

# Master of Science in Advanced Mathematics and Mathematical Engineering

---

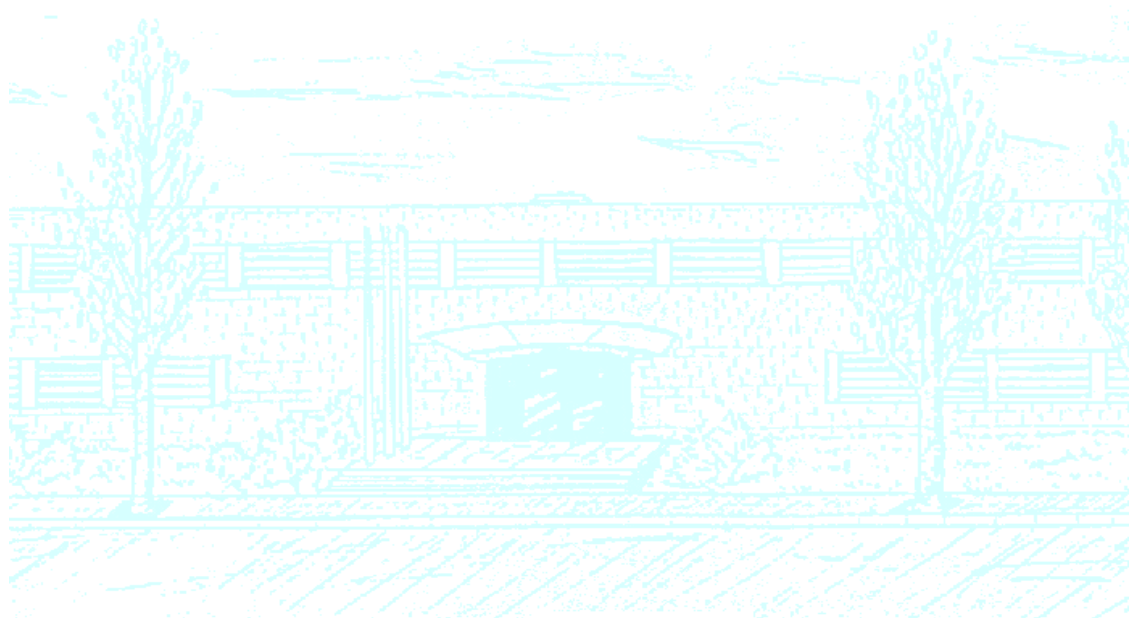
**Title:** Delaunay cylinders with constant non-local mean curvature

**Author:** Marc Alvinyà Rubió

**Advisor:** Xavier Cabré Vilagut

**Department:** Department of Mathematics

**Academic year:** 2016-2017





I would like to thank my thesis advisor Xavier Cabré for his guidance and support over the past three years. I would also like to thank my friends Tomàs Sanz and Óscar Rivero for their help and interest in my work.



## Abstract

The aim of this master's thesis is to obtain an alternative proof, using variational techniques, of an existence result for periodic sets in  $\mathbb{R}^2$  that minimize a non-local version of the classical perimeter functional adapted to periodic sets. This functional was introduced by Dávila, Del Pino, Dipierro and Valdinoci [20]. Our minimizers are periodic sets of  $\mathbb{R}^2$  having constant non-local mean curvature.

We begin our thesis with a brief review on the classical theory of minimal surfaces. We then present the non-local (or fractional) perimeter functional. This functional was first introduced by Caffarelli et al. [15] to study interphase problems where the interaction between particles are not only local, but long range interactions are also considered. Additionally, also using variational techniques, we prove the existence of solutions for a semi-linear elliptic equation involving the fractional Laplacian.

## Keywords

Minimal surfaces, non-local perimeter, fractional Laplacian, semi-linear elliptic fractional equation



## Contents

Chapter 1. Introduction	3
1.1. Main result	4
1.2. Outline of the thesis	5
Chapter 2. Classical minimal surfaces	7
2.1. Historical introduction	7
2.2. Functions of bounded variation and Caccioppoli sets	9
2.3. First and second variation of area	12
2.4. Regularity and minimal cones	14
2.5. The Bernstein problem and a conjecture of De Giorgi	18
Chapter 3. Non-local minimal surfaces	21
3.1. The fractional perimeter	21
3.2. Asymptotics of the fractional perimeter and a notion of non-local curvature	26
3.3. The non-local Allen-Cahn equation	27
3.4. Surfaces with constant non-local mean curvature	29
Chapter 4. The Dirichlet problem for the fractional Laplacian	31
4.1. The Dirichlet semi-linear problem	32
4.2. The periodic semi-linear problem	34
Chapter 5. The periodic fractional perimeter functional	38
Chapter 6. Delaunay cylinders with constant non-local mean curvature	44
Bibliography	48





## CHAPTER 1

### Introduction

This thesis concerns the non-local (or fractional) analogue of the classical periodic cylinders in  $\mathbb{R}^3$  with constant mean curvature found in 1841 by the French mathematician Charles-Eugène Delaunay [22]. These surfaces are critical points of the classical perimeter functional when a volume constraint is prescribed.

We will introduce the non-local analogue of the classical perimeter. The non-local or fractional perimeter functional was first introduced by Caffarelli et al. [16] to study interphase problems where the interaction between particles are not only local, but long range interactions are also considered. If we make a blow-down on the phase transition, we obtain that in the limit the level sets will approach a surface that minimizes the fractional perimeter functional. We will call them *non-local* or *fractional minimal surfaces*. Moreover, in [16] they also prove that such surfaces satisfy an Euler-Lagrange equation, namely *the non-local mean curvature equation* (that we call NMC equation for short) in reference to the classical mean curvature equation.

We will therefore consider surfaces with constant NMC that are periodic and cylindrically symmetric. They are the non-local analogue of the Delaunay cylinders.

The first existence and regularity result for non-local Delaunay surfaces was found by Cabré, Fall, Solà-Morales and Weth [10] where, using the implicit function theorem, they constructed periodic sets in  $\mathbb{R}^2$  with constant non-local mean curvature which bifurcate from a straight band.

Another way to construct such sets is variationally. In a subsequent paper of Dávila, Del Pino, Dipierro and Valdinoci [21] they consider a version of the fractional perimeter functional adapted to periodic sets. Namely, for a fractional parameter  $\alpha \in (0, 1)$  and a set  $E \subseteq \mathbb{R}^n$ , they consider the functional

$$P_\alpha(E) := \frac{1}{8} \int_{E \cap S} \int_{S \setminus E} \sum_{k \in \mathbb{Z}} \frac{dx dy}{|x - y + ke_1|^{n+\alpha}} \quad (1.0.1)$$

in the slab  $S := [-\pi, \pi] \times \mathbb{R}^{n-1}$ . Using variational techniques, they show the existence of codimension 1 surfaces of any dimension that minimizes the above functional among cylindrically decreasing symmetric competitors that are periodic in a given direction, assuming a volume constraint. Their method relies on a compactness result on the space of functions of bounded variation.

In their paper on constant NMC hypersurfaces in  $\mathbb{R}^n$  [15], Cabré, Fall and Weth pointed out that minimizers of the above periodic fractional perimeter functional (under a volume constraint) are in fact constant NMC hypersurfaces in a certain weak sense.

Nonetheless, because in their paper [21], Dávila, Del Pino, Dipierro and Valdinoci find minimizers among decreasing sets, their minimizers may not have constant NMC. Indeed, one could have a periodic set with lower “energy” that is not decreasing in the slab  $[0, \pi]$ .

Our objective will be to construct variationally the non-local analogue of the classical Delaunay surfaces in  $\mathbb{R}^3$ , but in  $\mathbb{R}^2$ . Throughout the thesis, we will sometimes refer to these surfaces as bands.

The main objective of this thesis will therefore be to derive a different and simplified approach to prove variationally the existence of such periodic surfaces in  $\mathbb{R}^2$ . As we shall see, we do not need to consider the class of minimizers to be decreasing, and hence we can show that the fractional perimeter functional adapted to periodic sets gives rise to constant NMC surfaces in a weak sense.

The simplification comes from finding a suitable expression of the periodic non-local perimeter. We can then show rather simply that a minimum exists in a particular functional space using a compact embedding and Fatou’s lemma. The main difference of our approach is that we do not work with the geometric problem of integrating over the set, but instead we reduce the dimension of the problem and end up working in one dimension.

### 1.1. MAIN RESULT

The main result of this thesis is presented next. We consider the periodic fractional perimeter functional for sets  $E := \{-u(x_1) < x_2 < u(x_1)\} \subset \mathbb{R}^2$ , which now reads

$$P_\alpha(u) := \frac{1}{8} \int_{-\pi}^{\pi} dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{|y_2| > u(y_1)} dy_2 \frac{1}{|x - y|^{2+\alpha}}, \quad (1.1.1)$$

with  $\alpha \in (0, 1)$ .

We consider the class of competitors  $\mathcal{A}$  given by the sets  $E \subseteq \mathbb{R}^2$  defined above for some non-negative, even  $2\pi$ -periodic function  $u : \mathbb{R} \rightarrow [0, +\infty]$  in the space  $W_{per}^{\alpha,1}(0, \pi)$  of even  $2\pi$ -periodic functions in the fractional Sobolev space  $W^{\alpha,1}(0, \pi)$ , satisfying a volume constraint  $\int_0^\pi u(x) dx = \mu$  for some given constant  $\mu > 0$ .

In this setting, we prove the existence of volume constrained minimizers of  $P_\alpha$  in  $\mathcal{A}$ . The proof is new and will appear in [14].

**Theorem 1.1.1.** *For any  $\mu > 0$  there exists a minimizer of  $P_\alpha$  in  $\mathcal{A}$ . More precisely, for any  $\mu > 0$  there exists a non-negative function  $u^* \in W_{per}^{\alpha,1}(0, \pi)$  such that  $\int_0^\pi u(x) dx = \mu$  and, for any non-negative function  $u \in W_{per}^{\alpha,1}$  with the same volume, we have that  $P_\alpha(u^*) \leq P_\alpha(u)$ . Moreover,  $u$  is a solution of*

$$H(u)(x) = a \text{ for all } x \in \mathbb{R} \text{ such that } u(x) > 0,$$

for a certain constant  $a \in \mathbb{R}$ , where  $H$  denotes the NMC.

## 1.2. OUTLINE OF THE THESIS

The work is organised as follows:

- In chapter 2 we present the classical theory of minimal surfaces and, in particular, the ideas behind the minimality of the Simons' cone, as well as that minimal surfaces are regular up to dimension 7. This will motivate and help to understand the subsequent chapters.
- In chapter 3 we introduce the notion of non-local perimeter functional and of non-local minimal surfaces. We provide with an existence result of minimizers. We compare this non-local setting with the classical theory and describe the main differences and similarities. We also discuss about surfaces with constant non-local mean curvature.
- In chapter 4 we study the Dirichlet problem for a semi-linear equation involving the fractional Laplacian operator. In particular, we prove the existence of weak solutions for the Dirichlet problem in bounded domains of  $\mathbb{R}^n$  under some restriction of the non-linearity, and for periodic functions defined in  $\mathbb{R}^2$ . These results has not been considered before and are new to this thesis. On a forthcoming work, Cabré, Mas and Solà-Morales [14] will study the semi-linear problem for periodic functions.
- In chapter 5 we begin our study of the fractional perimeter functional (1.1.1). We find a simplified expression of the functional and prove that, under a volume constraint, minimizers have constant non-local mean curvature.
- In chapter 6 we prove Theorem 1.1.1. The proof is new and will appear in [14].



## CHAPTER 2

### Classical minimal surfaces

This chapter is intended to be a brief survey on the classical theory of minimal surfaces. We start with an introduction to the main developments in the study of minimal surfaces, from the first formulation of the variational problem in the hands of Joseph Louis Lagrange, to the measure-theoretic ideas of De Giorgi and the important study of cones in  $\mathbb{R}^n$  of J. Simons.

We then dive further into the ideas of De Giorgi, extending the notion of perimeter for a larger class of sets and provide a very brief proof on the existence of minimal surfaces in this general setting. Next we comment on the regularity of minimal surfaces and give the ideas behind the minimality of the Simons' cone, whose proof by Bombieri, De Giorgi and Giusti in 1969 was the finishing line for the regularity theory of minimal surfaces.

Furthermore, and keeping within the chronological framework, we announce a famous conjecture established by De Giorgi in 1978 and its relation to minimal surfaces and, in particular, to the Bernstein problem.

#### 2.1. HISTORICAL INTRODUCTION

The problem of finding minimal surfaces, i.e. of finding the surface of least area among those bounded by a given curve, was one of the first considered after the foundation of the calculus of variations. It is called Plateau's problem, after the blind physicist who did beautiful experiments with soap films and bubbles. In his treatise *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* from 1873, Plateau described a multitude of experiments connected with the phenomenon of capillarity. Among other things, Plateau noted that every contour consisting of a single closed wire, whatever be its geometric form, bounds at least one soap film.

Minimal surface theory originated with the work of Lagrange who, in the 18-th century, considered the variational problem of finding the surface parametrised as  $\mathbf{x} = (x, y, h(x, y))$  of least area stretched across a given close contour. He derived the Euler-Lagrange equation for the solution

$$\frac{d}{dx} \left( \frac{h_x}{\sqrt{1+h_x^2+h_y^2}} \right) + \frac{d}{dy} \left( \frac{h_y}{\sqrt{1+h_x^2+h_y^2}} \right) = 0.$$

Or, computing the derivatives,

$$(1+h_y^2)h_{xx} - 2h_xh_yh_{xy} + (1+h_x^2)h_{yy} = 0.$$

He did not succeed in finding any solution beyond the plane until, in 1776, French mathematician Meusnier discovered that the helicoid and catenoid satisfy the equation, and that the differential expression corresponds to twice the mean curvature of the surface, concluding that surfaces with zero mean curvature are locally area minimising.

It was only in 1930 that a general solution was given to the problem of Plateau in 3-dimensional Euclidean space, with the independent papers of Douglas and Radó. Their methods were quite different; Radó's work held only for rectifiable simple closed curves, that is closed curves of finite length with no self intersections, whereas Douglas results holding for arbitrary simple closed curves. Both relied on setting up minimisation problems; Douglas considered the Dirichlet's integral and its relation to the area functional (see, for instance, the survey [27] by G. Jeremy and M. Micallef). In particular, he was awarded the Fields Medal in 1936 for this work. With the subsequent work of Courant, Morrey, etc. they extended the results to surfaces of higher genus, with many boundary components in general manifolds.

The extension of the problem to higher dimensions turns out to be much more difficult to study. It was not until thirty years later that the problem of Plateau was successfully attacked in its full generality using measure-theoretic methods. We will here introduce the ideas developed by De Giorgi. In his formalism, De Giorgi considered a hypersurface in  $\mathbb{R}^n$  as the boundary of a measurable set  $E$  whose characteristic function  $\varphi_E$  has distributional derivatives that are Radon measures of locally finite total variation, known as *Caccioppoli sets*. There were of course many other mathematicians who worked on minimal surfaces. For instance, one should name the work of Federer and Fleming, whose foundational paper *Normal and Integral Currents* established the existence of solutions to the Plateau problem in a very general setting.

We start now by presenting the problem of finding *non-parametric* minimal surfaces, i.e. finding a surface  $E$  which is the graph of a function  $u(x)$  defined in some bounded domain  $\Omega$  that is of minimal area. If  $u : \Omega \rightarrow \mathbb{R}$  is in the Sobolev space  $W^{1,1}(\Omega)$  of integrable function with weak derivatives, the area of its graph is given by

$$\mathcal{A}(E) = I(u) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx. \quad (2.1.1)$$

The fact that  $\partial E$  is prescribed reads now as  $u = u_0$  on  $\partial\Omega$ , where  $u_0$  is a given function. The variational problem is then to find

$$\inf \left\{ I(u) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx : u \in u_0 + W_0^{1,1}(\Omega) \right\}, \quad (2.1.2)$$

where  $W_0^{1,1}(\Omega) := \left\{ u \in W^{1,1}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\} \subseteq W^{1,1}(\Omega)$ .

Notice that, even though the function  $f(u) = \sqrt{1 + |Du|^2}$  is strictly convex and  $f(\xi) \geq |\xi|^p$  with  $p = 1$ , we cannot use the direct methods of the calculus of variations, since we are lead to work, because of the coercivity condition  $f(\xi) \geq |\xi|$ , in a non-reflexive space  $W^{1,1}(\Omega)$  and we therefore cannot expect the existence of a weakly

convergent subsequence of a minimising sequence. Indeed, in general, there is no minimizer of (2.1.2) in  $u_0 + W_0^{1,1}(\Omega)$ . We therefore need a different approach to deal with this problem.

Any  $C^2(\bar{\Omega})$  function  $u$  is a minimizer of (2.1.1) if and only if it is the solution of the *minimal surface equation* (also called first variation of area)

$$D_i \left\{ \frac{D_i u}{\sqrt{1 + |Du|^2}} \right\} = 0 \text{ in } \Omega \quad (2.1.3)$$

or more briefly

$$\operatorname{div} T(u) = 0, \text{ with } T(u) = Du(1 + |Du|^2)^{-1/2},$$

since the integrand  $\sqrt{1 + |Du|^2}$  is convex. Moreover, since it is strictly convex, the minimizer, if it exists, is unique.

The problem of finding solutions of the Dirichlet problem (2.1.3) is not generally solvable. When working in  $\mathbb{R}^2$ , it was proved by Bernstein that a solution exists for arbitrary data if  $\Omega$  is convex, but may fail to exist without the convexity of the domain, even if the boundary datum  $u_0$  is  $C^\infty$ . In a paper of Jenkins and Serrin, it was proved that the Dirichlet problem in  $n$  dimensions is always solvable if the mean curvature of  $\partial\Omega$  is nowhere negative.

We can instead consider a generalisation of the Dirichlet problem by not imposing the boundary condition  $u = u_0$  as a characterisation of the class of function competing to minimise the area  $\mathcal{A}$ , but rather we introduce it in the functional under consideration as a penalisation, and we look for a minimum of

$$\mathcal{J}(u) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx + \int_{\partial\Omega} |u - u_0| \, dH_{n-1},$$

where  $H_{n-1}$  stands for the  $(n - 1)$ -dimensional Hausdorff measure.

It is easily seen that a solution of the Dirichlet problem also minimizes  $\mathcal{J}$ . On the other hand, the new functional always has a minimum in the class  $BV(\Omega)$  of functions of bounded variation in  $\Omega$  (see Definition 2.2.3), independently of the mean curvature of the boundary. However, in general, the minimising function will not take the value  $u_0$  on  $\partial\Omega$ .

Unfortunately, the above problem is, geometrically, too restrictive. Indeed, any surface can be locally represented as a graph of a function, but is not the case globally. One is therefore led to consider more general surfaces, known as *parametric surfaces*. We refer the reader to Dacorogna's book [20, Chapter 5] for the study of minimal parametric surfaces.

## 2.2. FUNCTIONS OF BOUNDED VARIATION AND CACCIOPPOLI SETS

We will now present the approach of De Giorgi. As we already said, the key idea is to look at hypersurfaces in  $\mathbb{R}^n$  as boundaries of sets. De Giorgi defined then the perimeter in a more general setting. The advantage of defining the perimeter for a larger class of sets is the compactness in the space  $L^1$  of sets with finite perimeter.

As we shall see, it is then not difficult to show existence of solution to the problem of Plateau in this context of minimal boundaries.

Let us start by defining the functional space of functions with bounded variation.

**Definition 2.2.1.** Let  $\Omega \in \mathbb{R}^n$  be an open set and let  $f \in L^1(\Omega)$ . We define

$$\int_{\Omega} |Df| = \sup \left\{ \int_{\Omega} f \nabla \cdot g \, dx : g = (g_1, \dots, g_n) \in C_0^1(\Omega; \mathbb{R}^n), \text{ and } |g(x)| \leq 1 \text{ for } x \in \Omega \right\},$$

where  $\nabla \cdot g = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}$ .

**Remark 2.2.2.** If  $f \in C^1(\Omega)$ , then integration by parts gives

$$\int_{\Omega} f \nabla \cdot g \, dx = - \int_{\Omega} \sum_{i=1}^n \frac{\partial f}{\partial x_i} g_i \, dx = \langle \nabla f, g \rangle_{L^2(\Omega)}$$

for every  $g \in C_0^1(\Omega; \mathbb{R}^n)$ , so that, using Cauchy-Schwarz inequality

$$\int_{\Omega} |Df| = \int_{\Omega} |\nabla f|.$$

More generally, if  $f$  belongs to the Sobolev space  $W^{1,1}(\Omega)$  and  $\Omega$  has Lipschitz boundary, then

$$\int_{\Omega} |Df| = \int_{\Omega} |\nabla f| \, dx < \infty,$$

where  $\nabla f$  is the derivative of  $f$  in the weak sense.

**Definition 2.2.3** (Bounded Variation). A function  $f \in L^1(\Omega)$  is said to have bounded variation in  $\Omega$  if

$$\int_{\Omega} |Df| < \infty.$$

We define the space  $BV(\Omega)$  as the space of all functions in  $L^1(\Omega)$  with bounded variation.

**Remark 2.2.4.**  $W^{1,1}(\Omega) \subseteq BV(\Omega) \subseteq L^1(\Omega)$ . The fact that the two spaces are not equal can be seen with the next example. Suppose  $E \subseteq \mathbb{R}^n$  has  $C^2$  boundary and consider  $\varphi_E$ , the characteristic function of  $E$ . If in addition  $E$  is bounded, then

$$\int_{\Omega} \varphi_E \, dx = |E \cap \Omega| = \text{Lebesgue measure of } E \cap \Omega$$

and  $\varphi_E \in L^1(\Omega)$ . However,  $\varphi_E$  does not belong to  $W^{1,1}(\Omega)$ .

To see this, suppose  $g \in C_c^1(\Omega; \mathbb{R}^n)$ . Then, by Gauss-Green theorem,

$$\int_{\Omega} \varphi_E \nabla \cdot g \, dx = \int_E \nabla \cdot g \, dx = \int_{\partial E} \langle g, \nu \rangle \, dH^{n-1},$$

where  $\nu(x)$  is the outward unit normal to  $\partial E$  at  $x$  and  $H^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure. Now,  $|\nu(x)| = 1$ , so that, if  $|g(x)| \leq 1$  and  $g \in C_0^1(\Omega, \mathbb{R}^n)$ , then

$$\int_{\partial E} \langle g, \nu \rangle \, dH^{n-1} \leq H^{n-1}(\partial E \cap \Omega)$$



and hence

$$\int_{\Omega} |D\varphi_E| = \sup \left\{ \int_{\Omega} \varphi_E \nabla \cdot g \, dx : g \in C_c^1(\Omega; \mathbb{R}^n), |g(x)| \leq 1 \right\} \leq H^{n-1}(\Omega \cap \partial E) < \infty.$$

Thus  $\varphi_E \in BV(\Omega)$ , and in fact

$$\int_{\Omega} |D\varphi_E| = H^{n-1}(\partial E \cap \Omega).$$

We now state the lower semi-continuity of the semi-norm  $\int_{\Omega} |Df|$ . This result, along with the compactness theorem we are going to announce shortly after, are the key ingredients for the existence of minimal surfaces.

**Theorem 2.2.5** (Semi-continuity). *Let  $\Omega \in \mathbb{R}^n$  be an open set and  $\{f_i\}$  a sequence of functions in  $BV(\Omega)$  which converge in  $L^1_{loc}(\Omega)$  to a function  $f$ . Then*

$$\int_{\Omega} |Df| \leq \liminf_{j \rightarrow \infty} \int_{\Omega} |Df_j|. \quad (2.2.1)$$

PROOF. Let  $g \in C_0^1(\Omega; \mathbb{R}^n)$  such that  $|g| \leq 1$ . Then

$$\int_{\Omega} f \nabla \cdot g = \lim_{j \rightarrow \infty} \int_{\Omega} f_j \nabla \cdot g = \liminf_{j \rightarrow \infty} \int_{\Omega} f_j \nabla \cdot g \leq \liminf_{j \rightarrow \infty} \int_{\Omega} |Df_j|.$$

□

**Remark 2.2.6.** Under the norm  $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_{\Omega} |Df|$ , the functional space  $BV(\Omega)$  is a Banach space.

**Theorem 2.2.7** (Compactness). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  sufficiently regular (for instance, with a Lipschitz-continuous boundary). Then the set of functions uniformly bounded with the  $BV$ -norm are relatively compact in  $L^1(\Omega)$ .*

PROOF. See [26, Theorem 1.19].

□

We now define the perimeter of a set. As we already commented, with the above compactness result together with the semi-continuity theorem, we will prove the existence of minimising sets.

**Definition 2.2.8** (Perimeter and Caccioppoli Set). Let  $E$  be a Borel set and  $\Omega$  an open set in  $\mathbb{R}^n$ . We define the perimeter of  $E$  in  $\Omega$  as

$$P(E, \Omega) = \int_{\Omega} |D\varphi_E| = \sup \left\{ \int_E \nabla \cdot g \, dx : g \in C_0^1(\Omega; \mathbb{R}^n), |g(x)| \leq 1 \right\}. \quad (2.2.2)$$

If  $\Omega = \mathbb{R}^n$ , we denote  $P(E) = P(E, \mathbb{R}^n)$ . If a Borel set has locally finite perimeter, that is, if  $P(E, \Omega) < \infty$  for every bounded open set  $\Omega$ , then  $E$  is called a *Caccioppoli set*.

We say that a set  $E$  is of minimal perimeter in  $\mathbb{R}^n$  if it has minimal perimeter for every ball  $B_R$  of radius  $R > 0$ .

**Remark 2.2.9.** Recall that, for a sufficiently regular set  $E$ , we have

$$P(E, \Omega) = H^{n-1}(\Omega \cap \partial E).$$

We can finally state and prove the existence of minimal surfaces for the perimeter (2.2.2). As you can see, the proof is quite short and easy.

**Theorem 2.2.10** (Existence of minimal surfaces). *Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and let  $L$  be a Caccioppoli set. Then there exists a set  $E$  coinciding with  $L$  outside  $\Omega$  and such that*

$$\int_{\mathbb{R}^n} |D\varphi_E| \leq \int_{\mathbb{R}^n} |D\varphi_F|$$

for every  $F$  with  $F = L$  outside of  $\Omega$ .

**PROOF.** Since  $\Omega$  is bounded, there exists a real number  $R > 0$  such that  $\Omega \subset\subset B_R$ . Now, if  $F = L$  outside  $\Omega$ , then

$$\int_{\mathbb{R}^n} |D\varphi_F| = \int_{B_R} |D\varphi_F| + \int_{\mathbb{R}^n - B_R} |D\varphi_L|.$$

So we need only to show that there exists a set  $E$  in  $B_R$  coinciding with  $L$  outside of  $\Omega$  such that

$$\int_{B_R} |D\varphi_E| \leq \int_{B_R} |D\varphi_F|$$

for each set  $F$  in  $B_R$  coinciding with  $L$  outside of  $\Omega$ . Obviously, we have that  $\int_{B_R} |D\varphi_F|$  is bounded below by 0 and so, if  $\{E_j\}$  is a minimising sequence, we must have that  $\int_{B_R} |D\varphi_{E_j}|$  is uniformly bounded. Furthermore  $B_R$  is bounded so  $\int_{B_R} |D\varphi_{E_j}|$  is uniformly bounded. Hence  $\varphi_{E_j}$  is a uniformly bounded sequence in  $BV(\Omega)$  and, by compactness, there exists a subsequence, still denoted  $\{\varphi_{E_j}\}$ , which converges in  $L^1(B_R)$  to a function  $f$ . Since  $\varphi_{E_j}(x) \rightarrow f(x)$  for almost all  $x$  in  $B_R$ , and  $\varphi_{E_j}(x)$  is either 1 or 0 we may assume that  $f$  is the characteristic function of a set  $E$  (up to a subset of measure 0) which coincides with  $L$  outside of  $\Omega$ . Now, by the semi-continuity results, we see that  $E$  must provide the required minimum.  $\square$

Roughly speaking,  $\partial E$  minimizes the area among all surfaces with boundary  $\partial L \cap \partial \Omega$ .

### 2.3. FIRST AND SECOND VARIATION OF AREA

Suppose  $E \subseteq \mathbb{R}^n$  is a minimal set in  $B_1$ , and  $\{F_t\}$  is a one parameter family of diffeomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $F_0 = I := \text{identity}$  and the maps  $F_t - I$  have compact support in  $B_1$ . The sets

$$E_t = F_t(E) = \{F_t(x) : x \in E\}$$

must equal  $E$  outside  $B_1$  and so

$$\int_{B_1} |D\varphi_E| \leq \int_{B_1} |D\varphi_{E_t}|.$$

Then, assuming appropriate smoothness, we see that

$$\frac{d}{dt} A(t)|_{t=0} = 0 \tag{2.3.1}$$

and

$$\frac{d^2}{dt^2}A(t)|_{t=0} \leq 0. \quad (2.3.2)$$

The derivatives (2.3.1) and (2.3.2) are called *first* and *second variation of area* respectively.

We shall first give an expression for the first and second variation of area. We will then consider the case where  $E$  is a cone in  $\mathbb{R}^n$ , smooth everywhere except possibly at the origin. In particular, we will give the ideas behind the proof for the minimality of the Simons cone for  $n \geq 8$ . On the other hand, it was proved by Simons in 1968 that, for  $n < 8$ ,  $\partial E$  is a hyperplane (see Theorem 2.4.2).

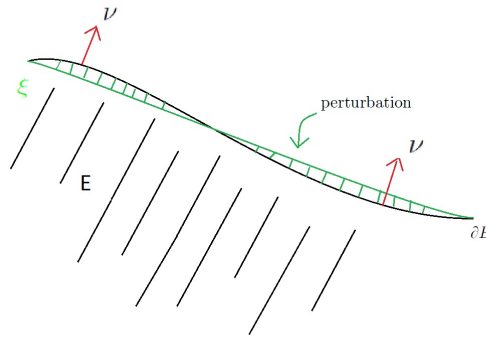


FIGURE 1. Small deformation perpendicular to the surface  $\partial E$

Choosing a particular deformation that shifts the original set  $E$  in a direction normal to the surface, i.e.  $\varphi_t = I + t \cdot \xi \cdot \nu$ , with  $\xi \in C_0^1$ ,  $\text{supp } \xi \subset B_R$ ,  $\nu$  is the unit exterior normal to  $\partial E$  and  $I$  stands for the identity (see Figure 1), we can obtain the following variation formulas:

$$\begin{aligned} \left\{ \frac{d}{dt} \int_A |D\varphi_{E_t}| \right\}_{t=0} &= \int_{\partial E} \mathcal{H} \xi dH_{n-1}, \\ \left\{ \frac{d^2}{dt^2} \int_A |D\varphi_{E_t}| \right\}_{t=0} &= \int_{\partial E} \left\{ |\delta\xi|^2 - (c^2 - \mathcal{H}^2)\xi^2 \right\} dH_{n-1} \end{aligned} \quad (2.3.3)$$

where  $\mathcal{H} = \mathcal{H}(x)$  is the mean curvature of the surface  $\partial E$  at  $x$ ,  $c^2 = c^2(x)$  is the sum of the squares of the principal curvatures of  $\partial E$  calculated at  $x$  (also the square of the norm of the second fundamental form) and  $\delta\xi$  is the vector  $(\delta_1\xi, \dots, \delta_n\xi)$ , with  $\delta_i = D_i - v_i \sum_{h=1}^n v_h D_h$  the *tangential derivatives* at  $x \in \partial E$ .

The reader should refer to the book of Enrico Giusti [26, Chapter 10] for more details.

## 2.4. REGULARITY AND MINIMAL CONES

In 1969 Bombieri, De Giorgi and Giusti proved that the *Simons' cone* (2.4.3) is a minimizer of the area, i.e. that any hypersurface which coincides with the cone outside a compact set  $K$  must have larger area in  $K$ . Thus providing the first example of a minimal surface with a singularity (at the origin).

The proof of Bombieri, De Giorgi and Giusti is quite involved. They use the nice tool of *calibrations* which, roughly speaking, are divergence free units vector fields that extends the normal field of a surface to the whole ambient space. Using the divergence theorem one finds that if such a field can be found, then the surface is minimal. The problem is then devoted to prove the existence of such vector field. Instead, we will explain the ideas of G. De Philippis and E. Paolini [32]. It utilises the notion of *sub-calibrations*, but only assumes knowledge of some basic facts about functions of bounded variation.

Since the blow-up<sup>1</sup> (or zoom in) of a minimal surface in every point is a minimal cone (if the point is non-singular, the cone is actually a half-space) the study of minimal cones is very important in the theory of minimal surfaces. If we could prove the non-existence of minimal cones with only a singularity in the origin in dimensions  $\leq n$ , the above would imply that the boundary of every minimal surface in dimension  $\leq n$  is regular. Otherwise we could blow up a singularity to arrive at some singular minimal cone. Unfortunately, this is only true for  $n \leq 7$ . In particular, the minimality of the Simons cone is by himself a very important step in the theory of regularity of minimal surfaces in higher dimension. Results of De Giorgi, Fleming, Almgren and Simons proved that an  $(n - 1)$ -dimensional minimal surface in  $\mathbb{R}^n$  is regular outside a singular set whose dimension is at most  $n - 8$ . The Simons cone is an example showing that the partial regularity results is optimal.

From now on, we will suppose that the set  $E$  is regular enough (it may not be bounded), and consider the perimeter of  $E$  in an open ball  $B_R$  as the  $(n - 1)$ -dimensional Hausdorff measure of  $\partial E \cap B_R$ . Therefore, a set  $E \subset \mathbb{R}^n$  is of minimal perimeter if and only if  $\forall B_R$  and  $\forall F \subset \mathbb{R}^n$  such that  $E \cap B_R^c = F \cap B_R^c$ , we have  $P(E; B_R) \leq P(F; B_R)$ .

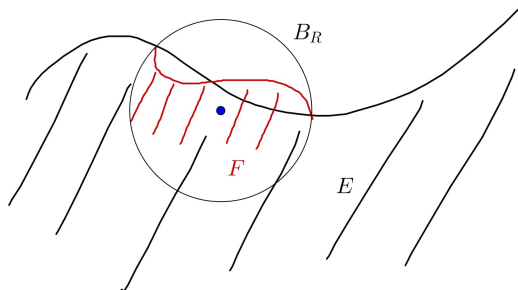


FIGURE 2.  $E$  is minimal among all  $F$  in  $B_R$

<sup>1</sup>For a set  $E \subset \mathbb{R}^n$  with  $0 \in \partial E$ , we call the set  $E_t := \{x \in \mathbb{R}^n | tx \in E\}$ ,  $t > 0$  a blow-up of  $E$  in 0.

**Remark 2.4.1.** A hyperplane is a set of minimal surface. The isoperimetric inequality states that a sphere has the smallest surface area per given volume.

We next state a very important theorem from James Simons [39] which excludes the existence (in low dimension) of singular minimising cones possessing only a vertex singularity.

**Theorem 2.4.2** (J. Simons). *Suppose  $E$  is a cone, such that  $\partial E$  is regular in  $\mathbb{R}^n - \{0\}$ . Suppose that  $\mathcal{H} \equiv 0$  and that the second variation of the area is non-negative. Then either  $\partial E$  is a hyperplane or  $n \geq 8$ .*

PROOF. See [26, Theorem 10.10]. □

In his work, J. Simons provided with the first example of stable surface with constant mean curvature and with a singularity (at the origin), the Simons' cone.

**Definition 2.4.3** (Simons Cone). Let  $E_S := \{(x', x'') \in \mathbb{R}^n \times \mathbb{R}^n : |x'|^4 - |x''|^4 < 0\} \subset \mathbb{R}^{2n}$ . The boundary of  $E$ ,  $\mathcal{C}_S := \{x_1^2 + \dots + x_n^2 = x_{n+1}^2 + \dots + x_{2n}^2\}$  is called the *Simons' cone*.

The mean curvature of the Simons cone vanishes at every point outside the origin. Therefore the first variation of the area vanishes along any deformation induced by compactly supported smooth vector fields. It was pointed out by Simons that this cone is also stable, meaning that the second variation of the area along smooth deformations is non-negative.

Let us now define the notion of *sub-calibration* which will be a key tool in the proof of the minimality of the Simons' Cone.

**Definition 2.4.4** (sub-calibration). Let  $E \subseteq \Omega$  be a measurable set such that the boundary  $\partial E \cap \Omega$  has  $C^2$  regularity. We say that a vector field  $\xi \in C^1(\Omega; \mathbb{R}^n)$  is a sub-calibration of  $E$  if it satisfies the following properties:

- (1)  $\xi(x) = \nu_E(x)$  is the exterior unit normal vector to  $\partial E$  for all  $x \in \partial E \cap \Omega$ .
- (2)  $\nabla \cdot \xi(x) \leq 0$  for all  $x \in E \cap \Omega$ .
- (3)  $|\xi(x)| \leq 1$  for all  $x \in \Omega$ .

Unlike the original approach with calibration, it is indeed very easy to find an explicit sub-calibration for our measurable set  $E$ , as next result points out.

**Proposition 2.4.5.** *Let  $u(x', x'') = \frac{|x'|^4 - |x''|^4}{4}$ , for  $(x', x'') \in \mathbb{R}^n \times \mathbb{R}^n$ . We have*

$$\nabla \cdot \frac{\nabla u}{|\nabla u|} = \frac{(|x'|^4 - |x''|^4) \left[ (n-1)|x'|^4 - (n+2)|x'|^2|x''|^2 + (n-1)|x''|^4 \right]}{|\nabla u|^3}. \quad (2.4.1)$$

Furthermore,  $\frac{\nabla u}{|\nabla u|}$  has the same sign as  $u(x', x'')$ . Hence, we have that the vector field  $\xi = \frac{\nabla u}{|\nabla u|}$  is a sub-calibration of the set  $E$ .

PROOF. The computation of the divergence is straightforward. To prove the second assumption, we just need to see that the term between brackets is non-negative. With the substitution  $t = \frac{|x'|^2}{|x''|^2}$  we obtain the relation

$$(n - 1)t^2 + (n + 2)t + (n - 1) \geq 0$$

which holds true for all  $t$  if and only if the discriminant

$$\Delta = (n + 2)^2 - 4(n - 1)^2 = 3n(4 - n) \leq 0.$$

Which in turn holds true if and only if  $n \geq 4$ . □

Notice that  $\mathcal{H}_S = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right)_{\mathcal{C}_S} = 0$ , i.e. the Simons cone is a critical surface of the perimeter.

We will now announce the minimality of the Simons cone as was first stated by Bombieri, De Giorgi and Giusti [8].

**Theorem 2.4.6** (Bombieri-De Giorgi-Giusti, 1969). *Let  $n \geq 4$ . The set  $E \subset \mathbb{R}^{2n}$  defined by*

$$x_1^2 + x_2^2 + \dots + x_n^2 \geq x_{m+1}^2 + x_{m+2}^2 + \dots + x_{2n}^2$$

*has an oriented boundary of least area.*

**Remark 2.4.7.** It is easy to see that the Simons' cone is not of minimal area in  $\mathbb{R}^2$ . A straight line will be of less perimeter than going through the origin (see Figure 3).

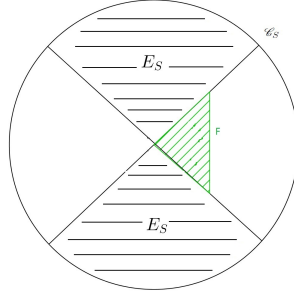
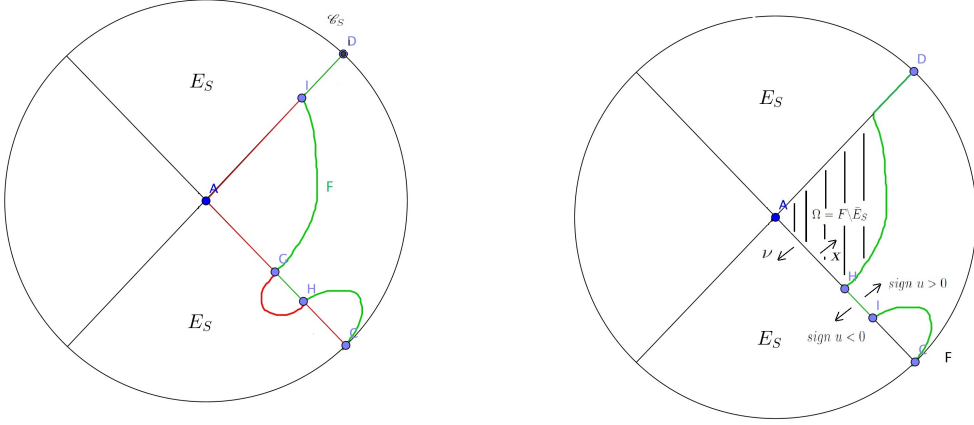


FIGURE 3. Non-minimality of the Simons' cone in  $\mathbb{R}^2$

**Remark 2.4.8.** A very heuristic argument for the minimality of  $\mathcal{C}_S$  is the following. We have  $\partial E$  a  $(n - 1)$ -dimensional surface. In polar coordinates the Jacobian for the Lebesgue measure is  $r^{n-1}drd\theta$ . If  $r \ll 1$ , then as  $n - 2$  increases,  $r^{n-2}$  decreases. Therefore, the things we do near the origin do not account much, and we may think that we want to go through the origin.

We shall now begin with the proof of Theorem 2.4.6.

FIGURE 4. Boundary of  $F'$  in green and  $F''$  in red

PROOF OF THEOREM 2.4.6 (C. DE PHILIPPS, E. PAOLINI, 2009). Take any competitor  $F$  and suppose it is regular enough. Consider the sets  $F' = F \setminus \bar{E}_S$  and  $F'' = E \setminus \bar{F}^c$ .

We claim the following:

**Claim 2.4.9.**  $P(E_S, B_R) \leq P(F', B_R)$  and  $P(E_S, B_R) \leq P(F'', B_R)$ .

We only prove the first inequality, the other being very similar.

PROOF CLAIM. From Proposition 2.4.5 we have  $\nabla \cdot \xi \geq 0$  in  $E_S^c \cap B_R$ . Hence,

$$0 \leq \int_{\Omega} \nabla \cdot \xi = \int_{\Omega} \langle \xi, \nu \rangle = \int_{\partial E_S \cap \bar{\Omega}} -1 + \int_{\partial F' \cap \bar{\Omega}} \langle \xi, \nu \rangle \Rightarrow P(E_S, B_R) \leq P(F', B_R),$$

where we have used the divergence theorem and the fact that  $\xi = \nu$  on the boundary of  $E_S$ .  $\square$

Now, we have  $P(F, B_R) = P(F', B_R) + P(F'', B_R) - P(F' \cap F'', B_R) \geq 2P(E_S, B_R) - P(E_S, B_R) = P(E_S, B_R)$ .  $\square$

Remember we have supposed that the class of competitors are smooth enough. For the general case, we use the notion of sub-minimal sets, which are minimal sets among all sets  $F \subseteq E$  such that  $E \setminus F \subset\subset A$ , for all bounded open sets  $A \subseteq \Omega$ . Then we prove in a similar way as before that the sequences of sets  $E_k$  and  $F_k^c$  defined as

$$E_k := \left\{ (x', x'') \in \mathbb{R}^n \times \mathbb{R}^n : u(x', x'') \leq -\frac{1}{4} \right\},$$

$$F_k := \left\{ (x', x'') \in \mathbb{R}^n \times \mathbb{R}^n : u(x', x'') \leq \frac{1}{4} \right\}$$

are both sub-minimal in  $\Omega$  and converge to  $\mathcal{C}_S$  and  $\mathcal{C}_S^c$  in  $L_{loc}^1(\Omega)$  respectively. It turns out that the  $L_{loc}^1$  limit of a sub-minimal set is also sub-minimal, and that if both

$E$  and  $E^c = \Omega \setminus E$  are sub-minimal in  $\Omega$ , then  $E$  is minimal in  $\Omega$ .

## 2.5. THE BERNSTEIN PROBLEM AND A CONJECTURE OF DE GIORGI

Is it true that  $C^2$  solutions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  of the minimal surface equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = 0$$

are necessarily affine functions?

S.N. Bernstein proved it true for  $n = 2$ . In fact, the above question is affirmative in dimension up to 7, but is instead negative for  $n \geq 8$ , as was shown by E. Bombieri, E. De Giorgi and E. Giusti [8], where, using the Simons cone they constructed a non-affine entire solution of the minimal surface equation.

Related to minimal surfaces we have a very important conjecture, named after Ennio De Giorgi, who conjectured that entire, bounded solutions of the Allen-Cahn equation (2.5.1) are one-dimensional, at least in dimension up to 8.

**Conjecture 2.5.1** (De Giorgi). *Let us consider a solution  $u \in C^2(\mathbb{R}^n)$  of*

$$-\Delta u = u - u^3 \tag{2.5.1}$$

*such that*

$$|u| \leq 1, \quad \partial_n u > 0$$

*in the whole  $\mathbb{R}^n$ . Is it true that all level sets  $\{u = \lambda\}$  of  $u$  are hyperplanes, at least if  $n \leq 8$ ?*

The goal is thus to establish the one-dimensional character or symmetry of  $u$ , namely, that  $u$  only depends on one variable or, equivalently, that the level sets of  $u$  are hyperplanes.

In 1997 Ghoussoub and Gui [25] proved De Giorgi conjecture for  $n = 2$  using a Liouville-type result developed by Berestycki, Caffarelli and Nirenberg in one of their papers on qualitative properties of solutions of semi-linear elliptic equations. Using similar techniques, Ambrosio and Cabré [4] extended these results to dimension  $n = 3$ . In 2003 the conjecture was proved by O. Savin in [34] for  $n \leq 8$  under the additional hypothesis

$$\lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1 \text{ for all } x' \in \mathbb{R}^{n-1}.$$

There is an heuristic argument that connects the conjecture of De Giorgi with the Bernstein problem for minimal graphs. Consider the functional  $F(u) = (1 - u^2)^2/4$ . Note that equation (2.5.1) is just the Euler-Lagrange equation of the energy

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4} \int_{\Omega} (1 - u^2)^2 \tag{2.5.2}$$



for bounded domains  $\Omega \subset \mathbb{R}^n$ . The first integral represents the kinetic energy, while the second one is the potential energy. With  $u$  as in the conjecture, consider the blow-down sequence

$$u_\epsilon(y) = u(y/\epsilon) \quad \text{for } y \in B_1 \subset \mathbb{R}^n,$$

and the penalised energy of  $u_\epsilon$  in  $B_1$ :

$$J_\epsilon(u_\epsilon) = \int_{B_1} \left\{ \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u_\epsilon) \right\} dy.$$

As  $\epsilon \rightarrow 0$ , the functionals  $J_\epsilon$  will  $\Sigma$ -converge to a functional which is finite only for characteristic functions with values  $\{-1, 1\}$  and equal (up to a multiplicative constant) to the area of the hypersurface of discontinuity. Heuristically, the sequence  $u_\epsilon$  is expected to converge to a characteristic function whose hypersurface of discontinuity  $S$  has minimal area or is at least stationary. The set  $S$  describes the behaviour at infinity of the level sets of  $u$ , and  $S$  is expected to be the graph of a function on  $\mathbb{R}^{n-1}$  (since the level sets of  $u$  are graphs due to hypothesis  $\partial_n u > 0$ ). The conjecture of De Giorgi states that the level sets are hyperplanes. The connection with the Bernstein problem is due to the fact that every minimal graph of a function defined on  $\mathbb{R}^m = \mathbb{R}^{n-1}$  is known to be a hyperplane whenever  $m \leq 7$ , i.e.  $n \leq 8$ .

Roughly speaking, one understands the behaviour of such a minimal surface in a neighbourhood of a point  $x_0$  by using the blow-up technique explained above. Dilating the picture more and more we end up with a limiting minimal surface defined in the whole space.



## CHAPTER 3

### Non-local minimal surfaces

In this chapter we introduce the notion of non-local minimal surfaces, i.e. sets that are minimizers of a non-local version of the classical perimeter. We will call it *fractional* or *non-local perimeter*. These surfaces can be interpreted as a non-infinitesimal version of classical minimal surfaces.

In the previous chapter we introduced the classical theory of minimal surfaces. We saw how minimal surfaces (or more generally surfaces with constant mean curvature) arise in physical situations where one has two phases interacting (eg. water and ice) and the energy of interaction is proportional to the area of the interface, which is due to the interaction between particles in both phases being negligible when they are far apart. Non-local minimal surfaces then describe phenomena where the interaction potential does not decay fast enough as particles get farther and farther apart, so that two particles on different phases contribute a non-trivial amount to the total interaction energy even if they are away from the interface.

Within these years, there has been a surge of activity in the study of fractional or non-local minimal surfaces. The efforts have been focused, mainly, in the regularity of the minimizers and in finding explicit examples. Still, apart from dimension 2, there is a lot to be understood, mainly for the classification of stable non-local minimal cones.

#### 3.1. THE FRACTIONAL PERIMETER

The notion of fractional perimeter was first introduced by Caffarelli, Savin and Roquejoffre [16]. Their work was motivated precisely by the structure of inter-phases that arise in classical field models when very long space correlation are present.

To introduce it in a soft way, we consider a measurable set  $E \subseteq \mathbb{R}^n$ , with  $n \geq 2$  and a bounded, open domain  $\Omega$ . For simplicity, we assume that the domain  $\Omega$  has smooth boundary.

**Definition 3.1.1.** Let  $A$  and  $B$  be two disjoint measurable sets. For any fixed  $s \in (0, 1/2)$ , we define the functional

$$J(A; B) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_A(x)\chi_B(y)}{|x - y|^{n+2s}} dx dy. \quad (3.1.1)$$

Clearly

$$\begin{aligned} J(A; B) &\geq 0, \quad J(A; B) = J(B; A), \\ J(A_1 \cup A_2; B) &= J(A_1; B) + J(A_2; B) \quad \text{for } A_1 \cap A_2 = \emptyset. \end{aligned}$$

**Definition 3.1.2** (Fractional perimeter). For a bounded set  $\Omega \subseteq \mathbb{R}^n$  and for a measurable set  $E \subset \mathbb{R}^n$  we define the fractional perimeter (or  $s$ -perimeter when we want to emphasise the fractional factor  $s$ )

$$Per_s(E; \Omega) := J(E \cap \Omega; \Omega \setminus E) + J(E \cap \Omega; (\mathbb{R}^n \setminus E) \cap (\mathbb{R}^n \setminus \Omega)) + J(E \setminus \Omega; \Omega \setminus E). \quad (3.1.2)$$

to be the “ $\Omega$ -contribution” for the  $H^s$ -norm of the characteristic function of  $E$  (see Remark 3.1.3 below).

It is non-local in the sense that it is not determined by the behaviour of  $E$  in a neighbourhood of  $\partial E$ . This is precisely a difficulty when studying non-local functionals. On the other hand, the functional is well defined for every measurable set. In particular, there is no need to introduce Caccioppoli sets in this case.

Roughly speaking, the fractional perimeter captures the interactions between a set  $E$  and its complement. These interactions occur in the whole space and are weighted by an homogeneous and rotationally invariant kernel with polynomial decay. We remove possible infinite contribution to the energy which come from infinity but which do not change the variational problem.

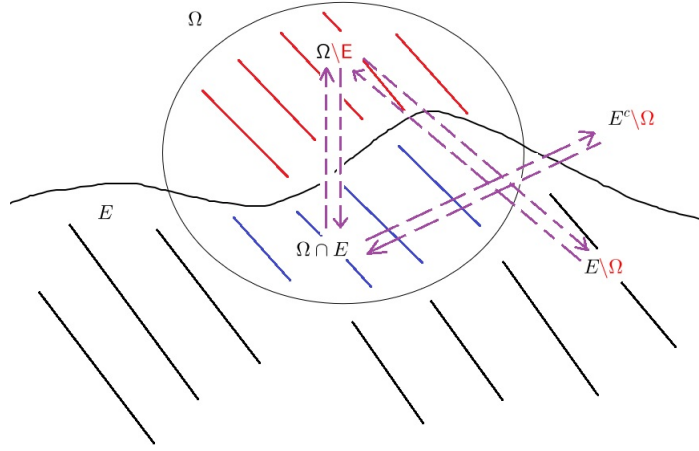


FIGURE 5. Fractional Perimeter

**Remark 3.1.3.** We will consider for  $s \in (0, 1/2)$  minimizers of the  $H^s$  semi-norm of the characteristic function  $\chi_E$  of a set  $E$  which is fixed outside a domain  $\Omega \subset \mathbb{R}^n$ ;

$$\begin{aligned} \|\chi_E\|_{H^s(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x - y|^{n+2s}} dx dy \\ &= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_E(x) \chi_{E^c}(y)}{|x - y|^{n+2s}} dx dy, \end{aligned}$$

where the integrals are taken in the principal value sense. Note that  $s < \frac{1}{2}$ ; no indicator function is in  $H^s$  for  $s \geq \frac{1}{2}$ . The semi-norm  $\|\chi_E\|_{H^s(\mathbb{R}^n)}$  makes sense if  $E$  is smooth and

bounded. To take into account unbounded sets, we remove the non-convergent part of  $\|\chi_E\|_{H^s(\mathbb{R}^n)}$ .

**Remark 3.1.4.** We remark that balls and cylinders have finite fractional perimeter on every bounded set  $\Omega$  of  $\mathbb{R}^n$ .

The functional in (3.1.2) naturally produces a minimisation problem:

**Definition 3.1.5** (Minimal set). We say that  $E$  is a minimizer for  $Per_s$  in  $\Omega$  if for any set  $F$  with  $F \cap \Omega^c = E \cap \Omega^c$  we have

$$Per_s(E; \Omega) \leq Per_s(F; \Omega).$$

We say that  $E$  is  $s$ -minimal in  $\mathbb{R}^n$  if it is  $s$ -minimal in any ball  $B_R \subset \mathbb{R}^n$  for any  $R > 0$ .

**Remark 3.1.6.** The set  $E \cap \Omega^c$  plays the role of “boundary data” for  $E \cap \Omega$ . If  $\Omega$  is a bounded Lipschitz domain, then  $Per_s(E; \Omega)$  is bounded by  $Per_s(E \setminus \Omega; \Omega) < \infty$ .

We are now going to prove some basic properties of  $s$ -minimizers. Namely, lower semi-continuity of the fractional perimeter and existence of  $s$ -minimizers.

**Proposition 3.1.7** (Lower semi-continuity). *If  $\chi_{E_n} \rightarrow \chi_E$  in  $L^1_{loc}$  then*

$$\liminf_{n \rightarrow \infty} J_\Omega(E_n) \geq J_\Omega(E).$$

PROOF. Recall that

$$J(A; B) = \iint_{A \times B} \frac{dx dy}{|x - y|^{n+2s}}.$$

It is clear that if  $\chi_{A_n} \rightarrow \chi_A$ ,  $\chi_{B_n} \rightarrow \chi_B$  in  $L^1_{loc}(\mathbb{R}^n)$  then any sequence contains a subsequence, say  $n_k$  such that for almost every  $(x, y)$

$$\chi_{A_{n_k}}(x) \chi_{B_{n_k}}(y) \rightarrow \chi_A(x) \chi_B(y).$$

Fatou’s lemma implies

$$\liminf_k J(A_{n_k}, B_{n_k}) \geq J(A, B).$$

□

**Theorem 3.1.8** (Existence of minimizers). *Let  $\Omega$  be a bounded Lipschitz domain and  $E_0 \subset \Omega^c$  be a given set. There exists a set  $E$ , with  $E \cap \Omega^c = E_0$  such that*

$$\inf_{F \cap \Omega^c = E_0} J_\Omega(F) = J_\Omega(E).$$

PROOF. The infimum is bounded since  $J_\Omega(E_0) < \infty$ . Let  $F_n$  be a minimising sequence. The  $H^s$  norms of the characteristic functions of  $F_n \cap \Omega$  are bounded. Thus, by compactness, there is a subsequence that converges in  $L^1(\mathbb{R}^n)$  to a set  $E \cap \Omega$ . Now the results follows from the lower semi-continuity. □

In [16] Caffareli et al. also proved that  $s$ -minimizers satisfy a suitable integral equation, that is the Euler-Lagrange equation corresponding to the functional in (3.1.2). Namely, suppose that  $E$  is  $s$ -minimal in  $\Omega$  and that  $x \in \Omega \cap (\partial E)$ . Then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x - y|^{n+2s}} dy = 0. \quad (3.1.3)$$

From the geometric point of view, (3.1.3) states that a suitable average of  $E$  (centred at any point of  $\partial E$ ) is balanced by the average of its complement. See Figure 6 below for a graphical interpretation. If the set is “sufficiently symmetric”, the number of 1’s and  $-1$ ’s is constant at every point of the boundary.

**Definition 3.1.9.** Let  $E$  be an open set in  $\mathbb{R}^n$  with  $C^2$ -boundary. Then for every  $x \in \partial E$ , the non-local or fractional mean curvature of  $\partial E$  at  $x$  (that we call NMC for short) is given by

$$H_E(x) = -P.V. \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x - y|^{n+2s}} dy := \lim_{\epsilon \searrow 0} \int_{|x-y| \geq \epsilon} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x - y|^{n+2s}} dy. \quad (3.1.4)$$

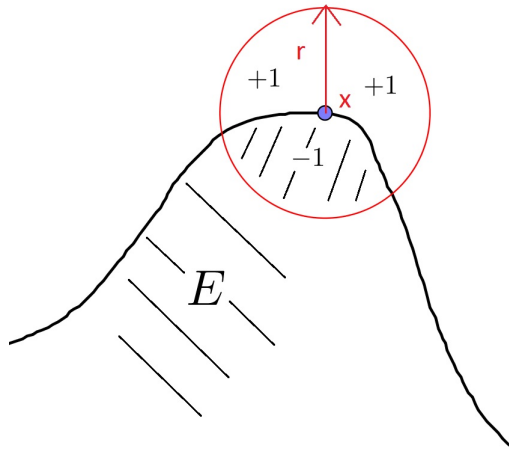


FIGURE 6. Non-local Minimal Curvature

Note that if  $\partial E$  is  $C^2$  in a neighbourhood of  $x$ , the NMC is well defined in the principal value sense. Because of the singularity of the denominator, a weaker notion is considered in the viscosity sense for non-smooth sets  $E$  (see [16, Theorem 5.1] for more details). In fact, it is sufficient for the boundary  $\partial E$  to be of class  $C^{1,\beta}$  for some  $\beta > 2s$  (in contrast with the classical mean curvature, which needs of at least  $C^2$  regularity).

**Remark 3.1.10.** As the kernel is invariant under Euclidean symmetries, we conclude for instance that any sphere  $\partial B_r(x)$  has constant non-local mean curvature. Note that the minus sign in front of the integral makes that balls and cylinders have constant positive NMC, in contraposition with the classical mean curvature.

With this notation,  $s$ -minimal surfaces have vanishing NMC, i.e. are critical sets of the NMC, and the analogy with the classical perimeter case is evident. To make

the analogy even stronger, we will see in the next section that the fractional perimeter converges to the classical perimeter, with good geometric and functional analytic properties.

The following is a more geometric expression for the NMC. Using an integration by parts on (3.1.4) and the fact that  $\nabla_y \cdot \left\{ (x-y)|x-y|^{-(n+2s)} \right\} = 2s|x-y|^{-(n+2s)}$  we arrive at

$$H_E(x) = -\frac{1}{s} P.V. \int_{\partial E} \frac{(x-y) \cdot \nu(y)}{|x-y|^{n+2s}} dy, \quad (3.1.5)$$

where  $\nu(y)$  denotes the outer unit normal to  $\partial E$ . We point out that the integral is absolutely convergent in the Lebesgue sense if  $\partial E$  is of class  $C^{1,\beta}$ ,  $\beta > 2s$ .

Another way of thinking of this is the following: using the notation  $\tilde{\chi}_E = \chi_E - \chi_{E^c}$  we have

$$\begin{aligned} H_E(x) &= -\frac{1}{2} P.V. \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x+y) - \tilde{\chi}_E(x-y)}{|y|^{n+2s}} dy \\ &= -\frac{1}{2} P.V. \int_{\mathbb{R}^n} \frac{\tilde{\chi}_E(x+y) - \tilde{\chi}_E(x-y) - 2\tilde{\chi}_E(x)}{|y|^{n+2s}} dy \\ &= \frac{(-\Delta)^s \tilde{\chi}_E(x)}{C(n,s)}, \end{aligned}$$

where  $C(n,s)$  is a dimensional constant depending on  $s$  and  $(-\Delta)^s$  denotes the fractional Laplacian operator. Using this suggestive representation, the Euler-Lagrange equation becomes

$$(-\Delta)^s \tilde{\chi}_E = 0 \text{ along } \partial E.$$

We will introduce the fractional Laplacian with more details in chapter 2.

As for the regularity of  $s$ -minimal surfaces, Caffarelli, Roquejoffre and Savin [16] gave the first regularity result for these sets, stating that, up to a singular closed set of finite Hausdorff dimension  $n-2$ , all  $s$ -minimal surfaces are locally  $C^{1,s}$  hypersurfaces. When  $s$  is close to  $\frac{1}{2}$ , Caffarelli and Valdinoci [17] proved that all non-local surfaces are smooth when the dimension of the ambient space is less or equal than 7. For  $n=2$ , Savin and Valdinoci [35] have proved that the only non-local minimal cones in  $\mathbb{R}^2$  are the trivial ones for all  $s \in (0, 1/2)$  (i.e. are half-planes). As a consequence, they obtain that the closed singular set of a non-local minimal surface has at most  $n-3$  Hausdorff dimension. This, together with the subsequent results of Barrios, Figalli and Valdinoci [6], leads, for  $s$  close to  $\frac{1}{2}$ , to their  $C^\infty$  regularity up to dimension  $n \leq 7$ .

To end with this introduction to the fractional perimeter, we state an important result by Serra et al. [18] that gives a universal perimeter estimate for stable sets of the fractional perimeter functional. Recall that a set  $E$  is called stable if the second variation of the functional is non-negative. In this case, a suitable weak formulation of stability is used (see [18, Definition 1.6]).

**Theorem 3.1.11.** *Let  $s \in (0, 1)$ ,  $R > 0$  and  $E$  be a stable set in the ball  $B_{2R}$  for the non-local  $s$ -perimeter functional. Then, the classical perimeter of  $E$  in  $B_R$  is bounded*

by  $CR^{n-1}$ , where  $C$  depends only on  $n$  and  $s$ . Moreover, the  $s$ -perimeter of  $E$  in  $B_R$  is bounded by  $CR^{n-s}$ .

Note that a universal perimeter estimate for local stable minimal surfaces is only known for the case of two-dimensional stable minimal surfaces that are *simply connected* and *immersed* in  $\mathbb{R}^3$ . Conversely, the perimeter estimate in Theorem 3.1.11 holds in every dimension and without topological constraints. Another remark is that the above estimate gives a control on the classical perimeter (i.e. the  $BV$ -norm of the characteristic function), which is stronger, both from the geometric and functional space perspective, than a control on the  $s$ -perimeter (recall that we have  $H^s(\Omega) \subseteq BV(\Omega)$  for a bounded set  $\Omega$ ).

### 3.2. ASYMPTOTICS OF THE FRACTIONAL PERIMETER AND A NOTION OF NON-LOCAL CURVATURE

The limiting behaviour of the fractional  $s$ -perimeter as  $s \rightarrow \frac{1}{2}^-$  and  $s \rightarrow 0^+$  turns out to be very interesting. Dávila, extending results by Bourgain, Brezis and Mironescu, showed that for a Borel set  $E \subset \mathbb{R}^n$  of finite perimeter in  $B_R$ ,

$$\lim_{s \rightarrow \frac{1}{2}^-} \left(\frac{1}{2} - s\right) Per_s(E; B_r) = \alpha_n P(E; B_r)$$

for almost any  $r \in (0, R)$ , with  $\alpha_n$  a constant depending on  $n$ . This implies that surfaces of minimal  $s$ -perimeter inherit the regularity properties of the classical minimal surfaces for  $s$  sufficiently close to  $1/2$  (see [17]). See also the paper of Ambrosio et al. [5] for an approach based of  $\Gamma$ -convergence.

The behaviour of  $Per_s$  as  $s \rightarrow 0^+$  is slightly more involved. In principle, the limit as  $s \rightarrow 0^+$  of  $Per_s$  is, at least locally, related to the Lebesgue measure. Nevertheless, the situation is complicated by the terms coming from infinity which, as  $s \rightarrow 0^+$  become of greater and greater importance. We refer to the paper of Dipierro et al. [23] for more details.

From now on we consider a set  $E \subseteq \mathbb{R}^n$  with  $C^2$  boundary. We introduce now the non-local objects that will play the role of directional and mean curvatures (see the survey [2] by Abatangelo and Valdinoci for a careful analysis of non-local mean curvature, with analogies and important differences with respect to the classical case).

**Definition 3.2.1.** The non-local mean curvature of  $\partial E$  at the point  $p \in \partial E$  is

$$H_s := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x - p|^{n+2s}} dx. \quad (3.2.1)$$

where  $\omega$  denotes the  $(n-2)$ -dimensional Hausdorff measure of the  $(n-2)$ -dimensional sphere.

Let  $e$  be any unit vector in the tangent space of  $\partial E$  at  $p$ .



**Definition 3.2.2.** We define the non-local directional curvature of  $\partial E$  at the point  $p \in \partial E$  in the direction  $e$  the quantity

$$K_{s,e} := \int_{\pi(e)} \frac{|y' - p'|^{n-2} \chi_E(y)}{|y - p|^{n+2s}} dy, \quad (3.2.2)$$

with  $\pi(e) := \left\{ y \in \mathbb{R}^n : y = \rho e + h\nu, \rho > 0, h \in \mathbb{R} \right\}$  the two-dimensional open half-plane,  $\nu$  the unit normal vector for  $\partial E$  at  $p$ .  $p = (p', p_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ ,  $y = (y', y_n)$ .

We endow  $\pi(e)$  with the induced two-dimensional Lebesgue measure, that is we define the integration over  $\pi(e)$  by the formula

$$\int_{\pi(e)} g(y) dy := \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh g(\rho e + h\nu).$$

**Remark 3.2.3.** Since the function  $\chi_E(x)/|x|^{n+2s}$  is not in the space  $L^1(\mathbb{R}^n)$ , the integrals (3.2.1) and (3.2.2) have to be taken in the principal value sense

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(p)} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x - p|^{n+2s}} dx.$$

**Theorem 3.2.4.** *In the above setting*

$$H_s = \frac{1}{\omega_{n-2}} \int_{S^{n-2}} K_{s,e} dH^{n-2}(e).$$

Namely, Theorem 3.2.4 states that the non-local mean curvature is the average of the non-local directional curvatures, thus providing a non-local counterpart of the classical mean curvature.

The direction that maximises the non-local directional curvature is not, in general, orthogonal to the one that minimizes it. A further remark is that, differently from the local case, in the non-local one it is not possible to calculate the mean curvature simply by taking the arithmetic mean of the principal curvatures.

**Theorem 3.2.5** (Asymptotics to  $\frac{1}{2}$ ). *For any  $e \in S^{n-2}$*

$$\lim_{s \nearrow \frac{1}{2}} (1 - 2s) K_{s,e} = K_e$$

and

$$\lim_{s \nearrow \frac{1}{2}} (1 - 2s) H_s = H,$$

where  $K_e$  (resp.,  $H$ ) is the directional curvature of  $E$  in the direction  $e$  (resp., the mean curvature of  $E$ ) at 0.

### 3.3. THE NON-LOCAL ALLEN-CAHN EQUATION

We saw in the previous chapter how classical minimal surfaces arise naturally in phase transition models. For instance, we saw that for the classical Allen-Cahn phase

transition model, minimizers of the functional

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + W(u(x)) \, dx, \text{ where } W(t) := \frac{(1-t^2)^2}{4} \text{ is a double-well potential} \quad (3.3.1)$$

satisfy the Allen-Cahn equation (2.5.1).

Heuristically, minimizers of the above energy have a strong tendency to stay close to  $\pm 1$ , which are the pure phases of the model, since these values kill the potential energy, while the gradient term forces the transition to occur with the least possible surface tension.

One interesting extension in the above setting is the study of long range interaction, which naturally leads to the analysis of phase transitions and interfaces of non-local type. To this end, we introduce an extension of the Allen-Cahn equation (2.5.1) from a local to a non-local setting.

Given an open domain  $\Omega \subset \mathbb{R}^n$  and the double-well potential  $W$ , we define the fractional Allen-Cahn equation

$$(-\Delta)^s u + W'(u) = 0 \text{ in } \Omega, \quad (3.3.2)$$

for  $s \in (0, 1)$ . The solutions are the critical points of the non-local energy

$$J(u) := \frac{1}{2} \iint_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx dy + \int_{\Omega} W(u(x)) \, dx, \quad (3.3.3)$$

up to normalisation constant. Comparing (2.5.2) and (3.3.3) we see that the kinetic energy is modified in order to take into account long range interactions. Notice that we have omitted the integral over  $\Omega^c \times \Omega^c$ , just as we did for the fractional perimeter, since we consider that the values of  $u$  are prescribed in this domain (in the local case, the values are prescribed on  $\partial\Omega$ ). Of course, the potential energy has local features, therefore the potential integrals are set over  $\Omega$  both in the local and the non-local case.

We may now proceed as the in classical case and consider a blow-up of  $u$ , namely  $u_\epsilon(x) = u(x/\epsilon)$ . It is also necessary to normalise by a multiplicative factor that depends on  $s$ . We end up obtaining the following functional

$$J_\epsilon(u) = \frac{1}{2} \iint \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx dy + \frac{1}{\epsilon(s)} \int W(u(x)) \, dx.$$

We have a phase transition function  $u_\epsilon$  whose level sets as  $\epsilon \searrow 0$  approach some  $\partial E$ . When  $s \in (0, 1/2)$  this  $\partial E$  is an  $s$ -minimal set, while for  $s \in [1/2, 1)$  it is a classical minimal surface (see Savin and Valdinoci [36]).

We also consider a fractional or non-local counterpart of the conjecture of De Giorgi. Namely, we consider the non-local Allen-Cahn equation (3.3.2) with  $u$  smooth, bounded and monotone in one direction, and wonder if it is also true, at least in low dimension, that  $u$  is one-dimensional. In this case, the conjecture was proved for  $n = 2$  and  $s = 1/2$  by Cabré and Solà-Morales [11]. In the case  $n = 2$  and for any  $s \in (0, 1)$ , the results is proved by Cabré and Sire [12] using the harmonic extension of the fractional

Laplacian. For  $n = 3$  a proof by Cabré and Cinti can be found in [13] for  $s \in [1/2, 1]$ . The conjecture is still open for  $n = 3$  and  $s \in [0, 1/2]$ , and for  $n \geq 4$ . In [28] Sire et al. proved the conjecture in  $n = 2$  for general, compactly supported fractional operators without using extension techniques. See also the master thesis of J.C. Felipe Navarro [24] for the one dimensional case for more general operators.

### 3.4. SURFACES WITH CONSTANT NON-LOCAL MEAN CURVATURE

We are now concerned with hypersurfaces of  $\mathbb{R}^n$  with constant non-local mean curvature. The first result we would like to point out is the non-local or fractional counterpart of the classical result by Alexandrov on the characterisation of spheres as the only closed embedded constant mean curvature hypersurface. This result was first stated and proved independently by Cabré et al. [10] and Ciruolo et al. [19]. The precise statement is the following (see [10, Theorem 1.1]).

**Theorem 3.4.1.** *Suppose that  $E$  is a non-empty bounded open set with  $C^{2,\beta}$ -boundary for some  $\beta > 0$  and with the property that  $H_E$  is constant on  $\partial E$ . Then  $E$  is a ball.*

This result states that every bounded (and a priori not necessarily connected) hypersurface without boundary and with constant nonlocal mean curvature must be a sphere.

In [10] they also prove the non-local counterpart of the classical results of Delaunay [22] on periodic cylinders with constant non-local mean curvature. They study sets  $E \subset \mathbb{R}^2$  with constant non-local mean curvature which have the form of bands or “cylinders” in the plane

$$E = \{(s_1, s_2) \in \mathbb{R}^2 : -u(s_1) < s_2 < u(s_1)\},$$

where  $u : \mathbb{R} \rightarrow (0, \infty)$  is a positive function. They prove the existence of a continuous branch of periodic bands that do not differ much from a straight band, all of them with the same constant NMC. Therefore they show that, in the non-local setting, these objects already exist in dimension 2. While they only exist in dimension 3 and higher in the classical constant mean curvature setting. In [15] Cabré, Fall and Weth prove the existence of such “perturbed” cylinders for dimension  $n \geq 2$ . Furthermore, they prove that they are indeed  $C^\infty$ .

Another way to construct such sets is variationally. This will be done in the subsequent chapters.



## CHAPTER 4

### The Dirichlet problem for the fractional Laplacian

Consider for simplicity a smooth and bounded function  $u$  (or more generally a function in the Schwartz space  $S(\mathbb{R}^n)$  of rapidly decreasing functions). The fractional Laplacian operator is defined as

$$(-\Delta)^s u(x) := C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} dy, \quad s \in (0, 1). \quad (4.0.1)$$

Notice that the above integral is well defined in the principal value sense. That is,

$$(-\Delta)^s u(x) = C_{n,s} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon} \frac{u(x) - u(x+y)}{|y|^{n+2s}} dy.$$

Note also that the integral above is well defined pointwise if, for instance, one considers  $u \in L^\infty(\mathbb{R}^n) \cap C_{loc}^\gamma(\mathbb{R}^n)$  for  $\gamma > 2s$ .

This operator, or more generally integro-differential operator of the form

$$-Lu(x) = \sum_{i,j} a_{ij} \partial_{ij} u + \sum_j b_j \partial_j u + \int_{\mathbb{R}^n} \{u(x+y) - u(x) - y \cdot \nabla u(x) \chi_{B_1}(y)\} d\nu(y)$$

arise naturally in the study of stochastic processes with jumps, or more precisely as the infinitesimal generator of a Lévy process. In particular, when the process has no diffusion or drift part, it is symmetric and the Lévy measure  $\nu(y)$  is absolutely continuous, the operator  $L$  can be written as

$$Lu(x) = P.V. \int_{\mathbb{R}^n} \{u(x) - u(x+y)\} K(y) dy$$

for some symmetric kernel  $K$ . The fractional Laplacian is the infinitesimal generator of a radially symmetric and stable Lévy process of order  $2s$ . Roughly speaking, a Lévy process represents the random motion of a particle whose successive displacements are independent and statistically identical over different time intervals of the same length.

The reader may refer to the author's bachelor thesis [3] and the nice survey of Bucur and Valdinoci [9, Chapter 1] for a more detailed introduction to the fractional Laplacian. See also the introductory paper by Valdinoci [41] for a presentation of the fractional Laplacian using a very intuitive probabilistic argument.

## 4.1. THE DIRICHLET SEMI-LINEAR PROBLEM

We consider here the semi-linear Dirichlet problem for a smooth bounded domain  $\Omega \subseteq \mathbb{R}^n$

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (4.1.1)$$

We shall prove that equation (4.1.1) is the Euler-Lagrange equation of the energy

$$\mathcal{E}(u) = \frac{1}{4} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy - \int_{\Omega} F(u), \quad (4.1.2)$$

where  $F$  is the primitive of  $f$ , i.e.  $F' = f$  (we may consider  $f$  to be continuous). Note that  $\mathcal{E}$  is well defined for all regular enough functions  $u$  which are bounded at infinity.

**Remark 4.1.1.** The first integral plays the role of the kinetic energy while the second integral is the potential energy.

**Proposition 4.1.2.** *Let  $u$  be a minimum of the functional (4.1.2). Then*

$$(-\Delta)^s u = f(u) \text{ in } \Omega. \quad (4.1.3)$$

Notice that we have not specified any regularity for the function  $u$ . We can suppose  $u$  to be regular enough.

**PROOF.** Let  $\varphi \in C_c^\infty(\Omega)$  and consider (small) perturbations  $u + \epsilon\varphi$ , with  $\epsilon$  small. If  $u$  is a minimizer of (4.1.1), then differentiating with respect to  $\epsilon$  at 0 we have

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left\{ \frac{1}{4} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(y) + \epsilon(\varphi(x) - \varphi(y))|^2}{|x - y|^{n+2s}} dx dy - \int_{\Omega} F(u + \epsilon\varphi) \right\} \\ &= \frac{1}{2} \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f(u)\varphi \\ &= \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)} \varphi(x) \frac{u(x) - u(y)}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f(u)\varphi \\ &= \int_{\Omega} dx \varphi(x) \left\{ \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy - f(u(x)) \right\} \\ &= \int_{\Omega} dx \varphi(x) \left\{ (-\Delta)^s u(x) - f(u(x)) \right\}. \end{aligned}$$

In the 3rd equality we have used the symmetry of the domain of integration in  $(x, y)$ . In the 4th equality we have used the fact that  $\varphi(x) \equiv 0$  if  $x \in \Omega^c$ .  $\square$

The minimizer of the functional  $\mathcal{E}$  among all functions with  $u = g$  in  $\mathbb{R}^n \setminus \Omega$  will therefore be a solution of (4.1.1) in a weak sense. In [37] Ros proves the existence and uniqueness of weak solutions of (4.1.1) when  $f$  only depends on the spacial variable  $x$  using Riesz representation theorem, assuming homogeneous Dirichlet conditions.

We will here show the existence of a weak solution when the non-linear term is sub-linear at infinity (assuming also homogeneous Dirichlet conditions). We start by defining the functional space in which we will work.

**Definition 4.1.3.** Let  $u \in H^s(\mathbb{R}^n)$ . We define the subspace  $H_0^s(\Omega)$  as

$$H_0^s := \{u \in H^s(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\}. \quad (4.1.4)$$

Equipped with the inner product

$$(u, v)_{H_0^s} := \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy, \quad (4.1.5)$$

the space  $H_0^s(\Omega)$  is a Hilbert space.

**Remark 4.1.4.** Notice that the functional  $\mathcal{E}$  can now be written as

$$\mathcal{E}(u) = \frac{1}{2} \|u\|_{H_0^s}^2 - \int_{\Omega} F(u). \quad (4.1.6)$$

**Theorem 4.1.5.** Let  $f$  be a continuous and sub-linear function at infinity, i.e.

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{t} = 0. \quad (4.1.7)$$

There exists a minimum  $u \in H_0^s(\mathbb{R}^n)$  of the functional  $\mathcal{E}$  in (4.1.2).

Essentially, the only assumption which is needed in order to prove the existence of minimizer is the fractional Poincaré inequality.

**Proposition 4.1.6** (Fractional Poincaré inequality). Let  $\Omega \subset \mathbb{R}^n$  be any bounded domain, and let  $u \in H_0^s(\Omega)$ . Then

$$\int_{\Omega} u^2 \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

PROOF. The result follows from Hölder's inequality and the fractional Sobolev embedding for  $p = 2$  and  $q = \frac{2n}{n-2s}$ .  $\square$

PROOF OF THEOREM 4.1.5. We will use a direct method of the calculus of variations to prove the existence of a minimizer.

Due to (4.1.7), there exists  $c > 0$ , which we consider small enough, such that

$$|f(t)| \leq c(1 + |t|) \quad \forall t \in \mathbb{R}. \quad (4.1.8)$$

Now, using (4.1.8) and Proposition 4.1.6 we have

$$\begin{aligned} \mathcal{E}(u) &= \frac{1}{2} \|u\|_{H_0^s}^2 - \int_{\Omega} F(u) \\ &\geq \frac{1}{2} \|u\|_{H_0^s}^2 - c \|u\|_{L^1(\Omega)} - \frac{c}{2} \|u\|_{L^2(\Omega)}^2 dx \\ &\geq \frac{1}{2} \|u\|_{H_0^s}^2 - c_1 \|u\|_{H_0^s} - c_2 \|u\|_{H_0^s}^2 \\ &= c_3 \|u\|_{H_0^s}^2 - c_1 \|u\|_{H_0^s}. \end{aligned}$$

Hence,  $\mathcal{E}$  is bounded from below and coercive. Consider a minimising sequence  $\{u_m\}_{m \in \mathbb{N}} \subset H_0^s(\Omega)$  of  $\mathcal{E}$ . Taking account of the coercivity of  $\mathcal{E}$ , the sequence  $\{u_m\}_{m \in \mathbb{N}}$  is necessarily bounded in  $H_0^s(\Omega)$ . Since the space  $H_0^s(\Omega)$  is reflexive, we can extract a subsequence, which for simplicity we still denote  $\{u_m\}_{m \in \mathbb{N}}$ , such that  $u_m$  weakly converges to  $u$  in  $H_0^s(\Omega)$ , i.e.

$$(u_m, \varphi)_{H_0^s(\Omega)} \rightarrow (u, \varphi)_{H_0^s(\Omega)} \text{ for all } \varphi \in H_0^s(\Omega).$$

It is straightforward to prove that the functional  $\int_{\Omega} F(u)$  is continuous from  $H_0^s(\Omega)$  with the weak topology to  $\mathbb{R}$ . For this, consider a weakly converging sequence  $v_m \rightharpoonup v$ . The sequence is uniformly bounded, i.e.  $\|v_m\|_{H_0^s(\Omega)} \leq M$  for all  $m \in \mathbb{N}$ . Using now the compact embedding  $H^s(\Omega) \subset\subset L^1(\Omega)$ , the sequence  $\{v_m\}$  is still uniformly bounded and moreover converges to  $v$  in  $L^1(\Omega)$ . Therefore, using Lebesgue's dominated convergence theorem and the fact that the function  $F$  is continuous, we have

$$\int_{\Omega} F(v_m) \rightarrow \int_{\Omega} F(v).$$

Moreover, by Hahn-Banach the norm  $\|\cdot\|_{H_0^s(\Omega)}$  is weakly lower semi-continuous.

Therefore, the functional  $\mathcal{E}$  is lower semi-continuous with the weak topology, i.e.

$$u_m \rightharpoonup u \text{ in } H_0^s(\Omega) \Rightarrow \liminf_{n \rightarrow \infty} \mathcal{E}(u_m) \geq \mathcal{E}(u).$$

Hence  $u \in H_0^s(\Omega)$  is a minimum of  $\mathcal{E}$ . □

Using a similar argument one can prove the existence of minimizers for the linear case (see the book of Molica, Radulescu and Servadei [7, Section 3.2]). Furthermore, in [7, Chapter 6], they show the existence of minimizers when the non-linear term  $f$  satisfies super-linear and sub-critical growths conditions at zero and at infinity. See also Abatangelo [1] for more general non-linearities using some sub- and super-solution methods.

## 4.2. THE PERIODIC SEMI-LINEAR PROBLEM

Let  $u$  be a  $2\pi$ -periodic function in  $\mathbb{R}$ . We study the periodic problem

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}. \tag{4.2.1}$$

Namely, we prove that the energy associated with this problem is

$$\mathcal{E}(u) = \frac{1}{4} \int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} - \int_{-\pi}^{\pi} F(u), \tag{4.2.2}$$

for  $F' = f$ .

Note that, in this case and following the semi-linear problem (4.1.1), the domain of integration for the kinetic energy would be  $(\mathbb{R} \times \mathbb{R}) \setminus ((-\pi, \pi)^c \times (-\pi, \pi)^c)$ . In our case, we do not consider the second contribution of  $(-\pi, \pi) \times (\mathbb{R} \setminus (-\pi, \pi))$ .



**Proposition 4.2.1.** *Assume  $u$  is a minimizer of the functional (4.2.2) among all  $2\pi$ -periodic functions defined in  $\mathbb{R}$ . Then*

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}.$$

Note that, just as before, the above equation will be a priori satisfied only in a weak sense.

**PROOF.** Let  $\varphi$  be a  $2\pi$ -periodic function (note that we do not need to consider  $\varphi \in C_c^\infty$  since we now work with a base) and consider  $u + \epsilon\varphi$ , with  $\epsilon$  small. We have

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy \frac{|u(x) - u(y) + \epsilon(\varphi(x) - \varphi(y))|^2}{|x - y|^{1+2s}} - \int_{-\pi}^{\pi} F(u + \epsilon\varphi) \\ &= \frac{1}{2} \int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy \frac{u(x) - u(y)}{|x - y|^{1+2s}} (\varphi(x) - \varphi(y)) - \int_{-\pi}^{\pi} f(u)\varphi \\ &= \int_{-\pi}^{\pi} dx \varphi(x) \left\{ \frac{1}{2} \int_{\mathbb{R}} dy \frac{u(x) - u(y)}{|x - y|^{1+2s}} (\varphi(x) - \varphi(y)) \right\} \\ &\quad + \int_{-\pi}^{\pi} dy \varphi(y) \left\{ \frac{1}{2} \int_{\mathbb{R}} dx \frac{u(y) - u(x)}{|x - y|^{1+2s}} (\varphi(x) - \varphi(y)) \right\} - \int_{-\pi}^{\pi} f(u)\varphi \\ &= A + B \end{aligned}$$

where  $A$  and  $B$  are the last integrals.

Now, making the change of variables

$$\left\{ \begin{array}{l} y - 2k\pi = z \\ dy = dz \end{array} \right\}, \left\{ \begin{array}{l} x - 2k\pi = \bar{z} \\ dx = d\bar{z} \end{array} \right\}$$

we can see that the expressions  $A$  and  $B$  above are equivalent:

$$\begin{aligned} B &= \sum_{k \in \mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} dy \varphi(y) \left\{ \frac{1}{2} \int_{-\pi}^{\pi} dx \frac{u(y) - u(x)}{|x - y|^{1+2s}} \right\} \\ &= \sum_{k \in \mathbb{Z}} dz \varphi(z) \left\{ \frac{1}{2} \int_{-(2k+1)\pi}^{-(2k-1)\pi} d\bar{z} \frac{u(z) - u(\bar{z})}{|z - \bar{z}|^{1+2s}} \right\} \\ &= \int_{-\pi}^{\pi} dz \varphi(z) \left\{ \frac{1}{2} \int_{\mathbb{R}} d\bar{z} \frac{u(z) - u(\bar{z})}{|z - \bar{z}|^{1+2s}} \right\} = A. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \int_{-\pi}^{\pi} dx \varphi(x) \left\{ \int_{\mathbb{R}} dy \frac{u(x) - u(y)}{|x - y|^{1+2s}} - f(u) \right\} \\ &= \int_{-\pi}^{\pi} dx \varphi(x) \left\{ (-\Delta)^s u - f(u) \right\}. \end{aligned}$$

In particular, we can extend the equality in the whole  $\mathbb{R}$  because of the periodicity of  $u$ .  $\square$

For the periodic case, the reader may refer to the author's bachelor thesis [3, Chapter 6] for the existence of a family of even and periodic solutions of the semi-linear problem under certain conditions for the non-linearity.



## CHAPTER 5

### The periodic fractional perimeter functional

In this chapter we consider sets  $E = \{-u(x_1) < x_2 < u(x_1)\} \subset \mathbb{R}^2$ , for non-negative (i.e.  $u \geq 0$ ), even  $2\pi$ -periodic functions  $u$ . We consider the periodic fractional perimeter functional

$$P_\alpha(E) := \frac{1}{8} \int_{E \cap \{-\pi < x_1 < \pi\}} \int_{E^c} \frac{dxdy}{|x - y|^{n+\alpha}}, \quad (5.0.1)$$

for  $0 < \alpha < 1$ .

This functional was introduced by Dávila et al. [21] to study periodic sets that are decreasing and cylindrically symmetric in a given direction. In their paper, they prove variationally the existence of a 1-periodic minimizer for every fixed volume within the slab  $\{(x, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : -\pi < x < \pi\}$  and for any dimension  $n \geq 2$ .

In this chapter we will prove that, in fact, the fractional perimeter functional adapted to periodic sets gives rise to constant NMC surfaces in a weak sense. For this, we consider periodic sets  $E$  defined above and the functional (5.0.1), which now reads

$$P_\alpha(u) := \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{|y_2| > u(y_1)} dy_2 \frac{1}{|x - y|^{2+\alpha}}, \quad (5.0.2)$$

**Remark 5.0.1.** Note that circles and bands have finite energy. To see this, consider a circle  $B$  centered at the origin of radius  $0 < r \leq \pi/2$ . Computing the functional evaluated at the ball using polar coordinates we obtain:

$$P_\alpha(B) \leq \frac{1}{4} \int_B \int_{\mathbb{R}^2 \setminus B} \frac{dxdy}{|x - y|^{2+\alpha}} = \pi r^{2-\alpha} < +\infty.$$

Using the same idea, we can bound the integral over a band with the integral above and see that bands also have finite energy.

Before that, let us give a simplified expression for the functional  $P_\alpha$ . This will considerably reduce our incoming work.

**Proposition 5.0.2.** *Consider the functions*

$$G(q) := \int_0^q d\tau (q - \tau)(1 + \tau^2)^{-\frac{2+\alpha}{2}}, \quad (5.0.3)$$

and

$$H(q) := G'(\infty)q - G(q). \quad (5.0.4)$$

The functional (5.0.1) can be expressed as

$$P_\alpha(u) = \frac{1}{2} \int_0^\pi dx \int_{\mathbb{R}} dy |x - y|^{-\alpha} \left\{ G\left(\frac{u(x) - u(y)}{|x - y|}\right) + H\left(\frac{u(x) + u(y)}{|x - y|}\right) \right\}. \quad (5.0.5)$$

Notice that we have reduced the dimension of the problem. Instead of two doubles integrals in  $\mathbb{R}^2$ , we are now left with two integrals in  $\mathbb{R}$ .

PROOF. We split the functional into two parts; an integration below and above the periodic bands, respectively:

$$\begin{aligned} P_\alpha(u) &= \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{-\infty}^{u(y_1)} dy_2 |x_1 - y_1|^{-(2+\alpha)} \left(1 + \left(\frac{y_2 - x_2}{|x_1 - y_1|}\right)^2\right)^{-\frac{2+\alpha}{2}} \\ &\quad + \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{u(y_1)}^{+\infty} dy_2 |x_1 - y_1|^{-(2+\alpha)} \left(1 + \left(\frac{y_2 - x_2}{|x_1 - y_1|}\right)^2\right)^{-\frac{2+\alpha}{2}}. \end{aligned}$$

Making the change of variables

$$\tau = \frac{y_2 - x_2}{|x_1 - y_1|}, \quad d\tau = \frac{1}{|x_1 - y_1|} dy_2 \quad (5.0.6)$$

and considering the continuous function

$$F(q) := \int_0^q \frac{d\tau}{(1 + \tau^2)^{\frac{2+\alpha}{2}}} \quad (5.0.7)$$

we obtain

$$\begin{aligned} P_\alpha(u) &= \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-(1+\alpha)} \int_{-\infty}^{-\frac{u(y_1)+x_2}{|x_1-y_1|}} d\tau (1 + \tau^2)^{-\frac{2+\alpha}{2}} \\ &\quad + \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-(1+\alpha)} \int_{\frac{u(y_1)-x_2}{|x_1-y_1|}}^{+\infty} d\tau (1 + \tau^2)^{-\frac{2+\alpha}{2}} \\ &= \frac{1}{4} \int_0^\pi dx_1 \int_{-u(x_1)}^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-(1+\alpha)} \left\{ F\left(-\frac{u(y_1) + x_2}{|x_1 - y_1|}\right) - F(-\infty) \right. \\ &\quad \left. + F(+\infty) - F\left(\frac{u(y_1) - x_2}{|x_1 - y_1|}\right) \right\}. \end{aligned}$$

The function  $F$  is odd and bounded at infinity ( $\tau^{2+\alpha}$  is integrable at infinity). Moreover, the function  $F\left(\frac{x_2 - u(y_1)}{|x_1 - y_1|}\right) - F\left(\frac{u(y_1) + x_2}{|x_1 - y_1|}\right)$  is even with respect to  $x_2$ . Combining all this we obtain

$$\begin{aligned} P_\alpha(u) &= \frac{1}{2} \int_0^\pi dx_1 \int_0^{u(x_1)} dx_2 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-(1+\alpha)} \left\{ F\left(\frac{x_2 - u(y_1)}{|x_1 - y_1|}\right) \right. \\ &\quad \left. - F\left(\frac{x_2 + u(y_1)}{|x_1 - y_1|}\right) + 2F(+\infty) \right\}. \quad (5.0.8) \end{aligned}$$

Consider now the function

$$\begin{aligned}
G(q) &:= \int_0^q F(p) dp = \int_0^q dp \int_0^p d\tau (1 + \tau^2)^{-\frac{2+\alpha}{2}} \\
&= \int_0^q d\tau + \int_\tau^q dp (1 + \tau^2)^{-\frac{2+\alpha}{2}} \\
&= \int_0^q d\tau (q - \tau)(1 + \tau^2)^{-\frac{2+\alpha}{2}},
\end{aligned} \tag{5.0.9}$$

and the changes of variables

$$\left\{ \begin{array}{l} \sigma_1 = \frac{x_2 - u(y_1)}{|x_1 - y_1|} \\ d\sigma_1 = \frac{dx_2}{|x_1 - y_1|} \end{array} \right\}, \left\{ \begin{array}{l} \sigma_2 = \frac{x_2 + u(y_1)}{|x_1 - y_1|} \\ d\sigma_2 = \frac{dx_2}{|x_1 - y_1|} \end{array} \right\}. \tag{5.0.10}$$

Notice that  $G' = F$ . Therefore, the function  $G$  has bounded derivative and thus  $G'(+\infty)$  is defined and equal to  $F(+\infty)$ . Moreover, the function  $G$  is even. Plugging all this into (5.0.8) we obtain

$$\begin{aligned}
P_\alpha(u) &= \frac{1}{2} \int_0^\pi dx_1 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-\alpha} \left\{ G\left(\frac{u(x_1) - u(y_1)}{|x_1 - y_1|}\right) - G\left(\frac{u(x_1) + u(y_1)}{|x_1 - y_1|}\right) \right. \\
&\quad \left. + 2F(+\infty) \frac{u(x_1)}{|x_1 - y_1|} \right\} \\
&= \frac{1}{2} \int_0^\pi dx_1 \int_{\mathbb{R}} dy_1 |x_1 - y_1|^{-\alpha} \left\{ G\left(\frac{u(x_1) - u(y_1)}{|x_1 - y_1|}\right) - G\left(\frac{u(x_1) + u(y_1)}{|x_1 - y_1|}\right) \right. \\
&\quad \left. + G'(\infty) \frac{u(x_1) + u(y_1)}{|x_1 - y_1|} \right\}.
\end{aligned}$$

To obtain the expression (5.0.5) we have only left to consider the function

$$H(q) := G'(\infty)q - G(q). \tag{5.0.11}$$

□

Let us briefly comment on the functions  $G$  and  $H$  that we have just used. We have  $G(0) = 0$ ,  $\lim_{x \rightarrow +\infty} G'(x) = G'(+\infty) < +\infty$  ( $G' = F$  which is bounded) and  $G''(x) = (1 + x^2)^{-\frac{2+\alpha}{2}} > 0$ . Therefore,  $G$  is a positive, even, strictly increasing and decreasing in the positive and negative axis respectively, and strictly convex function which starts as a quadratic function near the origin and becomes linear as we get away from it. On the other hand, we have  $H(0) = 0$ ,  $H''(x) = -G''(x) < 0$  and  $G(x) = G(x) - G(0) = G'(\xi)x < G'(+\infty)x$ . So  $H$  is an strictly increasing, bounded at  $+\infty$  and concave function.

From now on, we will only consider the simplified expression (5.0.5). The next results shows that the Euler-Lagrange equation for the periodic fractional perimeter (5.0.1) is the non-local mean curvature (3.1.4).

**Proposition 5.0.3.** *Let  $u$  be a positive, even  $2\pi$ -periodic function in  $\mathbb{R}$  which minimizes the functional (5.0.5) among all positive, even  $2\pi$ -periodic functions satisfying the following volume constraint:*

$$\frac{1}{\pi} \int_0^\pi u(x) dx = \mu, \text{ for some positive real number } \mu.$$

Then  $u$  is a solution of

$$H(u)(x) = a \text{ for all } x \in \mathbb{R}, \quad (5.0.12)$$

for some constant  $a \in \mathbb{R}$ , where  $H$  denotes the non-local mean curvature (3.1.4).

PROOF. Let  $\varphi$  be an even  $2\pi$ -periodic function in  $\mathbb{R}$  with  $\int_0^\pi \varphi = 0$  and consider a perturbation  $u + \epsilon\varphi$  so that  $\int_0^\pi u + \epsilon\varphi = 0$  and still  $u + \epsilon\varphi > 0$  (if  $\epsilon$  is small). Because  $u$  is a minimum we have

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} P_\alpha(u + \epsilon\varphi) = \frac{1}{2} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_0^\pi dx \int_{\mathbb{R}} dy |x-y|^{-\alpha} \left\{ G\left(\frac{u(x)-u(y)+\epsilon(\varphi(x)-\varphi(y))}{|x-y|}\right) \right. \\ &\quad \left. + H\left(\frac{u(x)+u(y)+\epsilon(\varphi(x)+\varphi(y))}{|x-y|}\right) \right\} \\ &= \frac{1}{2} \int_0^\pi dx \int_{\mathbb{R}} dy |x-y|^{-\alpha} \left\{ F\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{\varphi(x)-\varphi(y)}{|x-y|} + F(+\infty) \frac{\varphi(x)+\varphi(y)}{|x-y|} \right. \\ &\quad \left. - F\left(\frac{u(x)+u(y)}{|x-y|}\right) \frac{\varphi(x)+\varphi(y)}{|x-y|} \right\} \\ &= \int_0^\pi dx \varphi(x) \frac{1}{2} \int_{\mathbb{R}} dy |x-y|^{-(1+\alpha)} \left\{ F\left(\frac{u(x)-u(y)}{|x-y|}\right) - \left( F\left(\frac{u(x)+u(y)}{|x-y|}\right) - F(+\infty) \right) \right\} \\ &\quad + \int_{\mathbb{R}} dy \varphi(y) \frac{1}{4} \int_{-\pi}^\pi dx |x-y|^{-(1+\alpha)} \left\{ F\left(\frac{u(y)-u(x)}{|x-y|}\right) - \left( F\left(\frac{u(x)+u(y)}{|x-y|}\right) - F(+\infty) \right) \right\} \\ &= A + B \end{aligned}$$

where  $A$  and  $B$  are the last two integrals. Note that the factor  $1/4$  in  $B$  comes from the fact that we are now integrating in  $(-\pi, \pi)$ .

We shall now see that these two expressions are in fact equivalent:

$$\begin{aligned} B &= \sum_{k \in \mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} dy \varphi(y) \frac{1}{2} \int_0^\pi dx |x-y|^{-(1+\alpha)} \left\{ F\left(\frac{u(y)-u(x)}{|x-y|}\right) \right. \\ &\quad \left. - \left( F\left(\frac{u(x)+u(y)}{|x-y|}\right) - F(+\infty) \right) \right\}. \end{aligned}$$

Making the change of variables

$$\left\{ \begin{array}{l} \bar{y} = y - 2k\pi \\ d\bar{y} = dy \end{array} \right\}, \left\{ \begin{array}{l} \bar{x} = x - 2k\pi \\ d\bar{x} = dx \end{array} \right\} \quad (5.0.13)$$

we obtain

$$B = \int_{-\pi}^{\pi} d\bar{y} \varphi(\bar{y}) \left\{ \frac{1}{4} \sum_{k \in \mathbb{Z}} \int_{(2k-1)\pi}^{(2k+1)\pi} d\bar{x} |\bar{x} - \bar{y}|^{-(1+\alpha)} \left\{ F\left(\frac{u(\bar{x}) - u(\bar{y})}{|\bar{x} - \bar{y}|}\right) - \left( F\left(\frac{u(\bar{x}) + u(\bar{y})}{|\bar{x} - \bar{y}|}\right) - F(+\infty) \right) \right\} \right\},$$

which is equal to A.

Hence, using the expression in [10, Lemma 4.1] for the non-local mean curvature, we finally arrive at

$$\begin{aligned} 0 &= \int_0^{\pi} dx \varphi(x) \int_{\mathbb{R}} dy |x - y|^{-(1+\alpha)} \left\{ F\left(\frac{u(x) - u(y)}{|x - y|}\right) - \left( F\left(\frac{u(x) + u(y)}{|x - y|}\right) - F(+\infty) \right) \right\} \\ &= \int_0^{\pi} dx \varphi(x) \frac{1}{2} H(u)(x) \end{aligned}$$

for all  $\varphi$  with  $\int_0^{\pi} \varphi = 0$ . This gives us  $H(u)(x) = a$  for all  $x \in (0, \pi)$  in some weak sense, for some constant  $a \in \mathbb{R}$ . The result follows from the periodicity of  $u$ .  $\square$

Therefore, minimizers of (5.0.5) are surfaces with constant non-local mean curvature, at least in a weak sense; we do not know a priori if the minimizers of this functional are of class  $C^{1,\beta}$  for some  $\beta > \alpha/2$ . Note that the volume constraint excludes the trivial case  $u \equiv 0$ .





## Delaunay cylinders with constant non-local mean curvature

We consider the new functional, which we still denote  $P_\alpha$  for simplicity

$$P_\alpha(u) := \frac{1}{2} \int_0^\pi dx \int_{\mathbb{R}} dy |x-y|^{-\alpha} \left\{ G\left(\frac{|u(x)| - |u(y)|}{|x-y|}\right) + H\left(\frac{|u(x)| + |u(y)|}{|x-y|}\right) \right\} \geq 0. \quad (6.0.1)$$

In this case, if a function  $u$  minimizes the above functional, then the function  $|u|$  will also be a minimizer (and, in particular, we can take out the absolute values and recover our periodic fractional perimeter functional). Hence, we can consider non-negative minimizers  $u$  and thus the set

$$E = \{-u(x_1) < x_2 < u(x_1)\} \subset \mathbb{R}^2 \quad (6.0.2)$$

makes sense.

**Remark 6.0.1.** For functions  $u$  that are minimizers of the above functional, the Euler-Lagrange equation (5.0.12) holds at points  $x \in \mathbb{R}$  such that  $u(x) > 0$ .

We will look for non-negative, even  $2\pi$ -periodic minimizers in the fractional Sobolev space

$$W^{\alpha,1}(0, \pi) := \left\{ u \in L^1(0, \pi) : \frac{|u(x) - u(y)|}{|x - y|^{1+\alpha}} \in L^1((0, \pi) \times (0, \pi)) \right\}$$

i.e. and intermediary Banach space between  $L^1(0, \pi)$  and  $W^{1,1}(0, \pi)$ , endowed with the natural norm

$$\|u\|_{W^{\alpha,1}(0,\pi)} := \int_0^\pi |u| dx + \int_0^\pi \int_0^\pi \frac{|u(x) - u(y)|}{|x - y|^{1+\alpha}} dx dy,$$

where the term

$$[u]_{W^{\alpha,1}(0,\pi)} := \int_0^\pi \int_0^\pi \frac{|u(x) - u(y)|}{|x - y|^{1+\alpha}} dx dy \quad (6.0.3)$$

is the so-called *Gagliardo semi-norm* of  $u$ . We denote  $W_{per}^{\alpha,1}(0, \pi)$  the space of even  $2\pi$ -periodic functions in  $\mathbb{R}$  such that  $u \in W^{\alpha,1}(0, \pi)$ .

The space  $W^{\alpha,1}(0, \pi)$  can be compactly embedded into the Lebesgue space  $L^1(0, \pi)$  as the next (more general) result points out.

**Theorem 6.0.2.** *Let  $\alpha \in (0, 1)$ ,  $p \in [1, +\infty)$ ,  $q \in [1, p]$ ,  $\Omega \subset \mathbb{R}^n$  be a bounded extension domain<sup>1</sup> for  $W^{\alpha,p}$  and  $\mathcal{J}$  be a bounded subset of  $L^p(\Omega)$ . Suppose that*

$$\sup_{f \in \mathcal{J}} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha}} dx dy < +\infty.$$

*Then  $\mathcal{J}$  is pre-compact in  $L^q(\Omega)$ .*

PROOF. See [31, Theorem 7.1]. □

Thanks to this result we see that the subspace of non-negative functions is closed with the  $W^{\alpha,1}$  norm.

We have already proved that positive minimizers of (5.0.1) are critical points of the non-local mean curvature equation (5.0.12) (see Proposition 5.0.3). Let us now prove the existence of minimizers.

PROOF THEOREM 1.1.1. We denote by  $V$  the subspace of non-negative functions in  $W_{per}^{\alpha,1}(0, \pi)$ . Let  $\{u_k\} \subseteq V$  be a minimising sequence of  $P_{\alpha}$ , i.e.

$$\lim_{k \rightarrow \infty} P_{\alpha}(u_k) = \inf_{v \in W^{\alpha,1}(0, \pi)} P_{\alpha}(v).$$

We may assume that, for  $k$  sufficiently large, there exists a constant  $C > 0$  such that  $P(u_k) \leq C_1$ . In particular,

$$\int_0^{\pi} dx \int_{-\pi}^{\pi} dy |x - y|^{-\alpha} G\left(\frac{u_k(x) - u_k(y)}{|x - y|}\right) \leq C_1.$$

Because of the integrability of  $|t|^{-\alpha}$  near the origin, there exists a constant  $C_2 \geq C_1$  such that

$$\int_0^{\pi} dx \int_{-\pi}^{\pi} dy |x - y|^{-\alpha} \left\{ 1 + G\left(\frac{u_k(x) - u_k(y)}{|x - y|}\right) \right\} \leq C_2.$$

Now, the function  $G$  is positive, even and strictly convex. Moreover, near the origin it acts as a quadratic function but quickly converges to a linear function. This can be seen by studying the derivative  $G' = F$  in (5.0.7); the function  $F$  is bounded and gets closer to the upper bound fast, so that  $G'(x)$  approaches  $F(+\infty)$  quickly. The same happens in the negative axis since  $F$  is an odd function. Therefore, for a sufficiently large constant  $\beta > 0$  we have that

$$1 + G(x) \geq \frac{1}{\beta}|x|. \tag{6.0.4}$$

That is, our function can be bounded below by the linear function  $\frac{1}{\beta}|x|$ .

Thus, we arrive at

$$\frac{1}{\beta} \int_0^{\pi} dx \int_{-\pi}^{\pi} dy |x - y|^{-\alpha} \frac{u_k(x) - u_k(y)}{|x - y|} \leq C_2.$$

---

<sup>1</sup>An open set  $\Omega \subseteq \mathbb{R}^n$  is an extension domain for  $W^{\alpha,p}$  if there exists a positive constant  $C := C(n, p, \alpha, \Omega)$  such that for every function  $u \in W^{\alpha,p}(\Omega)$  there exists  $\tilde{u} \in W^{\alpha,p}(\mathbb{R}^n)$  with  $\tilde{u}(x) = u(x)$  for all  $x \in \Omega$  and  $\|\tilde{u}\|_{W^{\alpha,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{\alpha,p}(\Omega)}$ .

Which, in particular, means that the sequence  $\{u_k\}_k$  has uniformly bounded semi-norm  $[u_k]_{1,W^{\alpha,1}(0,\pi)} \leq C_3$  for some positive constant  $C_3$ . Furthermore, because of the volume constraint the  $L^1$  norm is equal to  $\mu\pi$ . Therefore, we can bound the  $W^{\alpha,1}$  norm of the minimising sequence, i.e.  $\|u_k\|_{W^{\alpha,1}(0,\pi)} \leq C$ , with some constant  $C > 0$  and some  $k \in \mathbb{N}$  sufficiently large. We can now use the compact embedding to extract a subsequence  $\{u_{k_j}\}$  strongly converging to a function  $u^* \in L^1(0, \pi)$ . In particular, since  $\{u_{k_j}\}$  converges almost everywhere to  $u^*$ , then  $\frac{u_{k_j}(x) - u_{k_j}(y)}{|x - y|^{1+\alpha}}$  also converges almost everywhere. Moreover, the limit  $u^*$  satisfies the volume constraint  $\mu > 0$ .

Here comes the tricky part. We can not use a direct method to show that the limit  $u^*$  is indeed in the space  $W^{\alpha,1}$ . This is because the functional space  $W^{s,1}(0, \pi)$  is not reflexive (see, for instance, Triebel's book [38, page 180] where the Besov space notation  $B^{s,1}$  is used). Fortunately for us, we can use Fatou's lemma to bound the semi-norm

$$[u^*]_{\pi, W^{\alpha,1}} = \int_0^\pi \int_{-\pi}^\pi \frac{|u^*(x) - u^*(y)|}{|x - y|^{1+\alpha}} dx dy \leq \liminf_{k \rightarrow +\infty} \int_0^\pi \int_{-\pi}^\pi \frac{|u_{k_j}(x) - u_{k_j}(y)|}{|x - y|^{1+\alpha}} dx dy \leq C.$$

Since  $u^*$  is even, we have  $[u^*]_{W^{\alpha,1}} = 2[u^*]_{\pi, W^{\alpha,1}} \leq C$ , which implies that  $\|u^*\|_{W^{\alpha,1}}$  is bounded, i.e.  $u^* \in W^{\alpha,1}(0, \pi)$ . Hence, since the subspace  $V$  of non-negative functions is closed in  $W_{per}^{\alpha,1}(0, \pi)$ , we have that  $u^* \in V$ .

We have left to prove that the functional evaluated  $u^*$  is indeed a minimum. For this, we can still use Fatou's lemma noticing that, since the functions  $G$  and  $H$  are both continuous from  $\mathbb{R}$  to  $[0, +\infty]$ , then

$$|x - y|^{-\alpha} G\left(\frac{u_{k_j}(x) - u_{k_j}(y)}{|x - y|}\right) \quad \text{and} \quad |x - y|^{-\alpha} H\left(\frac{u_{k_j}(x) + u_{k_j}(y)}{|x - y|}\right)$$

converge almost everywhere in  $(0, \pi) \times \mathbb{R}$ . Therefore, applying Fatou's lemma once again we obtain the lower semi-continuity of the functional  $P_\alpha$  in  $L^1$ , i.e.

$$P_\alpha(u^*) \leq \liminf_{k \rightarrow \infty} P_\alpha(u_{k_j}).$$

□



## Bibliography

- [1] Abatangelo, Nicola. *Large solutions for fractional Laplacian operators*, arXiv preprint arXiv:1511.00571 (2015).
- [2] Abatangelo, Nicola, and Enrico Valdinoci. *A notion of nonlocal curvature*, Numerical Functional Analysis and Optimization 35.7-9 (2014): 793-815.
- [3] Alvinyà, Marc. *PDE's with fractional diffusion*, Bachelor's degree thesis, UPCommons, 2015.
- [4] Ambrosio, Luigi, Cabré, Xavier, *Entire solutions of semilinear elliptic equations in  $R^3$  and a conjecture of De Giorgi*. J. American Math. Soc. 13, 2000, 725-739.
- [5] Ambrosio, Luigi, Guido De Philippis, and Luca Martinazzi. *Gamma-convergence of nonlocal perimeter functionals*, manuscripta mathematica 134.3 (2011): 377-403.
- [6] Barrera, Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces.*, arXiv preprint arXiv:1202.4606 (2012).
- [7] Bisci, Giovanni Molica, Vicentiu D. Radulescu, and Raffaella Servadei. *Variational methods for nonlocal fractional problems*, Vol. 162. Cambridge University Press, 2016.
- [8] Bombieri, Enrico, Ennio De Giorgi, and Enrico Giusti. *Minimal cones and the Bernstein problem*, Inventiones mathematicae 7.3 (1969): 243-268.
- [9] Bucur, Claudia, and Enrico Valdinoci. *Nonlocal diffusion and applications*, Vol. 20. Springer, 2016.
- [10] Cabré, Xavier, Mouhamed Moustapha Fall, Joan Solà-Morales, and Tobias Weth. *Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2016).
- [11] Cabré, Xavier, and Joan Solà-Morales. *Layer solutions in a half-space for boundary reactions*, Communications on pure and applied mathematics 58.12 (2005): 1678-1732.
- [12] Cabré, Xavier, and Yannick Sire. *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, Transactions of the American Mathematical Society 367.2 (2015): 911-941.
- [13] Cabré, Xavier, and Eleonora Cinti. *Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian*, arXiv preprint arXiv:1004.2866 (2010).
- [14] Cabré, Xavier, Albert Mas, and Joan Solà-Morales. *Stationary periodic solutions of fractional and dispersive equations: variational approach and associated Hamiltonian*, In preparation.
- [15] Cabré, Xavier, Mouhamed Moustapha Fall, and Tobias Weth. *Delaunay hypersurfaces with constant nonlocal mean curvature*, arXiv preprint arXiv:1602.02623 (2016).
- [16] Caffarelli, Luis, J.M. Roquejoffre, and Ovidiu Savin. *Nonlocal minimal surfaces.*, Communications on Pure and Applied Mathematics 63.9 (2010): 1111-1144.
- [17] Caffarelli, Luis, and E. Valdinoci. *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Advances in Mathematics 248 (2013): 843-871.
- [18] Cinti, Eleonora, Joaquim Serra, and Enrico Valdinoci. *Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces* arXiv preprint arXiv:1602.00540 (2016).
- [19] Ciraolo, Giulio, Alessio Figalli, Francesco Maggi, and Matteo Novaga *Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2016).
- [20] Dacorogna, Bernard. *Introduction to the Calculus of Variations*, World Scientific Publishing Co Inc, 2014.

- [21] Dávila, Juan, Manuel del Pino, Serena Dipierro, and Enrico Valdinoci *Nonlocal Delaunay surfaces*, *Nonlinear Analysis: Theory, Methods and Applications* 137 (2016): 357-380.
- [22] C. Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante*, *J. Math. Pures Appl.*, 6 (1841), 309-320.
- [23] Dipierro, Serena, Alessio Figalli, Giampiero Palatucci, and Enrico Valdinoci *Asymptotics of the  $s$ -perimeter as  $s$  searrow 0*, arXiv preprint arXiv:1204.0750 (2012).
- [24] Felipe Navarro, Juan Carlos. *A Liouville type result for fractional Schrödinger operators in 1D*, Master's degree thesis, UPCommons, 2017.
- [25] Ghossoub N., Gui C., *On a conjecture of De Giorgi and some related problems*, *Math. Ann.* 311 (1998), 481-491.
- [26] Giusti, Enrico. *Minimal surfaces and functions of bounded variation*, Springer, 1984.
- [27] Gray, Jeremy, and Mario Micalef. *The work of Jesse Douglas on minimal surfaces*, arXiv preprint arXiv:0710.5478 (2007).
- [28] Hamel, François, Xavier Ros-Oton, Yannick Sire, and Enrico Valdinoci *A one-dimensional symmetry result for a class of nonlocal semilinear equations in the plane*, *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* (2016).
- [29] Leoni, Giovanni. *A first course in Sobolev spaces*, Vol. 105. Providence, RI: American Mathematical Society, 2009.
- [30] Lieb, Elliott H., and Michael Loss. *Analysis, volume 14 of graduate studies in mathematics*, American Mathematical Society, Providence, RI, 4 (2001).
- [31] Di Nezza, Eleonora, Giampiero Palatucci, and Enrico Valdinoci. *Hitchhiker's guide to the fractional Sobolev spaces*, *Bulletin des Sciences Mathématiques* 136.5 (2012): 521-573.
- [32] G. De Philippis, E. Paolini, *A short proof of the minimality of Simons cone*, *Rend. Sem. Mat. Univ. Padova*, 121 (2009) pp. 233-241
- [33] Savin, Ovidiu. *Phase transitions, minimal surfaces and a conjecture of De Giorgi*, *Current developments in mathematics* (2009): 59-113.
- [34] Savin O., *Phase transitions: Regularity of flat level sets*, PhD Thesis UT Austin, 2003.
- [35] Savin, Ovidiu, and Enrico Valdinoci. *Regularity of nonlocal minimal cones in dimension 2*, *Calculus of Variations and Partial Differential Equations* (2013): 1-7.
- [36] Savin, Ovidiu, and Enrico Valdinoci. *Density estimates for a variational model driven by the Gagliardo norm*, *Journal de Mathématiques Pures et Appliquées* 101.1 (2014): 1-26.
- [37] Ros-Oton, Xavier. *Nonlocal elliptic equations in bounded domains: a survey*, arXiv preprint arXiv:1504.04099 (2015).
- [38] Triebel, Hans. *Theory of Function Spaces*, Springer, 1983.
- [39] J. Simons, *Minimal varieties in riemannian manifolds*, *Ann. of Math.*, 88 (1968) pp. 62-105
- [40] Valdinoci, Enrico. *A fractional framework for perimeters and phase transitions*, *Milan Journal of Mathematics* 81.1 (2013): 1-23.
- [41] Valdinoci, Enrico. *From the long jump random walk to the fractional Laplacian*, arXiv preprint ArXiv:0901.3261 1 (2009).