REGULARITY PROPERTIES OF FIBER DERIVATIVES ASSOCIATED WITH HIGHER-ORDER MECHANICAL SYSTEMS

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Abstract

The aim of this work is to study fiber derivatives associated to Lagrangian and Hamiltonian functions describing the dynamics of a higher-order autonomous dynamical system. More precisely, given a function in $T^*T^{(k-1)}Q$, we find necessary and sufficient conditions for such a function to describe the dynamics of a kth-order autonomous dynamical system, thus being a kth-order Hamiltonian function. Then, we give a suitable definition of (hyper)regularity for these higher-order Hamiltonian functions in terms of their fiber derivative. In addition, we also study an alternative characterization of the dynamics in Lagrangian submanifolds in terms of the solutions of the higher-order Euler-Lagrange equations.

Key words: *Higher-order systems; Lagrangian and Hamiltonian mechanics; Lagrangian submanifolds; Tulczyjew's triple.*

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1 Introduction

Higher-order dynamical systems play a relevant role in certain branches of theoretical physics, applied mathematics and numerical analysis. In particular, they appear in theoretical physics, in the mathematical description of relativistic particles with spin, string theories, Hilbert's Lagrangian for gravitation, Podolsky's generalization of electromagnetism and others, as well as in some problems of fluid mechanics and classical physics, and in numerical models arising from the discretization of dynamical control systems that preserve their inherent geometric structures. In these kinds of systems, the dynamics have explicit dependence on accelerations or higher-order derivatives of the generalized coordinates of position. The geometric tools used to study those systems have been developed mainly by M. de León, P.R. Rodrígues, D.J. Saunders and M. Crampin (among others) between the 70's and 90's in [18, 51, 52] (see also [2, 9, 10, 11, 19, 20, 35, 39], and references therein). These works are based in the ideas of the Lagrangian formalism introduced by J. Klein at the beginning of the 60's in [33]. In the aforementioned work, the Euler-Lagrange equations are obtained by J. Klein in a purely geometric way using the canonical geometric structures of the tangent and cotangent bundles, avoiding the use of variational calculus and exploiting the geometry of these dynamical systems.

The interest in higher-order dynamical systems has been growing up from the 90's due to the study of optimization and boundary value problems where the cost function involves higher-order derivatives, which may be modeled as variational problems with explicit dependence on higher-order derivatives of the generalized coordinates of position. These "higher-order variational problems" are of great interest for their useful applications in aeronautics, robotics, computer-aided design, air traffic control, trajectory planning, and, more generally, problems of interpolation and approximation of curves on Riemannian manifolds. These kinds of problems have been studied in [4, 5, 7, 30, 37, 41, 44] and more recently, in [22, 23, 24, 43] the development of variational principles which involve higher-order cost functions for optimization problems on Lie groups and their application in template matching for computational anatomy have been studied. These applications have produced a great interest in the study and development of new modern geometric tools and techniques to model properly higher-order variational problems, with the additional goal of obtaining a deepest understanding of the intrinsic properties of these problems. Some work in this line of research has been carried out recently in the following references, [13, 12, 49, 50, 40, 45, 46, 6, 31, 32, 28, 56].

Let us recall that the dynamics for a kth-order dynamical system can be obtained both by means of a Lagrangian function defined on a the kth-order tangent bundle $T^{(k)}Q$ of the smooth manifold Q that models the configuration space of the system, or by means of a Hamiltonian function defined on the cotangent bundle $T^*T^{(k-1)}Q$ (see [18] for details). The relation between Lagrangian and Hamiltonian dynamics can be studied either using the higher-order Legendre map [16, 17] or the Legendre-Ostrogradsky map [18]. These two well-known transformations are both derived from the Lagrangian function, and they provide a way to define a canonical Hamiltonian function, and hence to give a Hamiltonian formulation of the system. Nevertheless, for first-order dynamical systems it is known from [1] (Sections §3.5 and §3.6) that, starting from a Hamiltonian function in the cotangent bundle of the configuration space, it is possible to define a Lagrangian function in the tangent bundle describing the dynamics of the system, or to recover the starting Lagrangian if the Hamiltonian was defined using a Legendre map. The fundamental tool to do so is the fiber derivative of the Hamiltonian function. This same procedure can be carried out with a Hamiltonian function defined on $T^*T^{(k-1)}Q$, and a "Lagrangian function" can be defined in $TT^{(k-1)}Q$. This "Lagrangian function", however, may not be an actual Lagrangian function in the physical sense, since the generalized coordinates in the base manifold $T^{(k-1)}Q$ are generally not related to the fibered coordinates (the "velocities"). From the geometric point of view, this pretended "Lagrangian function" may not be defined in the holonomic submanifold $T^{(k)}Q \hookrightarrow TT^{(k-1)}Q$, which is the real domain for a kth-order Lagrangian function. This problem arises from the fact that the starting Hamiltonian function is not considered as a Hamiltonian function for kth-order system, but just as a Hamiltonian function for a first-order system defined on the cotangent bundle of a larger base manifold, since there is not an enforced relation between the momenta.

The discussion in the previous paragraph gives rises to two natural questions. First, what is a *higher-order Hamiltonian function*? And second, what does it mean for such a function to be *regular*? Observe that these concepts are clearly defined for first-order Lagrangian or Hamiltonian functions, and also for a higher-order Lagrangian functions (see [15] for the constrained first order case). In this work we pretend to give an answer to these questions. Indeed, in Section 4 we propose a definition of both concepts, always taking into account the particular case of Hamiltonian functions associated to a regular Lagrangian system. Moreover, we extend in a nontrivial way some results from [1], namely Propositions 3.6.7 and 3.6.8, and Theorem 3.6.9, to higher-order autonomous dynamical systems by means of the answers to the proposed questions.

For constrained systems, as well for singular Lagrangian functions, an alternative approach for a better understanding of the geometry involving mechanical systems was established by W.M. Tulczyjew: the so-called Tulczyjew's triple [54, 55], which makes strong use of Lagrangian submanifolds of suitable symplectic manifolds. Lagrangian submanifolds are of great interest in geometric mechanics, since they provide a way of describing both Lagrangian and Hamiltonian dynamics from a purely geometric and intrinsic point of view (see [54, 55]). In particular, let us recall that given a mechanical system described by a Lagrangian function $L\colon TQ \to \mathbb{R}$, the Lagrangian dynamics are "generated" by the Lagrangian submanifold $dL(TQ) \subset T^*TQ$. On the other hand, if the system is described by a Hamiltonian function $H\colon T^*Q \to \mathbb{R}$, then the Hamiltonian dynamics are "generated" by the Lagrangian submanifold $dH(T^*Q) \subset T^*T^*Q$. The relationship between these two formulations is provided by the so-called Tulczyjew's triple

$$T^*TQ \xleftarrow{\ \alpha_Q\ } TT^*Q \xrightarrow{\ \beta_Q\ } T^*T^*Q,$$

where α_Q and β_Q are both vector bundle isomorphisms, and T^*TQ, TT^*Q and T^*T^*Q are double vector bundles equipped with suitable symplectic structures.

If we consider a kth-order Lagrangian system described by a kth-order Lagrangian function $L\colon T^{(k)}Q \to \mathbb{R}$, a similar construction can be carried out (with some additional technical issues arising from the fact that $T^{(k)}Q$ is not a vector bundle in general, see [16, 17] for details), thus obtaining a Lagrangian submanifold in $TT^*T^{(k-1)}Q$, which completely determines the equations of motion for the dynamics. Moreover, these equations of motion are of Hamiltonian type if the Lagrangian system is regular. Our aim in this work is to study properties of fiber derivatives of functions defined on Lagrangian submanifolds, thus pursuing the research lines established in the works of M. de León and E. Lacomba [16, 17].

The paper is structured as follows. In Section 2 we introduce the necessary geometric background in order to make the paper as much selfcontained as possible. In particular, we review the definition and basic properties of symplectic manifolds and Lagrangian submanifolds, fiber derivatives of fiber preserving maps and their application to relate Lagrangian and Hamiltonian dynamics, higher-order tangent bundles and some of its canonical structures and, finally, a short review on the construction of Tulczyjew's triple for first-order dynamical systems. In Section 3 we study fiber derivatives of higher-order Lagrangian systems, relating the classical Legendre-Ostrogradsky map associated to a kth-order Lagrangian function L and the kth-order Legendre transformation defined on the Lagrangian submanifold $\Sigma_L \subset T^*TT^{(k-1)}Q$ generated by the Lagrangian L. To close this Section, we study the dynamics of the end of a thrown javelin as an illustrative example. Finally, in Section 4 we study the problem of giving an universal definition of higher-order Hamiltonian function, and the regularity properties of the fiber derivative associated with such a Hamiltonian.

2 Geometric background

In this Section we introduce the geometric structures and definitions that we use along this work. All the manifolds are real, second countable and C^∞. The maps and the structures are assumed to be C^∞. Sum over crossed repeated indices is understood. If M denotes a finite-dimensional smooth manifold, then $C^\infty(M)$, $\mathfrak{X}(M)$ and $\Omega^k(M)$ denote the sets of smooth functions, smooth vector fields and smooth k-forms on M, respectively.

2.1 Symplectic manifolds and Lagrangian submanifolds

Along this Subsection, M denotes a finite-dimensional smooth manifold. We refer to [8, 36, 57] for details and proofs.

Definition 1. *A* symplectic form *in M is a closed 2-form $\omega \in \Omega^2(M)$ which is nondegenerate, that is, for every $p \in M$, $i_{X_p}\omega_p = 0$ if, and only if, $X_p = 0$ where $X_p \in T_pM$. A* symplectic manifold *is a pair (M, ω), where M is a smooth manifold and ω is a symplectic form.*

Remark. If (M, ω) is a symplectic manifold, then the nondegeneracy of ω implies that M has even dimension, that is, $\dim M = 2n$. ◊

Definition 2. *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds, and $\Phi \colon M_1 \to M_2$ a diffeomorphism. Φ is a* symplectomorphism *if $\Phi^*\omega_2 = \omega_1$, and it is an* anti-symplectomorphism *if $\Phi^*\omega_2 = -\omega_1$, where $\Phi^*\omega_2$ denotes the pull-back of the 2-form ω_2 by the diffeomorphism Φ.*

A distinguished symplectic manifold is the cotangent bundle T^*Q of a n-dimensional smooth manifold Q. Let $\pi_Q \colon T^*Q \to Q$ be the canonical projection defined by $\pi_Q(\alpha_q) = q \in Q$, where $\alpha_q \in T_q^*Q$. The *Liouville 1-form*, denoted by $\theta_Q \in \Omega^1(T^*Q)$, is defined as

$$\langle (\theta_Q)_{\alpha_q}, X_{\alpha_q} \rangle = \langle \alpha_q, T_{\alpha_q}\pi_{T^*Q}(X_{\alpha_q}) \rangle \,, \text{ where } \alpha_q \in T^*Q \text{ and } X_{\alpha_q} \in T_{\alpha_q}T^*Q \,.$$

Observe that the Liouville 1-form satisfies $\alpha^*\theta_Q = \alpha$ for every $\alpha \in \Omega^1(Q)$. Then, one can define the *canonical symplectic form* of T_q^*Q, or *Liouville 2-form*, as $\omega_Q = -d\theta_Q \in \Omega^2(T^*Q)$. If $(U; (q^i))$, $1 \leqslant i \leqslant n$, are local coordinates in Q, the induced natural coordinates in $\pi_Q^{-1}(U) \subseteq T^*Q$ are (q^i, p_i), $1 \leqslant i \leqslant n$. In these coordinates, the local expression of the Liouville 1-form is $\theta_Q = p_i dq^i$, from where, the canonical symplectic form has the following coordinate expression $\omega_Q = dq^i \wedge dp_i$.

The existence of a nondegenerate 2-form on symplectic manifolds enables us to define some special submanifolds. In particular, we are interested in the study of *Lagrangian submanifolds*.

Definition 3. *Let (M, ω) be a symplectic manifold. An immersed submanifold $i_N \colon N \hookrightarrow M$ is a* Lagrangian submanifold *if $\dim N = \frac{1}{2}(\dim M)$ and $i_N^*\omega = 0$.*

Next, we introduce some particular Lagrangian submanifolds of the symplectic manifold (T^*Q, ω_Q). The first one is the image of a closed 1-form. Indeed, let $\lambda \in \Omega^1(Q)$ be a closed 1-form, and let us consider the subset $\Sigma_\lambda = \lambda(Q) \subset T^*Q$, which is a submanifold of T^*Q with canonical embedding $\lambda \colon Q \hookrightarrow T^*Q$. Then we have $\lambda^*\omega_Q = \lambda^*(-d\theta_Q) = -d\lambda^*\theta_Q = -d\lambda = 0$ since λ is closed. Hence, Σ_λ is a Lagrangian submanifold. If, moreover, λ is exact, that is, $\lambda = df$, with $f \in C^\infty(Q)$, we say that f is a *generating function* of the Lagrangian submanifold Σ_λ, and we denote it by Σ_f (see [57] for details).

There is a more general construction of Lagrangian submanifolds given by J. Śniatycki and W.M. Tulczyjew in [53] (see also [54, 55]) which we use in Subsection 3.1 to generate the dynamics of a higher-order dynamical system through Lagrangian submanifolds.

Theorem 1 (Śniatycki & Tulczyjew)**.** *Let Q be a smooth manifold, $\tau_Q \colon TQ \to Q$ its tangent bundle, $i_N \colon N \hookrightarrow Q$ a k-dimensional submanifold, and $f \colon N \to \mathbb{R}$ a smooth function. Then*

$$\begin{aligned} \Sigma_{f,N} &= \left\{ \mu \in T^*Q \mid \pi_Q(\mu) \in N \text{ and } \langle \mu, v \rangle = \langle df, v \rangle \text{ for every } v \in T_{\pi_Q(\mu)}N \right\} \\ &= \left\{ \mu \in T^*Q \mid i_N^*\mu = df \right\} \,, \end{aligned}$$

is a Lagrangian submanifold of (T^*Q, ω_Q).

Let (q^i), $1 \leqslant i \leqslant n$, be local coordinates in Q adapted to N, that is, such that N is locally defined by the constraints $q^{k+1} = \ldots = q^n = 0$. Then, the smooth function $f \colon N \to \mathbb{R}$ depends only on the coordinates $q^1, \ldots, q^k$, and the submanifold $\Sigma_{f,N} \hookrightarrow T^*Q$ is locally defined by

$$\Sigma_{f,N} = \left\{ (q^i, p_i) \in T^*Q \mid q^i = 0 \,,\ p_j = \frac{\partial f}{\partial q^j} \text{ for } k+1 \leqslant i \leqslant n-k \,,\, 1 \leqslant j \leqslant k \right\} .$$

Thus, it follows that $\dim \Sigma_{f,N} = n = \dim Q = \frac{1}{2}\dim T^*Q$. Moreover, taking into account the local expression of the canonical symplectic form ω_Q, if we denote $i_{\Sigma_{f,N}} \colon \Sigma_{f,N} \hookrightarrow T^*Q$ the canonical embedding, it follows that $i^*_{\Sigma_{f,N}}\omega_Q = 0$. Therefore, $\Sigma_{f,N}$ is a Lagrangian submanifold of the symplectic manifold (T^*Q, ω_Q) (see [54] for an intrinsic proof).

The importance of this result lies in the fact that Lagrangian submanifolds are associated to the dynamics of Lagrangian and Hamiltonian systems subject or not to constraints as we show in Subsection 3.1.

2.2 Fiber derivative of a Hamiltonian function

Along this Subsection, we consider a first-order dynamical system with n degrees of freedom whose configuration space is modeled by a n-dimensional smooth manifold Q, and let $H \in C^\infty(T^*Q)$ be a Hamiltonian function describing the dynamics of the system (see [1], §3.5 and §3.6, for details).

Definition 4. *Let $\pi\colon E \to M$ and $\rho\colon F \to M$ be vector bundles over the common base manifold M, and let $f\colon E \to F$ be a smooth fiber preserving map (not necessarily a vector bundle morphism). Let f_x denote $f|_{E_x}$, where $E_x = \pi^{-1}(x)$ is the fiber over $x \in M$. The* fiber derivative *of f is defined to be the map*

$$\begin{array}{rcl} \mathcal{F}f\colon E & \longrightarrow & \displaystyle\bigcup_{x\in M} L(E_x, F_x) \\ v_x & \longmapsto & Df_x(v_x) \end{array}$$

where $L(E_x, F_x)$ denotes the vector space of linear mappings from E_x to F_x.

Next, we apply the construction given in Definition 4 to the Hamiltonian function H. Let E be the cotangent bundle of Q, $\pi_Q\colon T^*Q \to Q$ the canonical projection, $F = Q \times \mathbb{R}$ the trivial vector bundle with projection $\mathrm{pr}_1\colon Q\times\mathbb{R} \to Q$, and f the map $\widetilde{H}\colon T^*Q \to Q\times\mathbb{R}$ defined by

$$\widetilde{H}(\alpha_q) = (\pi_Q(\alpha_q), H(\alpha_q)) \,.$$

The map $\widetilde{H}$ is smooth and fiber preserving, since

$$\mathrm{pr}_1(\widetilde{H}(\alpha_q)) = \mathrm{pr}_1(\pi_Q(\alpha_q), H(\alpha_q)) = \pi_Q(\alpha_q) \,,$$

although $\widetilde{H}$ is not a vector bundle morphism in general. Taking into account that $L(T^*_qQ, \mathbb{R}) = T^{**}_qQ \cong T_qQ$, the *fiber derivative of H*, denoted by $\mathcal{F}H\colon T^*Q \to TQ$, is defined as the fiber derivative of the map $\widetilde{H}$ in the sense of Definition 4.

Remark. This same procedure can be carried out with a Lagrangian function $L \in C^\infty(TQ)$. As it is well-known, the fiber derivative of L, $\mathcal{F}L\colon TQ \to T^*Q$, is the Legendre map leg_L relating the Lagrangian and Hamiltonian formalisms of dynamical systems (see [1], §3.5 for details). ♢

The map $\mathcal{F}H\colon T^*Q \to TQ$ is smooth and fiber preserving, that is, $\tau_Q \circ \mathcal{F}H = \pi_Q$. Let $(U, (q^i))$ be a local chart in Q, and (q^i, p_i) the induced natural coordinates in $\pi_Q^{-1}(U) \subseteq T^*Q$. Then, the coordinate expression of $\mathcal{F}H$ is determined by

$$\mathcal{F}H(q^i, p_i) = \left(q^i, \frac{\partial H}{\partial p_i}\right) \,,$$

from where we can observe that $\mathcal{F}H$ is smooth and fiber preserving.

Definition 5. *A Hamiltonian function $H \in C^\infty(T^*Q)$ is* regular *if the map $\mathcal{F}H\colon T^*Q \to TQ$ is a local diffeomorphism, and it is* hyperregular *if $\mathcal{F}H$ is a global diffeomorphism. Otherwise, the Hamiltonian function is said to be* singular.

Locally, the regularity condition for H is equivalent to

$$\det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)(\alpha_q) \neq 0\,, \text{ for every } \alpha_q \in T_q^*Q\,,$$

that is, a Hamiltonian function is regular if, and only if, its Hessian matrix with respect to the momenta is invertible at every point of T^*Q.

Next, we give a brief review of the relation between Hamiltonian and Lagrangian formalisms in terms of the fiber derivative of H (see [1] details). First, let us recall how to define a Lagrangian $L \in C^\infty(TQ)$ describing the dynamics of the system starting from a Hamiltonian function.

Proposition 1 ([1], Prop. 3.6.7)**.** *Let $H \in C^\infty(T^*Q)$ be a hyperregular Hamiltonian function, $\theta_Q \in \Omega^1(T^*Q)$ the Liouville 1-form, $\omega_Q \in \Omega^2(T^*Q)$ the canonical symplectic form and $X_H \in \mathfrak{X}(T^*Q)$ the unique vector field solution to the dynamical equation $i_{X_H}\omega_Q = dH$. The function $L \in C^\infty(TQ)$ defined by $L = \theta_Q(X_H)\circ \mathcal{F}H^{-1} - H \circ \mathcal{F}H^{-1}$ is hyperregular, and $\mathcal{F}L \equiv \operatorname{leg}_L = \mathcal{F}H^{-1}$.*

Observe that, up to this point, the Hamiltonian function $H \in C^\infty(T^*Q)$ could be any function defined on the cotangent bundle. Nevertheless, if $L \in C^\infty(TQ)$ is a hyperregular Lagrangian function describing the dynamics of the system, and $E_L = \Delta(L) - L \in C^\infty(TQ)$ is the energy of the system, with $\Delta \in \mathfrak{X}(TQ)$ being the Liouville vector field, then we can define a Hamiltonian function $H = E_L \circ \operatorname{leg}_L^{-1} \in C^\infty(T^*Q)$. Then, the following result holds.

Proposition 2 ([1]. Prop. 3.6.8 and Thm. 3.6.9)**.** *Let $L \in C^\infty(TQ)$ be a hyperregular Lagrangian and $H = E_L \circ \operatorname{leg}_L^{-1} \in C^\infty(T^*Q)$ the associated Hamiltonian function. Then H is hyperregular and $\mathcal{F}H = \operatorname{leg}_L^{-1}$. In addition, if $\tilde{L} = \theta_Q(X_H)\circ\mathcal{F}H^{-1} - H\circ\mathcal{F}H^{-1} \in C^\infty(TQ)$ is the hyperregular Lagrangian function associated to H by Proposition 1, then $\tilde{L} = L$.*

2.3 Tulczyjew's triple

In [54, 55], Tulczyjew established two identifications, the first one between TT^*Q and T^*TQ (useful to describe Lagrangian mechanics) and the second one between TT^*Q and T^*T^*Q (useful to describe Hamiltonian mechanics), giving rise to the so-called *Tulczyjew's triple*. In this Subsection we summarize these results. Along this Subsection, Q denotes a n-dimensional smooth manifold.

Let us recall that the double tangent bundle TTQ of a manifold Q is endowed with two vector bundle structures over the base TQ, given by the canonical projection $\tau_{TQ}\colon TTQ \to TQ$ arising from the tangent bundle structure, and the tangent map $T\tau_Q\colon TTQ \to TQ$ of τ_Q of the canonical projection $\tau_Q\colon TQ \to Q$ arising from the starting tangent bundle structure. These two structures are related by the *canonical flip* $\kappa_Q\colon TTQ \to TTQ$, which is an isomorphism of double vector bundles. If $(U;(q^i))$, $1 \leqslant i \leqslant n$, is a local chart in Q and $(q^i, v^i, \dot{q}^i, \dot{v}^i)$ the induced local coordinates in a suitable open set of TTQ, then κ_Q is given locally by $\kappa_Q(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, \dot{q}^i, v^i, \dot{v}^i)$. It is clear from this coordinate expression that κ_Q is an involution. From this, we can give the following definition.

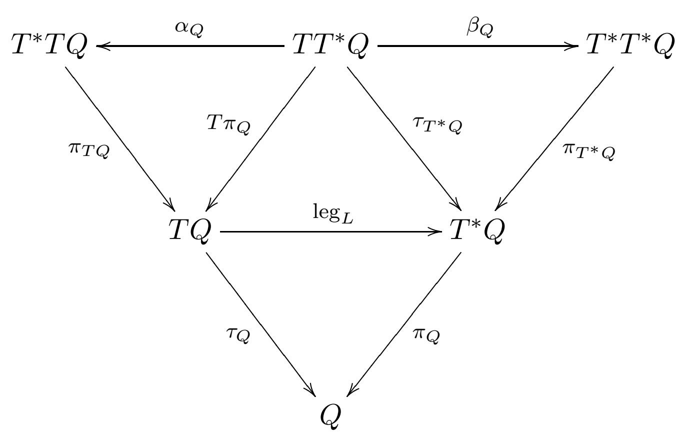
Definition 6. *The* Tulczyjew's isomorphisms *are the diffeomorphisms $\alpha_Q\colon TT^*Q \to T^*TQ$ and $\beta_Q\colon TT^*Q \to T^*T^*Q$ defined as follows:*

1. *α_Q is the dual map of κ_Q (as a double vector bundle morphism).*
2. *If $\omega_Q \in \Omega^2(T^*Q)$ is the canonical symplectic form, then $\beta_Q(X) = i_X\omega_Q\,,\ X \in TT^*Q$.*

Let $(U;(q^i))$ be a local chart in Q, and (q^i,p_i) the induced natural coordinates in $\pi_Q^{-1}(U) \subseteq T^*Q$. The induced natural coordinates in $\tau_{T^*Q}^{-1}(\pi_Q^{-1}(U)) \subseteq TT^*Q$ are $(q^i,p_i,\dot{q}^i,\dot{p}_i)$. In these coordinates, the maps α_Q and β_Q are given by $\alpha_Q(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,\dot{q}^i,\dot{p}_i,p_i)$ and $\beta_Q(q^i,p_i,\dot{q}^i,\dot{p}_i) = (q^i,\dot{q}^i,\dot{p}_i,p_i)$, respectively.

The map α_Q is a symplectomorphism when we consider on TT^*Q the symplectic structure given by the complete lift ω_Q^c of the canonical symplectic form ω_Q on T^*Q and on T^*TQ the canonical symplectic form ω_{TQ}. On the other hand, the map β_Q is an anti-symplectomorphism when we consider on TT^*Q the same symplectic structure ω_Q^c and we consider on T^*T^*Q the canonical symplectic structure ω_{T^*Q}.

The maps β_Q and α_Q give rise to the *Tulczyjew triple*, summarized in the following diagram



where $\mathrm{leg}_L \colon TQ \to T^*Q$ denotes the Legendre transformation associated to a given Lagrangian function $L \in C^\infty(TQ)$.

2.4 Higher-order tangent bundles

(See [18, 51] for details).

Let Q be a n-dimensional smooth manifold. We introduce an equivalence relation in the set $C^\infty(\mathbb{R},Q)$ of smooth curves $\gamma\colon \mathbb{R} \to Q$ as follows: given two curves $\gamma_1,\gamma_2\colon (-a,a) \to Q$, with $a > 0$, we say that γ_1 and γ_2 have *contact of order* k at $q_0 = \gamma_1(0) = \gamma_2(0)$ if there exists a local chart (U,φ) of Q such that $q_0 \in U$ and

$$\left.\frac{d^j}{dt^j}\right|_{t=0} (\varphi \circ \gamma_1(t)) = \left.\frac{d^j}{dt^j}\right|_{t=0} (\varphi \circ \gamma_2(t)) \ ,$$

for $j = 0,\ldots,k$. This is a well defined equivalence relation in $C^\infty(\mathbb{R},Q)$ and the equivalence class of a curve γ is denoted $[\gamma]_0^{(k)}$. The set of equivalence classes is denoted $T^{(k)}Q$, and it can be proved that it is a smooth manifold. Moreover, the map $\tau_Q^k\colon T^{(k)}Q \to Q$ defined by $\tau_Q^k\left([\gamma]_0^{(k)}\right) = \gamma(0)$ endows $T^{(k)}Q$ with a fiber bundle structure over Q, and therefore $T^{(k)}Q$ is called the *tangent bundle of order* k of Q, or k*th-order tangent bundle* of Q.

The manifold $T^{(k)}Q$ is endowed with some additional structure. In particular, for every $0 \leqslant r \leqslant k$ we define a surjective submersion $\tau_Q^{(r,k)}\colon T^{(k)}Q \to T^{(r)}Q$ as $\tau_Q^{(r,k)}\left([\gamma]_0^{(k)}\right) = [\gamma]_0^{(r)}$. It is easy to see that for every $0 \leqslant r \leqslant k$, the map $\tau_Q^{(r,k)}$ defines a fiber bundle structure. Moreover, we have that $T^{(1)}Q \equiv TQ$ is just the usual tangent bundle of Q, $T^{(0)}Q \equiv Q$ and $\tau_Q^{(0,k)} = \tau_Q^k$.

The *r-lift* of a smooth function $f \in C^\infty(Q)$, for $0 \leqslant r \leqslant k$, is the smooth function $f^{(r,k)} \in C^\infty(T^{(k)}Q)$ defined as

$$f^{(r,k)}\left([\gamma]_0^{(k)}\right) = \left.\frac{d^r}{dt^r}\right|_{t=0} (f \circ \gamma(t)) \, .$$

Of course, these definitions can be applied to functions defined on open sets of Q. Observe that the 0-lift of f coincides with f.

Local coordinates in $T^{(k)}Q$ are introduced as follows. Let (U,φ) a local chart in Q with coordinates (q^i), $1 \leqslant i \leqslant n$. Then the induced natural coordinates in the open set $(\tau_Q^k)^{-1}(U) \equiv T^{(k)}U \subseteq T^{(k)}Q$ are $\left(q_{(0)}^i, q_{(1)}^i, \ldots, q_{(k)}^i\right) \equiv \left(q_{(j)}^i\right)$ with $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k$, where $q_{(r)}^i = (q^i)^{(r,k)}$ for $0 \leqslant r \leqslant k$. Sometimes, we use the standard conventions $q_{(0)}^i \equiv q^i$, $q_{(1)}^i \equiv \dot{q}^i$ and $q_{(2)}^i \equiv \ddot{q}^i$.

The *canonical immersion* $j_k \colon T^{(k)}Q \to T(T^{(k-1)}Q)$ is defined as

$$j_k\left([\gamma]_0^{(k)}\right) = [\gamma^{(k-1)}]_0^{(1)} \, , \tag{1}$$

where $\gamma^{(k-1)}$ is the lift of the curve γ to $T^{(k-1)}Q$; that is, the curve $\gamma^{(k-1)} \colon \mathbb{R} \to T^{(k-1)}Q$ given by $\gamma^{(k-1)}(t) = [\gamma_t]_0^{(k-1)}$ where $\gamma_t(s) = \gamma(t+s)$. In the induced local coordinates of $T^{(k)}Q$, the map j_k is locally given by

$$j_k\left(q_{(0)}^i, q_{(1)}^i, q_{(2)}^i, \ldots, q_{(k)}^i\right) = \left(q_{(0)}^i, q_{(1)}^i, \ldots, q_{(k-1)}^i; q_{(1)}^i, q_{(2)}^i, \ldots, q_{(k)}^i\right) \, ,$$

from where we can deduce that in the induced natural coordinates $\left(q_{(j)}^i, v_{(j)}^i\right)$ of $TT^{(k-1)}Q$, the submanifold $T^{(k)}Q$ is defined locally by the $(k-1)n$ constraints $v_{(j)}^i = q_{(j+1)}^i$.

Denote by $\Omega^q(T^{(k)}Q)$ the real vector space of q-forms on $T^{(k)}Q$. In the exterior algebra of differential forms on $T^{(k)}Q$, denoted $\bigoplus_{q \geqslant 0} \Omega^q(T^{(k)}Q)$, we define an equivalence relation as follows: for $\alpha \in \Omega^q(T^{(k)}M)$ and $\beta \in \Omega^q(T^{(k')}Q)$,

$$\alpha \sim \beta \Longleftrightarrow \begin{cases} \alpha = \left(\tau_Q^{(k',k)}\right)^* (\beta) & \text{if } k' \leqslant k \\ \beta = \left(\tau_Q^{(k,k')}\right)^* (\alpha) & \text{if } k' \geqslant k. \end{cases}$$

Consider the quotient set $\Omega = \bigoplus_{k \geqslant 0} df^q(T^{(k)}Q) \, / \sim$, which is a commutative graded algebra. In this set we define the *Tulczyjew's derivation*, denoted by d_T, as follows: for every $f \in C^\infty(T^{(k)}Q)$ the function $d_T f \in C^\infty(T^{(k+1)}Q)$ is defined as $d_T f\left([\gamma]_0^{(k+1)}\right) = \left\langle d_{[\gamma]_0^{(k)}} f \, , j_{k+1}\left([\gamma]_0^{(k+1)}\right)\right\rangle$ where $j_{k+1} \colon T^{(k+1)}Q \to T(T^{(k)}Q)$ is the canonical immersion, and the covector $d_{[\gamma]_0^{(k)}} f \in T^*_{[\gamma]_0^{(k)}} T^{(k)}Q$ is the exterior derivative of f at $[\gamma]_0^{(k)} \in T^{(k)}Q$. Using the coordinate expression for j_{k+1}, the function d_T is given locally by

$$d_T f\left(q_{(0)}^i, \ldots, q_{(k+1)}^i\right) = \sum_{j=0}^{k} q_{(j+1)}^i \frac{\partial f}{\partial q_{(j)}^i}\left(q_{(0)}^i, \ldots, q_{(k)}^i\right) \, .$$

The map d_T extends to a derivation of degree 0 in Ω and, as $d_T d = d d_T$, it is determined by its action on functions and by the property $d_T(dq_{(j)}^i) = dq_{(j+1)}^i$.

Definition 7. *A curve $\psi \colon \mathbb{R} \to T^{(k)}Q$ is holonomic of type r, $1 \leqslant r \leqslant k$, if $\phi^{(k-r+1,k)} = \tau_Q^{(k-r+1,k)} \circ \psi$, where $\phi = \tau_Q^k \circ \psi \colon \mathbb{R} \to Q$; that is, the curve ψ is the lifting of a curve in Q up to $T^{(k-r+1)}Q$.*

From Definition 7, a vector field $X \in \mathfrak{X}(T^{(k)}Q)$ is a *semispray of type* r, $1 \leqslant r \leqslant k$, if every integral curve ψ of X is holonomic of type r. In the natural cordinates of $T^{(k)}Q$, the local expression of a semispray of type r is

$$X = q^i_{(1)} \frac{\partial}{\partial q^i_{(0)}} + q^i_{(2)} \frac{\partial}{\partial q^i_{(1)}} + \ldots + q^i_{(k-r+1)} \frac{\partial}{\partial q^i_{(k-r)}} + F^i_{(k-r+1)} \frac{\partial}{\partial q^i_{(k-r+1)}} + \ldots + F^i_{(k)} \frac{\partial}{\partial q^i_{(k)}} \, . \tag{2}$$

Remark. It is clear that every holonomic curve of type r is also holonomic of type s, for $s \geqslant r$. The same remark is true for semisprays. ◊

3 Geometric description of higher-order dynamical systems

In this Section we aim at studying fiber derivatives of higher-order Lagrangian systems, relating the classical Legendre-Ostrogradsky map and the kth-order Legendre transformation.

3.1 Higher-order Tulczyjew triple and dynamics generated by Lagrangian submanifolds

In this Subsection we explore some new results in the construction of the Tulczyjew's triple for higher-order dynamical systems of M. de León and E. Lacomba [16, 17]. In particular, we study fiber derivatives of higher-order Lagrangian systems, relating the classical Legendre-Ostrogradsky map associated to a kth-order Lagrangian function L and the kth-order Legendre transformation defined on the Lagrangian submanifold $\Sigma_L \subset T^*TT^{(k-1)}Q$ generated by the Lagrangian L. We show the theory with a simple but interesting example, the dynamics of the end of a thrown javelin.

Definition 8. *The* k^{th}-order Tulczyjew's isomorphism *is the map* $\beta_{T^{(k-1)}Q} \colon TT^*T^{(k-1)}Q \to T^*T^*T^{(k-1)}Q$ *defined by* $\beta_{T^{(k-1)}Q}(V) := i_V \omega_{T^{(k-1)}Q}$ *with* $V \in TT^*T^{(k-1)}Q$*, and* $\omega_{T^{(k-1)}Q}$ *being the canonical symplectic form of* $T^*T^{(k-1)}Q$*.*

Let (q^i), $1 \leqslant i \leqslant n$, be local coordinates in an open set $U \subset Q$, and $(q^i_{(j)})$, $0 \leqslant j \leqslant k-1$, the induced coordinates in $(\pi^{k-1}_Q)^{-1}(U) \subset T^{(k-1)}Q$ introduced in Section 2.4. Then, natural coordinates in $\left(\pi^{k-1}_Q \circ \pi_{T^{(k-1)}Q}\right)^{-1}(U) \subset T^*T^{(k-1)}Q$ are $\left(q^i_{(j)}, p^{(j)}_i\right)$, from where we deduce that the induced local natural coordinates in $TT^*T^{(k-1)}Q$ are $\left(q^i_{(j)}, p^{(j)}_i, \dot{q}^i_{(j)}, \dot{p}^{(j)}_i\right)$, with $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$. In these coordinates, the map $\beta_{T^{(k-1)}Q}$ is locally given by $\beta_{T^{(k-1)}Q}\left(q^i_{(j)}, p^{(j)}_i, \dot{q}^i_{(j)}, \dot{p}^{(j)}_i\right) = \left(q^i_{(j)}, \dot{q}^i_{(j)}, \dot{p}^{(j)}_i, p^{(j)}_i\right)$. This map is an anti-symplectomorphism when we consider $T^*T^*T^{(k-1)}Q$ endowed with the canonical symplectic structure and $TT^*T^{(k-1)}Q$ endowed with the symplectic structure given by the complete lift $\omega^c_{T^{(k-1)}Q}$ of the canonical symplectic form on $T^*T^{(k-1)}Q$.

The cotangent bundles $T^*T^*T^{(k-1)}Q$ and $T^*TT^{(k-1)}Q$ are examples of double vector bundles (see [29] for details). In particular, the double vector bundles $T^*T^*T^{(k-1)}Q$ and $T^*TT^{(k-1)}Q$ are canonically isomorphic via a vector bundle isomorphism over $T^*T^{(k-1)}Q$

$$\mathcal{R}_k \colon T^*TT^{(k-1)}Q \to T^*T^*T^{(k-1)}Q \, .$$

This map is an anti-symplectomorphism of symplectic manifolds (considering in both cotangent bundles the canonical symplectic structures), and also an isomorphism of double vector bundles.

It is completely determined by the condition

$$\left\langle \mathcal{R}_k(\alpha_u), W_{T^*\tau_{T^{(k-1)}Q}(\alpha_u)} \right\rangle = -\left\langle \alpha_u, \widetilde{W}_u \right\rangle + \left\langle W_{T^*\tau_{T^{(k-1)}Q}(\alpha_u)}, \widetilde{W}_u \right\rangle^T ,$$

for every $\alpha_u \in T^*_u TT^{(k-1)}Q$, $\widetilde{W}_u \in T_u TT^{(k-1)}Q$ and $W_{T^*\tau_{T^{(k-1)}Q}(\alpha_u)} \in TT^*T^{(k-1)}Q$ satisfying the relation $T\tau_{T^{(k-1)}Q}(\widetilde{W}_u) = T\pi_{T^{(k-1)}Q}(W_{T^*\tau_Q(\alpha_u)})$.

Here, $\langle\cdot,\cdot\rangle^T : TT^*T^{(k-1)}Q \times_{TT^{(k-1)}Q} TTT^{(k-1)}Q \to \mathbb{R}$ is the pairing defined by the tangent map of the usual pairing $\langle\cdot,\cdot\rangle : T^*T^{(k-1)}Q \times_{T^{(k-1)}Q} TT^{(k-1)}Q \to \mathbb{R}$, and the vector bundle projection $T^*\tau_{T^{(k-1)}Q} : T^*TT^{(k-1)}Q \to T^*T^{(k-1)}Q$ is characterized by $\langle T^*\tau_{T^{(k-1)}Q}(\alpha_u), w\rangle = \langle \alpha_u, w_u^\vee\rangle$ where $u, w \in T_{[q]^{(k-1)}_{(0)}}T^{(k-1)}Q$, $\alpha_u \in T^*_u TT^{(k-1)}Q$, and $w_u^\vee \in T_u TT^{(k-1)}Q$ is the vertical lift of the tangent vector w (see [21] for first order systems. The derivation for higher-order systems is derived straightforwardly from the definition given for first order systems).

Let (q^i), $1 \leqslant i \leqslant n$, be local coordinates in Q, and $(q^i_{(j)})$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k-1$, the induced coordinates in $T^{(k-1)}Q$ introduced in Section 2.4. Then, natural coordinates in $TT^{(k-1)}Q$ are $\left(q^i_{(j)}, v^i_{(j)}\right)$, from where we deduce that the induced natural coordinates in $T^*TT^{(k-1)}Q$ are $\left(q^i_{(j)}, v^i_{(j)}, p_i^{(j)}, \tilde{p}_i^{(j)}\right)$, with $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$. In these coordinates, the map $\mathcal{R}_k$ is locally given by

$$\mathcal{R}_k\left(q^i_{(j)}, v^i_{(j)}, p_i^{(j)}, \tilde{p}_i^{(j)}\right) = \left(q^i_{(j)}, \tilde{p}_i^{(j)}, -p_i^{(j)}, v^i_{(j)}\right) .$$

Then, composing $\beta_{T^{(k-1)}Q}$ with $\mathcal{R}_k^{-1}$ we obtain a map $\alpha_{T^{(k-1)}Q} : TT^*T^{(k-1)}Q \to T^*TT^{(k-1)}Q$, which is given in the natural coordinates $\left(q^i_{(j)}, p_i^{(j)}, \dot{q}^i_{(j)}, \dot{p}_i^{(j)}\right)$ in $TT^*T^{(k-1)}Q$ by

$$\alpha_{T^{(k-1)}Q}\left(q^i_{(j)}, p_i^{(j)}, \dot{q}^i_{(j)}, \dot{p}_i^{(j)}\right) = \left(q^i_{(j)}, \dot{q}^i_{(j)}, \dot{p}_i^{(j)}, p_i^{(j)}\right) . \tag{3}$$

This map is a symplectomorphism when we consider in $TT^*T^{(k-1)}Q$ the symplectic structure given by the complete lift $\omega^c_{T^{(k-1)}Q}$ of the canonical symplectic form $\omega_{T^{(k-1)}Q}$ on $T^*T^{(k-1)}Q$ and on $T^*TT^{(k-1)}Q$ the canonical symplectic form $\omega_{TT^{(k-1)}Q}$. The maps $\beta_{T^{(k-1)}Q}$ and $\alpha_{T^{(k-1)}Q}$ give rise to the *kth-order Tulczyjew triple*

$$T^*TT^{(k-1)}Q \xleftarrow{\alpha_{T^{(k-1)}Q}} TT^*T^{(k-1)}Q \xrightarrow{\beta_{T^{(k-1)}Q}} T^*T^*T^{(k-1)}Q,$$

Remark. The map $\alpha_{T^{(k-1)}Q} : TT^*T^{(k-1)}Q \to T^*TT^{(k-1)}Q$ can be obtained directly as the dual to the *canonical flip* $\kappa_{T^{(k-1)}Q} : TT^{(k-1)}Q \to TT^{(k-1)}Q$, which is an isomorphism of double vector bundle structures on $TT^{(k-1)}Q$ (see [17] for more details). We prefer to avoid the use of the canonical flip by using $\mathcal{R}_k^{-1}$, as in [28]. ◊

Now we introduce the dynamics using a suitable Lagrangian submanifold in $T^*TT^{(k-1)}Q$ and the kth-order Tulczyjew's triple. First, let $j_k : T^{(k)}Q \hookrightarrow TT^{(k-1)}Q$ be the canonical immersion defined in (1). Then, if $x \in T^{(k)}Q$, the map $j_k^* : T^*_{j_k(x)}(TT^{(k-1)}Q) \to T^*_x(T^{(k)}Q)$ is given by

$$j_k^*\mu = \mu \circ Tj_k , \text{ for every } \mu \in T^*_{j_k(x)}TT^{(k-1)}Q .$$

Using this map, if $L \in C^\infty(T^{(k)}Q)$ is a kth-order Lagrangian function, by Śniatycki and Tulczyjew's construction given in Theorem 1 we define a Lagrangian submanifold in the cotangent bundle $T^*TT^{(k-1)}Q$, endowed with the canonical symplectic structure, as follows

$$\Sigma_L = \left\{\mu \in T^*TT^{(k-1)}Q \mid j_k^*\mu = dL\right\} \hookrightarrow T^*TT^{(k-1)}Q .$$

The Lagrangian submanifold Σ_L fibers onto $j_k(T^{(k)}Q)$, it is locally parametrized by the $2kn$ coordinate functions $\left(q_{(0)}^i, \ldots, q_{(k)}^i, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}\right)$, $1 \leqslant i \leqslant n$, and it is immersed into $T^*TT^{(k-1)}Q$ as

$$\left\{\left(q_{(j)}^i\,;\, q_{(j+1)}^i\,;\, \frac{\partial L}{\partial q_{(0)}^i}, \frac{\partial L}{\partial q_{(1)}^i} - \tilde{p}_i^{(0)}, \ldots, \frac{\partial L}{\partial q_{(k-1)}^i} - \tilde{p}_i^{(k-2)}\,;\, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}, \frac{\partial L}{\partial q_{(k)}^i}\right)\right\}.$$

Therefore, taking this into account, the Lagrangian dynamics is given by the Lagrangian submanifold $N_L = \alpha_{T^{(k-1)}Q}^{-1}(\Sigma_L) \hookrightarrow TT^*T^{(k-1)}Q$. Locally, N_L is the set of elements in $TT^*T^{(k-1)}Q$ of the form

$$\left(q_{(j)}^i\,;\, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}, \frac{\partial L}{\partial q_{(k)}^i}\,;\, q_{(j+1)}^i\,;\, \frac{\partial L}{\partial q_{(0)}^i}, \frac{\partial L}{\partial q_{(1)}^i} - \tilde{p}^{(0)}, \ldots, \frac{\partial L}{\partial q_{(k-1)}^i} - \tilde{p}_i^{(k-2)}\right).$$

From this, the submanifold N_L determines the following set of differential equations

$$\frac{d}{dt}\tilde{p}_i^{(0)} = \frac{\partial L}{\partial q_{(0)}^i}, \tag{4}$$

$$\frac{d}{dt}\tilde{p}_i^{(j)} + \tilde{p}_i^{(j-1)} = \frac{\partial L}{\partial q_{(j)}^i}, \tag{5}$$

$$\frac{\partial L}{\partial q_{(k-1)}^i} - \tilde{p}_i^{(k-2)} = \frac{d}{dt}\left(\frac{\partial L}{\partial q_{(k)}^i}\right), \tag{6}$$

where $1 \leqslant j \leqslant k-2$ in (5), and $1 \leqslant i \leqslant n$ in every set. Differentiating the n equations (6) with respect to the time t, and replacing into equation (5) for $j = k-2$, we obtain the following equations

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_{(k)}^i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial q_{(k-1)}^i}\right) - \frac{\partial L}{\partial q_{(k-2)}^i} - \tilde{p}_i^{(k-3)}.$$

Differentiating the last set of equation with respect to the time t and replacing the result into (5) when $j = k-3$ we have

$$\frac{d^3}{dt^3}\left(\frac{\partial L}{\partial q_{(k)}^i}\right) = \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_{(k-1)}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_{(k-2)}^i}\right) + \frac{\partial L}{\partial q_{(k-3)}^i} - \tilde{p}_i^{(k-4)}.$$

Iterating the process $k-4$ times, we obtain the following set of n equations

$$\frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_{(k)}^i}\right) = \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial q_{(k-1)}^i}\right) - \frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial q_{(k-2)}^i}\right) + \ldots - \frac{d}{dt}\left(\frac{\partial L}{\partial q_{(1)}^i}\right) + \frac{d}{dt}\tilde{p}_i^{(0)}.$$

Using equations (4) we obtain the following n differential equations

$$\sum_{j=0}^{k}(-1)^j\frac{d^j}{dt^j}\left(\frac{\partial L}{\partial q_{(j)}^i}\right) = 0,$$

which are exactly the higher-order Euler-Lagrange equations for the higher-order Lagrangian function L (see [17]). From the computations and considerations given above, we have the following result.

Proposition 3. *The solutions of a kth-order Lagrangian system described by a kth-order Lagrangian function $L \in C^\infty(T^{(k)}Q)$ are curves $\mu\colon I \subset \mathbb{R} \to \Sigma_L$ satisfying*

$$\pi_{TT^{(k-1)}Q}\Big|_{\Sigma_L} \circ \mu = j_k \circ \gamma^{(k)}\,,$$

where $\gamma^{(k)}\colon I \to T^{(k)}Q$ is the k-lift of a curve $\gamma\colon I \to Q$, $j_k\colon T^{(k)}Q \to TT^{(k-1)}Q$ the canonical immersion, and $\pi_{TT^{(k-1)}Q}\Big|_{\Sigma_L}\colon \Sigma_L \to TT^{(k-1)}Q$ denotes the restriction of the canonical projection $\pi_{TT^{k-1}Q}\colon T^*TT^{(k-1)}Q \to TT^{(k-1)}Q$ to Σ_L.

Remark. Observe that the three spaces $T(T^*T^{(k-1)}Q)$, $T^*(T^*T^{(k-1)}Q)$ and $T^*(T^{(k-1)}Q)$ involved in the Tulczyjew triple are symplectic manifolds; the two maps $\alpha_{T^{(k-1)}Q}$ and $\beta_{T^{(k-1)}Q}$ involved in the construction are a symplectomorphism and an anti-symplectomorphism, respectively; and the dynamical equations (Euler-Lagrange and Hamilton equations) are the local equations defining the Lagrangian submanifolds

$$N_L = \alpha^{-1}_{T^{(k-1)}Q}(\Sigma_L) \quad\text{and}\quad S_H = \beta^{-1}_{T^{(k-1)}Q}\left(dH(T^*(T^{(k-1)}Q))\right)\,,$$

respectively. Moreover, the Lagrangian and Hamiltonian functions are not involved in the definition of the triple. In this sense, the triple is canonical. Finally, we would like to point out that the construction can be applied to an arbitrary Lagrangian function, not necessarily to a regular one. ♢

Remark (Higher-order variational constrained equations)**.** The natural extension for constrained (vakonomic) higher-order mechanical systems can be studied by considering the embedded submanifold $\mathcal{M} \subset T^{(k)}Q$ given by the vanishing of m independent constraint functions $\Phi^\alpha\colon T^{(k)}Q \to \mathbb{R}$, $\alpha = 1, \ldots, m$. We consider the extended Lagrangian $\mathcal{L} = L + \lambda_\alpha \Phi^\alpha$ which includes the Lagrange multipliers λ_α as a new extra variable. The equations of motion for the higher-order constrined variational problem are the higher-order Euler-Lagrange equations for $\mathcal{L}$, that is,

$$\sum_{r=0}^{k}(-1)^r \frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q^{(r)i}} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(r)i}}\right) = 0 \tag{7}$$

$$\Phi^\alpha\left(q^i_{(0)}, \ldots, q^i_{(k-1)}, q^i_{(k)}\right) = 0 \tag{8}$$

From the geometrical point of view, these kind of higher-order variationally constrained problems are determined by a submanifold $\mathcal{M} \subset T^{(k)}Q$ with inclusion $i : \mathcal{M} \hookrightarrow T^{(k)}Q$ and by a Lagrangian $L_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$. Using Theorem (1) we deduce that $\Sigma_{L_{\mathcal{M}}} \subset T^*TT^{(k-1)}Q$ is a Lagrangian submanifold. Moreover, using the Tulczyjew's symplectomorphism one induce a new Lagrangian submanifold $\alpha^{-1}_{T^{(k-1)}Q}(\Sigma_{L_{\mathcal{M}}}) \subset TT^*T^{(k-1)}Q$ which completely determines the constrained variational dynamics. The case of unconstrained mechanics is generated taking the whole space $T^{(k)}Q$ instead of $\mathcal{M}$ and a Lagrangian function $L : T^{(k)}Q \to \mathbb{R}$. Indeed, this procedure gives the correct dynamics for the higher-order constrained variational problem using the same ideas as in the previous subsection only by changing the Lagrangian $L : T^{(k)}Q \to \mathbb{R}$ by $\mathcal{L}$. This is basically because the powerful of Tulczyjew's triple does not depends on the Lagrangian function.

We assume that the restriction of the projection $(\tau_Q^{(k-1,k)})\Big|_{\mathcal{M}} : \mathcal{M} \to T^{(k-1)}Q$ is a submersion. Locally, this conditions means that the $m \times n$-matrix

$$\left(\frac{\partial(\Phi^1, \ldots, \Phi^m)}{\partial(q^1_{(k)}, \ldots, q^n_{(k)})}\right)$$

is of rank m at all points of $\mathcal{M}$.

Consequently, by the implicit function theorem, we can locally express the constraints (reordering coordinates if necessary) as

$$\phi^{\alpha}\left(q_{(0)}^{i},\ldots,q_{(k-1)}^{i},q_{(k)}^{a}\right)=q_{(k)}^{\alpha}\,,\ 1\leqslant\alpha\leqslant m\,,\ m+1\leqslant a\leqslant n\,,\ i=1,\ldots,n\,, \tag{9}$$

and therefore, we can define a Lagrangian function $L\in C^{\infty}(\mathcal{M})$ in these new adapted coordinates to $\mathcal{M}$. Observe that $\mathcal{M}\hookrightarrow T^{(k)}Q\hookrightarrow TT^{(k-1)}Q$, and let us denote by $i_{\mathcal{M}}\colon\mathcal{M}\hookrightarrow TT^{(k-1)}Q$ the composition of both inclusions. Now construct

$$\Sigma_{L,\mathcal{M}}=\left\{\mu\in T^*TT^{(k-1)}Q\mid i_{\mathcal{M}}^*\mu=dL\right\}$$

and proceeding as in the unconstrained case we obtain that the higher-order constrained dynamics in governed by the following set of n ordinary differential equations of order $2k$

$$\sum_{r=0}^{k}(-1)^r\frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q^{(r)i}}-\mu_{\alpha}\frac{\partial\Phi^{\alpha}}{\partial q^{(r)i}}\right)=0\,, \tag{10}$$

satisfying the constraints. Observe that $\mu\in T^*TT^{(k-1)}Q$ plays the role of Lagrange multipliers for the equations, forcing the dynamics to satisfy the constraints imposed by the submanifold $\mathcal{M}\subset T^{(k)}Q$. ◊

3.2 On the Legendre maps for higher-order dynamical systems

In this Subsection we introduce a Legendre transformation (a fiber derivative) $\mathbb{F}L\colon\Sigma_L\to T^*T^{(k-1)}Q$ in the Lagrangian submanifold generated by a kth-order Lagrangian function $L\in C^{\infty}(T^{(k)}Q)$ and we study its relationship with the Legendre-Ostrogradsky map $\operatorname{leg}_L\colon T^{(2k-1)}Q\to T^*T^{(k-1)}Q$.

Let Q be the configuration space of an autonomous dynamical system of order k with n degrees of freedom, and let $L\in C^{\infty}(T^{(k)}Q)$ be the Lagrangian function for this system. From the Lagrangian function L we construct the Poincaré-Cartan 1-form $\theta_L\in\Omega^1(T^{(2k-1)}Q)$, whose coordinate expression is

$$\theta_L=\sum_{r=1}^{k}\sum_{j=0}^{k-r}(-1)^j d_T^j\left(\frac{\partial L}{\partial q_{(r+j)}^{i}}\right)dq_{(r-1)}^{i}\,. \tag{11}$$

The Poincaré-Cartan 1-form $\theta_L\in\Omega^1(T^{(2k-1)}Q)$ allows to define the Legendre-Ostrogradsky map as follows:

Definition 9. *The* Legendre-Ostrogradsky map *associated to the kth-order Lagrangian function L is the fiber bundle morphism* $\operatorname{leg}_L\colon T^{(2k-1)}Q\to T^*T^{(k-1)}Q$ *over* $T^{(k-1)}Q$ *defined as follows: for every* $u\in TT^{(2k-1)}Q$,

$$\theta_L(u)=\left\langle T\tau_Q^{(k-1,2k-1)}(u)\,,\,\operatorname{leg}_L(\tau_{T^{(2k-1)}Q}(u)\right\rangle\,. \tag{12}$$

Besides the condition $\pi_{T^{(k-1)}Q}\circ\operatorname{leg}_L=\tau_Q^{(k-1,2k-1)}$ stated in the definition, the Legendre-Ostrogradsky map relates the Liouville form in $T^*T^{(k-1)}Q$ to the Poincaré-Cartan 1-form. That is, if $\theta_{T^{(k-1)}Q}\in\Omega^1(T^*T^{(k-1)}Q)$ is the canonical form of the cotangent bundle $T^*T^{(k-1)}Q$, then $\operatorname{leg}_L^*\theta_{T^{(k-1)}Q}=\theta_L$.

In the local coordinates $\left(q_{(j)}^i\right)$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant 2k-1$, of $T^{(2k-1)}Q$ introduced in Section 2.4, we define the following local functions

$$\hat{p}_i^{(r-1)} = \sum_{j=0}^{k-r} (-1)^j d_T^j \left(\frac{\partial L}{\partial q_{(r+j)}^i} \right) , \tag{13}$$

which are called the Jacobi-Ostrogradsky momenta. Observe that we have the following relation between $\hat{p}_i^{(r)}$ and $\hat{p}_i^{(r-1)}$

$$\hat{p}_i^{(r-1)} = \frac{\partial L}{\partial q_{(r)}^i} - d_T\left(\hat{p}_i^{(r)}\right) \, , \ \text{for } 1 \leqslant r \leqslant k-1 \, . \tag{14}$$

Remark. The relation (14) means that we can recover all the Jacobi-Ostrogradsky momenta coordinates from the set of highest order momenta $(\hat{p}_i^{(k-1)})$. ◊

Bearing in mind the local expression of the form θ_L, we can write $\theta_L = \hat{p}_i^{(j)} dq_{(j)}^i$. Let $(U;(q^i))$, $1 \leqslant i \leqslant n$, be a local chart of Q, and $\left(q_{(j)}^i\right)$, $0 \leqslant j \leqslant 2k-1$, the induced local coordinates in $(\tau_Q^{2k-1})^{-1}(U) \subset T^{(2k-1)}Q$ introduced in Section 2.4. From this, it is clear that the local expression of the Legendre-Ostrogradsky map leg_L is

$$\operatorname{leg}_L^*\left(q_{(r-1)}^i\right) = q_{(r-1)}^i \quad ; \quad \operatorname{leg}_L^*\left(p_i^{(r-1)}\right) = \hat{p}_i^{(r-1)} = \sum_{j=0}^{k-r} (-1)^j d_T^j \left(\frac{\partial L}{\partial q_{(r+j)}^i} \right) ,$$

where $1 \leqslant r \leqslant k$, that is,

$$\operatorname{leg}_L\left(q_{(0)}^i, \ldots, q_{(2k-1)}^i\right) = \left(q_{(0)}^i, \ldots, q_{(k-1)}^i, \hat{p}_i^{(0)}, \ldots, \hat{p}_i^{(k-1)}\right) .$$

Consider the tangent map $T\operatorname{leg}_L \colon T(T^{(2k-1)}Q) \to T(T^*T^{(k-1)}Q)$. Given an arbitrary point $[\gamma]_0^{(2k-1)} \in T^{(2k-1)}Q$, the tangent map of leg_L at $[\gamma]_0^{(2k-1)}$ is locally given by the following $2kn \times 2kn$ matrix

$$T_{[\gamma]_0^{(2k-1)}} \operatorname{leg}_L = \left(\begin{array}{cccc|cccc}
\mathrm{Id}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n \\
\mathbf{0}_n & \mathrm{Id}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0}_n & \mathbf{0}_n & \ldots & \mathrm{Id}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n \\
\hline
\dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(0)}^j} & \dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(k-1)}^j} & \dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(k)}^j} & \dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(k+1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(0)}}{\partial q_{(2k-1)}^j} \\
\dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(0)}^j} & \dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(k-1)}^j} & \dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(k)}^j} & \dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(k+1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(1)}}{\partial q_{(2k-1)}^j} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(0)}^j} & \dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(k-1)}^j} & \dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(k)}^j} & \dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(k+1)}^j} & \ldots & \dfrac{\partial \hat{p}_i^{(k-1)}}{\partial q_{(2k-1)}^j}
\end{array}\right) ,$$

where Id_n denotes the $n \times n$ identity matrix, $\mathbf{0}_n$ the $n \times n$ null matrix and $(\partial \hat{p}_i^{(r)} / \partial q_{(s)}^j)$ the $n \times n$ Jacobian matrix of the vector function $(\hat{p}_1^{(r)}, \ldots, \hat{p}_n^{(r)})$ with respect to the n variables

$(q^1_{(s)}, \ldots, q^n_{(s)})$. Moreover, using the relation (14) among the momenta function in combination with the coordinate expression of the Tulczyjew's derivation, a long but straightforward computation shows that

$$\frac{\partial \hat{p}_i^{(s)}}{\partial q^j_{(2k-1-s)}} = \frac{\partial^2 L}{\partial q^i_{(k)} \partial q^j_{(k)}},$$

from where we deduce that the antidiagonal $n \times n$ blocks in the lower right submatrix of $T_{[\gamma]_0^{(2k-1)}} \operatorname{leg}_L$ coincide with the Hessian matrix of the Lagrangian function. Therefore, it is clear that the Lagrangian function $L \in C^\infty(T^{(k)}Q)$ is regular if, and only if, the bundle morphism $\operatorname{leg}_L \colon T^{(2k-1)}Q \to T^*T^{(k-1)}Q$ is a local diffeomorphism. As a consequence of this, we have that if L is a kth-order regular Lagrangian then the set $(q^A_{(i)}, \hat{p}_A^{(i)})$, $0 \leqslant i \leqslant k-1$, is a set of local coordinates in $T^{(2k-1)}Q$, and $(\hat{p}_A^{(i)})$ are called the *Jacobi-Ostrogradsky momenta coordinates*.

Now, we introduce a Legendre transformation $\mathbb{F}L \colon \Sigma_L \to T^*(T^{(k-1)}Q)$ in the Lagrangian submanifold generated by a higher-order Lagrangian function.

Definition 10. *The higher-order Legendre transformation on the Lagrangian submanifold Σ_L, $\mathbb{F}L \colon \Sigma_L \to T^*T^{(k-1)}Q$, is the map defined by $\mathbb{F}L = \tau_{T^*(T^{(k-1)}Q)} \circ (\alpha_{T^{k-1}Q})^{-1}\big|_{\Sigma_L}$.*

In the natural coordinates $\left(q^i_{(0)}, \ldots, q^i_{(k)}, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}\right)$ of Σ_L introduced in Subsection 3.1, the map $\mathbb{F}L$ is locally determined by

$$\mathbb{F}L(q^i_{(0)}, \ldots, q^i_{(k)}, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}) = \left(q^i_{(0)}, \ldots, q^i_{(k-1)}, \tilde{p}_i^{(0)}, \ldots, \tilde{p}_i^{(k-2)}, \frac{\partial L}{\partial q^i_{(k)}}\right).$$

Definition 11. *A higher-order Lagrangian system determined by $L : T^{(k)}Q \to \mathbb{R}$ is regular if, and only if, $\mathbb{F}L$ is a local diffeomorphism.*

Remark. Observe that a higher-order Lagrangian system is regular if and only if $\left(\dfrac{\partial^2 L}{\partial q^i_{(k)} \partial q^j_{(k)}}\right)$ is a nondegenerate matrix. In such a case, since $\tilde{p}_i^{(k-1)} = \dfrac{\partial L}{\partial q^i_{(k)}}$, by the implicit function theorem, we can define the n coordinate functions $q^j_{(k)}$ as functions depending on $q^i_{(0)}, \ldots, q^i_{(k-1)}, \tilde{p}_i^{(k-1)}$; that is,

$$\tilde{q}^j_{(k)} = f(q^i_{(0)}, \ldots, q^i_{(k-1)}, \tilde{p}_i^{(k-1)}). \tag{15}$$

◊

Using the higher-order Legendre transformation we can give in an alternative way the solutions of the higher-order Lagrangian system as follows.

Proposition 4. *The solutions of a kth-order Lagrangian system described by a kth-order Lagrangian function $L \in C^\infty(T^{(k)}Q)$ are the curves $\mu \colon I \subset \mathbb{R} \to \Sigma_L$ satisfying*

$$\alpha^{-1}_{T^{(k-1)}Q}(\mu(t)) = \frac{d}{dt}\mathbb{F}L(\mu(t)),$$

where μ satisfies $\pi_{T^*TT^{k-1}Q}\big|_{\Sigma_L}(\mu(t)) = \gamma^{(k)}(t)$, and where $\gamma^{(k)}$ is the k-lift of a curve $\gamma \colon I \to Q$.

Next, we give an alternative characterization of the dynamics in the Lagrangian submanifold Σ_L in terms of the solution of the higher-order Euler-Lagrange equations.

Proposition 5. *A curve $\gamma\colon I \to Q$ is a solution of the higher-order Euler-Lagrange equations derived from a kth-order Lagrangian function $L \in C^\infty(T^{(k)}Q)$ if, and only if,*

$$\alpha_{T^{(k-1)}Q}\left(\frac{d}{dt}\left(\mathrm{leg}_L \circ \gamma^{(2k-1)}\right)\right) \in \Sigma_L \,,$$

where $\gamma^{(2k-1)}\colon I \to T^{(2k-1)}Q$ is the $(2k-1)$-lift of γ and $\mathrm{leg}_L\colon T^{(2k-1)}Q \to T^*(T^{(k-1)}Q)$ is the Legendre-Ostrogradsky map defined in (12)*.*

Proof. This proof is easy in coordinates. Let $(U;(q^i))$ be a local chart in Q, and let us denote $\gamma(t) = (q^i(t))$ in U. Then, the $(2k-1)$-lift of γ is given by $\gamma^{(2k-1)}(t) = (q^i_{(0)}(t), \ldots, q^i_{(2k-1)}(t))$, and we have

$$\left(\mathrm{leg}_L \circ \gamma^{(2k-1)}\right)(t) = \left(q^i_{(0)}(t), \ldots, q^i_{(k-1)}(t), \hat{p}_i^{(0)}(t), \ldots, \hat{p}_i^{(k-1)}(t)\right) \,,$$

where $\hat{p}^i_{(r)}$, $1 \leqslant i \leqslant n$ and $0 \leqslant r \leqslant k-1$, are the Jacobi-Ostrogradsky momenta coordinates defined in (13). Then, bearing in mind the coordinate expression (3) of $\alpha_{T^{k-1}Q}$ we have that

$$\alpha_{T^{(k-1)}Q}\left(\frac{d}{dt}\left(\mathrm{leg}_L \circ \gamma^{(2k-1)}\right)\right) = \left(q^i_{(j)}(t); q^i_{(j+1)}(t); \frac{d}{dt}\hat{p}_i^{(j)}(t); \hat{p}_i^{(j)}(t)\right) \,,$$

with $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$. Requiring $\alpha_{T^{(k-1)}Q}\left(\frac{d}{dt}\left(\mathrm{leg}_L \circ \gamma^{(2k-1)}\right)\right) \in \Sigma_L$, we obtain the following system of $(k+1)n$ differential equations on the component functions of $\gamma^{(2k-1)}$

$$\frac{d}{dt}\hat{p}_i^{(0)} = \frac{\partial L}{\partial q^i_{(0)}}, \quad \frac{d}{dt}\hat{p}_i^{(j)} + \hat{p}_i^{(j-1)} = \frac{\partial L}{\partial q^i_{(j)}}, \quad \hat{p}_i^{(k-1)} = \frac{\partial L}{\partial q^i_{(k)}}$$

with $1 \leqslant j \leqslant k-1$ in the second set of equations (which, observe, is exactly the relation (14) among the momenta). Combining these equations following the same patterns as in the end of Subsection 3.1, we obtain the higher-order Euler-Lagrange equations

$$\sum_{j=0}^{k} (-1)^j \frac{d^j}{dt^j}\left(\frac{\partial L}{\partial q^i_{(j)}}\right) = 0 \,. \qquad \square$$

From Propositions 4 and 5 it is clear that we can characterize the solutions of the Euler-Lagrange equations in purely geometric way by means of the either the higher-order Legendre map $\mathbb{F}L$ or the Legendre-Ostrogradsky map leg_L. Now, assume that the Lagrangian function $L \in C^\infty(T^{(k)}Q)$ is regular, so both maps $\mathbb{F}L\colon \Sigma_L \to T^*T^{(k-1)}Q$ and $\mathrm{leg}_L\colon T^{(2k-1)}Q \to T^*T^{(k-1)}Q$ are local diffeomorphisms. Then consider the map $\phi_L\colon T^{(2k-1)}Q \to \Sigma_L$ defined by the composition $\phi_L = \mathbb{F}L^{-1} \circ \mathrm{leg}_L$, which is locally given by

$$\phi_L\left(q^i_{(0)}, \ldots, q^i_{(2k-1)}\right) = \left(q^i_{(0)}, \ldots, q^i_{(k-1)}; \hat{p}_i^{(0)}, \ldots, \hat{p}_i^{(k-2)}, \frac{\partial L}{\partial q^i_{(k)}}\right) .$$

This map is a local diffeomorphism with local inverse $\phi_L^{-1} = \mathrm{leg}_L^{-1} \circ \mathbb{F}L$ in the corresponding open sets. If, moreover, L is hyperegular, then ϕ_L is a global diffeomorphism, from which we can recover the dynamics using the implicit function theorem and the inverse of the Tulczyjew's isomorphism. In addition, in this case we can establish the Hamiltonian formalism in $T^*T^{(k-1)}Q$ by defining a *canonical Hamiltonian function* $H \in C^\infty(T^*T^{(k-1)}Q)$ with coordinate expression

$$H(q^i_{(0)}, \ldots, q^i_{(k-1)}, p_i^{(0)}, \ldots, p_i^{(k-1)}) = \sum_{j=0}^{k-2} p_i^{(j)} q^i_{(j+1)} + p_i^{(k-1)} \tilde{q}^i_{(k)} - L(q^i_{(0)}, \ldots, q^i_{(k-1)}, \tilde{q}^i_{(k)}) \,,$$

where $\tilde{q}^i_{(k)}$ is given implicitly by (15). The canonical Hamiltonian function obtained in this way does not depend on the choice of the Legendre transformation used to derive it, and it depends only on the starting Lagrangian function. The corresponding Hamiltonian vector field X_H is determined by $i_{X_H}\omega_{T^{(k-1)}Q} = dH$. In this case we have that

$$\mathrm{Im}(X_H) = X_H(T^*T^{(k-1)}Q) = \beta^{-1}_{T^{k-1}Q}(dH(T^*T^{(k-1)}Q)) = \alpha^{-1}_{T^{(k-1)}Q}(\Sigma_L)\,.$$

In the singular case, the submanifold $\mathrm{Im}(dH)$ is not transversal with respect to $\pi_{T^*T^{(k-1)}Q}$, and therefore it is necessary to apply an integrability algorithm to find, if it exists, a subset where there are consistent solutions of the dynamics (see [26] and [27], for example).

3.3 An example: the dynamics of the end of a javelin

Let us consider the dynamical system that describes the motion of the end of a thrown javelin. This gives rise to a 3-dimensional second-order dynamical system, which is a particular case of the problem of determining the trajectory of a particle rotating about a translating center [14]. Let $Q = \mathbb{R}^3$ be the manifold modeling the configuration space for this system with coordinates $(q^1_{(0)}, q^2_{(0)}, q^3_{(0)}) = (q^i_{(0)})$. Using the induced coordinates in $T^{(2)}\mathbb{R}^3$, the Lagrangian function for this system is

$$L(q^i_{(0)}, q^i_{(1)}, q^i_{(2)}) = \frac{1}{2}\sum_{i=1}^{3}\left(\left(q^i_{(1)}\right)^2 - \left(q^i_{(2)}\right)^2\right)\,,$$

which is a regular Lagrangian function since the Hessian matrix of L with respect to the second-order velocities is

$$\left(\frac{\partial^2 L}{\partial q^j_{(2)}\partial q^i_{(2)}}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,.$$

The second-order Euler-Lagrange equations are

$$\frac{d^4}{dt^4}q^i_{(0)} + \frac{d^2}{dt^2}q^i_{(0)} = 0\,,$$

for $1 \leqslant i \leqslant 3$. The general solution for the second-order Euler-Lagrange equations is a curve $\gamma\colon I \subset \mathbb{R} \to \mathbb{R}^3$ with component functions $\gamma(t) = (\gamma^1(t), \gamma^2(t), \gamma^3(t))$ given by

$$\gamma^i(t) = c^i_1 + c^i_2 t + c^i_3\sin(t) + c^i_4\cos(t)\,,$$

with c^i_j constants, $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 4$.

The Jacobi-Ostrogradsky momenta are given by

$$\hat{p}^{(0)}_i = \frac{\partial L}{\partial q^i_{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q^i_{(2)}}\right) = q^i_{(1)} + q^i_{(3)}\,, \quad \hat{p}^{(1)}_i = \frac{\partial L}{\partial q^i_{(2)}} = -q^i_{(2)}\,.$$

Therefore, the Legendre-Ostrogradsky map is given by

$$\mathrm{leg}_L\left(q^i_{(0)}, q^i_{(1)}, q^i_{(2)}, q^i_{(3)}\right) = \left(q^i_{(0)}, q^i_{(1)}, \hat{p}^{(0)}_i, \hat{p}^{(1)}_i\right) = \left(q^i_{(0)}, q^i_{(1)}, q^i_{(1)} + q^i_{(3)}, -q^i_{(2)}\right)\,.$$

Consider a Lagrangian submanifold $\Sigma_L \hookrightarrow T^*TTQ$ associated with the Lagrangian function L. Local coordinates in the Lagrangian submanifold are $(q^i_{(0)}, q^i_{(1)}, q^i_{(2)}, p^{(0)}_i)$. The Legendre transformation on Σ_L is given locally by

$$\mathbb{F}L\left(q^i_{(0)}, q^i_{(1)}, q^i_{(2)}, p^{(0)}_i\right) = \left(q^i_{(0)}, q^i_{(1)}, p^{(0)}_i, \frac{\partial L}{\partial q^i_{(2)}}\right) = \left(q^i_{(0)}, q^i_{(1)}, p^{(0)}_i, -q^i_{(2)}\right)\,.$$

$\mathbb{F}L$ is a diffeomorphism, and thus the second-order system is regular. Therefore, we can define the second-order derivative in terms of the momenta as $q^i_{(2)} = -p^{(1)}_i$ and define a Hamiltonian function on T^*TQ, which is given in coordinates by

$$H\left(q^i_{(0)}, q^i_{(1)}; p^{(0)}_i, p^{(1)}_i\right) = p^{(0)}_i q^i_{(1)} - \frac{1}{2}\left(\left(q^i_{(1)}\right)^2 + \left(p^{(1)}_i\right)^2\right).$$

Hamilton's equations for this second-order dynamical system are

$$\frac{d}{dt}p^{(0)}_i = 0\,, \quad \frac{d}{dt}p^{(1)}_i = -p^{(0)}_i + q^i_{(1)}\,, \quad \frac{d}{dt}q^i_{(0)} = q^i_{(1)}\,, \quad \frac{d}{dt}q^i_{(1)} = -p^{(1)}_i\,.$$

From the equations given above, in the regular case, one can obtain in a straightforward way the second-order Euler-Lagrange equations.

4 On the definition and the regularity of a higher-order Hamiltonian function

In this Section we aim at studying the Hamiltonian functions that describe the dynamics of higher-order systems. More particularly, we want to define the notions of "higher-order Hamiltonian function" and "regularity" of these functions in a precise way.

Henceforth, we consider a kth-order dynamical system with n degrees of freedom, and let Q be a n-dimensional smooth manifold modeling the configuration space of this system. From the results in [18] and in Section 3 we know that the Hamiltonian phase space for this system is the cotangent bundle $T^*T^{(k-1)}Q$. Hence, let $H \in C^\infty(T^*T^{(k-1)}Q)$ be a Hamiltonian function describing the dynamics of the system.

4.1 Statement of the problems

Following the patterns in Section 2.2, let $\mathcal{F}H\colon T^*T^{(k-1)}Q \to TT^{(k-1)}Q$ be the fiber derivative of H. If $(U;(q^i))$, $1 \leqslant i \leqslant n$, is a local chart in Q and $(q^i_{(j)}, p^{(j)}_i)$, $0 \leqslant j \leqslant k-1$, are the induced local coordinates in a suitable open subset of $T^*T^{(k-1)}Q$, then the map $\mathcal{F}H$ is given locally by

$$\mathcal{F}H\left(q^i_{(0)}, \ldots, q^i_{(k-1)}; p^{(0)}_i, \ldots, p^{(k-1)}_i\right) = \left(q^i_{(0)}, \ldots, q^i_{(k-1)}; \frac{\partial H}{\partial p^{(0)}_i}, \ldots, \frac{\partial H}{\partial p^{(k-1)}_i}\right).$$

Moreover, the regularity condition for the Hamiltonian function H is, locally, equivalent to

$$\det\left(\frac{\partial^2 H}{\partial p^{(j)}_i \partial p^{(j')}_{i'}}\right)(\alpha_q) \neq 0\,, \text{ for every } \alpha_q \in T^*T^{(k-1)}Q\,.$$

Proposition 6. *Let $H \in C^\infty(T^*T^{(k-1)}Q)$ be a hyperregular Hamiltonian function, $\theta_{T^{(k-1)}Q} \in \Omega^1(T^*T^{(k-1)}Q)$ is the Liouville 1-form and $X_H \in \mathfrak{X}(T^*T^{(k-1)}Q)$ is the unique vector field solution to the dynamical equation*

$$i_{X_H}\omega_{T^{(k-1)}Q} = dH\,,$$

where $\omega_{T^{(k-1)}Q} = -d\theta_{T^{(k-1)}Q} \in \Omega^2(T^*T^{(k-1)}Q)$ is the canonical symplectic form. Then there exists a global diffeomorphism $L \in C^\infty(TT^{(k-1)}Q)$, locally determined by

$$L\left(q^i_{(j)}; v^i_{(j)}\right) = \tilde{p}^{(j)}_i \dot{q}^i_{(j)} - H\left(q^i_{(j)}; \tilde{p}^{(j)}_i\right),$$

where $\tilde{p}^{(j)}_i = (\mathcal{F}H^{-1})^* p^{(j)}_i \in TT^{(k-1)}Q$.

Proof. Using that the Hamiltonian function H is hyperregular and using Proposition 1 with $T^{(k-1)}Q$ as the base manifold, we define a hyperregular Lagrangian function (a global diffeomorphism) $L \in C^\infty(TT^{(k-1)}Q)$ as follows

$$L = \theta_{T^{(k-1)}Q}(X_H) \circ \mathcal{F}H^{-1} - H \circ \mathcal{F}H^{-1} \,,$$

where $\theta_{T^{(k-1)}Q} \in \Omega^1(T^*T^{(k-1)}Q)$ is the Liouville 1-form and $X_H \in \mathfrak{X}(T^*T^{(k-1)}Q)$ is the unique vector field solution to the dynamical equation

$$i_{X_H}\,\omega_{T^{(k-1)}Q} = dH \,,$$

where $\omega_{T^{(k-1)}Q} = -d\theta_{T^{(k-1)}Q} \in \Omega^2(T^*T^{(k-1)}Q)$ is the canonical symplectic form. In the induced local coordinates of $T^*T^{(k-1)}Q$, the Liouville 1-form is locally given by $\theta_{T^{(k-1)}Q} = p_i^{(j)} dq_{(j)}^i$. From where we deduce that the vector field $X_H \in \mathfrak{X}(T^*T^{(k-1)}Q)$ solution to the previous equation is locally given by

$$X_H = \frac{\partial H}{\partial p_i^{(j)}}\frac{\partial}{\partial q_{(j)}^i} - \frac{\partial H}{\partial q_{(j)}^i}\frac{\partial}{\partial p_i^{(j)}} \,. \tag{16}$$

Then, the function $\mathcal{F}H^*L = \theta_{T^{(k-1)}Q}(X_H) - H \in C^\infty(T^*T^{(k-1)}Q)$ is given in coordinates by

$$\mathcal{F}H^*L\left(q_{(j)}^i; p_i^{(j)}\right) = p_i^{(j)}\frac{\partial H}{\partial p_i^{(j)}} - H\left(q_{(j)}^i; p_i^{(j)}\right) \,.$$

From this, the coordinate expression of the Lagrangian function L in the induced natural coordinates $\left(q_{(j)}^i, v_{(j)}^i\right)$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k-1$, of $TT^{(k-1)}Q$ is

$$L\left(q_{(j)}^i; v_{(j)}^i\right) = \tilde{p}_i^{(j)} \dot{q}_{(j)}^i - H\left(q_{(j)}^i; \tilde{p}_i^{(j)}\right) \,,$$

where $\tilde{p}_i^{(j)} = (\mathcal{F}H^{-1})^* p_i^{(j)}$ are local functions in $TT^{(k-1)}Q$. □

It is important to point out that the Lagrangian function obtained may not be a kth-order Lagrangian function, in the physical sense: the coordinate functions in the basis $T^{(k-1)}Q$ may not be well-related to the coordinate functions in the fibers, in the sense that, in general, we have $q_{(j+1)}^i \neq \dot{q}_{(j)}^i$, that is, they are independent variables. From the geometric point of view, the problem is that L is not defined in the "holonomic" submanifold $j_k \colon T^{(k)}Q \hookrightarrow TT^{(k-1)}Q$. In fact, observe that, up to this point, the order of the system has not been taken into account at any step, since this condition is usually inherited from the Lagrangian formulation. That is, the function $H \in C^\infty(T^*T^{(k-1)}Q)$ is not considered as a Hamiltonian function for a kth-order dynamical system, but just as a Hamiltonian function for a first-order system defined on the cotangent bundle of a larger manifold, since we do not require any relation among the momenta. This issue gives rise to the first problem that we want to solve for Hamiltonian functions defined on $T^*T^{(k-1)}Q$.

Problem 1. *Given a function $H \in C^\infty(T^*T^{(k-1)}Q)$, find conditions on H such that it describes the dynamics of a kth-order dynamical system, that is, find a suitable definition of* kth-order Hamiltonian functions.

Remark. Observe that the equivalent problem in the Lagrangian formalism (find conditions on a function $L \in C^\infty(TT^{(k-1)}Q)$ such that L is a kth-order Lagrangian function) is solved straightforwardly, since there exists a distinguished "holonomic" submanifold $j_k \colon T^{(k)}Q \hookrightarrow TT^{(k-1)}Q$. Nevertheless, there is not such a submanifold in $T^*T^{(k-1)}Q$, and hence the problem is not trivial. ◊

Notice that a sufficient condition to ensure that L is indeed a kth-order Lagrangian function in the usual sense, that is, $L \in C^\infty(T^{(k)}Q)$ (as a submanifold of $TT^{(k-1)}Q$), is to require $\mathrm{Im}(\mathcal{F}H) \subseteq j_k(T^{(k)}Q)$. However, since $\dim T^{(k)}Q = (k+1)n < 2kn = \dim T^*T^{(k-1)}Q$ for $k > 1$, this requirement on the fiber derivative of H prevents the Hamiltonian function to be regular in the sense of Definition 5. Nevertheless, recall that the regularity condition for a kth-order Lagrangian function $L \in C^\infty(T^{(k)}Q)$ is locally equivalent to

$$\det\left(\frac{\partial^2 L}{\partial q^i_{(k)} \partial q^j_{(k)}}\right)\left([\gamma]^{(k)}_0\right) \neq 0\,, \text{ for every } [\gamma]^{(k)}_0 \in T^{(k)}Q\,,$$

with $1 \leqslant i,j \leqslant n$, instead of

$$\det\left(\frac{\partial^2 L}{\partial q^i_{(r)} \partial q^j_{(s)}}\right)\left([\gamma]^{(k)}_0\right) \neq 0\,, \text{ for every } [\gamma]^{(k)}_0 \in T^{(k)}Q\,,$$

with $1 \leqslant i,j \leqslant n$ and $1 \leqslant r,s \leqslant k$. That is, the Hessian of L is taken only with respect to the highest-order "velocities" $q^i_{(k)}$, and not with respect to all the "velocities". Therefore, we deduce that the regularity condition given in Definition 5 is not suitable for higher-order Hamiltonian functions, since too many "orders" of the momenta coordinates are taken into account. This gives rise to the second problem that we want to solve.

Problem 2. *To find a suitable definition of* regularity *for kth-order Hamiltonian functions $H \in C^\infty(T^*T^{(k-1)}Q)$ in terms of the fiber derivative of H such that Propositions 1 and 2 hold for kth-order dynamical systems.*

4.2 A particular case: the Hamiltonian function associated to a (hyper)regular Lagrangian system

In order to solve Problems 1 and 2 stated in the previous Subsection, we first consider the particular case of a well-known Hamiltonian that describes properly the dynamics of a higher-order system: the Hamiltonian function associated to a higher-order Lagrangian system.

Proposition 7. *Given a hyperregular kth-order Lagrangian function, $L \in C^\infty(T^{(k)}Q)$, there exists a unique kth-order Hamiltonian function associated to this Lagrangian given locally by*

$$H\left(q^i_{(0)},\ldots,q^i_{(k-1)};p^{(0)}_i,\ldots,p^{(k-1)}_i\right) = \sum_{j=0}^{k-2} q^i_{(j+1)}p^{(j)}_i + \tilde{q}^i_{(k)}p^{(k-1)}_i - L\left(q^i_{(0)},\ldots,q^i_{(k-1)},\tilde{q}^i_{(k)}\right)\,,$$

where $\tilde{q}^i_{(k)} = (\mathrm{leg}_L^{-1})^* q^i_{(k)}$. Moreover, $\mathrm{Im}(\mathcal{F}H) \subset j_k(T^{(k)}Q)$.

Proof. Let $\mathrm{leg}_L \colon T^{(2k-1)}Q \to T^*T^{(k-1)}Q$ be the Legendre-Ostrogradsky map defined in (12). From the Lagrangian function L we construct the Lagrangian energy $E_L \in C^\infty(T^{(2k-1)}Q)$, with coordinate expression

$$E_L\left(q^i_{(0)},\ldots,q^i_{(2k-1)}\right) = \sum_{r=1}^{k} q^i_{(r)} \sum_{j=0}^{k-r}(-1)^j d_T^j\left(\frac{\partial L}{\partial q^i_{(r+j)}}\right) - L\left(q^i_{(0)},\ldots,q^i_{(k)}\right)\,. \tag{17}$$

Then, since L is hyperregular, the Legendre-Ostrogradsky map is a diffeomorphism, and thus there exists a unique kth-order Hamiltonian function associated to this Lagrangian system defined by $H = E_L \circ \mathrm{leg}_L^{-1} \in C^\infty(T^*T^{(k-1)}Q)$. This Hamiltonian function is given locally by

$$H\left(q^i_{(0)},\ldots,q^i_{(k-1)};p^{(0)}_i,\ldots,p^{(k-1)}_i\right) = \sum_{j=0}^{k-2} q^i_{(j+1)}p^{(j)}_i + \tilde{q}^i_{(k)}p^{(k-1)}_i - L\left(q^i_{(0)},\ldots,q^i_{(k-1)},\tilde{q}^i_{(k)}\right)\,,$$

where $\tilde{q}^i_{(k)} = (\text{leg}_L^{-1})^* q^i_{(k)}$ are local functions in $T^*T^{(k-1)}Q$. Observe that the derivative of H with respect to a momenta coordinate $p_r^{(j)}$ gives

$$\frac{\partial H}{\partial p_r^{(j)}} = \begin{cases} q^r_{(j+1)} & \text{if } 0 \leqslant j \leqslant k-2\,, \\[2ex] \tilde{q}^r_{(k)} + p_i^{(k-1)} \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_r^{(k-1)}} - \dfrac{\partial L}{\partial q^i_{(k)}} \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_r^{(k-1)}} = \tilde{q}^r_{(k)} & \text{if } j = k-1\,, \end{cases}$$

where, since $T^*T^{(k-1)}Q = \text{Im}(\text{leg}_L)$, we have $p_i^{(k-1)} = \dfrac{\partial L}{\partial q^i_{(k)}}$, and therefore the last two terms in the above sums cancel each other. From this we deduce that the coordinate expression of the fiber derivative of H in this particular case is

$$\mathcal{F}H\left(q^i_{(0)}, \ldots, q^i_{(k-1)}; p_i^{(0)}, \ldots, p_i^{(k-1)}\right) = \left(q^i_{(0)}, \ldots, q^i_{(k-1)}; q^i_{(1)}, \ldots, q^i_{(k-1)}, \tilde{q}^i_{(k)}\right)\,. \tag{18}$$

It is clear from this coordinate expression that $\text{Im}(\mathcal{F}H) \subseteq T^{(k)}Q \stackrel{j_k}{\hookrightarrow} TT^{(k-1)}Q$. □

Let $\mathcal{F}H_o \colon T^*T^{(k-1)}Q \to T^{(k)}Q$ be the map defined by $\mathcal{F}H = \mathcal{F}H_o \circ j_k$, that is, the unique map such that the following diagram commutes

$$\begin{array}{ccc} T^*T^{(k-1)}Q & \xrightarrow{\ \mathcal{F}H\ } & TT^{(k-1)}Q \\ & \searrow^{\mathcal{F}H_o} & \uparrow j_k \\ & & T^{(k)}Q \end{array}$$

with coordinate expression

$$\mathcal{F}H_o\left(q^i_{(0)}, \ldots, q^i_{(k-1)}; p_i^{(0)}, \ldots, p_i^{(k-1)}\right) = \left(q^i_{(0)}, \ldots, q^i_{(k-1)}, \tilde{q}^i_{(k)}\right)\,.$$

Now, let us compute the coordinate expression of the tangent map of $\mathcal{F}H$ in an arbitrary point $\alpha_q \in T^*T^{(k-1)}Q$. Bearing in mind the coordinate expression (18) of $\mathcal{F}H$, the map $T_{\alpha_q}\mathcal{F}H$ is given in coordinates by the following $2kn \times 2kn$ real matrix

$$T_{\alpha_q}\mathcal{F}H = \left(\begin{array}{cccc|cccc} \text{Id}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \text{Id}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_n & \mathbf{0}_n & \ldots & \text{Id}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n \\ \hline \mathbf{0}_n & \text{Id}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_n & \mathbf{0}_n & \ldots & \text{Id}_n & \mathbf{0}_n & \mathbf{0}_n & \ldots & \mathbf{0}_n & \mathbf{0}_n \\ \dfrac{\partial \tilde{q}^i_{(k)}}{\partial q^j_{(0)}} & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial q^j_{(1)}} & \ldots & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial q^j_{(k-1)}} & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_j^{(0)}} & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_j^{(1)}} & \ldots & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_j^{(k-2)}} & \dfrac{\partial \tilde{q}^i_{(k)}}{\partial p_j^{(k-1)}} \end{array}\right),$$

where every entry is a $n \times n$ real matrix, and every block of the matrix has size $kn \times kn$. In particular, Id_n denotes the $n \times n$ identity matrix, $\mathbf{0}_n$ the $n \times n$ null matrix and, in the last row, we have $1 \leqslant i, j \leqslant n$. Observe that, in the most favorable case, the map $\mathcal{F}H \colon T^*T^{(k-1)}Q \to TT^{(k-1)}Q$ has rank $(k+1)n = \dim T^{(k)}Q$, that is, at the best the map $\mathcal{F}H_o \colon T^*T^{(k-1)}Q \to T^{(k)}Q$ is a submersion onto $T^{(k)}Q$. A long but straightforward calculation shows that

$$\left(\frac{\partial^2 H}{\partial p_i^{(k-1)} \partial p_j^{(k-1)}}\right) = \left(\frac{\partial \tilde{q}^i_{(k)}}{\partial p_j^{(k-1)}}\right) = \left(\frac{\partial \left(q^i_{(k)} \circ \text{leg}_L^{-1}\right)}{\partial p_j^{(k-1)}}\right) = \left(\frac{\partial^2 L}{\partial q^i_{(k)} \partial q^j_{(k)}}\right)^{-1},$$

which is a well-defined $n \times n$ matrix because L is (hyper)regular, and therefore the map $\mathcal{F}H_o$ is a submersion onto $T^{(k)}Q$. Observe that, as a consequence, $\mathcal{F}H_o$ admits local sections, that is, maps $\sigma\colon T^{(k)}Q \to T^*T^{(k-1)}Q$ satisfying $\mathcal{F}H_o \circ \sigma = \mathrm{Id}_{T^{(k)}Q}$.

The following result gives the illuminating key idea for the "true" definition of higher-order Hamiltonian function and regularity of a higher-order Hamiltonian function that we will exploit in the next Subsection.

Theorem 2. *Let $L : T^{(k)}Q \to \mathbb{R}$ be a kth-order Lagrangian function*

(i) If L is regular, there exists an open subset $U \subset T^{(2k-1)}Q$ such that

$$\mathcal{F}H_o|_U = \left(\tau_Q^{(k,2k-1)} \circ \mathrm{leg}_L^{-1}\right)\Big|_U .$$

(ii) If L is hyperregular then $\mathcal{F}H_o : T^*T^{(k-1)}Q \to T^{(k)}Q$ is a submersion onto $T^{(k)}Q$.

(iii) If L is hyperregular then $\mathcal{F}H_o$ admits a global section $\Upsilon : T^{(k)}Q \to T^*T^{(k-1)}Q$.

Proof. Let us consider the map $\mathcal{F}H_o \circ \mathrm{leg}_L\colon T^{(2k-1)}Q \to T^{(k)}Q$. In the natural coordinates $\left(q_{(j)}^i\right)$ of $T^{(2k-1)}Q$ ($1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k$), the local expression of this map is given by

$$\begin{aligned}(\mathcal{F}H_o \circ \mathrm{leg}_L)\left(q_{(0)}^i, \ldots, q_{(2k-1)}^i\right) &= \mathcal{F}H_o\left(q_{(0)}^i, \ldots, q_{(k-1)}^i; \hat{p}_i^{(0)}, \ldots, \hat{p}_i^{(k-1)}\right) \\ &= \left(q_{(0)}^i, \ldots, q_{(k-1)}^i, q_{(k)}^i\right) ,\end{aligned}$$

since $\tilde{q}_{(k)}^i = (\mathrm{leg}_L^{-1})^* q_{(k)}^i$. From this coordinate expression we deduce that for every point $[\gamma]_0^{(2k-1)} \in T^{(2k-1)}Q$ there exists an open subset $U \subseteq T^{(2k-1)}Q$ such that $[\gamma]_0^{(2k-1)} \in U$ and

$$(\mathcal{F}H_o \circ \mathrm{leg}_L)|_U = \tau_Q^{(k,2k-1)}\Big|_U ,$$

and, since $L \in C^\infty(T^{(k)}Q)$ is a regular Lagrangian function, the Legendre-Ostrogradsky map is a local diffeomorphism, we have

$$\mathcal{F}H_o|_U = \left(\tau_Q^{(k,2k-1)} \circ \mathrm{leg}_L^{-1}\right)\Big|_U .$$

Observe that, moreover, we assume that L is a hyperregular Lagrangian function. Therefore, the map $\mathrm{leg}_L^{-1}\colon T^*T^{(k-1)}Q \to T^{(2k-1)}Q$ is bijective and defined in the entire manifold $T^*T^{(k-1)}Q$. On the other hand, the map $\tau_Q^{(k,2k-1)}$ is surjective, from where we deduce that $\mathcal{F}H_o\colon T^*T^{(k-1)}Q \to T^{(k)}Q$ is surjective in this case, since we have

$$\mathcal{F}H_o = \tau_Q^{(k,2k-1)} \circ \mathrm{leg}_L^{-1} ,$$

that is, $\mathrm{Im}(\mathcal{F}H_o) = T^{(k)}Q$.

Observe that, in addition, there exists a map $\Upsilon\colon T^{(k)}Q \to T^*T^{(k-1)}Q$ defined by $\Upsilon = \mathrm{leg}_L \circ \Psi$, with $\Psi \in \Gamma\left(\tau_Q^{(k,2k-1)}\right)$ being a global section of $\tau_Q^{(k,2k-1)}$, which satisfies

$$\mathcal{F}H_o \circ \Upsilon = \tau_Q^{(k,2k-1)} \circ \mathrm{leg}_L^{-1} \circ \mathrm{leg}_L \circ \Psi = \tau_Q^{(k,2k-1)} \circ \Psi = \mathrm{Id}_{T^{(k)}Q} ,$$

that is, Υ is a global section of $\mathcal{F}H_o$. ∎

4.3 Higher-order Hamiltonian functions: definition and regularity

Bearing in mind the results obtained in the previous Section, we are now able to give a solution to Problems 1 and 2. From the last theorem the statement of Problem 1 and the calculations for the particular case of a Hamiltonian system associated to a hyperregular Lagrangian system, we can give the following definition.

Definition 12. *A function $H \in C^\infty(T^*T^{(k-1)}Q)$ is a* kth-order Hamiltonian function *if its fiber derivative, $\mathcal{F}H\colon T^*T^{(k-1)}Q \to TT^{(k-1)}Q$, takes values in the "holonomic" submanifold $j_k\colon T^{(k)}Q \hookrightarrow TT^{(k-1)}Q$, that is,* $\mathrm{Im}(\mathcal{F}H) \subseteq j_k(T^{(k)}Q)$.

In the induced natural coordinates $\left(q_{(j)}^i, p_i^{(j)}\right)$, $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$, of $T^*T^{(k-1)}Q$, taking into account that the submanifold $j_k\colon T^{(k)}Q \to TT^{(k-1)}Q$ is defined locally by the $(k-1)n$ constraints $v_{(j)}^i = q_{(j+1)}^i$ $(0 \leqslant j \leqslant k-2)$ and that the fiber derivative of H is a fiber bundle morphism, the condition for H to be a kth-order Hamiltonian function gives in coordinates

$$\frac{\partial H}{\partial p_i^{(j)}} = \mathcal{F}H^* v_{(j)}^i = \mathcal{F}H^* q_{(j+1)}^i = q_{(j+1)}^i\,, \text{ for every } 1 \leqslant i \leqslant n\,,\ 0 \leqslant j \leqslant k-2\,. \tag{19}$$

Observe that if $H \in C^\infty(T^*T^{(k-1)}Q)$ is a kth-order Hamiltonian function, then the fiber derivative $\mathcal{F}H$ of H induces a map $\mathcal{F}H_o\colon T^*T^{(k-1)}Q \to T^{(k)}Q$ defined as $\mathcal{F}H = j_k \circ \mathcal{F}H_o$. This map is given in coordinates by

$$\mathcal{F}H_o\left(q_{(0)}^i, \ldots, q_{(k-1)}^i; p_i^{(0)}, \ldots, p_i^{(k-1)}\right) = \left(q_{(0)}^i, \ldots, q_{(k-1)}^i, \frac{\partial H}{\partial p_i^{(k-1)}}\right)\,. \tag{20}$$

The map $\mathcal{F}H_o\colon T^*T^{(k-1)}Q \to T^{(k)}Q$ enables us to give a solution to Problem 2 in terms of its (local) "inverse map". Nevertheless, due to the restriction imposed by the dimensions of the manifolds involved, there is no inverse map to $\mathcal{F}H_o$ (even locally). Hence, instead of an inverse, we use the most similar approach, which consists in considering inverses in just one way, that is, sections. Therefore, following the patterns in [52], we give the following definition.

Definition 13. *A kth-order Hamiltonian function $H \in C^\infty(T^*T^{(k-1)}Q)$ is said to be* regular *if the map $\mathcal{F}H_o\colon T^*T^{(k-1)}Q \to T^{(k)}Q$ is a submersion onto $T^{(k)}Q$. If, moreover, $\mathcal{F}H_o$ admits a global section $\Upsilon\colon T^{(k)}Q \to T^*T^{(k-1)}Q$, the Hamiltonian function is said to be* hyperregular*. Otherwise, the Hamiltonian function is said to be* singular*.*

In the induced natural coordinates $\left(q_{(j)}^i, p_i^{(j)}\right)$, $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$, of $T^*T^{(k-1)}Q$ from the relation (19) we have

$$\frac{\partial^2 H}{\partial p_j^{(s)} \partial p_i^{(k-1)}} = \frac{\partial}{\partial p_i^{(k-1)}}\left(\frac{\partial H}{\partial p_j^{(s)}}\right) = \frac{\partial q_{(s+1)}^j}{\partial p_i^{(k-1)}} = 0\,, \tag{21}$$

where $1 \leqslant i,j \leqslant n$ and $0 \leqslant s \leqslant k-2$. From those and the local expression (20) of $\mathcal{F}H_o$ we deduce that its tangent map in an arbitrary point $\alpha_q \in T^*T^{(k-1)}Q$ is given locally by the following $(k+1)n \times 2kn$ real matrix

$$T_{\alpha_q}\mathcal{F}H_o = \left(\begin{array}{ccc|cccc}
\mathrm{Id}_n & \cdots & \mathbf{0}_n & \mathbf{0}_n & \cdots & \mathbf{0}_n & \mathbf{0}_n \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0}_n & \cdots & \mathrm{Id}_n & \mathbf{0}_n & \cdots & \mathbf{0}_n & \mathbf{0}_n \\
\hline
\dfrac{\partial^2 H}{\partial q_{(0)}^j \partial p_i^{(k-1)}} & \cdots & \dfrac{\partial^2 H}{\partial q_{(k-1)}^j \partial p_i^{(k-1)}} & \mathbf{0}_n & \cdots & \mathbf{0}_n & \dfrac{\partial^2 H}{\partial p_j^{(k-1)} \partial p_i^{(k-1)}}
\end{array}\right)\,, \tag{22}$$

where every entry is a $n\times n$ real matrix. In particular, Id_n denotes the $n\times n$ identity matrix, $\mathbf{0}_n$ the $n \times n$ null matrix and, in the last row, we have $1 \leqslant i,j \leqslant n$. Therefore, the local condition for a kth-order Hamiltonian function $H \in C^\infty(T^*T^{(k-1)}Q)$ to be a regular is

$$\det\left(\frac{\partial^2 H}{\partial p_j^{(k-1)}\partial p_i^{(k-1)}}\right)(\alpha_q) \neq 0\,, \text{ for every } \alpha_q \in T^*T^{(k-1)}Q\,.$$

Using both Definitions 12 and 13 we can now state and prove the analogous results to Propositions 1 and 2 in the higher-order setting. As for Proposition 1, we have the following result.

Proposition 8. *Let $H \in C^\infty(T^*T^{(k-1)}Q)$ be a regular kth-order Hamiltonian function, $\theta_{T^{(k-1)}Q} \in \Omega^1(T^*T^{(k-1)}Q)$ the Liouville 1-form, $\omega_{T^{(k-1)}Q} \in \Omega^2(T^*T^{(k-1)}Q)$ the canonical symplectic form, and $X_H \in \mathfrak{X}(T^*T^{(k-1)}Q)$ the unique vector field solution to the equation*

$$i_{X_H}\,\omega_{T^{(k-1)}Q} = dH\,. \tag{23}$$

Then the function $\theta_{T^{(k-1)}Q}(X_H) - H \in C^\infty(T^*T^{(k-1)}Q)$ is $\mathcal{F}H_o$-projectable, and the function $L \in C^\infty(T^{(k)}Q)$ defined by $(\mathcal{F}H_o)^*L = \theta_{T^{(k-1)}Q}(X_H) - H$ is a regular kth-order Lagrangian function. Moreover, for every $[\gamma]_0^{(2k-1)} \in T^{(2k-1)}Q$ there exists an open set $U \subseteq T^{(2k-1)}Q$ such that $[\gamma]_0^{(2k-1)} \in U$ and $(\mathcal{F}H_o \circ \mathrm{leg}_L)|_U = \tau_Q^{(k,2k-1)}\Big|_U$.

Proof. Following the patterns in [1], the easiest proof of this result is done in coordinates. Hence, let $\left(q_{(j)}^i, p_i^{(j)}\right)$, $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k-1$, be the induced natural coordinates in a suitable open set of $T^*T^{(k-1)}Q$. Along this proof we only consider these coordinates.

First, we must prove that the function $\theta_{T^{(k-1)}Q}(X_H) - H \in C^\infty(T^*T^{(k-1)}Q)$ is $\mathcal{F}H_o$-projectable, that is,

$$\mathrm{L}(Y)\left(\theta_{T^{(k-1)}Q}(X_H) - H\right) = 0\,, \text{ for every } Y \in \ker T\mathcal{F}H_o\,.$$

From the coordinate expression (22) of the tangent map $T\mathcal{F}H_o$ at a point $\alpha_q \in T^*T^{(k-1)}Q$, it is clear that a local basis for $\ker T\mathcal{F}H_o$ is given by

$$\ker T\mathcal{F}H_o = \left\langle \frac{\partial}{\partial p_i^{(0)}}, \ldots, \frac{\partial}{\partial p_i^{(k-2)}} \right\rangle\,.$$

On the other hand, from our calculations in Section 4.1 we know that the Hamiltonian vector fiels solution to equation (23) is given locally by (16), which in combination with the identities (19) gives

$$X_H = \sum_{l=0}^{k-2} q_{(l+1)}^i \frac{\partial}{\partial q_{(l)}^i} + \frac{\partial H}{\partial p_i^{(k-1)}} \frac{\partial}{\partial q_{(k-1)}^i} - \frac{\partial H}{\partial q_{(j)}^i} \frac{\partial}{\partial p_i^{(j)}}\,.$$

Thus, since the Liouville 1-form is given in coordinates by $\theta_{T^{(k-1)}Q} = p_i^{(j)} dq_{(j)}^i$, the function $\theta_{T^{(k-1)}Q}(X_H) - H \in C^\infty(T^*T^{(k-1)}Q)$ has the following coordinate expression

$$\theta_{T^{(k-1)}Q}(X_H) - H = \sum_{l=0}^{k-2} p_i^{(l)} q_{(l+1)}^i + p_i^{(k-1)} \frac{\partial H}{\partial p_i^{(k-1)}} - H\,.$$

Hence for every $Y = \partial/\partial p_r^{(s)}$, $1 \leqslant r \leqslant n$ and $0 \leqslant s \leqslant k-2$, we have

$$\begin{aligned} \mathrm{L}(Y)\left(\theta_{T^{(k-1)}Q}(X_H) - H\right) &= \mathrm{L}\left(\frac{\partial}{\partial p_r^{(s)}}\right)\left(\sum_{l=0}^{k-2} p_i^{(l)} q_{(l+1)}^i + p_i^{(k-1)}\frac{\partial H}{\partial p_i^{(k-1)}} - H\right) \\ &= q_{(s+1)}^r + p_i^{(k-1)}\frac{\partial^2 H}{\partial p_r^{(s)}\partial p_i^{(k-1)}} - \frac{\partial H}{\partial p_r^{(s)}} = 0\,, \end{aligned}$$

where, in the last equality, we have used the identities (19) and (21). Therefore, the function $\theta_{T^{(k-1)}Q}(X_H) - H \in C^\infty(T^*T^{(k-1)}Q)$ is $\mathcal{F}H_o$-projectable.

It is now a long but straightforward calculation to prove that the Lagrangian function $L \in C^\infty(T^{(k)}Q)$ defined by $(\mathcal{F}H_o)^*L = \theta_{T^{(k-1)}Q}(X_H) - H$ is regular. Indeed, using implicit differentiation and the chain rule, we have

$$\left(\frac{\partial^2 L}{\partial q_{(k)}^i \partial q_{(k)}^j}\right) = \left(\frac{\partial \hat{p}_j^{(k-1)}}{\partial q_{(k)}^i}\right) = \left(\frac{\partial p^j_{(k-1)} \circ \mathrm{leg}_L}{\partial q_{(k)}^i}\right) = \left(\frac{\partial^2 H}{\partial p_j^{(k-1)}\partial p_i^{(k-1)}}\right)^{-1}\,,$$

where $\hat{p}_j^{(k-1)}$ are n last Jacobi-Ostrogradsky momenta functions defined in (13). Therefore, since H is a kth-order regular Hamiltonian function, the Hessian of H with respect to the highest order momenta is invertible at every point of $T^*T^{(k-1)}Q$, and hence so is the Hessian of L with respect to the highest order velocities. Thus, L is regular.

Finally, let us consider the map $\mathcal{F}H_o \circ \mathrm{leg}_L \colon T^{(2k-1)}Q \to T^{(k)}Q$. In the natural coordinates of $T^{(2k-1)}Q$ the local expression of this map is given by

$$(\mathcal{F}H_o \circ \mathrm{leg}_L)\left(q_{(0)}^i, \ldots, q_{(2k-1)}^i\right) = \mathcal{F}H_o\left(q_{(0)}^i, \ldots, q_{(k-1)}^i, \hat{p}_i^{(0)}, \ldots, \hat{p}_i^{(k-1)}\right) = \left(q_{(0)}^i, \ldots, q_{(k-1)}^i, q_{(k)}^i\right)$$

since $\tilde{q}_{(k)}^i = \mathcal{F}H_o^* q_{(k)}^i$. From this coordinate expression we deduce that for every point $[\gamma]_0^{(2k-1)} \in T^{(2k-1)}Q$ there exists an open set $U \subseteq T^{(2k-1)}Q$ such that $[\gamma]_0^{(2k-1)} \in U$ and $(\mathcal{F}H_o \circ \mathrm{leg}_L)|_U = \left.\tau_Q^{(k,2k-1)}\right|_U$. □

Remark. Since $\mathcal{F}H_o$ is at the best a submersion we can not directly pull-back a function in $T^*T^{(k-1)}Q$ to $T^{(k)}Q$ by the inverse of $\mathcal{F}H_o$ (since there isn't), and hence we must first prove the $\mathcal{F}H_o$-projectability of the function. ◊

Remark. It is important to point out that in the conclusion of Proposition 8 we only prove the *regularity* of L, instead of its hyperregularity (as in Proposition 1). Indeed, even if the Hamiltonian function is hyperregular, we can only prove that leg_L is a surjective local diffeomorphism, but not the injectivity. ◊

Before stating and proving the analogous to Proposition 2, we need the following technical result, which we state and prove in a more general case than we need.

Lemma 1. *Let $L \in C^\infty(T^{(k)}Q)$ be a kth-order Lagrangian function, $\theta_L \in \Omega^1(T^{(2k-1)}Q)$ the associated Poincaré-Cartan 1-form and $E_L \in C^\infty(T^{(2k-1)}Q)$ the kth-order Lagrangian energy. If $X \in \mathfrak{X}(T^{(2k-1)}Q)$ is a semispray of type k, then $\left(\tau_Q^{(k,2k-1)}\right)^* L = \theta_L(X) - E_L$.*

Proof. This proof is easy in coordinates. Recall that the coordinate expression of a semispray of type r in $T^{(k)}Q$ is given by (2), from where we deduce that a semispray of type k X in $T^{(2k-1)}Q$ is given locally by

$$X = q_{(1)}^i \frac{\partial}{\partial q_{(0)}^i} + \ldots + q_{(k)}^i \frac{\partial}{\partial q_{(k-1)}^i} + F_{(k)}^i \frac{\partial}{\partial q_{(k)}^i} + \ldots + F_{(2k-1)}^i \frac{\partial}{\partial q_{(2k-1)}^i}\,.$$

Hence, bearing in mind the coordinate expression (11) of the Poincaré-Cartan 1-form, we deduce that the smooth function $\theta_L(X)$ is given locally by

$$\theta_L(X) = \sum_{r=1}^{k} q_{(r)}^i \sum_{j=0}^{k-r} (-1)^j d_T^j \left(\frac{\partial L}{\partial q_{(r+j)}^i} \right) .$$

Hence, bearing in minda the coordinate expression (17) of the Lagrangian energy E_L, we obtain

$$\begin{aligned} \theta_L(X) - E_L &= \sum_{r=1}^{k} q_{(r)}^i \sum_{j=0}^{k-r} (-1)^j d_T^j \left(\frac{\partial L}{\partial q_{(r+j)}^i} \right) - \sum_{r=1}^{k} q_{(r)}^i \sum_{j=0}^{k-r} (-1)^j d_T^j \left(\frac{\partial L}{\partial q_{(r+j)}^i} \right) + \left(\tau_Q^{(k,2k-1)} \right)^* L \\ &= \left(\tau_Q^{(k,2k-1)} \right)^* L . \end{aligned}$$

as claimed. □

Finally, as for Proposition 2, we have the following result.

Proposition 9. *Let $L \in C^\infty(T^{(k)}Q)$ be a hyperregular kth-order Lagrangian function and $H = E_L \circ \text{leg}_L^{-1} \in C^\infty(T^*T^{(k-1)}Q)$ the associated Hamiltonian function. Then H is a hyperregular kth-order Hamiltonian function and $\mathcal{F}H_o = \tau_Q^{(k,2k-1)} \circ \text{leg}_L^{-1}$. In addition, if $\tilde{L} \in C^\infty(T^{(k)}Q)$ is the kth-order Lagrangian function associated to H by Proposition 8, then $\tilde{L} = L$.*

Proof. The proof of H being a hyperregular kth-order Hamiltonian functions follows exactly the same patterns as the computations carried out in Section 4.2, and hence we omit them. Therefore, the only need to prove that if $\tilde{L} \in C^\infty(T^{(k)}Q)$ is the kth-order Lagrangian function associated to H by Proposition 8, then $\tilde{L} = L$. Observe that from the properties of the map $\mathcal{F}H_o^*$ we have

$$\mathcal{F}H_o^* \tilde{L} = (\tau_Q^{(k,2k-1)} \circ \text{leg}_L^{-1})^* \tilde{L} = \left(\text{leg}_L^{-1}\right)^* \left(\left(\tau_Q^{(k,2k-1)} \right)^* \tilde{L} \right) .$$

On the other hand, from the definition of the Hamiltonian function and the properties of the Legendre-Ostrogradsky map we have

$$\begin{aligned} \theta_{T^{(k-1)}Q}(X_H) - H &= \theta_{T^{(k-1)}Q}(X_H) - (E_L \circ \text{leg}_L^{-1}) = \left(\text{leg}_L^{-1}\right)^* \left(\theta_{T^{(k-1)}Q}(X_H) \circ \text{leg}_L - E_L \right) \\ &= \left(\text{leg}_L^{-1}\right)^* (\theta_L(X_L) - E_L) = \left(\text{leg}_L^{-1}\right)^* \left(\left(\tau_Q^{(k,2k-1)} \right)^* L \right) , \end{aligned}$$

where in the last step we have used Lemma 1, as the vector field X_L solution to the Lagrangian dynamical equation $i_{X_L} \omega_L = dE_L$ is a semispray of type 1 when L is a hyperregular higher-order Lagrangian function (see [18] for details). Equating these two expressions, we have

$$(\text{leg}_L^{-1})^* \left(\left(\tau_Q^{(k,2k-1)} \right)^* \tilde{L} \right) = \left(\text{leg}_L^{-1}\right)^* \left(\left(\tau_Q^{(k,2k-1)} \right)^* L \right) .$$

Now, since leg_L^{-1} is a global diffeomorphism and $\tau_Q^{(k,2k-1)}$ is the canonical projection, this last equality holds if, and only if, $\tilde{L} = L$. □

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