i\}$, with $S \subseteq N \setminus \{i\}$, is the difference $v(S \cup \{i\}) - v(S)$. Expression (2) shows that the allocation to each player by a semivalue ψ on G_N is an average of the marginal contributions to the coalitions to which it belongs, where the weighting coefficients p_{s+1} only depend on the coalition size.

Well known examples of semivalues are the Shapley value ϕ [19], for which $p_s = [n\binom{n-1}{s-1}]^{-1}$, and the Banzhaf value β [3, 16], for which $p_s = 2^{1-n}$. The Shapley value ϕ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \phi_i[v] = v(N)$ for every $v \in G_N$. It is worthy of mention that these two classical values are defined for each N.

Also, a parametric family of semivalues can be considered for each N. Given a real number $\alpha \in (0,1)$, the *binomial* semivalue ψ_α [17, 8] is defined on the set of TU games G_N by its weighting coefficients: $p_{\alpha,s} = \alpha^{s-1}(1-\alpha)^{n-s}$ for $s = 1, \ldots, n$. It is the unique family of semivalues whose weights are in geometric progression: $p_{\alpha,s+1}/p_{\alpha,s} = \alpha/(1-\alpha)$, $s = 1, \ldots, n-1$. In addition, n different binomial semivalues form a reference system on the set of all semivalues on G_N [1], so that, every semivalue can be written as a linear combination of them. The Banzhaf value is the binomial semivalue for $\alpha = 1/2$.

3. Extended semivalues and directed graphs

The solutions for TU games provided by the wide family of semivalues on G_N can be extended to solutions for generalized TU games in a same way as the classical Shapley value were extended by Nowak and Radzik in [15] for all games in Γ_N.

Definition 3.1. *Let ψ be a semivalue defined on G_N with weighting vector $(p_s)_{s=1}^n$. The extension on the set of generalized TU games Γ_N of semivalue ψ is the allocation rule defined by*

$$\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} \frac{p_{s+1}}{s!} \sum_{T \in H(S)} [v((T,i)) - v(T)] \quad \text{for all } i \in N \text{ and all } v \in \Gamma_N, \tag{3}$$

where (T,i) is the ordered coalition obtained from T adding element i at its end.

Remark 3.2. *By abuse of notation, we use the same letter ψ for the extended semivalue on Γ_N. It is clear that every extended semivalue ψ satisfies similar properties to A1–A4 but in the context of generalized TU games.*

A1'. Linearity. $\psi[\lambda u + \mu v] = \lambda \psi[u] + \mu \psi[v]$ *for all* $u, v \in \Gamma_N$ *and* $\lambda, \mu \in \mathbb{R}$.

A2'. Anonymity. $\psi_{\pi i}[\pi v] = \psi_i[v]$ *for all* $v \in \Gamma_N$, $i \in N$ *and* π *permutation of* N, *where game* πv *is defined by* $(\pi v)(\pi T) = v(T)$ *for all* $T \in H(S)$ *and* $S \subseteq N$.

A3'. Positivity. We call a game $v \in \Gamma_N$ *monotonic if* $v((T_1, T_2)) \geq v(T_1)$, *for all* $T_1 \in H(S_1)$, $T_2 \in H(S_2)$ *and* $S_2 \subseteq N \setminus S_1$. *If game* $v \in \Gamma_N$ *is monotonic, then* $\psi_i[v] \geq 0$ *for all* $i \in N$.

A4'. Projection. $\psi_i[v] = v(\{i\})$ *for all* $i \in N$ *and* $v \in \mathcal{A}_N$, *where* $\mathcal{A}_N$ *denotes the set of additive games in* Γ_N, *i.e., games* v *such that* $v((T_1, T_2)) = v(T_1) + v(T_2)$ *for all* $T_1 \in H(S_1)$ *and* $T_2 \in H(S_2)$, *if* $S_1 \cap S_2 = \emptyset$ *and* $S_1, S_2 \subseteq N$.

According to property A2, when a symmetric TU game is defined on a set N, the allocations to all players provided by each semivalue on G_N are coincident. Given a digraph D with set of nodes N, a symmetric TU game v is modified in a generalized TU game v^D where, in general, $v^D(T_1) \neq v^D(T_2)$ for $T_1, T_2 \in H(S)$ and $S \subseteq N$. Then, the allocations to the nodes of N as players in game v^D according to an extended semivalue are, in general, not coincident.

The allocation to each node through a symmetric TU game v modified by a digraph D depends on the geometry of the connections described by the modified game v^D and, also, on the amounts that weigh to the marginal contributions of each node; these amounts are related with the weighting vector of each semivalue. Both characteristics –geometry and weights– allow us to introduce several rankings among the nodes of digraphs as we establish in the following definition.

Definition 3.3. *Let v be a symmetric TU game defined on N and let ψ be a semivalue defined on G_N with weighting vector $(p_s)_{s=1}^n$. Given a digraph D with set of nodes N, we call accessibility of node $i \in N$ according to game v and semivalue ψ to the allocation obtained by player i in the modified game v^D according to the extended semivalue ψ defined on Γ_N:*

$$a_i[D; v, \psi] := \psi_i[v^D] = \sum_{S \subseteq N \setminus \{i\}} \frac{p_{s+1}}{s!} \sum_{T \in H(S)} [v^D((T, i)) - v^D(T)] \quad \text{for all } i \in N. \tag{4}$$

Example 3.4. *On the set of nodes $N = \{1, 2, 3\}$, given the digraph defined by $D = \{(1, 2), (2, 1), (3, 1), (2, 3), (3, 2)\}$, the modified game obtained from the symmetric TU game v_1 introduced in Example 2.4 is defined as follows:*

$$v_1^D(i) = 0 \ \forall i \in N, \ v_1^D(1, 3) = 0, \ v_1^D(i, j) = 1 \ \forall (i, j) \in D, \ v_1^D(1, 3, 2) = v_1^D(2, 1, 3) = 1$$

and $v_1^D(i, j, k) = 4 \ \forall (i, j, k) \in H(N), \ (i, j, k) \neq (1, 3, 2), (2, 1, 3)$.

In turn, according to digraph D, the modified games of games v_2 and v_3 in Example 2.4 take the same values as game v_1^D except:

$$v_2^D(i, j, k) = 3, \ v_3^D(i, j, k) = 2 \ \forall (i, j, k) \in H(N), \ (i, j, k) \neq (1, 3, 2), (2, 1, 3).$$

Independently of the above games, we can consider several semivalues on three-player games. For instance: the Shapley value ϕ with weighting vector $(p_1, p_2, p_3) = (1/3, \ 1/6, \ 1/3)$, the Banzhaf value β with weights $p_s = 1/4$, s=1, 2, 3, the binomial semivalue $\psi_{1/3}$ whose weighting vector is $(p_1, p_2, p_3) = (4/9, \ 2/9, \ 1/9)$ and, the binomial semivalue $\psi_{2/3}$ with $(p_1, p_2, p_3) = (1/9, \ 2/9, \ 4/9)$.

Table 1 shows some allocations to the nodes according to the corresponding extended semivalues and the modified games. A column offers the allocation and another column presents the percentage vector for comparison.

Several properties of the accessibility for specific selected games and all semivalues are described in the next proposition. A node i in a digraph D is called *inaccessible* if no other node $j \in N$ can be found such that $(j, i) \in D$. As it seems reasonable, we will propose conditions so that the accessibility of an inaccessible node takes null value, or the accessibility of a node does not decrease when an edge arriving to the considered node is added.

$\psi[v^D]$	$(\psi_1[v^D], \psi_2[v^D], \psi_3[v^D])$	%
$\phi[v_1^D]$	(4/3, 1, 2/3)	(44.44, 33.33, 22.22)
$\phi[v_2^D]$	(1, 5/6, 1/2)	(42.86, 35.71, 21.43)
$\beta[v_1^D]$	(5/4, 1, 5/8)	(43.48, 34.78, 21.74)
$\beta[v_2^D]$	(1, 7/8, 1/2)	(42.11, 36.84, 21.05)
$\beta[v_3^D]$	(3/4, 3/4, 3/8)	(40.00, 40.00, 20.00)
$\psi_{1/3}[v_1^D]$	(28/9, 20/9, 14/9)	(45.16, 32.26, 22.58)
$\psi_{2/3}[v_2^D]$	(4/3, 10/9, 2/3)	(42.86, 35.71, 21.43)

Table 1: Values of accessibility of the nodes in digraph D

Proposition 3.5. *Let D be a digraph defined on a finite set N ($D \subseteq D_N$). For every semivalue ψ defined on G_N:*

(i) if $v \in G_N$ is a zero-normalized TU game and $i \in N$ is an inaccessible node in D, then $a_i[D; v, \psi] = 0$;

(ii) if an edge leaving a node i is added, then $a_i[D \cup (i, j); v, \psi] = a_i[D; v, \psi]$, for every TU game $v \in G_N$;

(iii) if $v \in G_N$ is a monotonic and zero-normalized TU game, and an edge arriving to a node i is added, then $a_i[D \cup (j, i); v, \psi] \geq a_i[D; v, \psi]$;

(iv) the accessibility in the complete digraph D_N equals the payoff by the selected semivalue: $a_i[D_N; v, \psi] = \psi_i[v]$ for all $i \in N$, for every TU game $v \in G_N$.

Remark 3.6. *(a) From (iii), in addition to the condition of zero-normalized, regular semivalues (weighting coefficients $p_s > 0$ for $s = 1, \ldots, n$) and strictly monotonic TU games ($v(S) < v(T)$ whenever $S \subset T \subseteq N$) guarantee an increase of the accessibility for node i when a new edge (j, i) is added.*
(b) According to (iv), given a semivalue ψ acting on the set of TU games G_N, the concept of accessibility in a digraph offers an extension of the concept of solution defined by the considered semivalue.
(c) All statements in the above Proposition have been proved using the marginal contributions of each node to the ordered coalitions. This procedure allows us to differentiate the contribution of the modified game v^D to the accessibility with respect to the contribution due to the selected semivalue ψ. Now, we will generalize this work method.

Definition 3.7. *Let v be a TU game defined on a finite set N and let D be a digraph with set of nodes N. The vector of marginal contributions for a node $i \in N$ according to game v modified by digraph D is*

$$m_i(v^D) = (m_{i,1}(v^D), m_{i,2}(v^D), \ldots, m_{i,n}(v^D))$$

where each component is defined by

$$m_{i,s}(v^D) = \sum_{S' \subseteq N \setminus \{i\} : |S'| = s-1} \; \sum_{T \in H(S')} [v^D((T, i)) - v^D(T)] \quad \text{for } s = 1, \ldots, n. \tag{5}$$

Expression of accessibility (4) for each node $i \in N$ in a digraph $D \subseteq D_N$ can be rewritten in the following way:

$$a_i[D; v, \psi] = \psi_i[v^D] = \sum_{s=1}^{n} \frac{p_s}{(s-1)!} \sum_{S' \subseteq N \setminus \{i\}: |S'|=s-1} \sum_{T \in H(S')} [v^D((T,i)) - v^D(T)]. \quad (6)$$

On the other hand, from the weighting vector $(p_s)_{s=1}^n$ of a semivalue ψ defined on G_N, we can define the *weighting vector of the extended semivalue* ψ defined on the set of generalized games Γ_N by

$$\omega(\psi) = \left(\frac{p_1}{0!}, \frac{p_2}{1!}, \ldots, \frac{p_n}{(n-1)!}\right) \quad (7)$$

and then, the accessibility for node i in digraph D can be expressed as a scalar product:

$$a_i[D; v, \psi] = \omega(\psi) \cdot m_i(v^D). \quad (8)$$

The contribution of semivalue ψ to the accessibility of node i is collected by the weighting vector $\omega(\psi)$, whereas vector $m_i(v^D)$ summarizes both geometry of digraph D and marginal contributions of node i according to game v.

4. Accessibility decomposition

In this section we want to offer a systematic procedure of computation for the accessibility of the nodes in directed graphs based on the calculus over *oriented paths*, considered as elementary digraphs.

On a set of nodes N, a digraph P included in D_N is called an *oriented path* if

$$P = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{m-1}, i_m)\} \quad \text{with } i_1, i_2, \ldots, i_m \in N \text{ and } i_j \neq i_k \text{ if } j \neq k.$$

For short, we denote it by $P_{i_1 i_2 \ldots i_{m-1} i_m}$.

Definition 4.1. *Given a digraph $D \subseteq D_N$ and a node $i \in N$, we define digraph $D_{i]}$ as the union of all maximal oriented paths contained in D with last element node i.*

Example 4.2. *On the set of nodes $N = \{1, 2, 3, 4\}$, given the digraph defined by $D = \{(1,2), (1,3), (2,3), (2,4), (3,4), (4,1), (4,2)\}$, we have:*

$$D_{1]} = P_{241} \cup P_{2341} \qquad D_{2]} = P_{3412} \cup P_{1342}$$
$$D_{3]} = P_{2413} \cup P_{4123} \cup P_{423} \qquad D_{4]} = P_{124} \cup P_{134} \cup P_{1234}$$

Lemma 4.3. *Let D be a digraph defined on the set of nodes N and let v be a TU game defined on N. For the marginal contributions it is verified:*

(i) $m_i(v^D) = m_i(v^{D_{i]}})$ *for all* $i \in N$,

(ii) for every pair of paths P and P' with a same last node i,

(a) if $P \cap P'$ does not have any connected component containing node i, then

$$m_i(v^{P \cup P'}) = m_i(v^P) + m_i(v^{P'})$$

(b) if $P \cap P'$ has one connected component containing node i, $(P \cap P')_i$, then

$$m_i(v^{P \cup P'}) = m_i(v^P) + m_i(v^{P'}) - m_i(v^{(P \cap P')_i}).$$

The above Lemma shows that the marginal contributions of all nodes can be reduced to marginal contributions of last nodes in oriented paths. In this way, by symmetry property, given a digraph defined on a set N with n nodes, it suffices to know $n-1$ vectors of marginal contributions for each symmetric TU game v modified by oriented paths:

$$m_1(v^{P_{n \dots 1}}),\ m_1(v^{P_{n-1 \dots 1}}),\ \dots,\ m_1(v^{P_{21}}).$$

For simplicity of notation, we have chosen node 1 as last node in all oriented paths. The next propositions offer explicit expressions for these vectors of marginal contributions.

Proposition 4.4. *Let v be a symmetric and zero-normalized TU game defined on N, i.e., $v(S) = f(s)\ \forall S \subseteq N$ $(s = |S|)$ and $f(1) = 0$. If $P_{n \dots 1}$ is an oriented path joining all nodes in N, then the marginal contributions of node 1 in game v modified by $P_{n \dots 1}$ are given by*

$$\begin{aligned} &m_{1,1}(v^{P_{n \dots 1}}) = 0; \quad m_{1,2}(v^{P_{n \dots 1}}) = f(2);\\ &m_{1,k}(v^{P_{n \dots 1}}) = f(k) - f(k-1) + \sum_{j=2}^{k-1} (n-j-1) V_{n-j-1,k-j-1}[f(j) - f(j-1)], \end{aligned} \tag{9}$$

for $k = 3, \dots, n-1$, and

$$m_{1,n}(v^{P_{n \dots 1}}) = f(n) - f(n-1) + \sum_{j=2}^{n-2} (n-j-1)(n-j-1)![f(j) - f(j-1)].$$

Proposition 4.5. *Let v be a symmetric and zero-normalized TU game as defined in Proposition 4.4. If $P_{q \dots 1}$ is an oriented path involving some nodes in N $(2 \le q < n)$, then the marginal contributions of node 1 in game v modified by $P_{q \dots 1}$ are given by*

$$\begin{aligned} m_{1,k}(v^{P_{q \dots 1}}) &= m_{1,k}(v^{P_{n \dots 1}}), \quad \text{for } k = 1, \dots, q;\\ m_{1,k}(v^{P_{q \dots 1}}) &= \sum_{j=2}^{q-1} (n-j-1) V_{n-j-1,k-j-1}[f(j) - f(j-1)] +\\ &\quad + V_{n-q,k-q}[f(q) - f(q-1)], \quad \text{for } k = q+1, \dots, n-1, \end{aligned} \tag{10}$$

and $m_{1,n}(v^{P_{q \dots 1}}) = m_{1,n-1}(v^{P_{q \dots 1}})$.

Example 4.6. *On the set of nodes $N = \{1, 2, 3, 4\}$ we introduce the conferences game defined by $v(S) = f(s) = 2^s - s - 1\ \forall S \subseteq N$. In order to obtain all vectors of marginal contributions for this game v modified by any digraph defined on N, three vectors of marginal contributions are needed:*

$$m_1(v^{P_{4321}}) = (0, 1, 4, 8), \quad m_1(v^{P_{321}}) = (0, 1, 4, 4) \quad \text{and} \quad m_1(v^{P_{21}}) = (0, 1, 2, 2).$$

We now consider digraph $D = \{(1,2),(1,3),(2,3),(2,4),(3,4),(4,1),(4,2)\} \subseteq D_N$ *and, for each node in* N*, we determine its vector of marginal contributions for game* v *modified by* D*.*

For node 1, because $D_{1]} = P_{2341} \cup P_{241}$*, we have:*

$$\begin{aligned} m_1(v^D) = m_1(v^{D_{1]}}) &= m_1(v^{P_{2341}}) + m_1(v^{P_{241}}) - m_1(v^{P_{41}}) \\ &= (0,1,4,8) + (0,1,4,4) - (0,1,2,2) = (0,1,6,10). \end{aligned}$$

Similar computations follow for the remaining nodes:

$$\begin{aligned} m_2(v^D) &= m_2(v^{D_{2]}}) = m_2(v^{P_{1342}}) + m_2(v^{P_{3412}}) = (0,2,8,16); \\ m_3(v^D) &= m_3(v^{D_{3]}}) = m_3(v^{P_{2413}}) + m_3(v^{P_{4123}}) + m_3(v^{P_{423}}) - m_3(v^{P_{23}}) = (0,2,10,18); \\ m_4(v^D) &= m_4(v^{D_{4]}}) = m_4(v^{P_{1234}}) + m_4(v^{P_{134}}) - m_4(v^{P_{34}}) + m_4(v^{P_{124}}) = (0,2,10,14). \end{aligned}$$

Remark 4.7. *Expressions in Propositions 4.4 and 4.5 allow us to know the marginal contributions for the last node according to symmetric TU games modified by oriented paths. From property (i) in Lemma 4.3, the marginal contributions of intermediate nodes can be reduced to marginal contributions of last nodes:*

$$m_{i_k}(v^{P_{i_1 \dots i_k \dots i_m}}) = m_{i_k}(v^{(P_{i_1 \dots i_k \dots i_m})_{i_k]}}) = m_{i_k}(v^{P_{i_1 \dots i_k}}).$$

Remark 4.8. *We want now to pay attention, for instance, to the unions of oriented paths* $D_{3]}$ *and* $D_{4]}$ *in Example 4.2:* $D_{3]} = P_{2413} \cup P_{4123} \cup P_{423}$ *and* $D_{4]} = P_{124} \cup P_{134} \cup P_{1234}$*. Colloquially speaking, the structure of* $D_{3]}$ *can be obtained from the structure of* $D_{4]}$*, replacing an oriented path with 3 nodes –*P_{124}*– by an oriented path with 4 nodes –*P_{2413}*–.*

More precisely, let π *be a permutation of* N*. Given a digraph* D *with edges* (i,j) $i,j \in N$*,* $i \neq j$*, the digraph* D *transformed by* π *is* $\pi(D)$*, whose edges are* $(\pi(i),\pi(j))$ *for all* (i,j) *belonging to* D*. Digraphs* D *and* $\pi(D)$ *are isomorphic: they only differ on the label of their respective nodes. According to the above consideration, it is easy to see that there exists a permutation* π *defined on* N *so that* $D_{3]} \supset \pi(D_{4]})$*.*

The above comparison can be extended to all pairs of distinct unions of oriented paths $D_{i]}$*,* $i = 1,2,3,4$*, in Example 4.2. We have,*

$$D_{3]} \supset \pi(D_{2]}), \quad D_{3]} \supset \pi(D_{1]}) \quad \textit{and} \quad D_{4]} \supset \pi(D_{1]}),$$

where it is understood that each permutation π *is adequate to the studied comparison. This remark leads us to the following definition.*

Definition 4.9. *Let* D *be a digraph with set of nodes* N*. Node* i *is structurally more accessible than node* j *in digraph* $D \subseteq D_N$ *iff there exists a permutation* π *of* N *with* $\pi(j) = i$ *so that* $D_{i]} \supset \pi(D_{j]})$*.*

From now on, we will use this notation: given two vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ belonging to $\mathbb{R}^n$, we will say that $a \geq b$ iff $a_i \geq b_i$ $\forall i = 1, \dots, n$.

Proposition 4.10. *Let* D *be a digraph defined on a finite set* N *(*$D \subseteq D_N$*). For every symmetric and monotonic TU game* $v \in G_N$*, if node* i *is structurally more accessible than node* j *in digraph* D*, then* $a_i[D; v, \psi] \geq a_j[D; v, \psi]$ *for every semivalue* ψ *defined on* G_N*.*

Note that inclusion $D_{i]} \supset \pi(D_{j]})$ in Definition 4.9 is strict. In case of equality, $D_{i]} = \pi(D_{j]})$, with π permutation of N, we can say that nodes i and j are *equally accessible* and their respective accessibilities coincide for all semivalues on G_N.

According to the result obtained in the previous Proposition, it seems that nothing depends on the game and the semivalue selected to measure accessibility. On the contrary, a more larger structural accessibility of a node with respect to other node always results in a greater measure of accessibility, as it seems reasonable. The introduction of games and semivalues will allow the modification of the relationship between measures of accessibility, without changing the structural dominance of a node over another. However, not all pairs of nodes will be comparable according to the relation *structurally more accessible than.*

5. Oriented paths and convexity

Definition 5.1. *A cooperative TU game v on N is said to be convex if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2) - v(S_1 \cap S_2)$ $\forall S_1, S_2 \subseteq N$.*

Lemma 5.2. *Let v be a symmetric TU game defined on N, $v(S) = f(s)$ $\forall S \subseteq N$ with $s = |S|$. Convexity for symmetric game v is equivalent to conditions*

$$f(s) - f(s-1) \geq f(s-1) - f(s-2), \quad 2 \leq s \leq n = |N|. \tag{11}$$

In a set $N = \{1, 2, \ldots, n\}$ where a symmetric and zero-normalized TU game v is defined, we want to compare the marginal contributions of a given node when an edge is added to an oriented path. To do so, we consider oriented paths $P_{q\ldots 1}$ and $P_{q+1\ldots 1}$ for $2 \leq q \leq n-1$ and we focus on node 1. According to Proposition 4.5:

$$m_{1,k}(v^{P_{q\ldots 1}}) = m_{1,k}(v^{P_{q+1\ldots 1}}) = m_{1,k}(v^{P_{n\ldots 1}}),\ k = 1, \ldots, q.$$

The following proposition shows a relationship between the remaining marginal contributions and the convexity of game v.

Proposition 5.3. *Let v be a symmetric, monotonic and zero-normalized TU game on N. For all oriented paths $P_{q\ldots 1}$, $2 \leq q \leq n-1$, involving some nodes in N,*

$$m_{1,k}(v^{P_{q\ldots 1}}) \leq m_{1,k}(v^{P_{q+1\ldots 1}}),\ \forall k = q+1, \ldots, n \ \Leftrightarrow\ v \text{ is a convex game.}$$

According to this last Proposition and its previous comment, Eq. (8) allows us to formulate the following result.

Theorem 5.4. *For every symmetric, monotonic and zero-normalized TU game defined on N and every semivalue defined on G_N, by adjunction of previous nodes to oriented paths on N, the accessibility of their last nodes does not decrease if and only if the selected TU game is convex.*

In the previous statement, all inequalities can be replaced by strict inequalities. Convexity can be replaced by strict convexity, $f(s) - f(s-1) > f(s-1) - f(s-2)$, $2 \leq s \leq n = |N|$ and we can only consider regular semivalues (weighting coefficients $p_s > 0$, $1 \leq s \leq n$). Under these conditions, an increase of accessibility of the last nodes in oriented paths by adjunction of previous nodes is equivalent to strict convexity of the TU game.

Remark 5.5. *From the beginning, to define accessibility of nodes in digraphs, we have considered symmetric TU games, with the aim that the allocations depend on the geometry of the oriented network and they are not dependent on the label of each node.*

Other desired properties are satisfied asking some characteristics on the considered TU games. For instance, (i) null accessibility for inaccessible nodes is obtained by zero-normalized property of the game, (ii) no negative (positive) accessibility for the remaining nodes, by (strict) monotonicity, (iii) not decrease (increase) of accessibility for a node when an edge arriving to this node is added, also by (strict) monotonicity.

Now, it seems interesting to obtain an increase of accessibility of the last nodes in oriented paths by adjunction of previous nodes, so that the strict convexity of the selected games is also demanded.

All these required properties can be summarized for the symmetric games $v \in G_N$ *defined by means of* $v(S) = f(s)\ \forall S \subseteq N$ *according to three conditions:*

$$f(1) = 0; \quad f(2) > 0; \quad f(s) - f(s-1) > f(s-1) - f(s-2),\ 3 \le s \le n. \tag{12}$$

Example 5.6. *Example 2.4 revisited. The conferences game and the pairs game respectively defined by* $v_1(S) = f_1(s) = 2^s - s - 1$ *and* $v_2(S) = f_2(s) = s(s-1)/2$ *are strictly convex games. Nevertheless, the lengths game* $v_3(S) = f_3(s) = s - 1$ *as linear game is not strictly convex. In particular, this last game is the unique linear game verifying condition of zero-normalized, except a multiplicative constant. According to the previous remark, the two first games are convenient to compute weighted accessibility of nodes in oriented graphs.*

Lemma 5.7. *Let* $v \in G_N$ *be a symmetric, zero-normalized and linear TU game. Then, the vector of marginal contributions of the last node in game* v *modified by any oriented path does not depend on the considered path.*

Example 5.8. *We consider* $D = \{(1,2),(1,3),(2,3),(2,4),(3,4),(4,1),(4,2)\} \subseteq D_N$ *as defined in Example 4.6, with* $N = \{1,2,3,4\}$*, but we now consider the lengths game* $v(S) = f_3(s) = s - 1$ *for* $s = 1,\dots,4$*. All vectors of marginal contributions we need to compute accessibility of any node in* D *are coincident:* $m_1(v^{P_{4321}}) = m_1(v^{P_{321}}) = m_1(v^{P_{21}}) = (0,\ 1,\ V_{2,1},\ V_{2,2}) = (0,\ 1,\ 2,\ 4)$.

As we have worked in Example 4.6, the vectors of marginal contributions for each node in D *can be computed from the above vectors:*

$$m_1(v^D) = (0,\ 1,\ 2,\ 4) \quad \textit{and} \quad m_j(v^D) = (0,\ 2,\ 4,\ 8) \quad \textit{for } j = 2,3,4.$$

The vector of marginal contributions of each node is a multiple of vector $(0,\ 1,\ 2,\ 4)$ *and the factor exactly coincides with the number of edges arriving at the considered node. This way, given* ψ *any semivalue defined on four-player games by means of its weighting vector* (p_1, p_2, p_3, p_4)*, we compute the accessibility of the nodes in oriented graph* D *according to Eq. (8):*

$$a_1[D; v, \psi] = p_2 + p_3 + \frac{2}{3}p_4; \quad a_j[D; v, \psi] = 2p_2 + 2p_3 + \frac{4}{3}p_4,\ j = 2,3,4.$$

The percentage vector of accessibility (14.29, 28.57, 28.57, 28.57) *does not depend on the selected semivalue and its information is only taken of the number of edges arriving at each node.*

The above example can be easily generalized to every digraph and every linear game, where linear is employed as a synonym of a multiple of the lengths game defined by $f_3(s) = s - 1$. If we choose as test game a linear game, the action of the selected semivalue to compute accessibility is disabled. The families of test games we propose do not contain linear games; they are introduced in the next definition.

Definition 5.9. *Let G_N be the set of TU games with n players. We consider as test games to compute accessibility in digraphs with n nodes two families of symmetric games:*

(i) games of monomial type, $v_{\underline{r}}(S) = f_{\underline{r}}(s) = s^r - 1$, $\forall S \subseteq N$ *with* $r > 1$;

(ii) games of exponential type, $v_{\overline{r}}(S) = f_{\overline{r}}(s) = r^{s-1} - 1$, $\forall S \subseteq N$ *with* $r > 1$.

Each one of the games belonging to these families satisfies conditions (12) and it can be used as test game; also, every convex linear combination of them can be employed. Note that games $v_{\underline{2}}(S) = f_{\underline{2}}(s) = s^2 - 1$ and $v_{\overline{2}}(S) = f_{\overline{2}}(s) = 2^{s-1} - 1$ are quite close to the pairs game $v_2(S) = f_2(s) = s(s-1)/2$ and the conferences game $v_1(S) = f_1(s) = 2^s - s - 1$, respectively. In both cases, the linear component has been deleted.

6. Choosing semivalues

So far we have focused our attention on the cooperative games suggested to determine the accessibility of the nodes in a digraph. We will now pay attention on the semivalues involved according to its weighting coefficients. Given a semivalue ψ defined on the set G_N of TU games with n players, each coefficient p_s weighs marginal contributions to coalitions of size s, for $s = 1$ to $s = n$. To compute accessibility, we have extended each semivalue to the generalized games in Γ_N. The marginal contributions to ordered coalitions of size s are now weighted according to coefficients $p_s/(s-1)!$, as we have stated in Eq. (6).

Our purpose consists of choosing the coefficients of a semivalue defined on the set G_N of classic TU games so that the corresponding extended semivalue on Γ_N has a suitable distribution of weights on the ordered coalitions. That is, choosing in an appropriate way coefficients p_s, so that $p_s/(s-1)!$ respond to our claims. For instance, we suppose that, emulating to the Banzhaf value on classic TU games, we want to weigh every marginal contribution to the ordered coalitions with a same coefficient. In the following Definition, we introduce such a semivalue; before, a Lemma is needed.

Lemma 6.1. *The number of nonempty ordered coalitions in a set N is given by*

$$|\{T \in H(S) \mid \emptyset \neq S \subseteq N\}| = \lfloor e\, n! \rfloor - 1, \quad \text{for } |N| = n \geq 2,$$

where $\lfloor x \rfloor$ denotes the integer part of a positive number x.

Definition 6.2. *On the set G_N of TU games with n players, we define the semivalue $\widetilde{\beta}$ "emulating to the Banzhaf value" by means of its weighting coefficients:*

$$\widetilde{p}_s = \frac{(s-1)!}{\lfloor e\,(n-1)! \rfloor} \quad \text{for } s = 1, \ldots, n.$$

It is not difficult to check that the above coefficients $\widetilde{p}_s$ satisfy conditions (1), so that $\widetilde{\beta}$ belongs to the set of semivalues on G_N. When this semivalue is extended to generalized cooperative games in Γ_N, all coefficients weighting each marginal contribution to all ordered coalitions are coincident: $p_s/(s-1)! = p_{s'}/(s'-1)!$ for all $1 \leq s, s' \leq n$. The extended semivalue of $\widetilde{\beta}$ plays on Γ_N a similar role to the Banzhaf value β on the *classic* cooperative games of G_N. This motivates its definition.

The above procedure for the Banzhaf value can be extended to every semivalue ψ on G_N, as we stated in the next Definition.

Definition 6.3. *Let ψ be a semivalue on G_N with weighting vector $(p_s)_{s=1}^n$. We define the semivalue $\widetilde{\psi}$ "emulating to the semivalue ψ" by means of its weighting coefficients:*

$$\widetilde{p}_s = \frac{1}{(n-1)!\sum_{s=1}^n p_s/(n-s)!}(s-1)!p_s \quad \textit{for } s = 1, \ldots, n.$$

After some computations, one can see that coefficients $\widetilde{p}_s$ also satisfy conditions (1) characterizing semivalues on G_N. In addition, $p_{s'}\ \widetilde{p}_s/(s-1)! = p_s\ \widetilde{p}_{s'}/(s'-1)!$ for all $1 \leq s, s' \leq n$. This guarantees that the proportion between the coefficients weighting ordered coalitions of sizes s and s' according to the extended semivalue of $\widetilde{\psi}$ on games in Γ_N equals the proportion between the coefficients weighting non-ordered coalitions of sizes s and s' according to semivalue ψ defined on G_N.

7. A detailed example

Below, a detailed study of accessibility in a concrete example is offered. After the introduction of the digraph, several steps follow: (i) decomposition as union of all maximal oriented paths with a same last node; (ii) selection of some test games and determination of the corresponding vector of marginal contributions for each node; (iii) introduction of several semivalues according to suitable distributions of weights on the ordered coalitions and computation of accessibility in several cases and (iv) comments on the obtained results.

A competition digraph. A set of teams playing in a sports competition produces a series of dominance relations based on the result of each match. It seems thus natural that all obtained results are collected in a dominance digraph, so-called competition digraph D. We want to focus our attention in the *tournaments*, where each match has a winner and a loser. Here, the relation of dominance can be clearly translated in an oriented edge: if player j wins the match it played against i then, $(i, j) \in D$. Nevertheless, this definition of competition digraph is not unique; one can see a more general definition in Van den Brink and Borm [4], precisely in the case in which a draw can be a possible outcome of the match.

Basketball competitions are played in tournaments. The example we present is obtained at the European Basketball Championship (EuroBasket 2009), where, in the Qualifying Round, group F consisted of six players: Slovenia (1), Turkey (2), Serbia (3), Spain (4), Poland (5) and Lithuania (6). The relation of results of the 15 matches is: 1 wins to 2, 3, 5 and 6, 2 wins to 3, 4, 5, and 6, 3 wins to 4, 5 and 6, 4 wins to 1, 5 and 6, and 5 wins to 6. Fiba Europe (Erobasket 2009 Organizer) ranks the teams with 2 points for a win and 1 point for a loss. According to this system, vector $(9, 9, 8, 8, 6, 5)$ collects the official allocation to the teams of group F.

Now, this situation arising from a sports competition will be analyzed applying our accessibility measures. We first consider on the set of teams $N = \{1,2,3,4,5,6\}$ the competition digraph as it has been defined above:

$$D = \{(1,4),(2,1),(3,1),(3,2),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),\\ (6,3),(6,4),(6,5)\}$$

(i) For each team $i \in N$, the unions of maximal oriented paths $D_{i]}$ are:

$$D_{1]} = P_{654321} \cup P_{64321} \cup P_{65321} \cup P_{6321} \cup P_{65421} \cup P_{6421} \cup P_{6521} \cup P_{621} \cup P_{65431} \cup\\ P_{6431} \cup P_{6531} \cup P_{631} \cup P_{651} \cup P_{61},$$

$$D_{2]} = P_{651432} \cup P_{61432} \cup P_{65432} \cup P_{6432} \cup P_{6532} \cup P_{632} \cup P_{653142} \cup P_{63142} \cup P_{65142} \cup\\ P_{6142} \cup P_{6542} \cup P_{642} \cup P_{652} \cup P_{62},$$

$$D_{3]} = P_{652143} \cup P_{62143} \cup P_{65143} \cup P_{6143} \cup P_{6543} \cup P_{643} \cup P_{653} \cup P_{63},$$

$$D_{4]} = P_{653214} \cup P_{63214} \cup P_{65214} \cup P_{6214} \cup P_{65314} \cup P_{6314} \cup P_{6514} \cup P_{614} \cup P_{654} \cup P_{64},$$

$$D_{5]} = P_{65} \quad \text{and} \quad D_{6]} = \emptyset.$$

(ii) We now introduce two test games and compute the vectors of marginal contributions for the last node in oriented paths. For test game $v_{\underline{2}}(S) = f_{\underline{2}}(s) = s^2 - 1$:

$$m_1(v_{\underline{2}}^{P_{6\ldots1}}) = (0,3,14,44,90,92), \quad m_1(v_{\underline{2}}^{P_{5\ldots1}}) = (0,3,14,44,90,90),$$

$$m_1(v_{\underline{2}}^{P_{4\ldots1}}) = (0,3,14,44,88,88), \quad m_1(v_{\underline{2}}^{P_{321}}) = (0,3,14,42,84,84)$$

$$\text{and} \quad m_1(v_{\underline{2}}^{P_{21}}) = (0,3,12,36,72,72).$$

With these five vectors, all vectors of marginal contributions for the nodes in D can be obtained. For node 1:

$$m_1(v_{\underline{2}}^{D}) = m_1(v_{\underline{2}}^{D_{1]}}) = m_1(v_{\underline{2}}^{P_{6\ldots1}}) + 4m_1(v_{\underline{2}}^{P_{5\ldots1}}) + 4m_1(v_{\underline{2}}^{P_{4\ldots1}}) - m_1(v_{\underline{2}}^{P_{321}}) - 4m_1(v_{\underline{2}}^{P_{21}})\\ = (0,12,64,210,430,432).$$

Similar computations lead us to the vectors for the remaining nodes.

$$m_2(v_{\underline{2}}^{D}) = (0,12,62,202,416,420), \quad m_3(v_{\underline{2}}^{D}) = (0,9,44,140,286,288),$$

$$m_4(v_{\underline{2}}^{D}) = (0,12,58,186,380,382), \quad m_5(v_{\underline{2}}^{D}) = (0,3,12,36,72,72)$$

$$\text{and} \quad m_6(v_{\underline{2}}^{D}) = (0,0,0,0,0,0).$$

We repeat the same procedure for test game $v_{\overline{2}}(S) = f_{\overline{2}}(s) = 2^{s-1} - 1$.

$$m_1(v_{\overline{2}}^{P_{6\ldots1}}) = (0,1,5,17,38,46), \quad m_1(v_{\overline{2}}^{P_{5\ldots1}}) = (0,1,5,17,38,38),$$

$$m_1(v_{\overline{2}}^{P_{4\ldots1}}) = (0,1,5,17,34,34), \quad m_1(v_{\overline{2}}^{P_{321}}) = (0,1,5,15,30,30)$$

$$\text{and} \qquad m_1(v_{\underline{2}}^{P_{21}}) = (0, 1, 4, 12, 24, 24).$$

From the above vectors,

$$m_1(v_{\overline{2}}^{D}) = (0, 4, 24, 90, 200, 208), \quad m_2(v_{\overline{2}}^{D}) = (0, 4, 23, 85, 194, 210),$$

$$m_3(v_{\overline{2}}^{D}) = (0, 3, 16, 56, 124, 132), \quad m_4(v_{\overline{2}}^{D}) = (0, 3, 17, 63, 142, 150),$$

$$m_5(v_{\overline{2}}^{D}) = (0, 1, 4, 12, 24, 24) \quad \text{and} \quad m_6(v_{\overline{2}}^{D}) = (0, 0, 0, 0, 0, 0).$$

(iii) At this stage, we shall now proceed to choose semivalues. The first selected semivalue will be $\widetilde{\beta}$ which emulates the Banzahf value. In games with six players, $\lfloor e\ 5! \rfloor = 326$ so the weighting coefficients of $\widetilde{\beta}$ take values $p_s = (s-1)!/326$ for $s = 1, \ldots, 6$ and the unique weighting coefficient of the extended semivalue on games with ordered coalitions is 1/326.

The second selected semivalue is $\widetilde{\psi}_{1/3}$, who emulates the binomial semivalue $\psi_{1/3}$. Weighting vector of $\psi_{1/3}$:

$$(p_s)_{s=1}^{6} = (32/243,\ 16/243,\ 8/243,\ 4/243,\ 2/243,\ 1/243).$$

Weighting vector of semivalue $\widetilde{\psi}_{1/3}$, obtained according to Definition 6.3:

$$(\widetilde{p}_s)_{s=1}^{6} = (32/872,\ 16/872,\ 16/872,\ 24/872,\ 48/872,\ 120/872).$$

Weighting vector of the extended semivalue $\widetilde{\psi}$ defined on generalized games of Γ_N:

$$\omega(\widetilde{\psi}_{1/3}) = \frac{1}{872}\,(32,\ 16,\ 8,\ 4,\ 2,\ 1)\,.$$

For this competition digraph, we offer four measures of accessibility for the six nodes as rankings of the teams they represent. Table 2 shows the solution vectors computed from test games $v_{\underline{2}}$ and $v_{\overline{2}}$ and semivalues $\widetilde{\beta}$ and $\widetilde{\psi}_{1/3}$. At its end, two rows have been added: the first one contains the *classical* solution obtained by the eigenvector method, while the second one shows the official allocation given by Fiba Europe.

	solution vector
$a[D; v_{\underline{2}}, \widetilde{\beta}]$	(3.5215, 3.4110, 2.3528, 3.1227, 0.5982, 0.0000)
$a[D; v_{\underline{2}}, \widetilde{\psi}_{1/3}]$	(3.2523, 3.1514, 2.1972, 2.9151, 0.5780, 0.0000)
$a[D; v_{\overline{2}}, \widetilde{\beta}]$	(1.6135, 1.5828, 1.0153, 1.1503, 0.1994, 0.0000)
$a[D; v_{\overline{2}}, \widetilde{\psi}_{1/3}]$	(1.4037, 1.3601, 0.8945, 0.9977, 0.1927, 0.0000)
$eig(A)$	(0.6256, 0.5516, 0.3213, 0.4484, 0.0000, 0.0000)
Fiba Europe	(9.0000, 9.0000, 8.0000, 8.0000, 6.0000, 5.0000)

Table 2: Accessibility measures of the teams in competition digraph D

The classical solution for the problem of establishing a ranking among the nodes of an oriented network modeled by means of a digraph D requires matrix $A = (a_{ij})$, where a_{ij}

takes value 1 if (j,i) belongs to digraph D and value 0 otherwise. The solution is obtained according to the idea due to Wei [20] and Kendall [11], where the ranking among the nodes is based on the eigenvector of matrix A whose components are all positive.

For comparison, all six rankings are normalized in percentage vectors and presented in Table 3.

	percentage vector
$a[D; v_{\underline{2}}, \widetilde{\beta}]$	(27.08, 26.23, 18.09, 24.01, 4.60, 0.00)
$a[D; v_{\underline{2}}, \widetilde{\psi}_{1/3}]$	(26.89, 26.06, 18.17, 24.10, 4.78, 0.00)
$a[D; v_{\overline{2}}, \widetilde{\beta}]$	(29.01, 28.46, 18.26, 20.68, 3.59, 0.00)
$a[D; v_{\overline{2}}, \widetilde{\psi}_{1/3}]$	(28.95, 28.05, 18.45, 20.58, 3.97, 0.00)
$eig(A)$	(32.13, 28.33, 16.50, 23.03, 0.00, 0.00)
Fiba Europe	(20.00, 20.00, 17.78, 17.78, 13.33, 11.11)

Table 3: Normalized rankings for the teams in competition digraph D

(iv) A digraph competition with 6 teams allows us a total of 15 paired comparisons among them. In our comment we will pay special attention to the pairs of teams they obtain a same outcome made in the official classification.

Teams 1 and 2 are not structurally comparable in competition digraph D. Then, we compare vectors of marginal contributions. For test game $v_{\underline{2}}$,

$$m_1(v_{\underline{2}}^D) = (0, 12, 64, 210, 430, 432) \geq (0, 12, 62, 202, 416, 420) = m_2(v_{\underline{2}}^D),$$

so that the ranking will be favorable to team 1 according to game $v_{\underline{2}}$ and all semivalues, as one can be seen in rows 1 and 2 of Table 2. Nevertheless, for test game $v_{\overline{2}}$, the vectors of marginal contributions are not comparable:

$$m_1(v_{\overline{2}}^D) = (0, 4, 24, 90, 200, 208), \quad m_2(v_{\overline{2}}^D) = (0, 4, 23, 85, 194, 210).$$

In despite, according to both selected semivalues, team 1 also exceeds team 2 (rows 3 and 4). Note that only the last component of vector $m_2(v_{\overline{2}}^D)$ takes a greater value than the corresponding one of vector $m_1(v_{\overline{2}}^D)$. Selected semivalues $\widetilde{\beta}$ and $\widetilde{\psi}_{1/3}$ are not able to change the ranking among teams 1 and 2. It is easy to see that a semivalue with almost all weight in the last weighting coefficient may be able to change the ranking among both teams.

On the other hand, teams 3 and 4 are structurally comparable according to competition digraph D, since there exists π permutation of N with $D_{4]} \supset \pi(D_{3]})$. In this case, by Proposition 4.10, $a_4[D; v, \psi] \geq a_3[D; v, \psi]$ for every test game v and every semivalue ψ defined on G_N. Table 2 shows this inequality for the selected games and semivalues, where each row specifies a ratio for the accessibility measures among teams 3 and 4, based on each selected pair of test game and semivalue.

In competition digraph D, can not be found an eigenvector with all positive entries for the corresponding matrix A. We have selected, for comparison, an eigenvector $eig(A)$

with no negative entries. For teams 1 to 4, the ranking based on the eigenvector method coincides in all studied cases with our method based on test games and semivalues. Also, our null player, team 6, obtains in all cases value 0, including null entry of the considered eigenvector. Since team 5 is not a null player according to our development, its accessibility measures are strictly positive, whereas the eigenvector assigns null value to him. The eigenvector method offers to each node an allocation proportional to the sum of allocations to the nodes that link to him; in digraph D, only node 6 links to node 5 and then, the null allocation assigned to team 6 also induces null allocation to team 5.

8. Concluding remark

Using techniques of Game Theory, the development of this work allows us to offer several accessibility measures to the nodes of directed graphs. The main tool has been the cooperative games in generalized characteristic function form, since the treatment of oriented paths in digraphs requires to consider ordered coalitions. The method of obtaining accessibility measures is based on the choice of a pair consisting of a cooperative game, so-called test game, and a solution for cooperative games selected from the wide family of semivalues. All obtained measures for the nodes can be considered as exogenous procedures to compute accessibility in a digraph. In this way, our method allows us to emphasize some types of structural characteristics in the digraph and measure the position of its nodes according to them.

The main part of the paper has been devoted to desirable properties so that a game can be considered as a test game; it must be symmetric, zero-normalized, monotonic, convex. In addition, it is also considered the selection process for semivalues, according to desired criteria for their weighting coefficients. For each digraph, the present work offers a family of rankings for the accessibility of their nodes. If a node presents a better structurally position than another, all rakings allocate a more accessibility to the node in better position, but each selection of test game–semivalue modulates the ratio of allocation according to the characteristics of the selected pair. It is particularly interesting the case in which nodes are not structurally comparable in the digraph. Now, the pair formed by test game and semivalue is able to detect this situation, offering different rankings that depend on the features collected by the selected pair.

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10. Bibliography

[1] Amer, R., Giménez, J.M. (2003). Modification of semivalues for games with coalition structures. *Theory and Decision* **54**, 185–205.

[2] Amer, R., Giménez, J.M., Magaña, A. (2007). Accessibility in oriented networks. *European Journal of Operational Research* **180**, 700–712.

[3] Banzhaf, J.F. (1965). Weigthed voting doesn't work: A mathematical analysis. *Rut-*

gers Law Review **19**, 317–343.

[4] Brink, R. van den, Borm, P. (2002). Digraph competitions and cooperative games. *Theory and Decision* **53**, 327–342.

[5] Derks, J.J.M., Gilles, R.P. (1995). Hierarchical organization structures and constraints on coalition formation. *International Journal of Game Theory* **24**, 147–163.

[6] Dubey, P., Neyman, A., Weber, R.J. (1981). Value theory without efficiency. *Mathematics of Operations Research* **6**, 122–128.

[7] Gilles, R.P., Owen, G., Brink, R. van den. (1992). Games with permission structures: the conjunctive approach. *International Journal of Game Theory* **20**, 277–293.

[8] Giménez, J.M. (2001). *Contributions to the study of solutions for cooperative games* (in Spanish). Ph.D. Thesis. Technical University of Catalonia, Spain.

[9] Gómez, D. et al. (2001). Centrality and power in social networks. A game theoretic approach. In: *Proc. of the XXVI National Congress of Statistic and Operations Research*, Úbeda, Spain.

[10] Herings, P.J.J., Laan, G. van der, Talman, D. (2001). *Measuring the power of nodes in digraphs.* Discussion paper TI 2001-096/1, Tinbergen Institute, The Netherlands.

[11] Kendall, M.G. (1955). Further contributions to the theory of paired comparisons. *Biometrics* **11**, 43–62.

[12] Laan, G. van der, Brink, R. van den. (1998). Axiomatization of a class of share functions for n-person games. *Theory and Decision* **44**, 117–148.

[13] Laffond, G., Laslier, J.F., Le Breton, M. (1993). The bipartisan set of a tournament game. *Games and Economic Behavior* **5**, 182-201.

[14] Laslier, J.F. (1997). *Tournament solutions and majority voting.* Springer, Berlin.

[15] Nowak, A., Radzik, T. (1994). The Shapley value for n-person games in generalized characteristic function form. *Games and Economic Behavior* **6**, 150–161.

[16] Owen, G. (1975). Multilinear extensions and the Banzhaf value. *Naval Research Logistics Quarterly* **22**, 741–750.

[17] Puente, M.A. (2000). *Contributions to the representability of simple games and to the calculus of solutions for this class of games* (in Spanish). Ph.D. Thesis. Technical University of Catalonia, Spain.

[18] Sanchez, E., Bergantiños, G. (1997). On values for generalized characteristic functions. *OR Spektrum* **19**, 229–234.

[19] Shapley, L.S. (1953). A value for n-person games. In: Kuhn HW, Tucker AW (Eds), *Contributions to the Theory of Games II*, Princeton University Press, pp. 307–317.

[20] Wei, T.H. (1952). *The algebraic foundations of ranking theory.* Cambridge University Press. Cambridge.