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Abstract	We show that positive isometric averaging operators on the sequence space $\ell^2(\mathbb{Z}, \mu)$ are determined by very subtle arithmetic conditions on μ (even for very simple examples), contrary to what happens in the continuous case $L^2(\mathbb{R}^+)$, where any possible average value is realized by a suitable positive isometry.	
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1 Positive Isometric Averaging Operators 2 on $\ell^2(\mathbb{Z}, \mu)$

3 Santiago Boza  and Javier Soria

4 **Abstract.** We show that positive isometric averaging operators on the
5 sequence space $\ell^2(\mathbb{Z}, \mu)$ are determined by very subtle arithmetic condi-
6 tions on μ (even for very simple examples), contrary to what happens in
7 the continuous case $L^2(\mathbb{R}^+)$, where any possible average value is realized
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11 1. Introduction

12 Isometric properties for averaging operators have been considered in different
13 settings. For example, it is well known that the averaging Hardy operator

$$14 \quad Sf(x) = \frac{1}{x} \int_0^x f(t) dt$$

15 satisfies that [4]

$$16 \quad \|(S - I)f\|_{L^2} = \|f\|_{L^2}, \quad f \in L^2(\mathbb{R}^+). \quad (1)$$

17 Weighted versions of (1) on $L^2(w)$ can be found in [6]. With more generality,
18 the study of necessary and sufficient conditions for an operator T to be an
19 isometry on $L^2(\mathbb{R})$ have been obtained in [1], and a characterization was
20 given in terms of the restriction of T to characteristic functions of intervals.
21 These results were further extended in [3] to integral operators T defined on
22 $L^2(X)$, for a general measure space X , where the isometric condition on T
23 was characterized just by looking at the restriction of T to a certain class of
24 monotone functions. Other estimates for $T = S - I$ on $L^p(w)$, on monotone
25 functions, were studied in [2].

26 It is also worth noticing that the reason why the estimates are taken
27 with respect to the L^2 -norm is motivated by [7, 8], where it is proved that, in

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28 the setting of rearrangement invariant spaces, L^2 is the only space for which
 29 there exist nontrivial isometries.

30 In the discrete case of sequence spaces $\ell^2(\mathbb{Z})$, isometric averaging oper-
 31 ators are not so easily obtained. In particular, for the Cesàro operator

$$32 \quad S^d(\{a_n\}_{n \in \mathbb{N}})(m) = \frac{1}{m} \sum_{j=1}^m a_j,$$

33 it can be proved that the corresponding isometric property (1) does not hold.
 34 Motivated by this fact, we are interested in studying the existence of isome-
 35 tries T on $\ell^2(\mathbb{Z})$ which, on the one hand, are positive operators and, on the
 36 other, they are also λ -averaging operators (see Definition 1.3). To this end,
 37 given a positive operator T on $L^2(X)$, we need to define the action of T on
 38 constant functions:

39 **Definition 1.1.** *Let $T : L^2(X) \rightarrow L^2(X)$ be a positive linear operator. We*
 40 *define*

$$41 \quad T(\mathbf{1}) = \sup \{Tf : f \in L^2(X), 0 \leq f \leq 1\}.$$

42 *Remark 1.2.* We observe that if X is a finite measure space, then $\chi_X \in L^2(X)$
 43 and $T(\mathbf{1}) = T(\chi_X)$. In general, under no restrictions on either T or X , $T(\mathbf{1})$
 44 could be identically equal to infinity.

45 One of our main goals in this work is to describe, for a class of discrete
 46 measures μ on \mathbb{Z} (which includes the counting measure), the set of all possible
 47 averages for positive isometries. We will see that, even in this particular
 48 setting, these averages are determined by very subtle arithmetic conditions
 49 (see Theorem 3.1). Analogously, we can also consider the case when μ is a
 50 measure in \mathbb{N} (see Proposition 2.8). In Remark 1.5 we will motivate why we
 51 are restricting our attention to atomic measures.

52 To formulate the problem in a appropriate way and to fix the notation,
 53 we first give the following definition:

54 **Definition 1.3.** *Given a discrete and positive measure $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ on \mathbb{Z} , we*
 55 *will say that a positive operator T , defined for a given sequence $x \in \ell^2(\mathbb{Z}, \mu)$*
 56 *as*

$$57 \quad T(x)(j) = \sum_{k \in \mathbb{Z}} a_{jk} x_k \mu_k, \quad j \in \mathbb{Z}, \quad (2)$$

58 *is a λ -averaging operator, if it satisfies that*

$$59 \quad T(\mathbf{1}) \equiv \lambda \mathbf{1}. \quad (3)$$

60 *We will denote by $\sigma_A^+(\mu, \mathbb{Z}) = \sigma_A^+(\mu)$ the set of all values $\lambda \geq 0$ for*
 61 *which there exists a positive isometric operator such that (3) holds.*

62 *Remark 1.4.* A straightforward argument shows that (3) is equivalent to the
 63 condition

$$64 \quad \sum_{k \in \mathbb{Z}} a_{jk} \mu_k = \lambda, \text{ for all } j \in \mathbb{Z}, \quad (4)$$

65 which implies that the sequence $\{a_{jk}\}_{k \in \mathbb{Z}}$ is in $\ell^1(\mathbb{Z}, \mu)$, for any $j \in \mathbb{Z}$.

Clearly, the identity is a trivial example of an operator satisfying Definition 1.3, with $\lambda = 1$, and hence for all measures μ on \mathbb{Z} , $1 \in \sigma_A^+(\mu)$.

Observe also that $\lambda = 0$ would imply that $T \equiv 0$. Therefore, $\sigma_A^+(\mu) \subset (0, \infty)$. It also easily seen that $\sigma_A^+(\mu)$ is closed under multiplication (just by looking at the composition of the corresponding operators).

Without loss of generality, we are going to assume that μ is an infinite measure not vanishing at any $k \in \mathbb{Z}$. In fact, if μ is finite and T is a positive isometry with $T(\mathbf{1}) \equiv \lambda \mathbf{1}$, then

$$\lambda \left(\sum_{k \in \mathbb{Z}} \mu_k \right)^{1/2} = \|T(\mathbf{1})\|_{\ell^2(\mathbb{Z}, \mu)} = \|\chi_{\mathbb{Z}}\|_{\ell^2(\mathbb{Z}, \mu)} = \left(\sum_{k \in \mathbb{Z}} \mu_k \right)^{1/2},$$

which implies that $\lambda = 1$ and $\sigma_A^+(\mu) = \{1\}$. For this reason, from now on we will work with a measure $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$, with $0 < \mu_k < \infty$ (we can identify the support of μ with \mathbb{Z} and $\sum_{k \in \mathbb{Z}} \mu_k = \infty$).

Remark 1.5. We observe that the existence of λ -averaging positive isometric operators in \mathbb{R}^+ is trivial, since for any $\lambda > 0$, the dilation operator $T_\lambda f(x) = \lambda f(\lambda^2 x)$ satisfies all these properties: T_λ is a positive isometry in $L^2(\mathbb{R}^+)$ and $T_\lambda(\mathbf{1}) \equiv \lambda \mathbf{1}$, since if $f_N(x) = \chi_{(0, N)}(x)$, then

$$T_\lambda(f_N)(x) = \lambda \chi_{(0, N)}(\lambda^2 x) = \lambda \chi_{(0, N/\lambda^2)}(x) \rightarrow \lambda \chi_{\mathbb{R}^+}(x), \quad \text{as } N \rightarrow \infty.$$

Hence, $\lambda \chi_{\mathbb{R}^+} = \sup_N T_\lambda f_N \leq T_\lambda(\mathbf{1}) \leq \lambda \chi_{\mathbb{R}^+}$.

We see that this argument fails for \mathbb{Z} , since nontrivial dilations are never isometries on $\ell^2(\mathbb{Z})$.

In Sect. 2, we will prove some results about the possible values of the set $\sigma_A^+(\mu)$, when $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ is a general discrete positive measure. In particular, Theorem 2.2 is the main tool to translate the problem from an algebraic or matricial formulation into a geometrical property regarding some suitable partitions of the integers. Proposition 2.8 shows that we can transfer, in a canonical way, the averaging values from \mathbb{Z} to \mathbb{N} . In Sect. 3 we prove our main result (Theorem 3.1) which characterizes $\sigma_A^+(\mu)$ for all possible measures μ of the form $\mu_k = a \chi_A(k) + b \chi_B(k)$, $a, b > 0$, $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}$. In particular, $\sigma_A^+(\mu)$ is a countable set contained in $(0, 1]$, which depends on arithmetic properties of a/b and the cardinality of A and B .

In what follows we will use the following notation: for a given subset A of the integers, $|A|$ will denote the cardinality of A , and if $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ is a measure defined on \mathbb{Z} , $\mu(A) = \sum_{k \in A} \mu_k$.

2. Results on General Measures on \mathbb{Z}

We start by recalling the following result concerning isometries in an arbitrary real Hilbert space H , see [5].

Lemma 2.1. *Let H be a real Hilbert space, let $T : H \rightarrow H$, and let T^* denote its adjoint operator. Then T is an isometry, i.e., $\|Tx\| = \|x\|$ for any $x \in H$, if and only if $T^*T = I$.*

105 We are now going to apply this result to the discrete case $\ell^2(\mathbb{Z}, \mu)$, to
 106 obtain the main characterization needed in Sect. 3 to completely describe
 107 $\sigma_A^+(\mu)$:

108 **Theorem 2.2.** *A necessary and sufficient condition to find a positive and*
 109 *isometric λ -averaging operator T on $\ell^2(\mathbb{Z}, \mu)$ is the existence of a partition*
 110 *$\{I_j\}_{j \in \mathbb{Z}}$, of the set of integers \mathbb{Z} , for which*

$$111 \quad \frac{\mu(I_j)}{\mu_j} = \frac{1}{\lambda^2}, \text{ for all } j \in \mathbb{Z}. \quad (5)$$

112 *Proof.* Let T be a positive and isometric λ -averaging operator given as in
 113 (2). Then,

$$114 \quad T^*(x)(j) = \sum_{k \in \mathbb{Z}} a_{kj} x_k \mu_k, \quad x \in \ell^2(\mathbb{Z}, \mu).$$

115 As an application of Lemma 2.1, since T is an isometry then, for every $j \in \mathbb{Z}$,

$$116 \quad \sum_{l \in \mathbb{Z}} a_{lj}^2 \mu_l = \frac{1}{\mu_j}, \quad (6)$$

117 and for $j \neq k$,

$$118 \quad \sum_{l \in \mathbb{Z}} a_{lj} a_{lk} \mu_l = 0. \quad (7)$$

119 Condition (7) implies that, for all $j \in \mathbb{Z}$, there exists a unique k_j such
 120 that $a_{jk_j} > 0$ which, in combination with condition (4), shows that $a_{jk_j} =$
 121 λ/μ_{k_j} . Thus, the matrix satisfies that on each row j , just one element is
 122 different from zero, and on each column k , all the nonzero elements take the
 123 same value.

124 Thus, if we define $I_j = \{l \in \mathbb{Z}, a_{lj} \neq 0\}$, then it is easy to see that
 125 $\{I_j\}_{j \in \mathbb{Z}}$ is a partition of the integers. Finally, condition (6) combined with
 126 the equality $a_{lj} = \lambda/\mu_j$, if $l \in I_j$, gives us that for a fixed $j \in \mathbb{Z}$,

$$127 \quad \frac{1}{\mu_j} = \sum_{l \in I_j} a_{lj}^2 \mu_l = \sum_{l \in I_j} \frac{\lambda^2}{\mu_j^2} \mu_l = \frac{\lambda^2}{\mu_j^2} \sum_{l \in I_j} \mu_l,$$

128 which is (5).

129 Conversely, given a partition $\{I_j\}_{j \in \mathbb{Z}}$ and $\lambda > 0$ satisfying (5), we define

$$130 \quad a_{jk} = \begin{cases} \lambda/\mu_k, & \text{if } j \in I_k, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

131 Finally, if we define T as in (2), T is a positive and isometric λ -averaging
 132 operator; i.e., T satisfies (4), (6), and (7). \square

133 *Remark 2.3.* It is important to observe that the same partition of \mathbb{Z} , as a
 134 subset of $\mathcal{P}(\mathbb{Z})$, may, or may not, satisfy condition (5), depending on the
 135 numeration chosen. For example, if we take

$$136 \quad I_k = \begin{cases} \{0, 1\}, & k = 0 \\ \{k + 1\}, & k > 0 \\ \{k\}, & k < 0, \end{cases}$$

137 then, there is no measure μ and no $\lambda > 0$ for which (5) holds since, otherwise,
 138 for $k = 0$

139
$$\frac{\mu(I_0)}{\mu_0} = \frac{\mu_0 + \mu_1}{\mu_0} = 1 + \frac{\mu_1}{\mu_0} = \frac{1}{\lambda^2},$$

140 and, for $k < 0$,

141
$$\frac{\mu(I_k)}{\mu_k} = \frac{\mu_k}{\mu_k} = 1.$$

142 Hence, $\lambda = 1$ and $\mu_1 = 0$, which is a contradiction with the fact that $\mu_k > 0$,
 143 for all $k \in \mathbb{Z}$.

144 On the other hand, if we consider the same partition but numbered as
 145 follows

146
$$I_k = \begin{cases} \{0, 1\}, & k = 0 \\ \{-k\}, & k > 0 \\ \{-k + 1\}, & k < 0, \end{cases}$$

147 we can arbitrary take μ_0 and μ_1 , both strictly positive, such that

148
$$\frac{\mu(I_0)}{\mu_0} = \frac{\mu_0 + \mu_1}{\mu_0} = 1 + \frac{\mu_1}{\mu_0} := \alpha > 1.$$

149 For this α , if we define

150
$$\mu_k = \begin{cases} \alpha^{-2k-1}\mu_1, & k \leq -1 \\ \alpha^{2k-2}\mu_1, & k \geq 2, \end{cases}$$

151 then, condition (5) holds, with $\lambda = \alpha^{-1/2}$.

152 *Remark 2.4.* An example of the matricial representation of an operator T ,
 153 as in Theorem 2.2, where the sets I_j are intervals, is the following:

154
$$T \longleftrightarrow \begin{bmatrix} \dots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & \lambda/\mu_k & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & \lambda/\mu_k & 0 & \dots \\ \dots & \lambda/\mu_{k-1} & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \dots \\ \dots & \lambda/\mu_{k-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & \lambda/\mu_{k+1} & \dots \\ \dots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & \lambda/\mu_{k+1} & \dots \\ \dots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

155 For non necessarily positive isometries, this matricial representation is
 156 not longer true. For example, if we consider an isometric averaging convolu-
 157 tion operator in \mathbb{Z} , $T(a)(j) = (K * a)(j)$, $j \in \mathbb{Z}$, where $K \in \ell^1(\mathbb{Z})$, then Parseval's
 158 theorem give us that $|\widehat{K}(\theta)| = 1$, $\theta \in \mathbb{T}$. Since $T\mathbf{1}(j) = \lambda = \widehat{K}(0) > 0$, for

Author Proof

every $j \in \mathbb{Z}$, then $\lambda = 1$ (see also [3] for further information). In particular, if we take $\widehat{K}(\theta) = e^{i|\theta|}$, then it can be easily proved that

$$K(j) = \begin{cases} 1/2, & |j| = 1 \\ \frac{i}{\pi} \frac{1 + (-1)^j}{1 - j^2}, & |j| \neq 1, \end{cases}$$

and hence, the matrix of the operator T has coefficients $a_{j,k} = K(j - k)$, which do not satisfy condition (8).

We now show that there are some arithmetic restrictions on the measure μ to obtain a nontrivial $\sigma_A^+(\mu)$, together with some general properties of this set.

Proposition 2.5. *If $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ is a measure on \mathbb{Z} such that the cardinality of the set $A_\mu = \{k \in \mathbb{Z} : \mu_k \in \mathbb{R} \setminus \mathbb{Q}\}$ is finite and nonempty, then $\sigma_A^+(\mu) = \{1\}$.*

Proof. Due to the compatibility condition (5), we observe that, since there is only a finite number of irrational numbers in the sequence $\{\mu_k\}_{k \in \mathbb{Z}}$, the quotients

$$\mu(I_k) / \mu_k = \alpha$$

must be a constant rational number for all $k \in \mathbb{Z}$. For simplicity in the notation, let us assume that the indices $1 \leq k \leq N$ correspond to the values μ_k which are irrational. Since $\mu(I_k)$ is irrational, $1 \leq k \leq N$, and there are only N irrational values for μ then, for each such k , there exists a unique $\sigma(k) \in \{1, \dots, N\}$ such that $\sigma(k) \in I_k$. Clearly, σ is a permutation of the set $\{1, \dots, N\}$ and

$$\mu(I_k) = \mu_{\sigma(k)} + \beta_k = \alpha \mu_k,$$

for some $\beta_k \in \mathbb{Q}$, $1 \leq k \leq N$.

This system of linear equations can be written as

$$(\mathbb{I}_N + A)\vec{\mu} = \vec{\beta}, \tag{9}$$

where $\vec{\mu} = (\mu_1, \dots, \mu_N)^T$, $\vec{\beta} = (-\beta_1, \dots, -\beta_N)^T$, \mathbb{I}_N denotes the identity matrix of dimension N and A is a $N \times N$ matrix, depending on σ , such that $A^N = (-\alpha)^N \mathbb{I}_N$. Hence, it is enough to study whether $\mathbb{I}_N + A$ is invertible to conclude that $\vec{\mu}$ and $\vec{\beta}$ cannot satisfy (9).

Indeed, due to the properties of A and the Caley–Hamilton theorem, the minimal polynomial of $\mathbb{I}_N + A$ must divide $p(x) = (x - 1)^N - (-\alpha)^N$, and hence p is also its characteristic polynomial. From this observation we deduce that $\det(\mathbb{I}_N + A) = (-1)^N p(0) = 1 - \alpha^N$.

Then, $\mathbb{I}_N + A$ is invertible if and only if $\alpha \neq 1$ and hence, solving $\vec{\mu}$ from (9), we get that the components of $\vec{\mu}$ should be rational numbers, which is a contradiction since $\mathbb{I}_N + A$ is a matrix with rational coefficients, and $(\mathbb{I}_N + A)^{-1} \vec{\beta}$ is a vector in \mathbb{Q}^N . Thus, necessarily $\alpha = 1$ and in this case, the partition $I_k = \{k\}$ gives the average value $\lambda = 1$. \square

Proposition 2.6. *Let $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ be a measure on \mathbb{Z} satisfying that $\inf_{k \in \mathbb{Z}} \mu_k = m > 0$. Then, $\sigma_A^+(\mu) \subset (0, 1]$.*

198 *Proof.* Let λ be in $\sigma_A^+(\mu)$ and let $\{I_k\}_{k \in \mathbb{Z}}$ be as in Theorem 2.2. For any
 199 $\varepsilon > 0$, let us consider the set

200
$$A_\varepsilon = \{k \in \mathbb{Z} : m \leq \mu_k < m + \varepsilon\}.$$

201 Condition (5) implies that, for any $k \in A_\varepsilon$

202
$$\frac{1}{\lambda^2} = \frac{\mu(I_k)}{\mu_k} = \frac{\mu(I_k \cap A_\varepsilon) + \mu(I_k \cap A_\varepsilon^c)}{\mu_k} > \frac{m|I_k \cap A_\varepsilon| + (m + \varepsilon)|I_k \cap A_\varepsilon^c|}{m + \varepsilon}$$

 203
$$= |I_k \cap A_\varepsilon^c| + \frac{m}{m + \varepsilon}|I_k \cap A_\varepsilon|.$$

204 Since $\varepsilon > 0$ is arbitrary, and the sets $(I_k \cap A_\varepsilon^c)$ and $(I_k \cap A_\varepsilon)$ cannot be both
 205 simultaneously empty, then we obtain that $\lambda \leq 1$. \square

206 *Remark 2.7.* The following counterexample shows that this last statement
 207 is not true, in general, for discrete measures $\mu = \{\mu_k\}_{k \in \mathbb{Z}}$ whose infimum is
 208 equal to zero. To see this, just consider $\mu = \{\lambda_0^{2k}\}_{k \in \mathbb{Z}}$, $\lambda_0 > 1$ and take the
 209 partition $I_k = \{k - 1\}$, $k \in \mathbb{Z}$. Then,

210
$$\frac{\mu(I_k)}{\mu_k} = \frac{1}{\lambda_0^2},$$

211 and hence, $1 < \lambda_0 \in \sigma_A^+(\mu)$.

212 Observe that this example also shows that, in fact, for every $\lambda > 0$,
 213 there exists a measure μ such that $\lambda \in \sigma_A^+(\mu)$.

214 As we have already mentioned in the introduction, we can describe the
 215 averaging values in \mathbb{N} , similarly to the case of the integers, by means of a
 216 suitable change of indices:

217 **Proposition 2.8.** *There exists a bijection ψ between measures in \mathbb{Z} and mea-*
 218 *asures in \mathbb{N} such that, for every μ in \mathbb{Z}*

219
$$\sigma_A^+(\mu, \mathbb{Z}) = \sigma_A^+(\psi(\mu), \mathbb{N}).$$

220 *Proof.* First we notice that considering the bijection $\psi : \mathbb{Z} \rightarrow \mathbb{N}$ defined by

221
$$\psi(k) = \begin{cases} 2k + 1, & k \geq 0 \\ -2k, & k \leq -1, \end{cases}$$

222 and its corresponding inverse $\phi = \psi^{-1}$, we can obtain, from a positive mea-
 223 sure μ defined on \mathbb{Z} , a positive measure $\psi(\mu)$ defined on \mathbb{N} by means of

224
$$\psi(\mu)_j = \mu_{\phi(j)}, \quad j \in \mathbb{N},$$

225 and conversely.

226 Similarly, for a given sequence $a = \{a_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}, \mu)$, we obtain the
 227 sequence $\psi(a) := \{a_{\phi(j)}\}_{j \in \mathbb{N}} \in \ell^2(\mathbb{N}, \psi(\mu))$ and conversely, we have that
 228 $\phi(a) := \{a_{\psi(n)}\}_{n \in \mathbb{Z}}$, for $a \in \ell^2(\mathbb{N}, \psi(\mu))$. This is an isometric transformation
 229 from $\ell^2(\mathbb{Z}, \mu)$ onto $\ell^2(\mathbb{N}, \psi(\mu))$. Indeed,

230
$$\sum_{j=1}^{\infty} a_{\phi(j)}^2 \mu_{\phi(j)} = \sum_{k \in \mathbb{Z}} a_k^2 \mu_k.$$

231 By means of this correspondence, given an operator $T : \ell^2(\mathbb{Z}, \mu) \longrightarrow$
 232 $\ell^2(\mathbb{Z}, \mu)$ the following operator $\psi(T)$ on $\ell^2(\mathbb{N}, \psi(\mu))$ can be defined

233
$$\psi(T)(a)_j := T(\phi(a))_{\phi(j)}, \quad j \in \mathbb{N}.$$

234 If T is an isometric operator in $\ell^2(\mathbb{Z}, \mu)$, $\psi(T)$ is an isometry in $\ell^2(\mathbb{N}, \psi(\mu))$,
 235 since

236
$$\|\psi(T)(a)\|_{\ell^2(\mathbb{N}, \psi(\mu))}^2 = \sum_{j=1}^{\infty} T(\phi(a))_{\phi(j)}^2 \mu_{\phi(j)} = \sum_{k \in \mathbb{Z}} T(\phi(a))_k^2 \mu_k$$

 237
$$= \sum_{k \in \mathbb{Z}} \phi(a)_k^2 \mu_k = \sum_{k \in \mathbb{Z}} a_{\psi(k)}^2 \mu_k = \sum_{j=1}^{\infty} a_j^2 \psi(\mu)_j.$$

238 It is now easy to see that if $\lambda \in \sigma_A^+(\mu, \mathbb{Z})$, with T its associated posi-
 239 tive isometry, then $\lambda \in \sigma_A^+(\psi(\mu), \mathbb{N})$, and $\psi(T)$ defined in $\ell^2(\mathbb{N}, \psi(\mu))$, is
 240 the corresponding positive isometry. In fact, if $T(\mathbf{1}) \equiv \lambda \mathbf{1}$, then $\psi(T)(\mathbf{1}) =$
 241 $T(\phi(\mathbf{1})) = T(\mathbf{1}) \equiv \lambda \mathbf{1}$. The converse embedding $\sigma_A^+(\psi(\mu), \mathbb{N}) \subset \sigma_A^+(\mu, \mathbb{Z})$ is
 242 proved similarly. \square

243 **3. Case of $\mu_k = a\chi_A(k) + b\chi_B(k)$**

244 In this section we are going to give a complete description of the set $\sigma_A^+(\mu)$ in
 245 the case of a positive measure μ defined on \mathbb{Z} , taking two possible values. Since
 246 the compatibility condition (5) is homogeneous, the set $\sigma_A^+(\mu)$ is invariant
 247 under dilations on the measure and, therefore, we can assume without loss of
 248 generality that it has the form $\mu_k = r\chi_A(k) + \chi_B(k)$, where A, B is a partition
 249 of \mathbb{Z} , $r > 0$, and $k \in \mathbb{Z}$. Also, we can assume that B is an infinite set, since
 250 otherwise it would suffice to consider the measure $\mu_k = \chi_A(k) + r^{-1}\chi_B(k)$,
 251 $k \in \mathbb{Z}$.

252 Observe that Proposition 2.6 implies that, for these measures, $\sigma_A^+(\mu) \subset$
 253 $(0, 1]$. Moreover, if A is finite, Proposition 2.5 implies that r must be a rational
 254 number, and we will prove that, in this case, the set $\sigma_A^+(\mu) \subset \{1/\sqrt{n}\}_{n \in \mathbb{N}}$.
 255 However, we are going to see that, in general, the characterization of $\sigma_A^+(\mu)$ is
 256 more involved, and it strongly depends on the arithmetic properties of $r > 0$
 257 and $|A|$.

258 The main tool we are going to use is Theorem 2.2. We will reduce the
 259 condition for λ to be in $\sigma_A^+(\mu)$ (or, equivalently, the existence of a nonnegative
 260 isometric operator T for which $T(\mathbf{1}) = \lambda$) to finding a suitable partition of \mathbb{Z}
 261 satisfying (5).

262 **Theorem 3.1.** *Let $\{\mu_k\}_{k \in \mathbb{Z}}$ be a discrete and positive measure on \mathbb{Z} defined*
 263 *by $\mu_k = r\chi_A(k) + \chi_B(k)$, $k \in \mathbb{Z}$, where $r > 0$ and B is an infinite set.*

- 264 (i) *If A is finite and $r \notin \mathbb{Q}$, then $\sigma_A^+(\mu) = \{1\}$.*
 265 (ii) *Assume A is finite and $r = p/q \in \mathbb{Q}$, with $p, q \in \mathbb{N}$ and $(p, q) = 1$.*

266
$$\bullet \text{ If } |A| < q, \text{ then } \sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}.$$

267

268 • If $q \leq |A|$ and $q \mid |A|$, then

269
$$\sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{jq}} \right\}_{j \in \mathbb{N}} \cup \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}.$$

270 • If $q \leq |A|$ and $q \nmid |A|$, then $\sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}.$

271 (iii) Assume A is infinite.

272 • If $r \notin \mathbb{Q}$ satisfies that $ar^2 + br - c = 0$, where $b \in \mathbb{Z}$, $a, c \in \mathbb{N}$ and
273 $(a, b, c) = 1$, then

274
$$\sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{j(ar+b)+m}} \right\}_{j \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}} \cup \left\{ \frac{1}{\sqrt{n}} \right\}_{n \in \mathbb{N}}. \quad (10)$$

275 • In any other case we have that $\sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{n}} \right\}_{n \in \mathbb{N}}.$

276 *Proof.* The first part is a direct consequence of Proposition 2.5. To prove
277 (ii), we first introduce the following notation: for $\lambda \in \sigma_A^+(\mu)$, we denote by
278 $\alpha = 1/\lambda^2$. Let $\{I_k\}_{k \in \mathbb{Z}}$ be a partition of \mathbb{Z} . Then (5) implies that, for $k \in A$,

279
$$\frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = \alpha, \quad (11)$$

280 and for $k \in B$,

281
$$\frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = \alpha. \quad (12)$$

282 Since A is finite, there is some $k \in B$ such that $I_k \cap A = \emptyset$, and (12) implies
283 that $\alpha = |I_k \cap B| \in \mathbb{N}$. We now consider two possibilities: If $\alpha r \in \mathbb{N}$, then
284 $q \mid \alpha p$ and hence $q \mid \alpha$. On the other hand, if $\alpha r \notin \mathbb{N}$, from (11) we deduce
285 that, for all $k \in A$, $I_k \cap A \neq \emptyset$. Hence, since A is finite, the correspondence
286 between $k \in A$ and the sets I_k , with $k \in A$, must be a one to one mapping,
287 and therefore $|I_k \cap A| = 1$ and $|I_k \cap B| = |I_k| - 1$. Thus, from (11), we have
288 that

289
$$r|I_k \cap A| + |I_k \cap B| = r + |I_k| - 1 = \alpha r \Rightarrow (\alpha - 1)r \in \mathbb{N} \Rightarrow q \mid (\alpha - 1).$$

290 So we have obtained that $q \mid \alpha$ or $q \mid (\alpha - 1)$ are necessary conditions if
291 $\lambda \in \sigma_A^+(\mu)$. That is,

292
$$\sigma_A^+(\mu) \subset \left\{ \frac{1}{\sqrt{jq}} \right\}_{j \in \mathbb{N}} \cup \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}. \quad (13)$$

293 We will now consider each of the three cases of (ii):

294 Assume that $|A| < q$. If $q \mid \alpha$, since $\alpha = jq$, for some $j \in \mathbb{N}$, condition
295 (11) implies that, for $k \in A$,

296
$$p|I_k \cap A| + q|I_k \cap B| = jpq.$$

297 Thus, $|I_k \cap A|$ is a multiple of q , and therefore $|I_k \cap A| = 0$, for all $k \in A$,
298 since $|I_k \cap A| \leq |A| < q$. On the other hand, for $k \in B$, condition (12) implies

299
$$p|I_k \cap A| + q|I_k \cap B| = jq^2.$$

300 As before, this equation leads us to the condition $|I_k \cap A| = 0$, for all $k \in B$,
 301 which is a contradiction, since $\{I_k\}_k$ is a partition of \mathbb{Z} . Therefore, we have
 302 proved that $q \mid (\alpha - 1)$ and

$$303 \quad \sigma_A^+(\mu) \subset \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}. \quad (14)$$

304 Conversely, if $\alpha = jq + 1$, $j \in \mathbb{N} \cup \{0\}$, let us see that we can find a posi-
 305 tive isometric λ -averaging operator, which, by Theorem 2.2, is equivalent to
 306 finding a partition satisfying (5). Indeed, we construct $\{I_k\}_{k \in \mathbb{Z}}$ as follows:

307 If $k \in A$, we take $I_k = \{k\} \cup (I_k \cap B)$, with $|I_k| = pj + 1$, and if $k \in B$,
 308 we take $I_k \subset B$ and $|I_k| = \alpha$. It is clear that such a partition of \mathbb{Z} exists,
 309 since B is infinite. Finally, let us prove that (11) and (12), equivalently (5),
 310 hold:

311 If $k \in A$,

$$312 \quad \frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 1 + \frac{qpj}{p} = 1 + qj = \alpha,$$

313 and, if $k \in B$,

$$314 \quad \frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

315 which finally shows that

$$316 \quad \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}} \subset \sigma_A^+(\mu). \quad (15)$$

317 Therefore, using (14) and (15) we conclude the result.

318 Assume now that $q \leq |A|$ and $q \mid |A|$; that is, $|A| = sq$, for some $s \in \mathbb{N}$.
 319 Using (13), it suffices to prove that if $q \mid \alpha$ or $q \mid (\alpha - 1)$, then we can find a
 320 partition $\{I_k\}_{k \in \mathbb{Z}}$ satisfying both (11) and (12).

321 If $q \mid \alpha$, then $\alpha = jq$, $j \in \mathbb{N}$. Now, we set $\{I_k\}_{k \in \mathbb{Z}}$ as follows: Choose
 322 $k_1, \dots, k_s \in A$ and take $|I_{k_n} \cap A| = q$, $1 \leq n \leq s$ and $|I_{k_n} \cap B| = p(j - 1)$.

323 For $k \in A \setminus \{k_1, \dots, k_s\}$, take $I_k \subset B$, with $|I_k| = pj$. If $k \in B$, take
 324 $I_k \subset B$, with $|I_k| = \alpha$. Then, if $k \in \{k_1, \dots, k_s\}$,

$$325 \quad \frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = q + q(j - 1) = qj = \alpha.$$

326 If $k \in A \setminus \{k_1, \dots, k_s\}$

$$327 \quad \frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 0 + qj = \alpha.$$

328 If $k \in B$,

$$329 \quad \frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

330 which finally shows that

$$331 \quad \left\{ \frac{1}{\sqrt{jq}} \right\}_{j \in \mathbb{N}} \subset \sigma_A^+(\mu). \quad (16)$$

332 Still assuming that $q \leq |A|$ and $q \mid |A|$, we now consider the case $q \mid (\alpha - 1)$;
 333 that is, $\alpha = 1 + jq$, for some $j \in \mathbb{N} \cup \{0\}$. If $k \in A$, take $|I_k \cap A| = 1$ and
 334 $|I_k \cap B| = pj$. For $k \in B$, take $I_k \subset B$, with $|I_k| = \alpha$. If $k \in A$,

$$335 \quad \frac{\mu(I_k)}{\mu_k} = \frac{r|I_k \cap A| + |I_k \cap B|}{r} = 1 + \frac{qpj}{p} = 1 + qj = \alpha,$$

336 and, if $k \in B$,

$$337 \quad \frac{\mu(I_k)}{\mu_k} = r|I_k \cap A| + |I_k \cap B| = 0 + \alpha,$$

338 which shows

$$339 \quad \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}} \subset \sigma_A^+(\mu). \tag{17}$$

340 Then, using (13), (16), and (17), we have that

$$341 \quad \sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{jq}} \right\}_{j \in \mathbb{N}} \cup \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}.$$

342 To finish the proof of (ii) we now assume that A is finite, $q \leq |A|$, and
 343 $q \nmid |A|$. If $\{I_k\}_k$ is a partition associated to α , as in Theorem 2.2, using (11)
 344 and (12) it is easily seen that q has to divide $|I_k \cap A|$, for every $k \in \mathbb{Z}$, and
 345 hence, since $|A| = \sum_k |I_k \cap A|$, then q should also divide $|A|$, which is a
 346 contradiction. Thus, from (13) we get

$$347 \quad \sigma_A^+(\mu) \subset \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}. \tag{18}$$

348 Conversely, if $\alpha = jq + 1$, we define the same partition $\{I_k\}_{k \in \mathbb{Z}}$ as in the case
 349 $|A| < q$: If $k \in A$, we take $I_k = \{k\} \cup (I_k \cap B)$, with $|I_k| = pj + 1$, and if
 350 $k \in B$, we take $I_k \subset B$ and $|I_k| = \alpha$. Then, as before,

$$351 \quad \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}} \subset \sigma_A^+(\mu). \tag{19}$$

352 Thus, from (18) and (19) we conclude

$$353 \quad \sigma_A^+(\mu) = \left\{ \frac{1}{\sqrt{jq+1}} \right\}_{j \in \mathbb{N} \cup \{0\}}.$$

354 Finally, we give the proof of (iii) and we now assume that both A and
 355 B are infinite sets. First, if $\alpha \in \mathbb{N}$, we construct the partition in such a way
 356 that $I_k \subset A$, if $k \in A$ and $I_k \subset B$, if $k \in B$, with $|I_k| = \alpha$, for all $k \in \mathbb{Z}$.
 357 Using (11) and (12), we see that

$$358 \quad \{1/\sqrt{n}\}_{n \in \mathbb{N}} \subset \sigma_A^+(\mu). \tag{20}$$

359 To finish, we will prove the following claim:

360 $\sigma_A^+(\mu) \setminus \{1/\sqrt{n}\}_{n \in \mathbb{N}} \neq \emptyset$ if and only if r is an irrational number which is
 361 the positive root of a polynomial $ax^2 + bx - c$, where $b \in \mathbb{Z}$, $a, c \in \mathbb{N}$ and
 362 $(a, b, c) = 1$.

363 Indeed, if $\lambda \in \sigma_A^+(\mu) \setminus \{1/\sqrt{n}\}_{n \in \mathbb{N}}$ and $\alpha = 1/\lambda^2$, equations (11) and
 364 (12) imply that

365
$$\mu(I_k) = r\alpha = rm_k + c_k, \text{ for some } m_k, c_k \in \mathbb{N} \cup \{0\}, k \in A, \quad (21)$$

366
$$\mu(I_k) = \alpha = ra_k + l_k, \text{ for some } a_k, l_k \in \mathbb{N} \cup \{0\}, k \in B. \quad (22)$$

367 Since $\alpha \notin \mathbb{N}$, then $a_k, c_k \neq 0$ and hence $r \notin \mathbb{Q}$. In fact, if $r = p/q \in \mathbb{Q}$, with
 368 $(p, q) = 1$, we would obtain that

369
$$pq(m_k - l_k) = p^2a_k - q^2c_k,$$

370 and hence, p must divide c_k , which using (21) would imply that $\alpha \in \mathbb{N}$.

371 Note also that if we combine (21) and (22), we can prove that $a_{k'}r^2 +$
 372 $(l_{k'} - m_k)r - c_k = 0$, for every $k \in A$ and $k' \in B$. Conversely, if $r \notin \mathbb{Q}$ is
 373 the positive root of the polynomial $ax^2 + bx - c$, with $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$, we
 374 pick $j \in \mathbb{N}$, write $jb = l - m$, with $l, m \in \mathbb{N} \cup \{0\}$, and define the partition
 375 $\{I_k\}_{k \in \mathbb{Z}}$ as follows

376
$$|I_k \cap A| = m, \quad |I_k \cap B| = jc, \quad \text{if } k \in A,$$

 377
$$|I_k \cap A| = ja, \quad |I_k \cap B| = l, \quad \text{if } k \in B.$$

378 With this partition we have

379
$$\frac{\mu(I_k)}{\mu_k} = \begin{cases} \frac{mr + jc}{r}, & k \in A, \\ jar + l, & k \in B. \end{cases} \quad (23)$$

380 The fact that $(mr + jc)/r = jar + l$ shows that (23) satisfies the compatibility
 381 condition (5), and this proves the claim, since $jar + l \notin \mathbb{N}$. Moreover, we
 382 observe that $\alpha = jar + jb + m$, which gives us that

383
$$\sigma_A^+(\mu) \setminus \left\{ \frac{1}{\sqrt{n}} \right\}_{n \in \mathbb{N}} = \left\{ \frac{1}{\sqrt{j(ar + b) + m}} \right\}_{j \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}}. \quad (24)$$

384 Finally, if $r \notin \mathbb{Q}$ is the positive root of the polynomial $ax^2 + bx - c$, with
 385 $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$, (20) and (24) prove (10). On the other hand, if r is not
 386 as above, the claim and (20) show that $\sigma_A^+(\mu) = \{1/\sqrt{n}\}_{n \in \mathbb{N}}$. \square

387 **Example 3.2.** We now apply Theorem 3.1 to find $\sigma_A^+(\mu)$, for different sets A
 388 and concrete values of $r > 0$:

- 389 • If $A = \{0, 1, 2\}$ and $r = \sqrt{2}$, then $\sigma_A^+(\mu) = \{1\}$.
- 390 • If $A = \{0, 1, 2\}$ and $r = 1/4$, then $\sigma_A^+(\mu) = \{1/\sqrt{4j + 1}\}_{j \in \mathbb{N} \cup \{0\}}$.
- 391 • If $A = \{0, 1, 2\}$ and $r = 1$ (that is, μ is the counting measure in \mathbb{Z}), then
 392 $\sigma_A^+(\mu) = \{1/\sqrt{j}\}_{j \in \mathbb{N}}$.
- 393 • If $A = \{0, 1, 2\}$ and $r = 2/3$, then

394
$$\sigma_A^+(\mu) = \left\{ 1/\sqrt{3j} \right\}_{j \in \mathbb{N}} \cup \left\{ 1/\sqrt{3j + 1} \right\}_{j \in \mathbb{N} \cup \{0\}}.$$

- 395 • If $A = \{0, 1, 2\}$ and $r = 3/2$, then $\sigma_A^+(\mu) = \{1/\sqrt{2j + 1}\}_{j \in \mathbb{N} \cup \{0\}}$.

- 396 • If $A = \mathbb{N}$ and $r = (\sqrt{5}-1)/2$, which is a root of the polynomial x^2+x-1 ,
 397 then

$$398 \quad \sigma_A^+(\mu) = \left\{ \frac{\sqrt{2}}{\sqrt{j(\sqrt{5}+1)+2m}} \right\}_{j \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}} \cup \left\{ \frac{1}{\sqrt{j}} \right\}_{j \in \mathbb{N}}.$$

- 399 • If $A = \mathbb{N}$ and $r = (\sqrt{5}+3)/2$, which is a root of the polynomial x^2-3x+1 ,
 400 then $\sigma_A^+(\mu) = \{1/\sqrt{j}\}_{j \in \mathbb{N}}$.
 401 • If $A = \mathbb{N}$ and $r = \pi$, then $\sigma_A^+(\mu) = \{1/\sqrt{j}\}_{j \in \mathbb{N}}$.

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405 References

- 406 [1] Ash, P., Marshall Ash, J., Ogden, R.D.: A characterization of isometries. *J.*
 407 *Math. Anal. Appl.* **60**, 417–428 (1977)
 408 [2] Boza, S., Soria, J.: Solution to a conjecture on the norm of the Hardy operator
 409 minus the identity. *J. Funct. Anal.* **260**(4), 1020–1028 (2011)
 410 [3] Boza, S., Soria, J.: Isometries on $L^2(X)$ and monotone functions. *Math.*
 411 *Nach.* **287**, 160–172 (2014)
 412 [4] Brown, A., Halmos, P.R., Shields, A.L.: Cesàro operators. *Acta Sci. Math.*
 413 (Szeged) **26**, 125–137 (1965)
 414 [5] Cerdà, J.: Linear functional analysis. In: Graduate Studies in Mathematics, vol.
 415 116. American Mathematical Society, Providence, Real Sociedad Matemática
 416 Española, Madrid (2010)
 417 [6] Kaiblinger, N., Maligranda, L., Persson, L.E.: Norms in weighted L^2 -spaces and
 418 Hardy operators. In: Function Spaces, The Fifth Conference (Poznań, 1998),
 419 Lecture Notes in Pure and Appl. Math., vol. 213, pp. 205–216. Dekker, New
 420 York (2000)
 421 [7] Kalton, N.J., Randrianantoanina, B.: Surjective isometries on rearrangement-
 422 invariant spaces. *Q. J. Math. Oxford Ser. (2)* **45**(179), 301–327 (1994)
 423 [8] Zaidenberg, M.G.: A representation of isometries on function spaces. *Mat. Fiz.*
 424 *Anal. Geom.* **4**, 339–347 (1997)

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uncorrected proof