

NONLOCAL PHASE TRANSITIONS IN HOMOGENEOUS AND PERIODIC MEDIA

MATTEO COZZI⁽¹⁾, SERENA DIPIERRO^(2,3), AND ENRICO VALDINOCI^(2,3,4,5)

(1) – Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Diagonal 647, E-08028 Barcelona (Spain).

(2) – School of Mathematics and Statistics
University of Melbourne
Grattan Street, Parkville, VIC-3010 Melbourne (Australia).

(3) – School of Mathematics and Statistics
University of Western Australia
35 Stirling Highway, Crawley, WA-6009 Perth (Australia).

(4) – Weierstraß Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39, D-10117 Berlin (Germany).

(5) – Dipartimento di Matematica “Federigo Enriques”
Università degli studi di Milano
Via Saldini 50, I-20133 Milano (Italy).

E-mail addresses: matteo.cozzi@upc.edu, sdipierro@unimelb.edu.au, enrico@math.utexas.edu

In honor of Professor Paul Rabinowitz, with great esteem and admiration.

ABSTRACT. We discuss some results related to a phase transition model in which the potential energy induced by a double-well function is balanced by a fractional elastic energy. In particular, we present asymptotic results (such as Γ -convergence, energy bounds and density estimates for level sets), flatness and rigidity results, and the construction of planelike minimizers in periodic media.

Finally, we consider a nonlocal equation with a multiwell potential, motivated by models arising in crystal dislocations, and we construct orbits exhibiting symbolic dynamics, inspired by some classical results by Paul Rabinowitz.

1. INTRODUCTION

Goal of this paper is to collect into a homogeneous and original presentation a series of results about the fractional Allen-Cahn (or scalar Ginzburg-Landau) equation.

The classical model for this equation arises in the study of phase coexistence and it has several applications in material sciences. In its basic version, the model aims to describe the phase separation occurring in some media. The two phases can be described by a state parameter function $u : \mathbb{R}^n \rightarrow [-1, 1]$. In this setting, $n \in \mathbb{N}$ is the dimension of the ambient space and the values -1 and 1 for u represent the “pure phases” of the system.

The total energy of the system is supposed to be made of two terms: a “potential” term $\mathcal{W}(u)$, which forces minimizers to stay “as close as possible to the pure phases”, and an “elastic” (or, with a slight abuse of a terminology borrowed from similar setting in Hamiltonian dynamics, “kinetic”) term $\mathcal{K}(u)$, which “prevents the formation of unnecessary interfaces”.

2010 *Mathematics Subject Classification.* 35R11, 82B26.

Key words and phrases. Nonlocal Ginzburg-Landau-Allen-Cahn equation, De Giorgi conjecture, planelike minimizers, chaotic orbits.

More precisely, given a (bounded, smooth) set $\Omega \subset \mathbb{R}^n$ (the “container”), and a smooth function $W : \mathbb{R} \rightarrow [0, +\infty]$ (the “potential”) which has nondegenerate minima at ± 1 (i.e., say, $W(\pm 1) = 0 < W(r)$ for any $r \in \mathbb{R} \setminus (-1, 1)$, with $W''(\pm 1) > 0$), we set

$$\mathcal{W}(u) := \int_{\Omega} W(u(x)) dx.$$

In the classical case, the formation of interfaces is penalized by a local elastic term of the form

$$(1.1) \quad \mathcal{K}(u) := \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 dx,$$

which leads to the total energy functional

$$(1.2) \quad \mathcal{E}(u) := \mathcal{K}(u) + \mathcal{W}(u) = \int_{\Omega} \frac{|\nabla u(x)|^2}{2} + W(u(x)) dx,$$

whose critical points are solutions of the partial differential equation

$$(1.3) \quad -\Delta u(x) + W'(u(x)) = 0, \quad \text{for any } x \in \Omega.$$

Recently, some attention has been devoted to nonlocal phase transition models, in order to capture the long term interactions between particles and to describe the boundary effects, see e.g. [AB98, Gon09, SV12b]. Here, we consider a phase transition model driven by a nonlocal energy of fractional type, which can be described as follows. Particles are supposed to interact according to a kernel, which we take invariant under translations and rotations, scale invariant and with polynomial decay. More concretely, we set

$$(1.4) \quad K(y) := \frac{1}{|y|^{n+2s}},$$

with $s \in (0, 1)$ and consider, as elastic energy, the quantity

$$(1.5) \quad \mathcal{K}(u) := \frac{1}{2} \int_{Q_{\Omega}} |u(x) - u(y)|^2 K(x - y) dx dy,$$

where

$$Q_{\Omega} := \mathbb{R}^{2n} \setminus (\Omega^c)^2 = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega).$$

Here above, $\Omega^c := \mathbb{R}^n \setminus \Omega$ denotes the complementary set.

By comparing (1.1) with (1.5), we see that we have replaced the classical seminorm in the Sobolev space H^1 with a seminorm in the fractional Sobolev space H^s , with $s \in (0, 1)$. As for the domain of integration, the idea, both in (1.1) and in (1.5), is that the values of the state parameter u are fixed outside the container Ω , so they should not really contribute to an effective energy and the energy should only take into account contributions which “see the container Ω ”. In this sense, the integral in (1.1) takes into account the whole of the space \mathbb{R}^n , with the exception of the contributions that lie outside the container Ω (that is, the integral in (1.1) ranges in $\mathbb{R}^n \setminus (\Omega^c) = \Omega$). In the same spirit, the energy in (1.5), which is a double integral, takes into account all the interactions in the whole of the space $\mathbb{R}^n \times \mathbb{R}^n$, with the exception of the ones which only involve points outside the container Ω (that is, the integral in (1.5) ranges in $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$, which indeed coincides with Q_{Ω}).

With these observations, when the elastic energy in (1.1) is replaced by the nonlocal one in (1.5), the total energy in (1.2) is replaced by its fractional analogue

$$(1.6) \quad \mathcal{E}(u) := \mathcal{K}(u) + \mathcal{W}(u) = \frac{1}{2} \int_{Q_{\Omega}} |u(x) - u(y)|^2 K(x - y) dx dy + \int_{\Omega} W(u(x)) dx,$$

whose critical points are solutions of the partial differential equation

$$(1.7) \quad (-\Delta)^s u(x) + W'(u(x)) = 0, \quad \text{for any } x \in \Omega,$$

which can be seen as a nonlocal counterpart of (1.3). Here, as customary in the literature involving fractional operators, we are using the notation $(-\Delta)^s$ to denote the fractional Laplacian, i.e. the integrodifferential operator given (up to multiplicative dimensional constants that we neglect here) by

$$\int_{\mathbb{R}^n} (2u(x) - u(x+z) - u(x-z)) K(z) dz.$$

We refer to [Lan72, Ste70, Sil05, DNPV12] for an introduction to the fractional Laplacian.

Here, we aim to discuss the theory of nonlocal phase transitions, as described by the energy functional in (1.6) and by the pseudodifferential equation in (1.7), basically discussing the similarities and the important differences with respect to the classical theory, especially in the light of Γ -convergence, density estimates, rigidity and flatness results and periodic and quasiperiodic structures arising in periodic media.

2. Γ -CONVERGENCE AND DENSITY ESTIMATES

To study the asymptotics of the solutions of (1.7), it is convenient to look at the spacial scaling $x \rightarrow \frac{x}{\varepsilon}$. Correspondingly, one can appropriately rescale the energy functional as

$$\mathcal{E}_\varepsilon(u) := \begin{cases} \mathcal{K}(u) + \varepsilon^{-2s}\mathcal{W}(u) & \text{if } s \in (0, 1/2), \\ |\log \varepsilon|^{-1}\mathcal{K}(u) + |\varepsilon \log \varepsilon|^{-1}\mathcal{W}(u) & \text{if } s = 1/2, \\ \varepsilon^{2s-1}\mathcal{K}(u) + \varepsilon^{-1}\mathcal{W}(u) & \text{if } s \in (1/2, 1). \end{cases}$$

When we want to emphasize the dependence of the energy functional (or of the energy contributions) on the container Ω , we will write, respectively, $\mathcal{W}(u; \Omega)$, $\mathcal{K}(u; \Omega)$, $\mathcal{E}(u; \Omega)$ and $\mathcal{E}_\varepsilon(u; \Omega)$.

In this notation, if $u_\varepsilon(x) := u(x/\varepsilon)$ and $\Omega_\varepsilon := \varepsilon\Omega$, we have that

$$\begin{aligned} \mathcal{W}(u_\varepsilon; \Omega_\varepsilon) &= \varepsilon^n \mathcal{W}(u; \Omega) \\ \text{and } \mathcal{K}(u_\varepsilon; \Omega_\varepsilon) &= \varepsilon^{n-2s} \mathcal{K}(u; \Omega) \end{aligned}$$

and so

$$\mathcal{E}_\varepsilon(u_\varepsilon; \Omega_\varepsilon) = \varepsilon^{n-\min\{2s, 1\}} \mathcal{E}(u; \Omega)$$

if $s \in (0, 1) \setminus \{1/2\}$, and

$$\mathcal{E}_\varepsilon(u_\varepsilon; \Omega_\varepsilon) = \frac{\varepsilon^{n-1}}{|\log \varepsilon|} \mathcal{E}(u; \Omega)$$

if $s = 1/2$ (this additional logarithmic factor is indeed very typical for fractional problems related to the square root of the Laplacian).

The scale of this functional is chosen in such a way that the following Γ -convergence result holds (see [SV12]):

Theorem 2.1. *As $\varepsilon \searrow 0$, the functional \mathcal{E}_ε Γ -converges to*

$$\mathcal{E}_0(u) := \begin{cases} \mathcal{K}(u; \Omega) & \text{if } s \in (0, 1/2) \text{ and } u = \chi_E - \chi_{E^c} \text{ a.e. in } \Omega \text{ for some } E \subseteq \Omega, \\ c_\star \text{Per}(E, \Omega) & \text{if } s \in [1/2, 1) \text{ and } u = \chi_E - \chi_{E^c} \text{ a.e. in } \Omega \text{ for some } E \subseteq \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

It is worth to point out that Theorem 2.1 may be rephrased by saying that when $s \in [1/2, 1)$ the fractional phase transition energy Γ -converges to the classical perimeter functional – as it happens indeed in the classical case $s = 1$. The classical counterpart when $s = 1$ of Theorem 2.1 was indeed one of the founding results of Γ -convergence, see [MM77].

On the other hand, when $s \in (0, 1/2)$, the fractional phase transition energy converges to the fractional perimeter functional that was introduced in [CRS10]. This suggests that the fractional parameter $s = 1/2$ provides a threshold for the asymptotic behavior of nonlocal phase transitions: above such threshold, the behavior of the interfaces at a large scale somehow “localizes”, since such behavior is dictated by the classical, and thus local, perimeter minimization; but below such threshold the behavior of the interfaces at a large scale fully maintains its nonlocal properties, since it is driven by a nonlocal perimeter functional.

Of course, Γ -convergence is a very elegant and effective method to deal with the asymptotics of functionals and it fits well with the calculus of variations and with the problems of minimization. On the other hand, it gives little information on the geometric properties of the solutions of the equation. In the classical case, to overcome this difficulty, a theory of “density estimates” has been developed in [CC95]. Namely, the goal of this theory is to establish energy bounds and bounds in measure theoretic sense for the level sets of minimizers, with the goal of showing that minimizers behave “like one-dimensional solutions” at least in terms of energy and in terms of measure occupied by their interface.

In the fractional framework, we have that the energy of minimizers of \mathcal{E}_ε is locally uniformly bounded, according to the following result:

Theorem 2.2. *Let $\vartheta_1 \in (0, 1)$. If u_ε minimizes \mathcal{E}_ε in $B_{1+\varepsilon}$ and $|u_\varepsilon(0)| < \vartheta_1$, then*

$$(2.1) \quad c \leq \mathcal{E}_\varepsilon(u_\varepsilon, B_1) \leq C,$$

with $C > c > 0$ only depending on n , s and W .

The upper energy bound in (2.1) was proved in [SV14] and the lower bound in [CV16].

A counterpart of the energy bounds in Theorem 2.2 is a density estimate, which says, roughly speaking, that the measure of the interface of minimizers of \mathcal{E}_ε is locally of size comparable to ε . The precise result goes as follows:

Theorem 2.3. *Let $\vartheta_1, \vartheta_2 \in (0, 1)$. If u_ε minimizes \mathcal{E}_ε in B_1 and $|u_\varepsilon(0)| < \vartheta_1$, then*

$$(2.2) \quad c\varepsilon \leq |\{|u_\varepsilon| < \vartheta_2\} \cap B_1| \leq C\varepsilon,$$

with $C > c > 0$ only depending on n , s and W .

The upper density estimate in (2.2) was proved in [SV14] and the lower bound in [CV16]. We remark that the connection between the energy bound in (2.1) and the measure theoretic bound in (2.2) is particularly close in the local case (i.e. for $s = 1$) and also in the weakly nonlocal case (i.e. when $s \in (1/2, 1)$), since, roughly speaking, the potential energy is capable of measuring the size of the interface in a sufficiently sharp way. On the other hand, in the strongly nonlocal case (i.e. when $s \in (0, 1/2)$), the potential energy does not suffice for this scope and one has to carefully take into account the interaction in the elastic energy, possibly at any scale, to detect the dominant contributions. In particular, the case $s = 1/2$ for these estimates turns out to be the most delicate one, since the precise bounds involve a logarithmic correction.

We stress that the optimal bounds obtained in (2.1) and (2.2) not only provide the uniform convergence of the level sets of the minimizers to their limit interface (see [CC95]), but also provide the cornerstone for the construction of planelike solutions in periodic media for which the oscillation from the reference plane is of the same order of the size of periodicity of the media (this feature will be more thoroughly detailed in Section 4). For this, we also remark that both Theorems 2.2 and 2.3 hold for more general kernels K and potentials W (see [CV16]).

3. RIGIDITY AND FLATNESS RESULTS

A byproduct of the optimal bounds in (2.1) and (2.2) is that minimizers behave as if they were one-dimensional functions (that is, for functions of only one Euclidean variable, with sufficient decay at infinity, one can check “by hands” that formulas (2.1) and (2.2) hold true).

A natural question in this setting is whether global one dimensional solutions of (1.7) indeed exist. That is, up to normalization, if there exists a function $u_0 : \mathbb{R} \rightarrow (-1, 1)$ that satisfies

$$(3.1) \quad \begin{aligned} (-\Delta)^s u_0(x) + W'(u_0(x)) &= 0, & \text{for any } x \in \mathbb{R}, \\ \lim_{x \rightarrow \pm\infty} u_0(x) &= \pm 1. \end{aligned}$$

We stress that when $s = 1$, the existence of such “transition layer” u_0 is obvious, since the problem boils down to an ordinary differential equation, which can be integrated explicitly (by multiplying the equation by $u_0'(x)$ and taking the antiderivative – or equivalently by using the Law of Conservation of Energy).

On the other hand, differently from the classical case, when $s \in (0, 1)$ the existence of such solution u_0 is a rather delicate business, and it has been established, using variational and energy methods, in [PSV13], [CS15] and, in further generality, in [CP16].

Of course, given the estimates in (2.1) and (2.2), one may wonder under which additional conditions (if any) we can say that solutions of (1.7) are indeed one-dimensional, i.e., up to normalizations, are of the form $u(x) = u_0(x_n)$, or, say, the level sets of u are hyperplanes. In the classical case $s = 1$ this was in fact the content of a beautiful conjecture by Ennio De Giorgi in [DG79], which can be stated as follows:

Conjecture 3.1. *Let $u \in C^2(\mathbb{R}^n, [1, 1])$ satisfy*

$$(3.2) \quad -\Delta u = u - u^3$$

and

$$\partial_{x_n} u > 0$$

in the whole of \mathbb{R}^n . Is it true that all the level sets of u are hyperplanes, at least if $n \leq 8$?

We refer to [FV09] for a detailed account of the available results related to Conjecture 3.1; here, we would like to discuss the fractional analogue of Conjecture 3.1 when equation (3.2) is replaced by its nonlocal counterpart

$$(-\Delta)^s u = u - u^3$$

with $s \in (0, 1)$. In this case, a positive answer to this problem was given in [CSM05] when $n = 2$ and $s = 1/2$, in [SV09, CS15] when $n = 2$ and $s \in (0, 1)$, in [CC10] when $n = 3$ and $s = 1/2$ and in [CC14] when $n = 3$ and $s \in (1/2, 1)$ (see also [SV13] and [BV16] for different proofs, also related to nonlocal minimal surfaces).

Hence, all in all, at the moment, to the best of our knowledge, the state of the art on the nonlocal analogue of Conjecture 3.1 can be summarized by the following result:

Theorem 3.2. *Let $u \in C^2(\mathbb{R}^n, [1, 1])$ satisfy*

$$(-\Delta)^s u = u - u^3$$

and $\partial_{x_n} u > 0$

in the whole of \mathbb{R}^n . Assume also that

$$(3.3) \quad \begin{aligned} & \text{either } n = 2 \text{ and } s \in (0, 1), \\ & \text{or } n = 3 \text{ and } s \in [1/2, 1). \end{aligned}$$

Then all the level sets of u are hyperplanes.

As a matter of fact, Theorem 3.2 holds true for a very general class of equations under condition (3.3), see [CSM05, SV09, CS15, CC10, CC14, SV13, BV16]. Of course, it would be very desirable to go beyond condition (3.3), or to provide counterexamples.

It is also suggestive to compare the threshold $s = 1/2$ in (3.3) with the different behavior of the Γ -limit described by Theorem 2.1. In any case, it is not clear whether or not condition (3.3) reflects somehow the different behavior of local and nonlocal minimal surfaces. See also [BV16] for further discussions on this point.

4. PLANELIKE MINIMIZERS IN PERIODIC MEDIA

A classical topic in differential geometry (resp., dynamical systems) is to look for periodic and quasiperiodic solutions of problems set in periodic media which lie at a bounded distance from any fixed hyperplane (resp. which possess a rotation number in average). For instance, in [Mor24, Hed32] the case of the plane with a Riemannian metric was taken into account, establishing the existence of minimal geodesics which stay at a bounded distance from any prescribed straight line (depending on the arithmetic properties of the slope of this line, the geodesics turn out to be either periodic or quasiperiodic).

A similar problem in higher codimension turns out to be, in general, ill-posed, since [Hed32] provided an example of a metric in \mathbb{R}^3 which does not have geodesics at bounded distance from a particular direction. Nevertheless, a similar problem can be efficiently set in higher dimension, provided that one takes into consideration objects of codimension 1, such as minimal hypersurfaces (namely, hypersurfaces which minimize a periodic perimeter functional) that lie at bounded distance from any fixed hyperplane. We refer to [Ban90, CdLL01, AB01] and references therein for these types of problems and, for instance, to [Mat93] for related problems in the setting of dynamical systems.

The construction of periodic and quasiperiodic objects in periodic media has also a long and important tradition in partial differential equations, see [Mos86, RS11] and the references therein. For instance, in [Val04] the classical Ginzburg-Landau-Allen-Cahn equation (3.2) is set in a periodic medium, and one constructs global solutions which are minimal, and either periodic or quasiperiodic, and whose level sets stay at a bounded distance from a prescribed hyperplane. Roughly speaking, from a physical point of view, one obtains in this way some phase coexistence in an infinite periodic medium whose interface is a flat hyperplane, up to a uniformly bounded error.

Goal of this section is to describe in detail some of the results concerning periodic and quasiperiodic minimizers for the nonlocal phase transition equation in a periodic medium. For this, we take into consideration the following heterogeneous variant of the total energy (1.6):

$$(4.1) \quad \mathcal{E}(u) = \mathcal{E}(u; \Omega) := \frac{1}{2} \int_{Q_\Omega} |u(x) - u(y)|^2 K(x - y) dx dy + \int_{\Omega} Q(x) W(u(x)) dx,$$

with K as in (1.4), and the corresponding Euler-Lagrange equation

$$(-\Delta)^s u(x) + Q(x) W'(u(x)) = 0, \quad \text{for any } x \in \mathbb{R}^n.$$

In order to model a non-homogeneous periodic environment, the potential W appears now modulated by a measurable function $Q : \mathbb{R}^n \rightarrow [Q_*, Q^*]$, with $Q^* \geq Q_* > 0$, which is periodic with respect to a discrete lattice of step $\tau \geq 1$, that is

$$Q(x + k) = Q(x), \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and any } k \in \tau\mathbb{Z}^n.$$

As we just said, the energy \mathcal{E} embodies the presence of an underlying heterogeneous medium by means of the multiplicative correction Q in the potential term. Note that the model is sensitive to the periodicity scale of the medium through the factor τ .

Of course, a broader setting can be taken into account, for instance by considering a more general periodic potential or by letting the kernel K be space-dependent as well. This latter generalization is particularly interesting as it encompasses models in which the interaction between two particles of the system does not depend only on their distance, but may vary (albeit in a periodic way) as the particles occupy different places in the space.

Here, we choose to favor a not too involved exposition and thus to stick to the simpler model provided by (4.1). For a presentation of our contributions in a wider generality, we refer the interested reader to [CV15, CV16].

The result that we shortly discuss addresses the existence of a *planelike* minimizer for \mathcal{E} in the whole space \mathbb{R}^n . That is, a function $u : \mathbb{R}^n \rightarrow [-1, 1]$ that minimizes \mathcal{E} in Ω for any bounded set $\Omega \subset \mathbb{R}^n$ (a so-called *class A minimizer*) and whose intermediate level set $\{|u| < 9/10\}$ lies at a bounded distance from any fixed hyperplane of \mathbb{R}^n . By construction, the minimizer enjoys a suitable periodicity or quasi-periodicity property, depending on whether the slope of the associated hyperplane is rational or not.

When Q is constant, the energy is translation-invariant and, as a result, *planar* minimizers exist (recall the discussion at the beginning of Section 3). Under the presence of a non-trivial modulation Q , such construction cannot be carried out and, in general, one-dimensional minimizers do not exist. However, the following fact holds true.

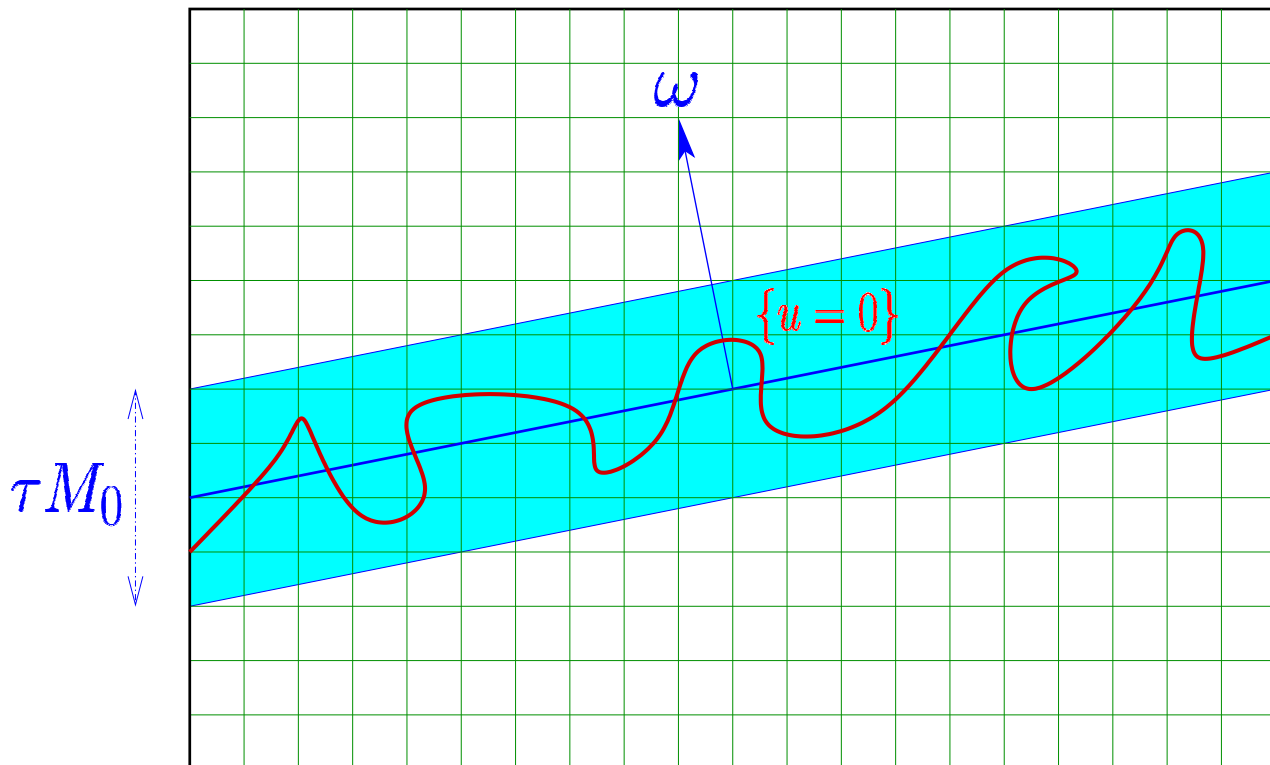


FIGURE 1. A planelike minimizer as given in Theorem 4.1.

Theorem 4.1. *There exists a universal constant $M_0 > 0$, such that, given any $\omega \in \mathbb{R}^n \setminus \{0\}$, we can construct a class A minimizer u for \mathcal{E} satisfying*

$$\left\{ x \in \mathbb{R}^n : |u(x)| < \frac{9}{10} \right\} \subset \left\{ x \in \mathbb{R}^n : \frac{\omega}{|\omega|} \cdot x \in [0, \tau M_0] \right\}.$$

Furthermore, u enjoys the following quasi-periodicity properties:

- if $\omega \in \tau\mathbb{Q}^n \setminus \{0\}$, then u is \sim -periodic, i.e. it respects the equivalence relation \sim defined in \mathbb{R}^n by

$$(4.2) \quad x \sim y \quad \text{if and only if} \quad y - x \in \tau\mathbb{Z}^n \text{ and } \omega \cdot (y - x) = 0;$$

- if $\omega \in \mathbb{R}^n \setminus \tau\mathbb{Q}^n$, then u is the locally uniform limit of a sequence of periodic class A minimizers.

Theorem 4.1 provides the aforementioned existence of planelike minimizers for the nonlocal energy (4.1), see Figure 1.

We stress that the size of the strip where the (essential) transition of a minimizer occurs is proportional to the periodicity scale τ of the medium. The proportionality constant M_0 is universal, in the sense that it depends only on the structural constants involved in the model, i.e. n , s , Q_* and Q^* .

Besides being interesting in itself, this fact plays a crucial role in deducing from Theorem 4.1 a similar construction of planelike minimal surfaces for a periodic nonlocal perimeter (see [CV16] for more details).

Also, the value $9/10$, that has been used to identify an interface region for the minimizer, obviously plays no particular role and may be indeed replaced by any $\vartheta \in (0, 1)$. However, the constant M_0 would then depend on ϑ as well.

The rest of this section contains a sketch of the proof of Theorem 4.1. The general strategy adopted is shaped on the one designed in [CdLL01] for a model described by a periodic surface energy, and developed in [Val04], where the argument is used to deal with a local, non-homogeneous Allen-Cahn-Ginzburg-Landau functional similar to (1.2).

We begin by addressing the case of a direction $\omega \in \tau\mathbb{Q}^n \setminus \{0\}$. We denote by $\widetilde{\mathbb{R}}^n$ any fundamental domain of the quotient space \mathbb{R}^n / \sim , where the relation \sim is given in (4.2), and consider the class of admissible

functions

$$\mathcal{A}_\omega^M := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^n) : u \text{ is } \sim\text{-periodic, } u(x) \geq \frac{9}{10} \text{ if } \frac{\omega}{|\omega|} \cdot x \leq 0 \text{ and } u(x) \leq -\frac{9}{10} \text{ if } \frac{\omega}{|\omega|} \cdot x \geq M \right\},$$

for any large $M > 0$. A straightforward application of the Direct Method of the Calculus of Variations gives the existence of global minimizers of the auxiliary functional

$$\mathcal{F}(u) := \frac{1}{2} \int_{\tilde{\mathbb{R}}^n} \int_{\tilde{\mathbb{R}}^n} |u(x) - u(y)|^2 K(x - y) dx dy + \int_{\tilde{\mathbb{R}}^n} Q(x) W(u(x)) dx,$$

within the set \mathcal{A}_ω^M , at least¹ if $s \in (1/2, 1)$. So we denote by \mathcal{M}_ω^M the set of such minimizers.

Observe that \mathcal{F} differs from the energy $\mathcal{E}(\cdot, \tilde{\mathbb{R}}^n)$, given in (4.1), for the sole fact that the latter contains the term integrated over $\tilde{\mathbb{R}}^n \times (\tilde{\mathbb{R}}^n)^c$ counted twice, namely

$$(4.3) \quad \begin{aligned} Q_{\tilde{\mathbb{R}}^n} &= (\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n) \cup (\tilde{\mathbb{R}}^n \times (\tilde{\mathbb{R}}^n)^c) \cup ((\tilde{\mathbb{R}}^n)^c \times \tilde{\mathbb{R}}^n) \\ \text{while } \tilde{\mathbb{R}}^n \times \mathbb{R}^n &= (\tilde{\mathbb{R}}^n \times \tilde{\mathbb{R}}^n) \cup (\tilde{\mathbb{R}}^n \times (\tilde{\mathbb{R}}^n)^c). \end{aligned}$$

The necessity of considering the auxiliary functional \mathcal{F} is peculiar to the nonlocal setting considered here and is mostly due to the fact that the functional \mathcal{E} does not behave well with respect to the periodic structure induced by \sim .

In more concrete terms, given a function u , one can consider its \sim -periodic extension \tilde{u} , defined for a.e. $x \in \mathbb{R}^n$ as

$$\tilde{u}(x) := u(\tilde{x}), \quad \text{where } \tilde{x} \text{ is the only element of } \tilde{\mathbb{R}}^n \text{ such that } x \sim \tilde{x}.$$

Then, it holds in general that

$$(4.4) \quad \mathcal{E}(u; \tilde{\mathbb{R}}^n) \neq \mathcal{E}(\tilde{u}; \tilde{\mathbb{R}}^n).$$

However, from (4.3), noticing that the potential term has a local character, we obtain that

$$(4.5) \quad \begin{aligned} &\mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(\tilde{u}) \\ &= \frac{1}{2} \int_{Q_{\tilde{\mathbb{R}}^n}} |u(x) - u(y)|^2 K(x - y) dx dy - \frac{1}{2} \int_{\tilde{\mathbb{R}}^n} \int_{\mathbb{R}^n} |\tilde{u}(x) - \tilde{u}(y)|^2 K(x - y) dx dy \\ &= \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} |u(x) - u(y)|^2 K(x - y) dx dy - \frac{1}{2} \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} |\tilde{u}(x) - \tilde{u}(y)|^2 K(x - y) dx dy. \end{aligned}$$

In particular, if u is \sim -periodic (hence $u = \tilde{u}$),

$$(4.6) \quad \mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(\tilde{u}) = \frac{1}{2} \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} |u(x) - u(y)|^2 K(x - y) dx dy.$$

In addition, we have that

$$(4.7) \quad \begin{aligned} &\text{any minimizer } u \text{ of } \mathcal{F} \text{ in the class } \mathcal{A}_\omega^M \text{ (i.e. any } u \in \mathcal{M}_\omega^M) \\ &\text{is a minimizer of the energy } \mathcal{E} \text{ with respect to perturbations} \\ &\text{supported inside the quotiented strip} \end{aligned}$$

$$\tilde{\mathcal{S}}_\omega^M := \tilde{\mathbb{R}}^n \cap \mathcal{S}_\omega^M, \quad \text{with } \mathcal{S}_\omega^M := \left\{ x \in \mathbb{R}^n : \frac{\omega}{|\omega|} \cdot x \in [0, M] \right\}.$$

This fact, which is of key importance and ultimately motivates the introduction of the functional \mathcal{F} , is based on the following computation: let ϕ be a smooth function supported inside $\tilde{\mathcal{S}}_\omega^M$ and let $v := u + \phi$,

¹We stress that the case in which $s \in (0, 1/2]$ can be treated afterwards via a limiting argument, approximating K with kernels truncated at infinity. The difficulties arise essentially for the fact that, when $s \in (0, 1/2]$, the functional \mathcal{F} is identically equal to $+\infty$ on \mathcal{A}_ω^M , due to the ‘‘fat’’ tails of the kernel.

with $u \in \mathcal{A}_\omega^M$. Then, ϕ vanishes outside $\tilde{\mathbb{R}}^n$ and $\tilde{v} = \tilde{u} + \tilde{\phi} = u + \tilde{\phi}$. Consequently, if $x \in (\tilde{\mathbb{R}}^n)^c$ and $y \in \tilde{\mathbb{R}}^n$, we have that

$$\begin{aligned} & |v(x) - v(y)|^2 - |\tilde{v}(x) - \tilde{v}(y)|^2 \\ &= |u(x) - u(y) - \phi(y)|^2 - |u(x) + \tilde{\phi}(x) - u(y) - \phi(y)|^2 \\ &= |u(x) - u(y)|^2 + \phi^2(y) - 2(u(x) - u(y))\phi(y) \\ &\quad - |u(x) + \tilde{\phi}(x) - u(y)|^2 - \phi^2(y) + 2(u(x) + \tilde{\phi}(x) - u(y))\phi(y) \\ &= |u(x) - u(y)|^2 - |u(x) + \tilde{\phi}(x) - u(y)|^2 + 2\tilde{\phi}(x)\phi(y). \end{aligned}$$

Hence, recalling (4.5) and (4.6),

$$\begin{aligned} & \left(\mathcal{E}(v; \tilde{\mathbb{R}}^n) - \mathcal{F}(\tilde{v}) \right) - \left(\mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(u) \right) \\ (4.8) \quad &= \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} \left(|v(x) - v(y)|^2 - \frac{1}{2}|\tilde{v}(x) - \tilde{v}(y)|^2 - \frac{1}{2}|u(x) - u(y)|^2 \right) K(x - y) dx dy \\ &= \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} \left(\frac{1}{2}|v(x) - v(y)|^2 - \frac{1}{2}|u(x) + \tilde{\phi}(x) - u(y)|^2 + \tilde{\phi}(x)\phi(y) \right) K(x - y) dx dy. \end{aligned}$$

Also, using the \sim -periodicity and changing variables $x \mapsto x - k$, $y \mapsto y + k$,

$$\begin{aligned} & \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} |u(x) + \tilde{\phi}(x) - u(y)|^2 K(x - y) dx dy \\ &= \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \omega \cdot k = 0}} \int_{\tilde{\mathbb{R}}^n + k} \int_{\tilde{\mathbb{R}}^n} |u(x) + \tilde{\phi}(x) - u(y)|^2 K(x - y) dx dy \\ &= \sum_{\substack{k \in \mathbb{Z}^n \setminus \{0\} \\ \omega \cdot k = 0}} \int_{\tilde{\mathbb{R}}^n} \int_{\tilde{\mathbb{R}}^n - k} |u(x) + \tilde{\phi}(x) - u(y)|^2 K(x - y) dx dy \\ &= \int_{\tilde{\mathbb{R}}^n} \int_{(\tilde{\mathbb{R}}^n)^c} |u(x) + \tilde{\phi}(x) - u(y)|^2 K(x - y) dx dy \\ &= \int_{\tilde{\mathbb{R}}^n} \int_{(\tilde{\mathbb{R}}^n)^c} |u(x) + \phi(x) - u(y)|^2 K(x - y) dx dy \\ &= \int_{\tilde{\mathbb{R}}^n} \int_{(\tilde{\mathbb{R}}^n)^c} |v(x) - v(y)|^2 K(x - y) dx dy. \end{aligned}$$

So we insert this information into (4.8), exchanging the roles of x and y , thanks to the even symmetry of K , and we find that one term simplifies, yielding to the identity

$$(4.9) \quad \left(\mathcal{E}(v; \tilde{\mathbb{R}}^n) - \mathcal{F}(\tilde{v}) \right) - \left(\mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(u) \right) = \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} \tilde{\phi}(x)\phi(y) K(x - y) dx dy.$$

Now, we take a competitor w for u and we define $\phi_1 := (w - u)_+ \geq 0$ (resp. $\phi_2 := (w - u)_- \geq 0$) and $v_1 := u - \phi_1$ (resp. $v_2 := u - \phi_2$). In this way, we deduce from (4.9) that

$$(4.10) \quad \left(\mathcal{E}(v_i; \tilde{\mathbb{R}}^n) - \mathcal{F}(\tilde{v}_i) \right) - \left(\mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(u) \right) = \int_{(\tilde{\mathbb{R}}^n)^c} \int_{\tilde{\mathbb{R}}^n} \tilde{\phi}_i(x)\phi_i(y) K(x - y) dx dy \geq 0,$$

for $i \in \{1, 2\}$.

Notice also that $\max\{u, w\} = v_1$ and $\min\{u, w\} = v_2$. Therefore, after a simple computation we get that

$$\mathcal{E}(v_1; \tilde{\mathbb{R}}^n) + \mathcal{E}(v_2; \tilde{\mathbb{R}}^n) \leq \mathcal{E}(u; \tilde{\mathbb{R}}^n) + \mathcal{E}(w; \tilde{\mathbb{R}}^n).$$

This and (4.10) give that

$$(4.11) \quad \mathcal{E}(u; \tilde{\mathbb{R}}^n) + \mathcal{E}(w; \tilde{\mathbb{R}}^n) \geq 2 \left(\mathcal{E}(u; \tilde{\mathbb{R}}^n) - \mathcal{F}(u) \right) + \mathcal{F}(\tilde{v}_1) + \mathcal{F}(\tilde{v}_2).$$

Now, if u is a minimizer of \mathcal{F} in the class \mathcal{A}_ω^M , we have that $\mathcal{F}(u) \leq \mathcal{F}(\tilde{v}_i)$, for $i \in \{1, 2\}$, and this information, combined with (4.11), gives that $\mathcal{E}(w; \tilde{\mathbb{R}}^n) \geq \mathcal{E}(u; \tilde{\mathbb{R}}^n)$. This establishes (4.7).

Now, while the functions in \mathcal{M}_ω^M are indeed minimizers of \mathcal{E} inside $\tilde{\mathcal{S}}_\omega^M$ thanks to (4.7), there is still no evidence of why they should extend their minimizing properties beyond such domain, and in fact in general they do not. In addition, the set of minimizers \mathcal{M}_ω^M is typically made up of more than just one element. This lack of uniqueness may lead to a corresponding lack of symmetry and rigidity in the elements of \mathcal{M}_ω^M , and prevent them from being class A minimizers.

For these reasons, we direct our attention to a specific element of the set \mathcal{M}_ω^M , namely the *minimal minimizer*.

The minimal minimizer u_ω^M of the class \mathcal{M}_ω^M is defined as

$$u_\omega^M(x) := \inf_{u \in \mathcal{M}_\omega^M} u(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

It is not hard to prove that u_ω^M is unique and belongs to \mathcal{M}_ω^M . Therefore, by considering u_ω^M we select the element of \mathcal{A}_ω^M with the lowest energy \mathcal{F} and, at the same time, having an interface with the tightest possible oscillation. As a matter of fact, this double optimality translates into the following two nice features that are enjoyed by the minimal minimizer:

- (i) the *doubling* or *no-symmetry-breaking property*, i.e. u_ω^M is a minimizer also within functions which exhibit a periodicity of multiple period;
- (ii) the *Birkhoff property*, i.e. the level sets of u_ω^M and its translations along vectors $k \in \tau\mathbb{Z}^n$ are well-ordered and have no non-trivial intersections.

The consequences entailed by these facts are twofold. On the one hand, the doubling property implies that u_ω^M is a minimizer for \mathcal{E} with respect to all compact perturbations occurring inside the strip \mathcal{S}_ω^M . On the other hand, the Birkhoff property extends such minimizing character to the whole space \mathbb{R}^n , if the width M of the strip is sufficiently large.

More precisely, by combining the energy and density estimates of Theorems 2.2 and 2.3, we obtain the existence of a “clean” ball B inside \mathcal{S}_ω^M over which u_ω^M is, say, smaller than $-9/10$. The scale invariance of such estimates ensures that the radius of this ball is a universal fraction of M . Then, the Birkhoff property allows us to translate B around \mathcal{S}_ω^M (in a discrete way) and clean out a full substrip of width comparable to τ , provided that $M \geq M_0\tau$, for some large universal constant $M_0 > 0$. This says that the minimal minimizer u_ω^M starts attaining values below $-9/10$ well before meeting the upper constraint $\{\omega \cdot x = M|\omega|\}$. Such *unconstrainedness* is the key observation that leads to deducing that u_ω^M is a class A minimizer for \mathcal{E} , which thereafter follows almost immediately.

The proof of Theorem 4.1 is essentially complete for the case of a direction $\omega \in \tau\mathbb{Q}^n \setminus \{0\}$. When instead $\omega \in \mathbb{R}^n \setminus \mathbb{Q}^n$, we consider a sequence $\{u_k\}$ of periodic planelike minimizers, corresponding to rational approximating directions $\{\omega_k\} \subset \tau\mathbb{Q}^n \setminus \{0\}$ of ω . Uniform Hölder estimates combined with the fact that the value M_0 does not depend on the chosen direction then allow us to take the limit in k and obtain a planelike class A minimizer with interface confined in a strip orthogonal to ω .

In conclusion, with Theorem 4.1 we are able to prove the existence of planelike minimizers for the energy \mathcal{E} , even if its nonlocal nature prevents it from matching the underlying periodic medium (recall (4.4)) as well as in the classical, local case treated in [Val04].

On top of that, the techniques described in this section are flexible enough to be adapted to other nonlocal periodic models, which arise, for instance, in connection with fractional perimeter functionals and long-range Ising models (see [CV16, CDV16]).

5. MULTIWELL POTENTIALS AND CHAOTIC ORBITS

Among the many others, one of the main achievements of Paul Rabinowitz consists in providing a clear and elegant framework in which the chaotic behavior of many equations of great physical importance can be rigorously detected and deeply understood. We would like to point out one application of the theory

that he developed in the framework of Hamiltonian dynamics to nonlocal equations. For this, we consider an equation of the type

$$(5.1) \quad (-\Delta)^s u(x) + a(x) V'(u(x)) = 0, \quad \text{for any } x \in \mathbb{R}.$$

Here, V is a smooth multiwell potential with a discrete set of nondegenerate minima and a is a smooth function (indeed, a more general setting, also comprising systems of equations, may be taken into account). The reader may compare equations (1.7) and (5.1): in some sense, equation (5.1) is taking into account the possibility of drifting from one minimum of V to another one (as well as the layer solution u_0 in (3.1) connects the minima of the two-well potential W). In this sense, the modulation function a provides the possibility of favoring this kind of multiple jumps.

Equation (5.1) also arises naturally in the study of crystal dislocation dynamics: in this framework, the function u can be interpreted as the discrepancy between the rest position of an atom and its actual position, see e.g. Section 2 of [DPV15] for simple physical motivations.

The result that we present here has been obtained in [DPV] and can be interpreted as a fractional counterpart of the classical results in [RCZ00].

Roughly speaking, these results aim to provide a *symbolic dynamics* for the solutions of equation (5.1) (under suitable “nondegeneracy assumptions” on a). That is, one considers the discrete space consisting of the equilibria of V and finds a solution of (5.1) which induces a shift operator on such space. Namely, given a sequence of minima of V in a prescribed order, one finds a solution of (5.1) which gets close to each of these minima of V , in the required order.

More precisely, we have the following result:

Theorem 5.1. *Assume that $V \in C^2(\mathbb{R}, [0, +\infty))$ is even, with $V(r+k) = V(r)$, for any $r \in \mathbb{R}$ and $k \in \mathbb{Z}$. Assume also that $V(k) = 0$ for any $k \in \mathbb{Z}$, that $V(r) > 0$ and $V''(0) > 0$.*

Let $a(x) := a_1 + a_2 \cos(\varepsilon x)$, with $a_1 > a_2 > 0$ and $\varepsilon > 0$ small enough.

Let $\zeta_1, \dots, \zeta_N \in \mathbb{Z}$. Then, there exist $b_1 < \dots < b_{2N-2} \in \mathbb{R}$ and a solution u of (5.1) such that

$$\begin{aligned} \lim_{x \rightarrow -\infty} u(x) &= \zeta_1, \\ |u(x) - \zeta_1| &< \frac{1}{10} \quad \text{for any } x \in (-\infty, b_1], \\ |u(x) - \zeta_{i+1}| &< \frac{1}{10} \quad \text{for any } x \in [b_{2i}, b_{2i+1}] \text{ for any } i \in \{1, \dots, N-2\}, \\ |u(x) - \zeta_N| &< \frac{1}{10} \quad \text{for any } x \in [b_{2N-2}, +\infty) \\ \text{and} \quad \lim_{x \rightarrow +\infty} u(x) &= \zeta_N. \end{aligned}$$

The situation stated in Theorem 5.1 is described in Figure 2.

Here is a sketch on how Theorem 5.1 can be deduced from the results in [DPV]. First of all, we may suppose that

$$(5.2) \quad |\zeta_{i+1} - \zeta_i| = 1 \text{ for all } i \in \{1, \dots, N-1\}.$$

Indeed, this may be obtained just by adding the intermediate integers in the original sequence ζ_1, \dots, ζ_N .

Now we claim that

$$(5.3) \quad \zeta_{i+1} \in \mathcal{A}(\zeta_i),$$

for any $i \in \{1, \dots, N-1\}$, where the admissible class $\mathcal{A}(\zeta_i)$ is defined in Section 8 of [DPV] (roughly speaking, $\mathcal{A}(\zeta_i)$ contains all the integers ζ which can be connected to ζ_i by a constrained orbit from a neighborhood of ζ_i to a neighborhood of ζ with minimal possible action).

To prove (5.3), we suppose that $\zeta_{i+1} = \zeta_i + 1$ (recall (5.2); the case $\zeta_{i+1} = \zeta_i - 1$ is analogous). Let $\zeta \in \mathcal{A}(\zeta_i)$. Since, by Lemma 8.1 in [DPV], we have that $2\zeta_i - \zeta$ also belongs to $\mathcal{A}(\zeta_i)$, by possibly

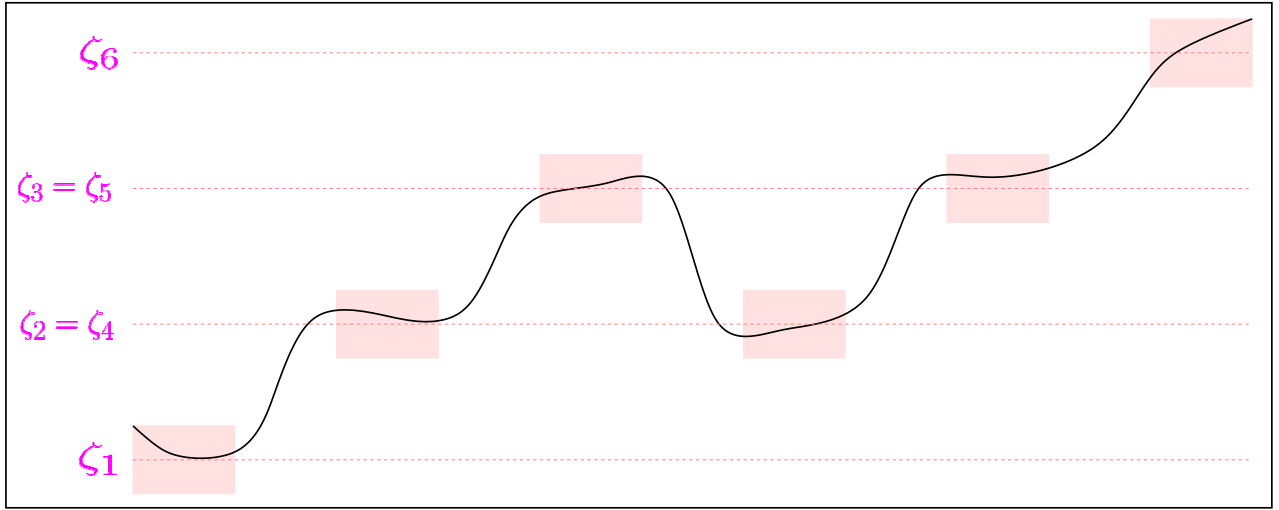


FIGURE 2. The multibump solution constructed in Theorem 5.1.

replacing ζ with $2\zeta_i - \zeta$ we may and do suppose that $\zeta > \zeta_i$. That is, $\zeta \geq \zeta_i + 1$. Now, if $\zeta = \zeta_i + 1$, we have that

$$\mathcal{A}(\zeta_i) \ni \zeta = \zeta_i + 1 = \zeta_{i+1},$$

and so (5.3) is proved.

Hence we may focus on the case in which $\zeta \geq \zeta_i + 2$. In this case, given $r \in (0, 1/4]$, if we have an orbit u_i such that $|u_i(x) - \zeta_i| \leq r$ for any $x \leq b_1$ and $|u_i(x) - \zeta| \leq r$ for any $x \geq b_2$, we can define

$$u_i^*(x) := \min\{u_i(x), \zeta_i + 1\} = \min\{u_i(x), \zeta_{i+1}\}.$$

Then, if $x \leq b_1$, we have that

$$u_i(x) \leq \zeta_i + r < \zeta_i + 1,$$

hence $u_i^*(x) = u_i(x) \in [\zeta_i - r, \zeta_i + r]$. In addition, if $x \geq b_2$, then

$$u_i(x) \geq \zeta - r \geq \zeta_i + 2 - r > \zeta_i + 1,$$

thus $u_i^*(x) = \zeta_i + 1 \in [\zeta_{i+1} - r, \zeta_{i+1} + r]$. This gives that u_i^* is a constrained orbit from a neighborhood of ζ_i to a neighborhood of ζ_{i+1} .

What is more, $V(u_i^*(x)) \leq V(u_i(x))$ for any $x \in \mathbb{R}$; also, for any $x, y \in \mathbb{R}$,

$$|u_i^*(x) - u_i^*(y)| \leq |u_i(x) - u_i(y)|.$$

As a consequence, the action of u_i^* is less than or equal to the action of u_i , hence ζ_{i+1} is admissible, thus proving (5.3).

Now, from (5.3) and Theorem 9.3 in [DPV] we obtain Theorem 5.1 here.

We underline that an important difference between the result in Theorem 5.1 and the classical ones for Hamiltonian systems lies in the glueing methods. Indeed, in the classical case, the technique of cutting-and-pasting different trajectories is abundantly used (typically, to construct suitable competitors for lowering the action functional). In the nonlocal case, this method may lead to additional difficulties, since the action of the new orbit obtained by a cut-and-paste of two trajectories is not simply the sum of the two actions of the original trajectories, since nonlocal interactions take place in the elastic part of the functional, which may be in fact the dominant contribution when the fractional parameter s is small (think once more to the asymptotics in Theorem 2.1).

To overcome such a difficulty, in [DPV] we introduced a “clean interval” method. Namely, one has to perform the cut-and-paste techniques always at points in which the trajectories meet in a “very flat” way (that is the oscillations of the two trajectories need to be appropriately small in a sufficiently large interval). This fact, combined with suitable “elliptic estimates”, allows us to estimate the remainder terms and to efficiently adapt the dynamical systems methods also to nonlocal cases.

ACKNOWLEDGMENTS

This work has been supported by the ERC grant 277749 “EPSILON Elliptic PDE’s and Symmetry of Interfaces and Layers for Odd Nonlinearities”, the PRIN grant 201274FYK7 “Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations” and the Alexander von Humboldt Foundation.

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