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On Perfect and Quasiperfect Dominations in Graphs

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Abstract.
A subset $S \subseteq V$ in a graph $G = (V, E)$ is a k-quasiperfect dominating set (for $k \geq 1$) if every vertex not in S is adjacent to at least one and at most k vertices in S. The cardinality of a minimum k-quasiperfect dominating set in G is denoted by $\gamma_{1k}(G)$. Those sets were first introduced by Chellali et al. (2013) as a generalization of the perfect domination concept and allow us to construct a decreasing chain of quasiperfect dominating numbers $n \geq \gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G)$ in order to indicate how far is G from being perfectly dominated. In this paper we study properties, existence and realization of graphs for wich the chain is short, that is, $\gamma_{12}(G) = \gamma(G)$. Among them, one can find cographs, claw-free graphs and graphs with extremal values of $\Delta(G)$.

1. Introduction

All the graphs considered here are finite, undirected, simple, and connected. Given a graph $G = (V, E)$, the *open neighborhood* of $v \in V$ is $N(v) = \{u \in V; uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v is the number of neighbors of v, i.e., $\deg(v) = |N(v)|$. The *maximum degree* of G, denoted by $\Delta(G)$, is the largest degree among all vertices of G. Similarly, it is defined the *minimum degree* $\delta(G)$. For undefined basic concepts we refer the reader to introductory graph theoretical literature as [6].

Given a graph G, a subset S of its vertices is a *dominating set* of G if every vertex v not in S is adjacent to at least one vertex in S, or in other words $N(v) \cap S \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of cardinality $\gamma(G)$ is called a *γ-code* [14].

The most efficient way for a set S to dominate occurs when every vertex not in S is adjacent to exactly one vertex in S. In that case, S is called a *perfect dominating set*, which were introduced in [8] and studied in [2, 4, 5, 11–13, 15, 16] under different names. We denote by $\gamma_{11}(G)$ the minimum cardinality of a perfect dominating set of G and called it the *perfect domination number*. A perfect dominating set of cardinality $\gamma_{11}(G)$ is called a *γ_{11}-code*.

Not always is possible to achieve perfection, so it is natural to wonder if we can obtain something close to it. In [7], the authors defined a generalization of perfect dominating sets called a *k-quasiperfect dominating set* for $k \geq 1$ (*γ_{1k}-set* for short). Such a set S is a dominating set where every vertex not in S is adjacent

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to at most k vertices of S (see also [10, 18]). The *k-quasiperfect domination number* $\gamma_{1k}(G)$ is the minimum cardinality of a γ_{1k}-set of G and a γ_{1k}-*code* is a γ_{1k}-set of cardinality $\gamma_{1k}(G)$. Certainly, γ_{11}-sets and $\gamma_{1\Delta}$-sets are respectively perfect dominating and dominating sets. Thus, given a graph G of order n and maximum degree Δ, one can construct the following decreasing chain of quasiperfect domination parameters:

$$n \geq \gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G)$$

For any graph G, the values in this chain give us an idea about how far is G from being perfectly dominated. Particularly, in this work we focus our attention when the chain is *short*, or in other words $\gamma_{12}(G) = \gamma(G)$. The next result, obtained in [7], provides a variety of families for which the chain is short.

Theorem 1. *If G is a graph of order n verifying at least one of the following conditions:*

1. $\Delta(G) \geq n - 3$;
2. $\Delta(G) \leq 2$;
3. *G is a cograph;*
4. *G is a claw-free graph;*

then $\gamma_{12}(G) = \gamma(G)$.

The paper is organized as follows: in the next section we introduce several well-known and technical results that will be useful for the rest of the paper. The next two sections deal with the study of the cases of Theorem 1, thus Section 3 is devoted to the extremal degree families and Section 4 to cographs and claw-free graphs.

2. Basic and general results

In this section, we review some results founded in the literature about quasiperfect parameters as well as introduce some basic technical results that will be useful in the rest of the paper. The next table summarizes the values of parameters under consideration for some simple families of graphs:

	paths	cycles	cliques	stars	bicliques	wheels
G	P_n	C_n	K_n	$K_{1,n-1}$	$K_{p,n-p}$	W_n
n	$n \geq 3$	$n \geq 4$	$n \geq 2$	$n \geq 4$	$2 \leq p \leq n-p$	$n \geq 3$
$\Delta(G)$	2	2	$n-1$	$n-1$	$n-p$	$n-1$
$\gamma_{11}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$	1	1	2	1
$\gamma_{12}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1
$\gamma(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1

Table 1: Quasiperfect domination parameters of some basic graphs.

The next result provides a number of basic technical facts:

Proposition 1. *Let $G = (V, E)$ a graph of order n. In the following, $\Delta(G) = \Delta$, $\gamma(G) = \gamma$, $\delta(G) = \delta$ and let k and r be two positive integers such that $1 \leq k \leq \Delta$ and $k \leq r \leq n$:*

1. *If $\gamma \leq \Delta$, then $\gamma_{1\gamma}(G) = \ldots = \gamma_{1\Delta}(G) = \gamma$;*
2. $\gamma_{1\delta}(G) < n$;
3. $\gamma_{11}(G) = 1$ *if and only if* $\Delta = n - 1$.
4. *Let S be a γ_{1k}-set of G and $v \in V$. If $|N(v) \cap S| > k$ then $v \in S$.*
5. *Let S be a γ_{1k}-set of G and let K be a clique of G. If $|V(K) \cap S| > k$ then $V(K) \subseteq S$.*

Proof.

1. Assume S is a γ-code. Then, for any vertex v not in S, it is clear that $1 \le |N(v) \cap S| \le \gamma \le i \le \Delta$. Hence, S is an γ_{1i}-set, and consequently $\gamma_{1i}(G) \le \gamma$.
2. Now let $v \in V$ with $\deg(v) = \delta$. Since $1 \le |N(v) \cap S| \le |N(v)| = \delta$, the set $S = V \setminus \{v\}$ is a $\gamma_{1\delta}$-set and consequently, $\gamma_{1\delta}(G) \le n - 1 < n$.
3. Assume that $S = \{v\}$ is a γ_{11}-code of G. Then v is a universal vertex, i.e., $\deg(v) = n - 1$. Conversely, let $v \in V$ with $\deg(v) = n - 1$. Since $N(v) = V \setminus \{v\}$, the set $\{v\}$ is a γ_{11}-code.
4. If S is a γ_{1k}-set, then no vertex outside S can have more than k neighbors in S. So if a vertex v verifies $N|(v) \cap S| > k$ then it belongs to S.
5. Similarly as the previous case, if there exists a clique K in G with more than k vertices in S, then any vertex in $V(K) \setminus S$, if exists, has more than k neighbors in S so they should be in S or S cannot be a γ_{1k}-set. Consequently $V(K) \subseteq S$.

□

From the computational point of view, it is important that k-quasiperfect domination numbers can be expressed in terms of an integer program. The formulation is as follows: given a vertex subset $S \subseteq V(G) = \{v_1, \dots, v_n\}$ the characteristic column vector $X_S = (x_i)$ for $1 \le i \le n$ satisfies $x_i = 1$ if $v_i \in S$ and $x_i = 0$ otherwise. Now S is a dominating set if $|N(v_i) \cap S| \ge 1$ for $v_i \notin S$ or, equivalently, if $N \cdot X_S \ge 1_n - X$ (see [14]). The set S is a k-quasiperfect dominating set if it is dominating and $|N(v_i) \cap S| \le k$ for $v_i \notin S$. Since clearly, $|N(v_i) \cap S| \le n - 1$ for $v_i \in S$, this two conditions can be expressed as $N \cdot X \le k1_n + (n - k - 1)X$. Thus the final formulation for the k-quasiperfect domination number $\gamma_{1k}(G)$:

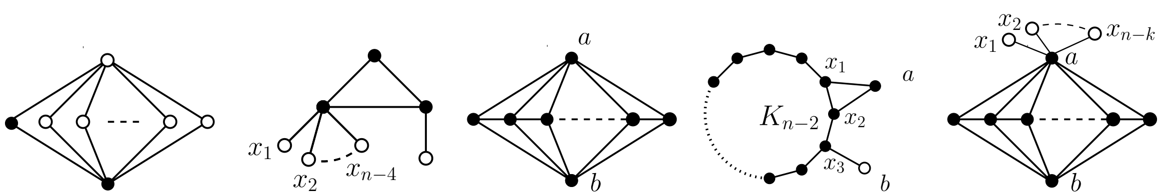
$$\begin{aligned} \gamma_{1k}(G) &= \min \sum_{i=1}^{n} x_i \\ \text{subject to } & N \cdot X \ge 1_n - X \\ & N \cdot X \le k1_n + (n - k - 1)X \\ & \text{with } x_i \in \{0, 1\} \end{aligned}$$

3. Extremal degree families

Extremal values of the maximum degree $\Delta(G)$ leads to a short quasiperfect domination chain as it was stated in Theorem 1. In this section, we examine the relationship between extremal values of the maximum degree and the quasiperfect domination parameters. Note that if $\Delta(G) \le 2$, then G is claw-free which will be considered in Subsection 4.2. On the other hand, $\Delta(G) = n - 1$ only for complete graphs. Hence, this section is divided into the following remaining extremal cases: $\Delta(G) = n - 2$, $\Delta(G) = n - 3$ and $\Delta(G) = 3$.

3.1. $\Delta(G) = n - 2$

As it was pointed out in Theorem 1, if $\Delta(G) = n - 2$ then $\gamma_{12}(G) = \gamma(G)$ and in this case $\gamma(G) = 2$. So the only question that remains open is whether or not there exists a graph under these conditions with any value of $\gamma_{11}(G)$. The following result answers this question:



(a) $\gamma_{11}(G) = 2$ (b) $\gamma_{11}(G) = 3$ (c) $\gamma_{11}(G) = n$ (d) $\gamma_{11}(G) = n - 1$ (e) $n - \gamma_{11}(G) \ge 2$

Figure 1: Graphs of order n, maximum degree $\Delta = n - 2$ and any possible value k of γ_{11}.

Theorem 2. *Let k, n be positive integers such that $n \geq 4$ and $2 \leq k \leq n$. Then, there exists a graph G of order n such that $\Delta(G) = n-2$ and $\gamma_{11}(G) = k$ if and only if $(k, n) \notin \{(3,4),(4,4),(4,5),(5,5)\}$.*

Proof. The only graphs of order 4 and $\Delta(G) = 2$ are the cycle C_4 and the path P_4, and in both cases $\gamma_{11}(G) = 2$. On the other hand, there are eight graphs of order 5 and maximum degree $\Delta(G) = 3$, all of them having either $\gamma_{11}(G) = 2$ or $\gamma_{11}(G) = 3$.

Thus, the only remained case is when $n \geq 6$.

When $k = 2$, the graph $G = K_{2,n-2}$ (see Figure 1(a)) has $\Delta(G) = n-2$ and $\gamma_{11}(G) = 2$ since the black vertices form a γ_{11}-code.

For the case $k = 3$, we consider the graph in Figure 1(b) where the three black vertices are a γ_{11}-code.

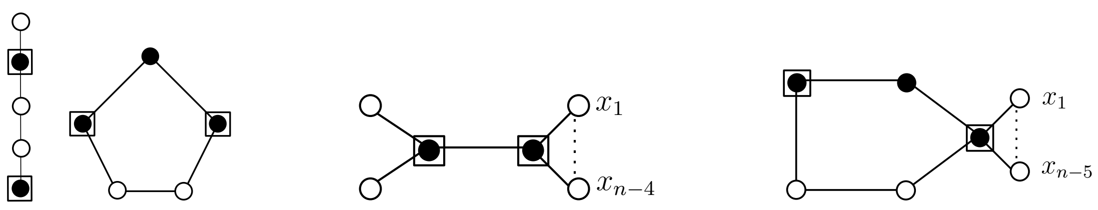
For the case $k = n$, we construct the graph $P_{k-2} \vee \bar{K}_2$ showed in Figure 1(c). Let $V(P_{n-2}) = \{1, \ldots, n-2\}$ and $V(\bar{K}_2) = \{a, b\}$. Obviously, $\Delta(G) = n-2$. Let us see that $\gamma_{11}(G) = n$, that is, the unique γ_{11}-set is $V(G)$. Let S be a γ_{11}-set of G, then $|S| \geq 2$. We distinguish tree cases:

- If $\{a, b\} \subset S$ then $|N(i) \cap S| > 1$ for all $i \in V(P_{n-2})$. By Proposition 1, $V(P_{n-2}) \subset S$, that is, $S = V(G)$.
- If $\{i, j\} \subset S$, for some $\{i, j\} \subset V(P_{n-2})$, then $|N(a) \cap S| > 1$ and $|N(b) \cap S| > 1$. Then $\{a, b\} \subset S$. By previous item, then $S = V(G)$.
- If $\{a, i\} \subset S$ for some $i \in P_{n-2}$, then $|N(i+1) \cap S| > 1$ or $|N(i-1) \cap S| > 1$. In any case $|V(P_{n-2}) \cap S| > 1$, and then, by previous item, $S = V(G)$.

The following case occurs when $k = n-1$. We construct the graph in Figure 1(d). Note that the set of black vertices is a γ_{11}-set with cardinality $n-1$. We claim that any γ_{11}-code should contain those vertices. Let S be a γ_{11}-code. Since S is dominating, $S \cap \{x_1, x_2, a\} \neq \emptyset$ and $S \cap \{x_3, b\} \neq \emptyset$. Observe that $\{a, b\}$ is not a dominating set, and if S contains two vertices of $V(K_{n-2}$ then it contains all the vertices of the clique plus the vertex a. Thus, it only remains two check the cases $\{x_1, b\} \subseteq S$ (the case $\{x_2, b\} \subseteq S$ is analogous) and $\{x_3, a\} \subseteq S$. However in the former case, we have that $x_2 \in S$ and in the later, $x_1 \in S$. Hence in any case, there are two vertices of the clique in S and consequently all the black vertices belong to S.

Finally, suppose $n-k \geq 2$ and $k \geq 4$. The graph we have to consider is the one depicted in Figure 1(e). We denote $W = V(P_{k-2} \vee \bar{K}_2)$ and $V(\bar{K}_2) = \{a, b\}$. Obviously $\Delta(G) = n-2$, $\{a, b\}$ is a γ-code and also a γ_{1k}-code for $k \geq 2$. On the other hand, W is a γ_{11}-set of G and $|W| = k$. It only remains to prove that any γ_{11}-set of G contains W.

Let S be a γ_{11}-set of G, then its cardinality is at least 2. As S dominates all the vertices, $|W \cap S| \geq 1$. If $|W \cap S| \geq 2$ then, analogously as an above case, we obtain $W \subset S$. We suppose that $|W \cap S| = 1$. In this case, as $V \setminus W$ do not dominates b, then it is necessary that $W \cap S \subset N[b]$ and then $a \notin S$ and $\{x_1, \ldots, x_{n-k}\} \subset S$ because S is a dominating set. But in this case, $|N(a) \cap S| > 2$, that is a contradiction. Finally, we conclude that $\gamma_{11}(G) = |W| = k$. □



(a) P_5 and C_5 (b) $|V(G)| = n \geq 6, \gamma = \gamma_{11} = 2$ (c) $|V(G)| = n \geq 6, \gamma = 2, \gamma_{11} = 3$

Figure 2: Some graphs with order $n \geq 5$ and maximum degree $n-3$

3.2. $\Delta(G) = n - 3$

For this case, as in the previous subsection, it holds true $\gamma_{12}(G) = \gamma(G)$. However, either $\gamma(G) = 2$ or 3, and for both cases we characterize whether or not there exists a graph for any value of $\gamma_{11}(G)$.

Theorem 3. *Let (k, n) be a pair of integers such that $2 \leq k \leq n$ and $n \geq 5$. Then, there exists a graph G such that $|V(G)| = n$, $\Delta(G) = n - 3$, $\gamma(G) = 2$ and $\gamma_{11}(G) = k$, if and only if $(k, n) \notin \{(4, 5), (5, 5), (4, 6), (5, 6), (6, 6)\}$.*

Proof. The unique graphs with 5 vertices that satisfy $\Delta = 5 - 3 = 2$ are the path P_5 and the cycle C_5, (see Figure 2(a)). It is clear that $\gamma(P_5) = \gamma_{11}(P_5) = 2$ (squared black vertices) and $\gamma(C_5) = 2$ (squared vertices), $\gamma_{11}(C_5) = 3$ (black vertices).

Graphs in Figure 2(b) satisfy $n \geq 6$, $\Delta = n - 3$, $\gamma = \gamma_{11} = 2$, and squared black vertices are both a γ-code and a γ_{11}-code. On the other hand, graphs in Figure 2(c) satisfy $n \geq 6$, $\Delta = n - 3$, $\gamma = 2$ and $\gamma_{11} = 3$, where the pair of squared vertices is a γ-code and the set of black vertices is a γ_{11}-code. Note that there is no graph G satisfying $n = 6$, $\Delta = 6 - 3 = 3$ and $\gamma_{11} \geq 4$ (see Theorem 5).

Finally, in Table 2, we show examples of graphs with $n \geq 7$, $\Delta(G) = n - 3$, $\gamma = 2$ and $\gamma_{11} \geq 4$. □

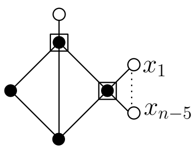
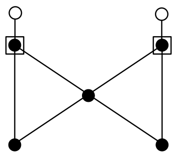
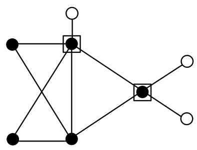
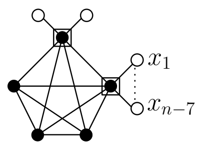
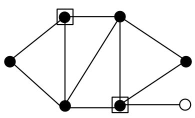
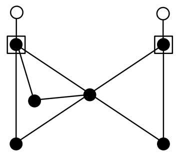
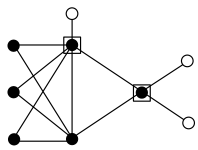
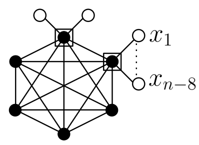
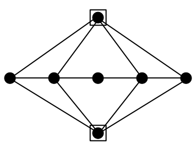
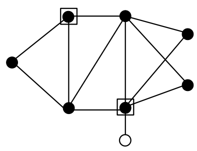
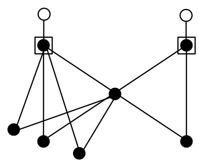
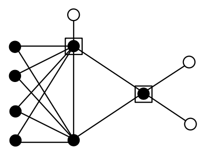
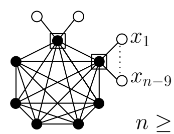
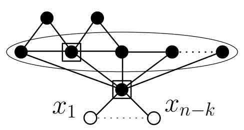
γ_{11} \ n	7	8	9	10	≥11
4		$n \geq 7$			
5				$n \geq 9$	
6				 $n \geq 10$	
7					 $n \geq 11$
≥8	no sense	 $8 \leq \gamma_{11} = k \leq n$			

Table 2: Examples of graphs with $n \geq 7$, $\Delta(G) = n - 3$, $\gamma = 2$ and $\gamma_{11} \geq 4$. The pair of squared vertices in each graph is a γ-code and black vertices form a γ_{11}-code.

Theorem 4. *Let (k, n) be a pair of integers such that $3 \leq k \leq n$ and $n \geq 6$. Then, there exists a graph G such that $|V(G)| = n$, $\Delta(G) = n-3$, $\gamma(G) = 3$ and $\gamma_{11}(G) = k$, if and only if $(k, n) \notin \{(4, 6), (5, 6), (6, 6), (5, 7), (6, 7), (7, 7), (8, 8)\}$.*

Proof. Graphs in Figure 3(a) satisfies $n \geq 6$, $\Delta = n-3$, $\gamma = \gamma_{11} = 3$, and squared black vertices are both a γ-code and a γ_{11}-code. In Figure 3(b), we have an example of graphs with $n \geq 7$, $\Delta = n-3$, $\gamma = \gamma_{11} = 4$. Note that there is no graph G satisfying $n = 6$, $\Delta = 6-3 = 3$ and $\gamma_{11} \geq 4$ (see Subsection 3.3). There exist 16 non-isomorphic graphs with 7 vertices, maximum degree 4, domination number 3 and at most 2 vertices of degree 1, and 46 non-isomorphic graphs with 8 vertices, maximum degree 5, domination number 3 and with no vertices of degree 1. By inspection, we have checked that there is no case belonging to $(5,7), (6,7), (7,7), (8,8)$.

Finally, in Table 3, we show examples of graphs with $n \geq 7$, $\Delta = n-3$, $\gamma = 3$ and $\gamma_{11} = k \geq 4$. □

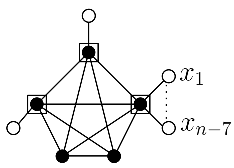
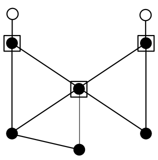
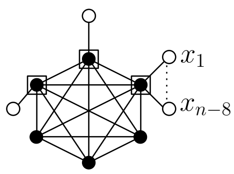
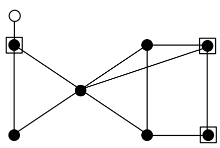
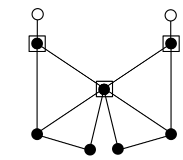
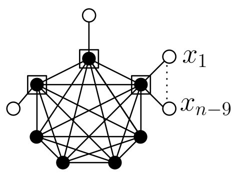
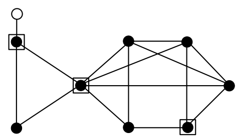
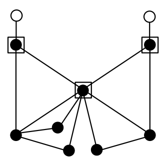
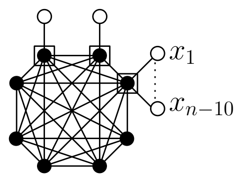
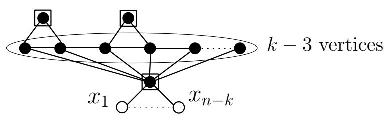
γ_{11} \ n	8	9	10	≥11
5	 $n \geq 8$			
6		 $n \geq 9$		
7			 $n \geq 10$	
8	does not exist			 $n \geq 11$
≥9	no sense	 $9 \leq \gamma_{11} = k \leq n$		

Table 3: Examples of graphs with $n \geq 7$, $\Delta(G) = n-3$, $\gamma = 3$ and $\gamma_{11} \geq 4$. The triplet of squared vertices in each graph is a γ-code and black vertices form a γ_{11}-code.

3.3. $\Delta(G) = 3$

Note that if $\Delta(G) \leq 2$, then the graph G is claw-free which will be studied in Subsection 4.2. Hence, here we focus our attention on $\Delta(G) = 3$.

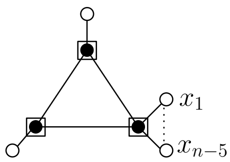
In [7] it is shown that $\Delta(G) \leq 4$ implies $\gamma_{12}(G) \leq n-1$. We use similar techniques in the case $\Delta(G) = 3$ to prove that $\gamma_{11}(G) \leq n-3$.

Lemma 1. *Let G be a graph of order n and $\Delta(G) = 3$ such that at least one of the following conditions holds:*

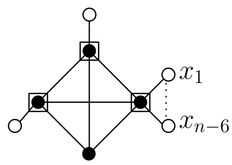
1. *G contains an induced cycle C such that all of its vertices have degree 3.*
2. *There exist two vertices $u, v \in V(G)$ with $\deg(u) = \deg(v) = 2$, $d(u, v) \geq 2$ and there is an induced path P joining them such that all of its vertices, other than u and v, have degree 3.*

Then $\gamma_{11}(G) \leq n - 3$.

Proof. If G satisfies condition 1, the set $V(G) \setminus V(C)$ is a γ_{11}-set, and if G satisfies condition 2, the set $V(G) \setminus V(P)$ is a γ_{11}-set, so in both cases we obtain $\gamma_{11}(G) \leq n - 3$. □



(a) $|V(G)| = n \geq 6$, $\gamma = \gamma_{11} = 3$



(b) $|V(G)| = n \geq 7$, $\gamma = \gamma_{11} = 4$

Figure 3: Small cases for Theorem 3.2

Theorem 5. *Let G be a graph of order n and $\Delta(G) = 3$, other than the bull graph. Then $\gamma_{11}(G) \leq n - 3$. Moreover the bull graph B satisfies $\gamma_{11}(B) = 3 = n - 2$.*

Proof. To begin with, note that vertices belonging to the triangle of the bull graph B (see Figure 3.3) are a γ_{11}-code, and hence $\gamma_{11}(B) = 3$.

Suppose next that G is a tree. It is clear that $\Delta(G) = 3$ implies that there are at least three leaves $\{l, m, k\}$ in G, and then $V(G) \setminus \{l, m, k\}$ is a γ_{11}-set. So we may assume that G (strictly) contains at least one induced cycle C. We consider different situations regarding this cycle.

Case 1. *The cycle C contains two vertices a and b with $\deg(a) = \deg(b) = 2$, $d(a, b) \geq 2$.*

It is clear that C contains at least a vertex w of degree 3, so walking from w along the cycle towards both directions, we find two vertices u and v in C, such that $\deg(u) = \deg(v) = 2$, $d(u, v) \geq 2$ and all vertices, other than u and v, in the induced path on the cycle between them, have degree 3. So G satisfies condition 2 of Lemma 1.

Case 2. *The cycle C contains exactly two vertices a and b with $\deg(a) = \deg(b) = 2$ and they satisfy $d(a, b) = 1$.*

There are three possible situations:

1. There exists a vertex $w \in V(G)$ with $\deg(w) = 1$. Then it is clear that w is neither a neighbor of a nor of b, and so $V(G) \setminus \{a, b, w\}$ is a γ_{11}-set.
2. There exists a vertex $w \in V(G) \setminus \{a, b\}$ with $\deg(w) = 2$. In this case $w \notin V(C)$ and $d(a, w) \geq 2, d(b, w) \geq 2$. Having in mind that all vertices in C, different from a and b have degree 3, going along an induced path from a (or b) to w, it is possible to find vertices $u, v \in V(G)$ such that $\deg(u) = \deg(v) = 2$, $d(u, v) \geq 2$ and all vertices, other than u and v, in the induced path between them, have degree 3. So G satisfies condition 2 of Lemma 1.
3. Any vertex $w \in V(G) \setminus \{a, b\}$ satisfies $\deg(w) = 3$. Then there must be another cycle D in G, different from C, and it is clear that $a, b \notin V(D)$. So G satisfies condition 1 of Lemma 1.

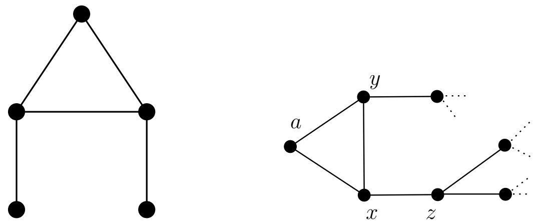
Case 3. *The cycle C contains exactly one vertex a with $\deg(a) = 2$.*

Firstly, if there exists $w \in V(G) \setminus \{a\}$ with $\deg(w) = 2$, going along an induced path from a to w, we can find vertices $u, v \in V(G)$ that ensure G satisfies condition 2 of Lemma 1. So suppose now that any vertex $w \in V(G) \setminus \{a\}$ has degree either 1 or 3. If there is another cycle D in G, different from C, then G satisfies condition 1 of Lemma 1. Therefore assume that G is an unicyclic graph, with cycle C.

1. If C has at least 4 vertices, from the fact that it has just one vertex of degree 2 it follows that there are three or more vertices of degree 3. Thus G has at least three leaves and therefore $\gamma_{11}(G) \leq n - 3$.
2. If $C = C_3$, the cycle with 3 vertices, using that G is not the bull graph (see Figure 4(b)), it is clear that vertex z, the neighbor of x not in C_3, is not a leaf so it has degree 3 (remember that we have assumed that no vertex other than a has degree 2). Then there are at least three leaves in G and $\gamma_{11}(G) \leq n - 3$.

Case 4. *All vertices in C have degree 3.*

Here G satisfies condition 1 of Lemma 1. □



(a) Bull graph. (b) Not a bull.

Figure 4:

Note that the upper bound in Theorem 5 is tight, for instance $\gamma_{11}(K_{1,3}) = 1 = n - 3$. However this bound can be improved in some special cases.

Proposition 2. *Let G be a cubic graph other than the complete graph with four vertices K_4. Then $\gamma_{11}(G) \leq n - 4$.*

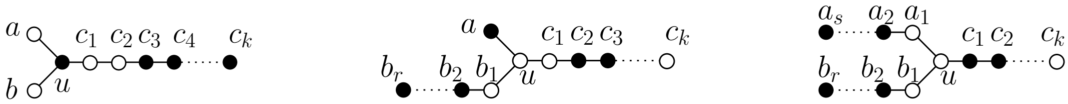
Proof. Let G be a cubic graphs other than K_4. If G has an induced cycle with at least 4 vertices, then using condition 1 of Lemma 1 we obtain $\gamma_{11}(G) \leq n - 4$. Suppose on the contrary that G contain no induced cycle of length greater of equal than 4 so G is a chordal graph. It is well known that chordal graphs have a perfect elimination ordering, so we order the vertex set $V(G) = \{v_1, \ldots, v_n\}$ in that way. Then for any vertex, its neighbors occurring after it in the order form a clique. Applying this property to v_1 we obtain that its three neighbors form a triangle, so $G = K_4$. □

Proposition 3. *Let T be a tree with order $n \geq 7$ and $\Delta(G) = 3$. Then $\gamma_{11}(T) \leq n - 4$.*

Proof. If T has at least two vertices of degree 3, then it has at least four leaves and we are done. So suppose that there exists an unique vertex $u \in V(T)$ with $deg(u) = 3$. We denote by A, B, C the three sets of vertices of the connected components of $T \setminus \{u\}$, with $|A| \leq |B| \leq |C|$. If $A = \{a\}$ and $B = \{b\}$ then $C = \{c_1, c_2, c_3, \ldots, c_k\}$ with $k \geq 4$ and we define $S = \{u, c_3, \ldots, c_k\}$ (see Figure 5(a)). If $A = \{a\}$ and $B = \{b_1, \ldots b_r\}$ with $r \geq 2$ then $C = \{c_1, \ldots c_k\}$ with $k \geq 3$ and we define $S = \{a, b_2, \ldots, b_r, c_2, \ldots, c_{k-1}\}$ (see Figure 5(b)). Finally if $A = \{a_1, \ldots a_s\}$ with $s \geq 2$ then $B = \{b_1, \ldots b_r\}$ with $r \geq 2$ and $C = \{c_1, \ldots c_k\}$ with $k \geq 2$ and we define $S = \{a_2, \ldots, a_s, b_2, \ldots, b_r, c_1, \ldots, c_{k-1}\}$ (see Figure 5(c)). In all cases S is a γ_{11}-set of T with $|S| = n - 4$. □

4. Cographs and claw-free graphs

Theorem 1 provides different conditions for a graph to have a short quasiperfect domination chain. Certain extremal values of the maximum degree guarantees that, as we have just studied in Section 3. However in this section, we are interested in those cases in which the graph belongs to special classes, namely cographs and claw-free graphs.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 5: Black vertices are in S and white vertices are not.

4.1. Cographs

Cographs are inductively defined as follows [3, 9]:

- Every single vertex graph is a cograph.
- If G_1 and G_2 are two cographs, then their disjoint union is a cograph.
- The join graph $G_1 \vee G_2$ of two cographs is a cograph. The join graph $G_1 \vee G_2$ is obtained from their disjoint union by adding all edges between vertices of G_1 and G_2.

Note that if G is a connected cograph then it is the join of two cographs $G = G_1 \vee G_2$. The next result gives us the values of $\gamma_{11}(G)$ where G is the join of two graphs.

Theorem 6. *Let $G = G_1 \vee G_2$ be a graph of order n. Then,*

1. $\gamma_{11}(G) = 1$ *if and only if* $\gamma(G) = 1$.
2. $\gamma_{11}(G) = 2$ *if and only if both G_1 and G_2 have at least an isolated vertex.*
3. $\gamma_{11}(G) = n$ *in other case.*

Proof. The first claim is obvious. From now on, we assume that there is no universal vertex in G.

For the second claim, let $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ be isolated vertices in G_1 and G_2 respectively, then it is clear that $\{u_1, u_2\}$ is a γ_{11}-code of G. Conversely, suppose that G_1 (without lost of generality) has no isolated vertex. Let S be a γ_{11}-code of G, then S must contain at least one vertex $v_1 \in V(G_1)$ and one vertex $v_2 \in V(G_2)$. Let x be a neighbor of v_1 in G_1, then it has at least two neighbors in S, so $x \in S$ and $\gamma_{11}(G) \geq 3$.

Finally if G has no universal vertex and G_1 has no isolated vertices, we know that $\gamma_{11}(G) \geq 3$. Let S be a γ_{11}-code of G, then there are at least two vertices of S in $V(G_i)$, for some $i \in \{1, 2\}$ which implies $V(G_j) \subset S$ for $j \neq i$. Note that $|V(G_j)| \geq 2$, because there is no universal vertex in G, so there are at least two vertices of S in $V(G_j)$ and also $V(G_i) \subset S$, as desired. □

Note that a connected cograph is the join of two cographs. Thus, the above theorem applies also to those graphs.

Corollary 1. *Let $G = G_1 \vee G_2$ be a connected cograph without universal vertices. Then, $\gamma_{11}(G) = 2$ if both G_1 and G_2 have at least an isolated vertex, and $\gamma_{11}(G) = n$ in any other case.*

4.2. Claw-free graphs

Claw-free graphs, also known as $K_{1,3}$-free graphs, is another graph family where $\gamma = \gamma_{12}$ according to Theorem 1. The next result provides examples of claw-free graphs for a great variety of different values for γ, γ_{11} and n.

Theorem 7. *Let h, k, n be integers such that $4 \leq n$, $2 \leq h \leq k \leq n$ satisfying $h + k \leq n$ or $3h + k + 1 \leq 2n$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$.*

Proof. First suppose that $h+k \leq n$. Let $r = n-(h+k)+2$. We consider the graph G formed by two complete graphs of order r and k, sharing exactly one vertex v, and $h-1$ vertices of degree 1 pending from distinct vertices $u_1,\dots,u_{h-1}$ of the complete graph K_k different from v (see Figure 6(a)). Note that G is a claw-free graph and since $r \geq 2$, the sets $\{u_1,\dots,u_{h-1},v\}$ and $V(K_k)$ are respectively a γ-code with h vertices and a γ_{11}-code with k vertices.

Suppose now that $3h+k+1 \leq 2n$ and $h+k > n$. Let $r = n-h$ and $s = n-k$. Then,

$$2n \geq 3h+k+1 \Rightarrow (n-h)+(n-k) \geq 2h+1 \Rightarrow r+s > 2h \Rightarrow r > 2h-s = s+2(h-s)$$

where $s = n-k \geq 0$ and $h-s = h-(n-k) = (k+h)-n \geq 1$. Therefore we construct the following graph G: consider the graph K_r, and $s+2(h-s)$ different vertices of K_r, $u_1,\dots,u_s,v_1,\dots,v_{h-s},w_1,\dots,w_{h-s}$. Attach a vertex of degree 1 to each vertex $u_1,\dots,u_s$ and consider $h-s$ vertices $x_1,\dots,x_{h-s}$ of degree 2, where x_i is adjacent to v_i and w_i (see Figure 6(b)). Notice that G is a claw-free graph and since $r > s+2(h-s)$, the sets $\{u_1,\dots,u_s,v_1,\dots,v_{h-s}\}$ and $V(K_r)\cup\{x_1,\dots,x_{h-s}\}$ are respectively a γ-code of G with h vertices and a γ_{11}-code of G with $n-s=k$ vertices. □

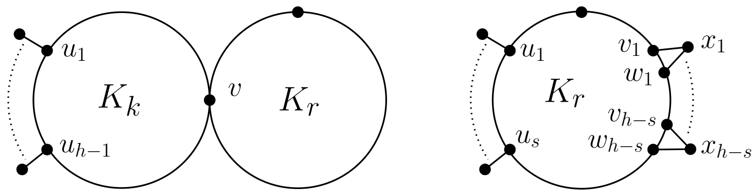


Figure 6: Claw-free graphs on the proof of Theorem 7.

Conditions $h+k \leq n$ or $3h+k+1 \leq 2n$ in Theorem 7 are sufficient to ensure that there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$. There are some cases where the reverse is also true. For instance, if G is a claw-free graph with $\gamma(G) = \frac{n}{2}$, n even, then G is the cycle C_4 or G is the corona graph of a complete graph K_m (see [1]) and $k = \gamma_{11}(G) = \frac{n}{2}$, so $h+k \leq n$. Also in the following proposition we show that they are necessary conditions, with just two exceptions, in the case of graphs with small order. So we think that the reverse of Theorem 7 could be true in a wider range of cases.

Proposition 4. *Let h,k,n be integers such that $4 \leq n \leq 7$ and $2 \leq h \leq k \leq n$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$ if and only if $h+k \leq n$ or $3h+k+1 \leq 2n$ or $(h,k,n) = (2,6,6)$.*

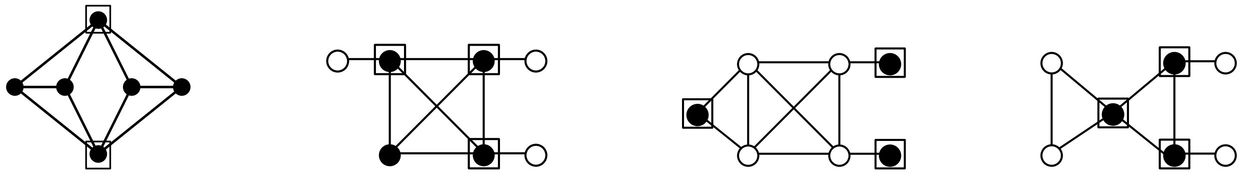
Proof. Firstly, if h,k,n satisfy hypothesis then, using Theorem 7 we obtain the desired graphs, except in case $(h,k,n) = (2,6,6)$, that is shown in Figure 7(a).

Conversely suppose that G is a claw-free graph with orden n and such that $\gamma(G) = h$ (so $h \leq \frac{n}{2}$) and $\gamma_{11}(G) = k$ with $4 \leq n \leq 7$. If $\Delta(G) = n-1$ then $h = k = 1$, which not our case. If $\Delta(G) = 2$, then G must be the n-cycle or the n-path, with $4 \leq n \leq 7$, and it is easy to check that $h+k \leq n$ in all cases. This completely solve the case $n = 4$. In the remaining cases we classify graphs using the maximum degree $3 \leq \Delta(G) \leq n-2$.

If $n = 5$ then $h = 2$. The case we have to check is $\Delta(G) = 3$, so G is the bull graph that satisfy $h = 2$, $k = 3$ or G is not the bull graph and using Theorem 5, $h = 2$, $k = n-3 = 2$. In both cases $h+k \leq n$.

If $n = 6$ then $2 \leq h \leq 3$. If $\Delta(G) = 3$, Theorem 5 gives $k \leq n-3 = 3$ that implies $h+k \leq n$. If $\Delta(G) = 4$ then $h = 2$ and we distinguish to options: $k = 6$ implies $(h,k,n) = (2,6,6)$ and $k \leq 5$ means $3h+k+1 \leq 3\cdot 2+5+1 = 12 = 2n$.

If $n = 7$ again $2 \leq h \leq 3$. In the case $\Delta(G) = 3$, using Theorem 5, we obtain $k \leq n-3 = 4$ so $h+k \leq n$. If $\Delta(G) = 4$ and $h = 2$ then $3h+k+1 \leq 3\cdot 2+7+1 = 2n$. On the other hand it is easy to check ([17]) that there are exactly three claw-free graphs of order 7 with $\Delta(G) = 4$ and $h = 3$ (see Figures 7(b), 7(c), 7(d)) and they satisfy $3 \leq k \leq 4$, so $h+k \leq n$. Finally $\Delta(G) = 5$ implies $h = 2$ and $3h+k+1 \leq 3\cdot 2+7+1 = 2n$. □



(a) $(h, k, n) = (2, 6, 6)$

(b) $n = 7$, $\Delta = 4$, $h = 3$, $k = 4$

(c) $n = 7$, $\Delta = 4$, $h = 3$, $k = 3$

(d) $n = 7, \Delta = 4, h = 3$, $k = 3$

Figure 7: Squared vertices are a γ-code and black vertices are a γ_{11}-code.

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