

Fuzzy implication functions based on powers of continuous t-norms

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Abstract

The modification (relaxation or intensification) of the antecedent or the consequent in a fuzzy “If, Then” conditional is an important asset for an expert in order to agree with it. The usual method to modify fuzzy propositions is the use of Zadeh’s quantifiers based on powers of t-norms. However, the invariance of the truth value of the fuzzy conditional would be a desirable property when both the antecedent and the consequent are modified using the same quantifier. In this paper, a novel family of fuzzy implication functions based on powers of continuous t-norms which ensure the aforementioned property is presented. Other important additional properties are analyzed and from this study, it is proved that they do not intersect the most well-known classes of fuzzy implication functions.

Keywords: Fuzzy implication function, continuous t-norm, powers of t-norms, fuzzy negation.

1. Introduction

The study of fuzzy implication functions has experienced sensational growth in the last decades due to their applications in approximate reasoning and fuzzy control [3, 4]. Indeed, these operators are used to model fuzzy conditionals and to perform fuzzy inference processes through Modus Ponens and Modus Tollens rules. Beyond these fields, the applications of fuzzy implication functions extend to other domains such as image processing, fuzzy relational equations, fuzzy DI-subsethood measures, computing with words, data mining and rough sets (see [3, 4] and the references within).

Going back to the approximate reasoning field, although some other approaches exist based on the use of not functionally expressible fuzzy implications [11], usually the truth value of a fuzzy conditional of the form $P \rightarrow Q$, where P and Q are fuzzy propositions, is functionally expressed from the truth values of the initial propositions P and Q . Of course, this task is carried out through the so-called fuzzy implication functions. There are lots of different models to construct fuzzy implication functions (see [2, 3]) and in fact this variety becomes necessary to have a wide range of possibilities in order to find the more adequate in each context (see [19]). Indeed, each family of fuzzy implication functions satisfies a subset of the most important additional properties: the exchange principle, the ordering property, the law of importation, among many others which have been studied recently in detail [2, 3, 15].

There is a special property not usually required on fuzzy implication functions, but closely related to approximate reasoning. To introduce this property let us recall the following classical example given in [16] based on tomatoes. It is reasonable to think that the following fuzzy propositions:

If the tomato is red, then it is ripe.
If the tomato is very red, then it is very ripe.
If the tomato is little red, then it is little ripe.

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should have the same truth value. These fuzzy conditionals involve some linguistic modifiers such as *very* or *little* which modify (intensify or relax, respectively) the involved fuzzy propositions. A usual method to model these linguistic modifiers is through the use of the so-called Zadeh's powering modifiers [6]. In particular, potential functions are used in such a way that, for all $x \in [0, 1]$, "*very* x " is computed as x^2 , or "*little* x " is computed as $x^{1/2}$. Although to define these linguistic modifiers Zadeh makes use of the product t-norm it is clear that the same task can be done through powers of any continuous t-norm [21].

The previous property has not been studied yet for the most well-known fuzzy implication functions. In fact, most of them fail to fulfill it. Thus, the main goal of this paper is to introduce a novel family of fuzzy implication functions based on powers with respect to any continuous t-norm which satisfies the required property. In this way, the contribution of this paper is not only the proposal of another family of fuzzy implication functions through a novel construction method (see [2, 4], the references therein and the recent papers [1, 12, 13, 14, 18, 20]), but mainly the relationship of this family with the property of invariance with respect to linguistic modifiers modeled through powers of t-norms.

The structure of the paper is as follows. In Section 2, the basic concepts and results related to powers of t-norms and fuzzy implication functions will be collected. Then, the definition of the novel family of fuzzy implication functions based on powers of t-norms is presented in Section 3. The structure of these operators depending on the chosen t-norm and some examples are given. In Section 4, it is proved that this family fulfills the invariance with respect to linguistic modifiers modeled through powers of t-norms. In addition, other additional properties of fuzzy implication functions are studied as well as their intersection with the most well-known families. The paper ends with a section devoted to the conclusions and future work.

2. Preliminaries

We will suppose the reader is familiar with basic results on t-norms, t-conorms, fuzzy negations and fuzzy implication functions (see [7, 21] for more details on t-norms and see [2, 3, 4, 5] for more details on fuzzy implication functions). In this section we will recall only some concepts about fuzzy implication functions and also about powers with respect to continuous t-norms to make the paper as self-contained as possible.

2.1. Fuzzy implication functions

First, we recall the definition of a fuzzy implication function.

Definition 1 ([3, 5]). A binary operation $I : [0, 1]^2 \rightarrow [0, 1]$ is said to be a *fuzzy implication function* if it satisfies:

(I1) $I(x, z) \geq I(y, z)$ when $x \leq y$, for all $z \in [0, 1]$.

(I2) $I(x, y) \leq I(x, z)$ when $y \leq z$, for all $x \in [0, 1]$.

(I3) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

It follows from the definition that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$ whereas the symmetrical values $I(x, 0)$ and $I(1, x)$ are not derived from the definition. Fuzzy implication functions can satisfy additional properties usually coming from tautologies in crisp logic. Let us recall here some of the most usual ones.

Definition 2 ([3, 5]). Let I be a fuzzy implication function.

- The function N_I defined by $N_I(x) = I(x, 0)$ for all $x \in [0, 1]$, is called the *natural negation* of I and it is always a fuzzy negation.

- I can additionally satisfy the following properties:

1. Exchange Principle:

$$I(x, I(y, z)) = I(y, I(x, z)), \quad \text{for all } x, y, z \in [0, 1]. \quad (\text{EP})$$

2. Law of importation with respect to a t-norm T :

$$I(T(x, y), z) = I(x, I(y, z)), \quad \text{for all } x, y, z \in [0, 1]. \quad (\text{LI}_T)$$

3. Left-neutrality principle:

$$I(1, y) = y \quad \text{for all } y \in [0, 1]. \quad (\text{NP})$$

4. Ordering Property:

$$x \leq y \iff I(x, y) = 1 \quad \text{for all } x, y \in [0, 1]. \quad (\text{OP})$$

5. Identity Principle:

$$I(x, x) = 1 \quad \text{for all } x \in [0, 1]. \quad (\text{IP})$$

6. The *contrapositive symmetry* with respect to a fuzzy negation N ,

$$I(x, y) = I(N(y), N(x)), \quad \text{for all } x, y \in [0, 1]. \quad (\text{CP(N)})$$

2.2. Powers with respect to continuous t-norms

We will suppose that all t-norms T used in this section and all along the paper are continuous. For more details about the results included in this section see [21] where powers with respect to continuous t-norms were fully studied.

From the associativity of any t-norm T , integer powers with respect to T can be defined in the usual way, that is,

$$x_T^{(n)} = T(\overbrace{x, x, \dots, x}^{n \text{ times}}) \quad \text{for all } x \in [0, 1], n \in \mathbb{Z}^+ \text{ and } n \geq 2,$$

with the conventions $x_T^{(1)} = x$ and $x_T^{(0)} = 1$ for all $x \in [0, 1]$.

Similarly, n -th roots and rational powers of an element $x \in [0, 1]$ with respect to a t-norm T are defined as

$$x_T^{(\frac{1}{n})} = \sup\{z \in [0, 1] \mid z^{(n)} \leq x\}, \quad x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)^{(m)}$$

for all $m, n \in \mathbb{Z}^+$.

Lemma 1 ([21]). Consider $k, m, n \in \mathbb{Z}^+$ and let T be a continuous t-norm. Then $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$ for all $x \in [0, 1]$.

From the continuity of T , rational powers with respect to T can be extended to irrational powers through the following definition.

Definition 3 ([21]). Let T be a continuous t-norm and $r \in \mathbb{R}^+$ a positive real number. Consider $\{a_n\}_{n \in \mathbb{Z}^+}$ a sequence of rational numbers such that $\lim_{n \rightarrow \infty} a_n = r$. For all $x \in [0, 1]$, the power $x_T^{(r)}$ is defined as

$$x_T^{(r)} = \lim_{n \rightarrow \infty} x_T^{(a_n)}.$$

The continuity of T ensures both, the existence of the limit and the independence of the considered sequence $\{a_n\}_{n \in \mathbb{Z}^+}$. It is immediate to check that $0 \leq x_T^{(r)} \leq 1$ and $x_T^{(r)} \leq y_T^{(r)}$ whenever $x \leq y$ for all $x, y \in [0, 1]$ and $r \in \mathbb{R}^+$.

When the selected t-norm is Archimedean, the expressions of these powers only depend on the additive generator of the t-norm.

Proposition 2 ([21]). Let T be a continuous Archimedean t-norm with additive generator t . Then

$$x_T^{(r)} = t^{-1}(\min\{t(0), rt(x)\}) \quad \text{for all } x \in [0, 1] \text{ and } r \geq 0.$$

Example 1. In the cases of the three basic continuous t-norms we have that for all $x \in [0, 1]$:

- When $T(x, y) = T_L(x, y) = \max\{x + y - 1, 0\}$ is the Łukasiewicz t-norm, then $x_{T_L}^{(r)} = \max\{0, 1 - r + rx\}$.
- When $T(x, y) = T_P(x, y) = xy$ is the Product t-norm, then $x_{T_P}^{(r)} = x^r$.
- When $T(x, y) = T_M(x, y) = \min\{x, y\}$ is the Minimum t-norm, then $x_{T_M}^{(r)} = \begin{cases} x & \text{if } r > 0, \\ 1 & \text{if } r = 0. \end{cases}$

3. Implication functions based on powers of continuous t-norms

In this section, the formal definition of the family of fuzzy implication functions based on powers of t-norms will be presented. In addition, some examples of these operators will be given as well as the structure of these fuzzy implication functions depending on the structure of the underlying t-norm.

Let us begin by analyzing first the idea behind the formal definition that will be presented later on (see Definition 4). As we have already mentioned in the introduction, fuzzy implication functions are usually the logical operators that manage fuzzy “If, Then” conditionals in fuzzy logic and approximate reasoning¹.

Specifically, the truth value of a fuzzy conditional of the form $P \rightarrow Q$, where P and Q are fuzzy propositions, is functionally expressed from the truth values of the initial propositions P and Q , as follows

$$\mu_{P \rightarrow Q}(x, y) = I(\mu_P(x), \mu_Q(y)),$$

where $I : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a fuzzy implication function. Let us consider the following example of a fuzzy conditional:

“If the price of the computer is around 450 euros, then it is very good.”

It is quite sure that an expert will not agree with this sentence. Probably the expert will require to relax the consequent and will suggest to change it by *good* or even by *correct*. A usual method to modify the consequent in this way can be through the use of the Zadeh’s linguistic powering modifiers [6]. Indeed, the linguistic modifiers can be modeled as x^r where $x \in [0, 1]$ and $r \in \mathbb{R}^+$ in such a way that the consequent is relaxed if $r < 1$ or intensified if $r > 1$.

Based on this idea, the truth value of $x \rightarrow y$ could be defined as the highest possible power of y such that y up to this power becomes greater than or equal to x . For instance, $(0.8 \rightarrow 0.64) = \frac{1}{2}$ because of $0.64^{\frac{1}{2}} = 0.8$. Although to define these linguistic modifiers Zadeh makes use of the product t-norm, any continuous² t-norm T can be suitable to compute its powers and to model the linguistic modifier.

From this previous idea we can formalize the definition as follows.

Definition 4. A binary operator $I : [0, 1]^2 \rightarrow [0, 1]$ is said to be a T -power based implication if there exists a continuous t-norm T such that

$$I(x, y) = \sup\{r \in [0, 1] \mid y_T^{(r)} \geq x\} \quad \text{for all } x, y \in [0, 1].$$

If I is a T -power based implication, then it will be denoted by I^T .

Let us prove first that the defined function I^T is a fuzzy implication in the sense of Definition 1, for any continuous t-norm T .

Proposition 3. Let T be a continuous t-norm and I^T its power based implication. Then I^T is a fuzzy implication function.

PROOF. It is clear from the definition that I^T is decreasing with respect to the first variable and increasing with respect to the second one. Moreover, we also have:

- $I^T(0, 0) = \sup\{r \in [0, 1] \mid 0_T^{(r)} \geq 0\} = \sup[0, 1] = 1.$
- $I^T(1, 1) = \sup\{r \in [0, 1] \mid 1_T^{(r)} \geq 1\} = \sup[0, 1] = 1.$
- $I^T(1, 0) = \sup\{r \in [0, 1] \mid 0_T^{(r)} \geq 1\} = \sup\{0\} = 0. \quad \square$

¹A different approach based in the use of not functionally expressible fuzzy implications was pointed out in [11].

²As it is recalled in the preliminaries (Section 2.2), powers of t-norms have been studied in detail for the case of continuous t-norms and for this reason we limit ourselves to this case. It is possible to extend such study to more general t-norms like for instance left-continuous t-norms, but this is a question that we will not deal with in this paper and that we will leave for a future work.

Let us see now the general expression of the T -power based implications depending on the structure of the continuous t -norm T used in the process.

Proposition 4. *Let T be a continuous t -norm and I^T its power based implication.*

- *If $T = T_M$ is the minimum t -norm, then I^{T_M} agrees with the Rescher implication, that is:*

$$I^{T_M}(x, y) = I_{RS}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x > y. \end{cases}$$

- *If T is an Archimedean t -norm with additive generator t , then*

$$I^T(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{t(x)}{t(y)} & \text{if } x > y, \end{cases}$$

with the convention that $\frac{a}{+\infty} = 0$ for all $a \in [0, 1]$.

- *If T is an ordinal sum t -norm of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$, where T_i is an Archimedean t -norm with additive generator t_i for all $i \in I$, then*

$$I^T(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{t_i \left(\frac{x - a_i}{b_i - a_i} \right)}{t_i \left(\frac{y - a_i}{b_i - a_i} \right)} & \text{if } x, y \in [a_i, b_i] \text{ and } x > y, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. From Proposition 7 we already know that $I^T(x, y) = 1$ for all $x \leq y$ in all the cases. Thus, we only need to prove the result for values x, y such that $x > y$ and we will do it case by case.

- i) If $T = T_M$, taking $x > y$ we have that $y_T^{(r)} = y < x$ for all $r > 0$ and so,

$$I^T(x, y) = \sup\{r \in [0, 1] \mid y_T^{(r)} \geq x\} = \sup\{0\} = 0$$

for all $x > y$.

- ii) Suppose now that T is an Archimedean t -norm with additive generator t . In this case, using Proposition 2 we have

$$y_T^{(r)} \geq x \iff t^{-1}(\min\{t(0), rt(y)\}) \geq x \iff rt(y) \leq t(x).$$

Now the result follows trivially when $t(y) \neq +\infty$ and the formula holds also when $t(y) = +\infty$ (i.e., when $y = 0$ and T is strict) with the convention $\frac{a}{+\infty} = 0$.

- iii) Suppose now that T is an ordinal sum of Archimedean t -norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$. Take $x > y$ and let us distinguish two cases:

- If there is some idempotent element of T , say α , such that $x > \alpha > y$. In this case we have

$$y_T^{(r)} \leq \alpha_T^{(r)} = \alpha < x \quad \text{for all } r > 0$$

and we obtain again

$$I^T(x, y) = \sup\{r \in [0, 1] \mid y_T^{(r)} \geq x\} = \sup\{0\} = 0.$$

- If there is some $i \in I$ such that $x, y \in [a_i, b_i]$, then we have $\frac{x-a_i}{b_i-a_i} > \frac{y-a_i}{b_i-a_i}$ and applying a similar reasoning to the one of item (ii) to the Archimedean t-norm T_i we obtain

$$I^{T_i} \left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \right) = \sup \left\{ r \in [0, 1] \mid \left(\frac{y-a_i}{b_i-a_i} \right)_{T_i}^{(r)} \geq \frac{x-a_i}{b_i-a_i} \right\} = \frac{t_i \left(\frac{x-a_i}{b_i-a_i} \right)}{t_i \left(\frac{y-a_i}{b_i-a_i} \right)}.$$

However, if $x, y \in [a_i, b_i]$ we have that

$$T(x, y) = a_i + (b_i - a_i) T_i \left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i} \right)$$

and then, it is not difficult to check that

$$y_T^{(r)} = \left(\frac{y-a_i}{b_i-a_i} \right)_{T_i}^{(r)}.$$

Finally, the result follows from the fact that

$$y_T^{(r)} \geq x \iff \left(\frac{y-a_i}{b_i-a_i} \right)_{T_i}^{(r)} \geq \frac{x-a_i}{b_i-a_i}.$$

□

The structure of the T -power based implications in the cases when the continuous t-norm is Archimedean or an ordinal sum can be viewed in Figure 1.

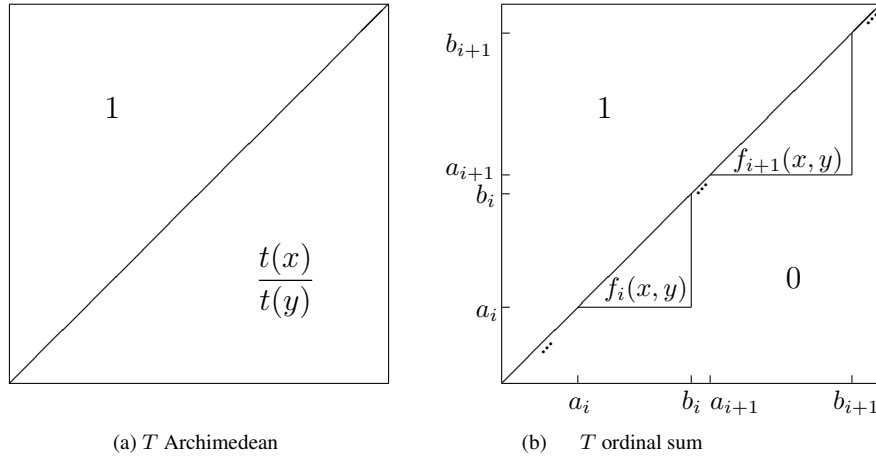


Figure 1: Structure of T -power based implications I^T when T is Archimedean and when T is an ordinal sum of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$, where $f_i(x, y)$ stands by $f_i(x, y) = \frac{t_i \left(\frac{x-a_i}{b_i-a_i} \right)}{t_i \left(\frac{y-a_i}{b_i-a_i} \right)}$ for all $i \in I$.

The following example gives the expression of the T -power based implications for some concrete t-norms.

Example 2. Let us consider the two classical Archimedean t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ as well as the ordinal sum t-norm given by $T = (\langle 0, 0.5, T_{\mathbf{P}} \rangle, \langle 0.5, 1, T_{\mathbf{L}} \rangle)$.

- If $T = T_{\mathbf{P}}$ is the Product t-norm then

$$I^{T_{\mathbf{P}}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{\log x}{\log y} & \text{if } x > y. \end{cases}$$

- If $T = T_{\mathbf{L}}$ is the Łukasiewicz t-norm then

$$I^{T_{\mathbf{L}}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1-x}{1-y} & \text{if } x > y. \end{cases}$$

- If $T = (\langle 0, \frac{1}{2}, T_{\mathbf{P}} \rangle, \langle \frac{1}{2}, 1, T_{\mathbf{L}} \rangle)$, then

$$I^{T_{\mathbf{P}}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{\log(2x)}{\log(2y)} & \text{if } x, y \in [0, \frac{1}{2}] \text{ and } x > y, \\ \frac{2-2x}{2-2y} & \text{if } x, y \in [\frac{1}{2}, 1] \text{ and } x > y, \\ 0 & \text{otherwise.} \end{cases}$$

To finish the study about the structure of these fuzzy implication functions, we present the following result that will be useful along the paper. It proves that there is a close relationship between the t-norm T and its T -power based implication. Recall before that an element $x \in [0, 1]$ is said to be T -idempotent when $T(x, x) = x$.

Proposition 5. *Let T be a continuous t-norm and I^T its power based implication. For all $x \in [0, 1]$ the following items hold:*

- i) x is T -idempotent if, and only if, $I^T(x, y) = 0$ for all $y < x$.
- ii) If $I^T(y, x) = 0$ for all $y > x$ then x is T -idempotent.
- iii) If $([a_i, b_i], T_i)$ is an Archimedean summand of T , then T_i is strict if, and only if, $I^T(x, a_i) = 0$ for all $x > a_i$.

PROOF. The equivalences stated in items (i) and (ii) are straightforward from the structure of I^T , see Proposition 4.

On the other hand, it is clear that $I^T(x, a_i) = 0$ for all $x \geq b_i$. However, for all x such that $a_i < x < b_i$ we have

$$I^T(x, a_i) = 0 \iff \frac{t_i\left(\frac{x-a_i}{b_i-a_i}\right)}{t_i(0)} = 0 \iff t_i(0) = +\infty.$$

that is, if, and only if, T_i is strict. □

4. Properties of T -power based implications

In this section, an in-depth analysis of T -power based implications will be carried out. In the first part of the section, we will prove that this family fulfills the invariance with respect to linguistic modifiers modeled through powers of t-norms. In a second part, we will study other additional properties which can be required to a fuzzy implication function.

4.1. Invariance with respect to linguistic modifiers

As we have already commented in the introduction, there is a special property not usually required on fuzzy implications, but closely related to approximate reasoning. This condition is the invariance with respect to linguistic modifiers modeled through powers of t-norms which allows to assign the same truth value to fuzzy conditionals where both the antecedent and the consequent are modified by the same power of t-norm (see the classical example of tomatoes recalled in the introduction). This property is satisfied by all T -power based implications I^T introduced in this paper, as it will be proved in this subsection.

First of all, let us formalize this property.

Definition 5. Let I be a fuzzy implication function. It is said that I is *invariant with respect to T -powers*, or simply that it is *power invariant* when

$$I^T(x, y) = I^T\left(x_T^{(r)}, y_T^{(r)}\right) \quad (PI_T)$$

holds for all $r > 0$ and for all $x, y \in [0, 1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$.

At this point, we can prove the result.

Proposition 6. *Let T be a continuous t -norm and I^T its power based implication. Then I^T is power invariant.*

PROOF. If $x \leq y$, then $x_T^{(r)} \leq y_T^{(r)}$ and in this case

$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(x, y) = 1.$$

Thus, we only need to prove the (PI_T) property for values x, y such that $x > y$. We will do it depending on the t -norm T .

- If $T = T_M$, x is T -idempotent for all $x \in [0, 1]$ and the result is trivial because $x_T^{(r)} = x$ for all $r > 0$.
- If T is Archimedean with additive generator t , take $x > y$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$. Then $x_T^{(r)} > y_T^{(r)}$ and

$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(t^{-1}(rt(x)), t^{-1}(rt(y))) = \frac{rt(x)}{rt(y)} = \frac{t(x)}{t(y)} = I^T(x, y).$$

- Consider T an ordinal sum of Archimedean t -norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$, where each T_i has additive generator t_i for all $i \in I$. Take again $x > y$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$ and let us distinguish two cases:

- If there is some T -idempotent element α such that $x > \alpha > y$ then $x_T^{(r)} > \alpha > y_T^{(r)}$ and

$$I^T(x_T^{(r)}, y_T^{(r)}) = I^T(x, y) = 0.$$

- If there is some $i \in I$ such that $a_i \leq y < x \leq b_i$ then it is also $a_i \leq y_T^{(r)} < x_T^{(r)} \leq b_i$ and

$$\begin{aligned} I^T(x_T^{(r)}, y_T^{(r)}) &= I^T\left(a_i + (b_i - a_i)t_i^{-1}\left(rt_i\left(\frac{x-a_i}{b_i-a_i}\right)\right), a_i + (b_i - a_i)t_i^{-1}\left(rt_i\left(\frac{y-a_i}{b_i-a_i}\right)\right)\right) \\ &= \frac{rt_i\left(\frac{x-a_i}{b_i-a_i}\right)}{rt_i\left(\frac{y-a_i}{b_i-a_i}\right)} = \frac{t_i\left(\frac{x-a_i}{b_i-a_i}\right)}{t_i\left(\frac{y-a_i}{b_i-a_i}\right)} = I^T(x, y). \end{aligned}$$

□

4.2. Other additional properties

Let us study some other properties. It will be determined which additional properties usually required to a fuzzy implication function are satisfied by power based implications. Whenever they do not satisfy a property in general, the conditions to ensure that they fulfill it will be given.

There are some properties of T -power based implications that can be easily derived from the given definition.

Proposition 7. *Let T be a continuous t -norm and I^T its power based implication. Then*

- i) I^T satisfies (OP) and so it also satisfies (IP) .
- ii) $I^T(1, y) = 0$ for all $y < 1$. Consequently I^T never satisfies (NP) .

PROOF. Let us prove the two statements.

- i) Using Definition 4 we obtain:

$$I^T(x, y) = 1 \iff \sup\{r \in [0, 1] \mid y_T^{(r)} \geq x\} = 1 \iff y_T^{(1)} = y \geq x$$

from which we deduce (OP) and consequently also (IP) .

1
2
3
4 ii) Consider $y < 1$. First of all, if $y = 0$, since I^T is a fuzzy implication function by Proposition 3 we have that
5 $I^T(1, 0) = 0$. Otherwise, taking into account that $T(x, y) = 1$ if, and only if, $x = y = 1$, we get

$$6 \quad I^T(1, y) = \sup\{r \in [0, 1] \mid y_T^{(r)} = 1\} = \sup\{0\} = 0$$

7
8 for all $0 < y < 1$, which proves the second item. □

9
10 Taking into account that all the most usual classes of fuzzy implication functions, namely R -implications, (S, N) -
11 implications, QL and D -implications, and Yager's implications, satisfy (NP) , the previous proposition ensures in
12 particular that T -power based implications never agree with them. In fact, there is only one fuzzy implication function
13 from Table 1.3 in [3], which contains the most well-known of these operators, that belongs to the family of T -power
14 based implications. Specifically, the Rescher implication I_{RS} introduced in [17] is also obtained as the power based
15 implication derived from the minimum t-norm as it has already been proved in Proposition 4.

16 Let us continue by studying the natural negation of I^T that again depends on the structure of the continuous t-norm
17 T .

18
19 **Proposition 8.** *Let T be a continuous t-norm and I^T its power based implication. The natural negation of I^T , N_{I^T} ,
20 is given by:*

21 i) *If T is the Minimum t-norm or T is a strict Archimedean t-norm then $N_{I^T} = N_{\mathbf{D1}}$ is the Gödel negation, that
22 is:*

$$23 \quad N_{I^T}(x) = N_{\mathbf{D1}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

24
25
26 ii) *If T is a non-strict Archimedean t-norm with additive generator t then*

$$27 \quad N_{I^T}(x) = \frac{t(x)}{t(0)} \quad \text{for all } x \in [0, 1].$$

28
29
30 iii) *If T is an ordinal sum of Archimedean t-norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$, where each T_i has additive
31 generator t_i for all $i \in I$,*

- 32 - *If $a_i \neq 0$ for all $i \in I$ or there is some $i_0 \in I$ such that $a_{i_0} = 0$ and T_{i_0} is strict then $N_{I^T} = N_{\mathbf{D1}}$.*
- 33 - *If there is some $i_0 \in I$ such that $a_{i_0} = 0$ and T_{i_0} is non-strict then*

$$34 \quad N_{I^T}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{t_{i_0}\left(\frac{x}{b_{i_0}}\right)}{t_{i_0}(0)} & \text{if } 0 < x \leq b_{i_0}, \\ 0 & \text{if } x > b_{i_0}. \end{cases}$$

35
36
37 PROOF. The result is straightforward taking $y = 0$ in Proposition 4. □

38
39 Moreover, from the above proposition we immediately deduce the following result which determines when the
40 natural negation of these fuzzy implication functions is strong.

41
42 **Corollary 9.** *Let T be a continuous t-norm and I^T its power based implication. Then its natural negation N_{I^T} is
43 strong if, and only if, T is a non-strict Archimedean t-norm such that its normalized³ additive generator t is involutive
44 (i.e., such that $t^2 = id$).*

45
46 PROOF. It is clear that when T is either the Minimum t-norm, a strict t-norm or an ordinal sum of Archimedean
47 t-norms, its natural negation can not be strong. Otherwise, if T is a non-strict Archimedean t-norm with normalized
48 additive generator t , we have that

$$49 \quad N_{I^T}(N_{I^T}(x)) = t(t(x))$$

50 and the result follows. □

51
52
53
54
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56
57 ³An additive generator t of a non-strict Archimedean t-norm is said to be normalized when $t(0) = 1$.

Example 3. To illustrate the previous result, let us consider the following two examples:

- i) Let us consider the t-norm T given by the ordinal sum $T = (\langle 0, 1/2, T_L \rangle, \langle 1/2, 1, T_P \rangle)$. A straightforward computation using Propositions 4 and 8 shows that the T -power based implication I^T is given in this case by:

$$I^T(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1-2x}{1-2y} & \text{if } x, y \in [0, 1/2] \text{ and } x > y, \\ \frac{\log(2x-1)}{\log(2y-1)} & \text{if } x, y \in [1/2, 1] \text{ and } x > y, \\ 0 & \text{otherwise,} \end{cases}$$

and its natural negation is given by:

$$N_{I^T}(x) = \begin{cases} 1 - 2x & \text{when } x \leq 1/2, \\ 0 & \text{when } x > 1/2. \end{cases}$$

In Figure 2 the structure of this T -power based implication I^T as well as its natural negation N_{I^T} .

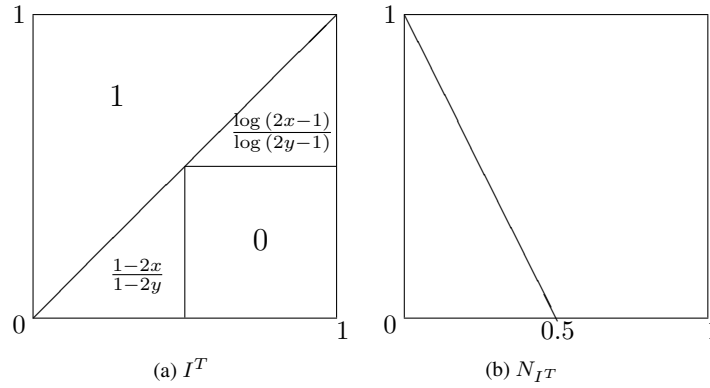


Figure 2: Structure of I^T and its natural negation N_{I^T} when $T = (\langle 0, 1/2, T_L \rangle, \langle 1/2, 1, T_P \rangle)$ is the t-norm given in Example 3-(i).

- ii) Consider now the Łukasiewicz t-norm T_L . The expression of its power based implication was already given in Example 2-(ii). Note that its natural negation is strong since $N_{I^{T_L}}(x) = N_C(x) = 1 - x$, the classical negation.

Next, we want to deal with another important property of fuzzy implication functions, the contraposition with respect to a fuzzy negation N . Again the result will depend on the structure of the t-norm T . However, to prove the general result we need first some preliminary results. Let us denote by Idemp_T the set of all T -idempotent elements.

Lemma 10. *Let T be a continuous t-norm, I^T its power based implication and N a fuzzy negation. If I^T satisfies $CP(N)$ then the following items hold:*

- i) N is strictly decreasing.
- ii) The natural negation of I^T is the Gödel negation.
- iii) If N is continuous then x is T -idempotent if, and only if, $N(x)$ is T -idempotent (that is, $N(\text{Idemp}_T) = \text{Idemp}_T$).
- iv) If N is continuous and T is an ordinal sum of Archimedean t-norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$ then all T_i must be strict.

PROOF. To prove (i), suppose that there exist $x > y$ such that $N(x) = N(y)$. Since I^T satisfies $CP(N)$, we have that $I^T(x, y) = I^T(N(y), N(x))$, but by (OP), $I^T(N(y), N(x)) = 1$ and consequently, $I^T(x, y) = 1$ from which we deduce $x \leq y$ leading to a contradiction.

To prove (ii) take $x > 0$. From the previous step we necessarily have $N(x) < 1$ and so, taking $y = 0$ in the contraposition property, we obtain

$$N_{I^T}(x) = I^T(x, 0) = I^T(1, N(x)) = 0,$$

where the last equality is due to Proposition 7–(ii).

To prove item (iii) note that from Proposition 5, $CP(N)$ and the fact that N is continuous and strictly decreasing by item (i) we have:

$$\begin{aligned} x \text{ is } T\text{-idempotent} &\implies I^T(x, y) = 0 \text{ for all } y < x \\ &\implies I^T(N(y), N(x)) = 0 \text{ for all } N(y) > N(x) \\ &\implies I^T(z, N(x)) = 0 \text{ for all } z > N(x) \\ &\implies N(x) \text{ is } T\text{-idempotent.} \end{aligned}$$

Reciprocally, if $N(x)$ is T -idempotent then $I^T(N(x), y) = 0$ for all $y < N(x)$. By $CP(N)$ we obtain $I^T(N^{-1}(y), x) = 0$ for all $N^{-1}(y) > x$ which implies that x is T -idempotent by Proposition 5–(ii).

Finally, to prove (iv) suppose that N is continuous and T is an ordinal sum of Archimedean t -norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$. First of all, by the previous point, $N(a_i)$ is T -idempotent for all $i \in I$. Now, for all $i \in I$ and for all $x > a_i$, we have by $CP(N)$ that

$$I^T(x, a_i) = I^T(N(x), N(a_i))$$

but since $N(x) < N(a_i)$ and $N(a_i)$ is T -idempotent, by Proposition 5–(ii), $I^T(N(x), N(a_i)) = 0$. Thus, $I^T(x, a_i) = 0$ for all $x > a_i$ and T_i must be strict by Proposition 5–(iii). \square

Now we can give the characterization of all T -power based implications that satisfy $CP(N)$ with respect to any fuzzy negation, except for ordinal sum t -norms in which case we need to assume continuity of the fuzzy negation N .

Proposition 11. *Let T be a continuous t -norm, I^T its power based implication and N a fuzzy negation. The following statements hold:*

i) *If T is the Minimum t -norm then I^T satisfies $CP(N)$ if, and only if, N is strictly decreasing.*

ii) *If T is strict with additive generator t then I^T satisfies $CP(N)$ if, and only if, N is a strong negation given by*

$$N(x) = t^{-1} \left(\frac{k}{t(x)} \right) \quad \text{for all } x \in [0, 1], \quad (1)$$

for some positive constant k .

iii) *If T is a non-strict Archimedean t -norm then I^T never satisfies $CP(N)$.*

iv) *If N is continuous and T is an ordinal sum of Archimedean t -norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$, where each T_i has additive generator t_i for all $i \in I$, then I^T satisfies $CP(N)$ if, and only if, T_i is strict for all $i \in I$ and N satisfies the following properties:*

a) *N is strict with $N(\text{Idemp}_T) = \text{Idemp}_T$.*

b) *For all $i \in I$ there is some $j \in I$ such that N is given by*

$$N(x) = a_j + (b_j - a_j)t_j^{-1} \left(\frac{k_i}{t_i \left(\frac{x - a_i}{b_i - a_i} \right)} \right), \quad \text{for some positive constant } k_i, \quad (2)$$

for all $x \in [a_i, b_i]$.

PROOF. Let us prove the result item by item.

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4 i) If I^{T_M} satisfies $CP(N)$ the fuzzy negation N is strictly decreasing by Lemma 10-(i). Conversely, since N
5 is strictly decreasing we have $x \leq y$ if, and only if, $N(y) \leq N(x)$ and the result follows directly from the
6 structure of I^{T_M} (see Proposition 4).

7
8 ii) Suppose now that T is strict with additive generator t . If I^T satisfies $CP(N)$, we already know that N is strictly
9 decreasing by Lemma 10-(i) and then taking x, y such that $1 > x > y > 0$ we have $N(y) > N(x)$. Thus,

$$\frac{t(N(y))}{t(N(x))} = I^T(N(y), N(x)) = I^T(x, y) = \frac{t(x)}{t(y)}.$$

10
11
12 Consequently, there is a positive constant k such that for all $x \in]0, 1[$ we have

$$t(x)t(N(x)) = k \implies N(x) = t^{-1}\left(\frac{k}{t(x)}\right).$$

13
14 Since $N(0) = 1$ and $N(1) = 0$ and assuming that $\frac{k}{+\infty} = 0$ we can write

$$N(x) = t^{-1}\left(\frac{k}{t(x)}\right) \quad \text{for all } x \in [0, 1].$$

15
16
17 Conversely, it is an easy computation to prove that I^T satisfies $CP(N)$ with respect to the negation N given by
18 Equation (1).

19
20
21 iii) If T is a non-strict Archimedean t-norm then N_{IT} does not agree with the Gödel negation (see Proposition 8)
22 and the result follows from Lemma 10-(ii).

23
24
25 iv) Suppose now that N is continuous and T is given by the ordinal sum $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$. If I^T satisfies
26 $CP(N)$ we already know that all T_i are strict and that condition (a) holds by Lemma 10-(i), (iii) and (iv).
27 This ensures that for each $i \in I$ there is some $j \in I$ such that $N([a_i, b_i]) = [a_j, b_j]$. Moreover, for all
28 $b_i > x > y > a_i$ we have $a_j < N(x) < N(y) < b_j$ and then

$$\frac{t_j\left(\frac{N(y)-N(b_i)}{N(a_i)-N(b_i)}\right)}{t_j\left(\frac{N(y)-N(b_i)}{N(a_i)-N(b_i)}\right)} = I^T(N(y), N(x)) = I^T(x, y) = \frac{t_i\left(\frac{x-a_i}{b_i-a_i}\right)}{t_i\left(\frac{y-a_i}{b_i-a_i}\right)}.$$

29
30
31 Since $N(a_i) = b_j$ and $N(b_i) = a_j$, similarly as in point (ii), there is a positive constant k_i such that for all
32 $x \in [0, 1]$ we have

$$t_i\left(\frac{x-a_i}{b_i-a_i}\right)t_j\left(\frac{N(x)-a_j}{b_j-a_j}\right) = k_i,$$

33
34
35 from which we deduce

$$N(x) = a_j + (b_j - a_j)t_j^{-1}\left(\frac{k_i}{t\left(\frac{x-a_i}{b_i-a_i}\right)}\right) \quad \text{for all } x \in [a_i, b_i].$$

36
37
38 Conversely, if conditions (a), (b) hold let us prove that I^T satisfies $CP(N)$ by distinguishing some cases.

39
40
41 – If $x \leq y$ then $N(y) \leq N(x)$ and

$$I^T(N(y), N(x)) = I^T(x, y) = 1.$$

42
43
44 – If $x > y$ and there is some T -idempotent element α with $x > \alpha > y$ then $N(y) > N(\alpha) > N(x)$ and

$$I^T(N(y), N(x)) = I^T(x, y) = 0.$$

- 1
2
3
4 – If $x > y$ and there is some $i \in I$ such that $b_i \geq x > y \geq a_i$. In this case there is some $j \in I$ such that
5 $a_j \leq N(y) < N(x) \leq b_j$ and then $I^T(x, y)$ and $I^T(N(y), N(x))$ are respectively given by,

$$6 \quad I^T(x, y) = \frac{t_i \left(\frac{x-a_i}{b_i-a_i} \right)}{t_i \left(\frac{y-a_i}{b_i-a_i} \right)}, \quad \text{and} \quad I^T(N(y), N(x)) = \frac{t_j \left(\frac{N(y)-N(b_i)}{N(a_i)-N(b_i)} \right)}{t_j \left(\frac{N(y)-N(b_i)}{N(a_i)-N(b_i)} \right)}.$$

7
8
9
10 Now, condition (b) ensures that the equality $I^T(N(y), N(x)) = I^T(x, y)$ also holds in this case. \square

11
12 **Remark 1.** Some remarks emerge regarding the fuzzy negation given by Equation (2).

- 13
14 i) For all $i \in I$, it holds that $N(a_i) = b_j$ and $N(b_i) = a_j$.
15 ii) When T has an odd finite number of idempotent elements, the middle one must necessarily be the fixed point
16 of the strict negation N .
17 iii) Since N is a fuzzy negation, if there exist i_1 and i_2 such that $a_{i_1} \leq b_{i_1} \leq a_{i_2} \leq b_{i_2}$, it must be

$$18 \quad a_{j_2} = N(b_{i_2}) \leq b_{j_2} = N(a_{i_2}) \leq a_{j_1} = N(b_{i_1}) \leq b_{j_1} = N(a_{i_1}).$$

19
20
21 **Example 4.** Let us consider two different t-norms and their corresponding power based implications in order to
22 compute the fuzzy negations with which they satisfy $CP(N)$.

- 23
24 1. First, we consider the Product t-norm $T_{\mathbf{P}}$. Its corresponding power-based implication was already computed
25 in Example 2. Using Proposition 11, this fuzzy implication function satisfies $CP(N)$ only with the following
26 family of strong fuzzy negations

$$27 \quad N_k(x) = e^{\frac{k}{\log x}}$$

28 for some positive constant k .

- 29
30 2. Now consider the t-norm T given by the ordinal sum $T = (\langle 0, 1/2, T_{\mathbf{P}} \rangle, \langle 1/2, 1, T_{\mathbf{P}} \rangle)$ and its power based
31 implication whose expression is as follows

$$32 \quad I^T(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{\log(2x)}{\log(2y)} & \text{if } x, y \in [0, 1/2] \text{ and } x > y, \\ \frac{\log(2x-1)}{\log(2y-1)} & \text{if } x, y \in [1/2, 1] \text{ and } x > y, \\ 0 & \text{otherwise,} \end{cases}$$

33
34
35
36
37
38 Using Proposition 11 and assuming continuous fuzzy negations N , this fuzzy implication function satisfies
39 $CP(N)$ with the following family of strict fuzzy negations:

$$40 \quad N_{k_1, k_2}(x) = \begin{cases} \frac{1}{2} + \frac{1}{2} e^{\frac{k_1}{\log(2x)}} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} e^{\frac{k_2}{\log(2x-1)}} & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

41 for some positive constants k_1 and k_2 .

42 In Figure 3 the structure of this T -power based implication I^T as well as one of the fuzzy negations with which
43 it satisfies $CP(N)$ are depicted.

44
45
46
47
48 With respect to other usual properties like the exchange principle and the law of importation, power based impli-
49 cations never satisfy them. We collect these negative results in the following proposition.

50
51 **Proposition 12.** Let T be a continuous t-norm and I^T its power based implication. The following statements hold:

- 52
53 i) I^T does not satisfy (EP).
54
55 ii) I^T does not satisfy $(LI_{T'})$ with respect to any t-norm T' .
56
57

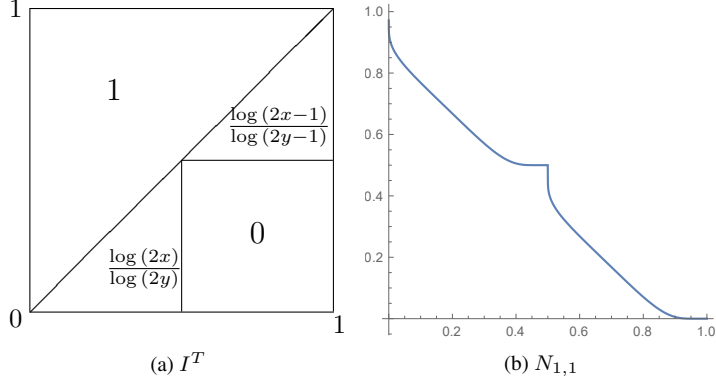


Figure 3: Structure of I^T and the fuzzy negation $N_{1,1}$ with which I^T satisfies $CP(N)$ when $T = \langle (0, 1/2, T_{\mathbf{P}}), \langle 1/2, 1, T_{\mathbf{P}} \rangle \rangle$ is the t-norm given in Example 4-(ii).

PROOF. We only need to prove item (i) because item (ii) immediately follows from it (see Remark 7.3.1 in [3]).

Suppose that I^T satisfies (EP). Since it also satisfies (OP) by Proposition 7-(i), it should satisfy also (NP) by Lemma 1.3.4 in [3]. However this is a contradiction with Proposition 7-(ii). Thus, I^T does not satisfy (EP). \square

Finally, another important property is T -transitivity. Recall that we have already proved that T -power based implications are in fact reflexive relations (i.e., $I^T(x, x) = 1$ for all $x \in [0, 1]$). In case they also satisfy T -transitivity, that is,

$$T(I^T(x, y), I^T(y, z)) \leq I^T(x, z) \quad \text{for all } x, y, z \in [0, 1], \quad (3)$$

we will obtain T -preorders on $[0, 1]$. T -preorders were introduced by Zadeh in [22] and are very important fuzzy relations, since they fuzzify the concept of preorder on a set. Thus, let us deal now with the T -transitivity property.

Proposition 13. *Let T be a continuous t-norm and I^T its power based implication. Then I^T satisfies T -transitivity if, and only if, one of the following cases hold:*

- $T = T_{\mathbf{M}}$.
- T is Archimedean with $T \leq T_{\mathbf{P}}$.
- T is an ordinal sum of Archimedean t-norms of the form $T = \langle \langle a_i, b_i, T_i \rangle \rangle_{i \in I}$, where $T_i \leq T_{\mathbf{P}}$ for all $i \in I$.

PROOF. Let us prove the result depending on the t-norm T .

- If $T = T_{\mathbf{M}}$, the Minimum t-norm, $I^{T_{\mathbf{M}}}$ is the Rescher implication and the result is just a matter of simple computation.
- Let us consider an Archimedean t-norm T with additive generator t . First, if $x \leq y$ or $y \leq z$, the result is obvious using (OP) and the monotonicities of a fuzzy implication function. Otherwise, we have that I^T satisfies T -transitivity if, and only if,

$$t^{(-1)} \left(t \left(\frac{t(x)}{t(y)} \right) + t \left(\frac{t(y)}{t(z)} \right) \right) \leq \frac{t(x)}{t(z)}.$$

Taking now $a = \frac{t(x)}{t(y)}$ and $b = \frac{t(y)}{t(z)}$, the previous inequality is equivalent to

$$t(ab) \leq t(a) + t(b), \quad \text{for all } a, b \in [0, 1],$$

which in turn is equivalent to $T \leq T_{\mathbf{P}}$.

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4 • Let us consider now an ordinal sum t-norm T of Archimedean t-norms of the form $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$,
5 where each T_i has additive generator t_i . First, if $x \leq y$ or $y \leq z$, the result is obvious using (OP) and the
6 monotonicities of a fuzzy implication function. Otherwise, consider $x > y > z$ and let us distinguish two cases:

- 7
8 – If there is some idempotent element of T , say α , such that $x > \alpha > y$ or $y > \alpha > z$, the result holds
9 since in this case $T(I^T(x, y), I^T(y, z)) = 0$.
10
11 – If there is some $i \in I$ such that $x, y, z \in [a_i, b_i]$, then we have that I^T satisfies T -transitivity if, and only
12 if,

$$13 \quad t_i^{(-1)} \left(t_i \left(\frac{t_i(x)}{t_i(y)} \right) + t_i \left(\frac{t_i(y)}{t_i(z)} \right) \right) \leq \frac{t_i(x)}{t_i(z)}.$$

14 A similar argument to the previous item concludes that the previous inequality is equivalent to

$$15 \quad t_i(ab) \leq t_i(a) + t_i(b), \quad \text{for all } a, b \in [0, 1], i \in I,$$

16 and thus, to $T_i \leq T_{\mathbf{P}}$ for all $i \in I$. □

17
18
19
20 **Example 5.** The Product $T_{\mathbf{P}}$ t-norm and the Łukasiewicz t-norms are examples of Archimedean t-norms which
21 generates T -power based implications satisfying the T -transitivity. In addition, the ordinal sum t-norms $T_1 =$
22 $(\langle 0, 1/2, T_{\mathbf{L}} \rangle, \langle 1/2, 1, T_{\mathbf{P}} \rangle)$ and $T_2 = (\langle 0, 1/2, T_{\mathbf{P}} \rangle, \langle 1/2, 1, T_{\mathbf{P}} \rangle)$ which have been used previously in this paper
23 are also valid choices to generate T -power based implications satisfying the T -transitivity.
24

25 5. Conclusions and future work

26
27 In this paper, we have introduced a new class of fuzzy implication functions based on the use of powers of a
28 continuous t-norm T . This family satisfies a useful property in approximate reasoning such as the invariance with
29 respect to powers of t-norms which have been usually used to model linguistic modifiers. The fulfillment of this
30 property ensures that the fuzzy conditionals of examples such as the classical one of tomatoes [16] have the same
31 truth value. Moreover, other additional properties have been studied. In particular, power based implications always
32 satisfy (OP) and (IP) and the conditions under which they satisfy the contrapositive symmetry and the T -transitivity
33 have been determined. The fact that these fuzzy implication functions do not satisfy the left neutrality principle proves
34 that they constitute a new family of these operators which does not intersect the most well-known families.
35

36 This paper constitutes only a first step in the study of this new family of fuzzy implication functions. Several open
37 problems are still open and they will be milestones in our future work. We can highlight:

- 38
39 • To axiomatically characterize power based implications. We are convinced that the invariance property with
40 respect to powers of t-norms will play a key role in the result.
41
42 • To study other important additional properties such as the Modus Ponens and the Modus Tollens. These prop-
43 erties could be studied not only with respect to the same t-norm T , but they could be generalized to any other
44 t-norm T_1 (or more general conjunctive such as a conjunctive uninorm), see [8, 9].
45
46 • This paper has dealt only with continuous t-norms. An interesting future research line would be to extend this
47 study to left-continuous t-norms, whose powers can be easily adapted.
48

49 Acknowledgement

50
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