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# On (non-)exponential decay in generalized thermoelasticity with two temperatures

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## Abstract

We study solutions for the one-dimensional problem of the Green-Lindsay and the Lord-Shulman theories with two temperatures. First, existence and uniqueness of weakly regular solutions are obtained. Second, we prove the exponential stability in the Green-Lindsay model, but the non-exponential stability for the Lord-Shulman model.

*Keywords:* Two-temperatures; generalized thermoelasticity; exponential decay; Green-Lindsay theory; Lord-Shulman theory

## 1 Introduction

The usual theory of heat conduction based on Fourier's law implies the instantaneous propagation of heat waves. This fact is not well accepted from the viewpoint of physics because it contradicts the causality principle. Accordingly, a big interest has been developed to propose alternative constitutive equations to the Fourier law. We recall the classical formulations of Lord-Shulman [9] and Green-Lindsay theories [5], which are based on the Cattaneo-Maxwell theory heat conduction. This is the case when the heat equation is hyperbolic.

Thermoelasticity with two temperatures is one of the non-classical theories of thermomechanics of elastic solids. The main difference of this theory with respect to the classical one is in the thermal dependence. The theory was proposed by Chen, Gurtin and Williams (see [1], [3], [6]) and several authors have dedicate its attention to this problem (Ieşan [7], Chen et al. [2], [16], Quintanilla [11], [12], among others). In this paper where elastic effects are taken into consideration we deal with the two models proposed by Youssef [18]. They correspond to the two-temperature modifications of the Green-Lindsay and Lord-Shulman theories. Uniqueness and instability of solutions was obtained in [10].

First, the well-posedness will be proved in spaces with only combined, hence less regularity than known for the classical single-temperature case. Then we prove that the solutions uniformly decay exponentially for the Green-Lindsay theory, but the decay is slow – not exponential - for the Lord-Shulman case. This is a surprising aspect of this paper providing another interesting example for a situation where the change from Fourier's to Cattaneo-Maxwell's law leads to a loss of exponential stability, cp. [14] for the classical exponentially stable single-temperature case, and [4, 13, 15] for other examples of loss of exponential stability for plates or Timoshenko type models.

The one-dimensional system of equations that governs the deformations of a centrosymmetric thermoelastic material in the theory of Green and Lindsay with two temperatures is

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + a(\theta_x + \alpha\dot{\theta}_x) \\ h\ddot{\theta} + d\dot{\theta} - a\dot{u}_x = k\phi_{xx} \\ \phi - \theta = mk\phi_{xx}. \end{cases} \quad (1.1)$$

Here,  $u$  is the displacement,  $\theta$  is the temperature and  $\phi$  is the conductive temperature,  $\rho$  is the mass density and  $a, \alpha, h, d, \mu, m$  and  $k$  are constitutive constants. We will assume

$$a \neq 0, \alpha > 0, \rho > 0, k > 0, m > 0, h > 0, \mu > 0 \text{ and } \alpha d \geq h. \quad (1.2)$$

In fact, the last inequality is a consequence of the entropy inequality of Green and Lindsay (see [5]). We study the solutions of the system (1.1) in  $B \times J$ , where  $B = [0, \pi]$  and  $J = [0, \infty)$ . We assume the homogeneous Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = \phi(0, t) = \phi(\pi, t) = 0, \quad t \in J \quad (1.3)$$

together with the initial conditions

$$u(x, 0) = u^0, \dot{u}(x, 0) = v^0, \theta(x, 0) = \theta^0, \dot{\theta}(x, 0) = \psi^0, \quad x \in B. \quad (1.4)$$

We consider the isomorphism  $Id - mk\Delta: \phi \rightarrow \phi - mk\phi_{xx} = \theta$ . This operator acts on  $W^{2,2}(B) \cap W_0^{1,2}(B)$  and take values in  $L^2$ . We denote by  $\Phi(\theta) = \phi$  the inverse operator. In view of the boundary conditions, we have

$$\|\theta\|^2 = \|\phi\|^2 + 2mk\|\phi_x\|^2 + m^2k^2\|\phi_{xx}\|^2 \quad (\text{norm in } L^2). \quad (1.5)$$

We also consider the problem determined by the boundary conditions

$$u(0, t) = u(\pi, t) = \phi_x(0, t) = \phi_x(\pi, t) = 0, \quad t \in J. \quad (1.6)$$

In this case, the functional  $\Phi$  acts on  $L_*^2(B)$  and takes values in  $W^{2,2}(B) \cap L_*^2(B) \cap \{\phi, \phi_x(0) = \phi_x(\pi) = 0\}$ , where  $L_*^2(B) = \{\theta \in L^2(B), \int_0^\pi \theta dx = 0\}$ . The functional  $\Phi$  is also an isomorphism. At the same time equality (1.5) holds.

The one-dimensional system of equations that governs the deformations of a thermoelastic material in the theory of Lord and Shulman with two temperatures is

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + a\theta_x \\ h_1\dot{\hat{\theta}} - a\dot{u}_x = k\phi_{xx} \\ \phi - \theta = mk\phi_{xx}, \end{cases} \quad (1.7)$$

where  $\hat{f} = f + d_1\dot{f}$ , together with the boundary conditions (1.3) and initial conditions (1.4). For the coefficients we assume

$$a \neq 0, \rho > 0, k > 0, m > 0, h_1 > 0, \mu > 0 \text{ and } d_1 > 0. \quad (1.8)$$

Section 2 is devoted to the Green-Lindsay theory with two temperatures. We prove the existence and uniqueness of solutions as well as exponential stability of solutions. Section 3 has a similar structure, but for the Lord-Shulman theory with two temperatures. Here, however, we prove the maybe unexpected slow, non-exponential decay of the solutions.

## 2 Green-Lindsay theory

We write the system (1.1) as

$$\begin{cases} \dot{u} = v, & \dot{v} = \frac{1}{\rho} [\mu u_{xx} + a(\theta_x + \alpha\psi_x)] \\ \dot{\theta} = \psi, & \dot{\psi} = \frac{1}{h} [av_x - d\psi + k\Phi(\theta)_{xx}] \end{cases} \quad (2.1)$$

and (1.1)<sub>3</sub>. In this section we prove the existence, uniqueness and exponential decay for the problem proposed by the system (2.1) with initial conditions (1.4) and boundary conditions (1.3). We also point out how to extend the results for the boundary conditions (1.6) easily.

We denote by  $\mathcal{H}$  the Hilbert space

$$\{(u, v, \theta, \psi) : u \in W_0^{1,2}(B), v, \theta, \psi \in L^2(B)\} \quad (2.2)$$

with inner product

$$\begin{aligned} \langle (u, v, \theta, \psi), (u^*, v^*, \theta^*, \psi^*) \rangle &:= \frac{1}{2} \int_0^\pi \left[ \rho v \bar{v}^* + \mu u_x \bar{u}_x^* + \frac{h}{\alpha} (\theta + \alpha\psi) \overline{(\theta^* + \alpha\psi^*)} \right. \\ &\quad \left. + \left( d - \frac{h}{\alpha} \right) \theta \bar{\theta}^* + \alpha k \Phi(\theta)_x \overline{\Phi(\theta^*)_x} + mk^2 \alpha \Phi(\theta)_{xx} \overline{\Phi(\theta^*)_{xx}} \right] dx. \end{aligned} \quad (2.3)$$

We define

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu}{\rho} D^2 & 0 & \frac{a}{\rho} D & \frac{a\alpha}{\rho} D \\ 0 & 0 & 0 & I \\ 0 & \frac{a}{h} D & \frac{k}{h} D^2 \Phi & -\frac{d}{h} \end{pmatrix} \quad (2.4)$$

where  $I$  is the identity operator and  $D$  denotes the derivative with respect to  $x$ . (1.1) does not provide regularity for  $\psi, \theta$ , therefore the term  $\mu u_{xx} + a(\theta_x + \alpha\psi_x)$  in (2.1) has to be interpreted as  $D(\mu u_x + a(\theta + \alpha\psi))$ . Separate regularity like  $u_{xx}, \theta_x, \psi_x \in L^2(B)$  is not available.

Our problem can be written as the following Cauchy problem in the Hilbert space  $\mathcal{H}$ :

$$\frac{d\omega}{dt} = \mathcal{A}\omega, \quad \omega_0 = (u_0, v_0, \theta_0, \psi_0), \quad (2.5)$$

where  $\omega = (u, v, \theta, \psi)$ . The domain  $\mathcal{D}$  of  $\mathcal{A}$  is the set of  $\omega \in \mathcal{H}$  such that  $\mathcal{A}\omega \in \mathcal{H}$ . It is a dense subspace of  $\mathcal{H}$ .

For the boundary conditions (1.6) we have to work with the Hilbert space

$$\{(u, v, \theta, \psi), u \in W_0^{1,2}(B), v \in L^2(B), \theta, \psi \in L_*^2(B)\}$$

with the same inner product as in (2.3) and matrix operator  $\mathcal{A}$  as in (2.4).

### 2.1 Existence and uniqueness of solutions

**Theorem 2.1** *The operator  $\mathcal{A}$  is dissipative, and  $\text{Range}(\mathcal{A}) = \mathcal{H}$ .*

*Proof.* Taking into account the evolutive equations and the boundary conditions, we have the dissipativity by observing

$$\begin{aligned} \text{Re}\langle \mathcal{A}\omega, \omega \rangle &= \frac{1}{2} \text{Re} \left\{ \int_0^\pi \left[ [\mu u_{xx} + a(\theta_x + \alpha\psi_x)] \bar{v} + \mu v_x \bar{u}_x + \left( \frac{h}{\alpha} \psi + av_x - d\psi + k\Phi(\theta)_{xx} \right) \overline{(\theta + \alpha\psi)} \right. \right. \\ &\quad \left. \left. + \left( d - \frac{h}{\alpha} \right) \psi \bar{\theta} + \alpha k \Phi(\psi)_x \overline{\Phi(\theta)_x} + mk^2 \alpha \Phi(\psi)_{xx} \overline{\Phi(\theta)_{xx}} \right] dx \right\} \\ &= \frac{1}{2} \int_0^\pi \left[ (h - \alpha d) |\psi|^2 - k |\phi_x|^2 - k^2 m |\phi_{xx}|^2 \right] dx. \end{aligned} \quad (2.6)$$

Moreover, for  $f = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ , the equation  $\mathcal{A}\omega = f$  is solved by  $\omega = (u, v, \theta, \psi) \in \mathcal{D}$ , where  $v := f_1$ ,  $\psi := f_3$ , and

$$\begin{aligned} \theta(x) := \phi - mk\phi_{xx} = & -\frac{a}{k} \int_0^x f_1 ds + \frac{d}{k} \int_0^x \int_0^s f_3 d\tau ds + \frac{h}{k} \int_0^x \int_0^s f_4 d\tau ds \\ & + \frac{x}{\pi k} \left[ a \int_0^\pi f_1 ds - d \int_0^\pi \int_0^s f_3 d\tau ds - h \int_0^\pi \int_0^s f_4 d\tau ds \right] \\ & - m[-aDf_1 + df_3 + hf_4], \end{aligned} \quad (2.7)$$

$$\begin{aligned} u(x) := & -\frac{a}{\mu} \int_0^x \theta ds + \frac{\rho}{\mu} \int_0^x \int_0^s f_2 d\tau ds - \frac{a\alpha}{\mu} \int_0^x f_2 ds \\ & + \frac{x}{\pi\mu} \left[ a \int_0^\pi \theta ds - \rho \int_0^\pi \int_0^s f_3 d\tau ds + a\alpha \int_0^\pi f_2 ds \right]. \quad \square \end{aligned} \quad (2.8)$$

As a consequence of Theorem 2.1 and the Lumer-Phillips corollary to the Hille-Yosida Theorem [8] we obtain the well-posedness.

**Theorem 2.2** *The operator  $\mathcal{A}$  generates a contraction semigroup  $\{e^{t\mathcal{A}}\}_{t \geq 0}$ , and for  $\omega_0 \in \mathcal{D}$  there exists a unique solution  $\omega \in C^0([0, \infty), \mathcal{D}) \cap C^1([0, \infty), \mathcal{H})$ .*

The proof of Theorem 2.1 can be also adapted to the boundary conditions (1.6). The main difference will consist in the new expressions for  $\theta(x)$  and  $u(x)$ , but it can be done in a direct way. Therefore, the existence of the semigroup proposed at Theorem 2.2 can be also obtained for the conditions (1.6).

## 2.2 Exponential decay

To prove the exponential stability of the solutions we use the following characterization, going back to Gearhart, Huang and Prüß (see [8]).

**Theorem 2.3** *Let  $\{e^{t\mathcal{A}_*}\}_{t \geq 0}$  be a  $C_0$ -semigroup of contractions generated by the operator  $\mathcal{A}_*$  in the Hilbert space  $\mathcal{H}_*$ . Then the semigroup is exponentially stable if and only if  $i\mathbb{R} \subseteq \varrho(\mathcal{A}_*)$  (resolvent set) and*

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|(i\beta I - \mathcal{A}_*)^{-1}\| < \infty, \quad \beta \in \mathbb{R}. \quad (2.9)$$

**Theorem 2.4** *The operator  $\mathcal{A}$  generates a semigroup which is exponentially stable.*

*Proof.* Since  $0 \in \varrho(\mathcal{A})$ , following the arguments in ([8], p. 25), we assume that the imaginary axis is not contained in the resolvent set. Then there exists a real number  $\varpi \neq 0$  with  $\|\mathcal{A}^{-1}\|^{-1} \leq |\varpi| < \infty$  such that the set  $\{i\lambda, |\lambda| < |\varpi|\}$  is in the resolvent of  $\mathcal{A}$  and  $\sup\{\|(i\lambda I - \mathcal{A})^{-1}\|, |\lambda| < |\varpi|\} = \infty$ . Therefore, there exists a sequence of real numbers  $\lambda_n$  with  $\lambda_n \rightarrow \varpi$ ,  $|\lambda_n| < |\varpi|$  and a sequence of unit norm vectors  $\omega_n = (u_n, v_n, \theta_n, \psi_n)$  in the domain of the operator  $\mathcal{A}$  such that

$$\|(i\lambda_n I - \mathcal{A})\omega_n\| \rightarrow 0. \quad (2.10)$$

This implies

$$i\lambda_n u_n - v_n \rightarrow 0 \text{ in } W^{1,2}, \quad (2.11)$$

$$i\lambda_n v_n - \frac{1}{\rho} (\mu D^2 u_n + aD\theta_n + a\alpha D\psi_n) \rightarrow 0 \text{ in } L^2, \quad (2.12)$$

$$i\lambda_n \theta_n - \psi_n \rightarrow 0 \text{ in } L^2, \quad (2.13)$$

$$i\lambda_n\psi_n - \frac{1}{h} (aDv_n + kD^2\Phi(\theta_n) - d\psi_n) \rightarrow 0 \text{ in } L^2. \quad (2.14)$$

Considering  $\text{Re}\langle (i\lambda_n I - \mathcal{A})\omega_n, \omega_n \rangle$  then (2.6) implies  $\|\psi_n\| \rightarrow 0$ ,  $\|\phi_{n,x}\| \rightarrow 0$  and  $\|\phi_{n,xx}\| \rightarrow 0$  in  $L^2$ . From equation (2.13),  $\|\theta_n\| \rightarrow 0$  in  $L^2$ . Taking into account that  $\Phi(\theta) = \phi$ , and removing from (2.14) the terms that tend to 0, we get that  $\frac{a}{h\lambda_n}Dv_n \rightarrow 0$ . Multiplying (2.12) by  $\frac{\rho}{\lambda_n}v_n$  we obtain

$$i\rho\|v_n\|^2 + \mu\langle Du_n, \frac{1}{\lambda_n}Dv_n \rangle + a\langle \theta_n, \frac{1}{\lambda_n}Dv_n \rangle + a\alpha\langle \psi_n, \frac{1}{\lambda_n}Dv_n \rangle \rightarrow 0. \quad (2.15)$$

Thus,  $\|v_n\|^2 \rightarrow 0$ . The next step is to multiply (2.12) by  $u_n$  and, since  $Du_n$  is bounded, we get  $\rho\langle i\lambda_nv_n, u_n \rangle + \mu\|Du_n\|^2 \rightarrow 0$ . Using (2.11),  $-\rho\|v_n\|^2 + \mu\|Du_n\|^2 \rightarrow 0$  and then,  $\|Du_n\|^2 \rightarrow 0$ . Finally,  $Du_n \rightarrow 0$  in  $L^2$ . These behaviors contradict the hypothesis that  $\omega_n$  has norm 1.

Now, (2.9) is proved by a similar argument. If it is not true, there exist a sequence  $\lambda_n$  with  $|\lambda_n| \rightarrow \infty$  and a sequence of unit norm vectors  $\omega_n = (u_n, v_n, \theta_n, \psi_n)$  in the domain of the operator  $\mathcal{A}$  such that (2.10) holds. We can now follow the arguments used previously when  $(\lambda_n)_n$  is bounded.  $\square$

The proof of Theorem 2.4 can be adapted to the boundary conditions (1.6). Therefore we have obtained the exponential decay of solutions for the boundary conditions (1.3) and also (1.6).

### 3 Lord-Shulman theory

We re-write (1.7) as a first order system

$$\begin{cases} \dot{\hat{u}} = \hat{v}, & \dot{\hat{v}} = \frac{1}{\rho}[\mu\hat{u}_{xx} + a\theta_x + ad_1\psi_x] \\ \dot{\theta} = \psi, & \dot{\psi} = \frac{1}{h_1d_1}[a\hat{v}_x + k\Phi(\theta)_{xx} - h_1\psi] \end{cases} \quad (3.1)$$

and (1.1)<sub>3</sub>. Again in this section we give the existence and uniqueness of solutions for the boundary conditions (1.3). The extension for the boundary conditions (1.6) can be done in a similar way as in Section 2. However, in this section we prove the slow decay of solutions only for the boundary conditions (1.6). One suspects a similar result for the boundary conditions (1.3), but our arguments can not be extended to this case.

In analogy to Section 2, we denote by  $\mathcal{H}_1$  the Hilbert space

$$\{(\hat{u}, \hat{v}, \theta, \psi) : \hat{u} \in W_0^{1,2}, \hat{v}, \theta, \psi \in L^2\} \quad (3.2)$$

with inner product

$$\begin{aligned} \langle (\hat{u}, \hat{v}, \theta, \psi), (\hat{u}^*, \hat{v}^*, \theta^*, \psi^*) \rangle := & \frac{1}{2} \int_0^\pi \left[ \rho\hat{v}\overline{\hat{v}^*} + \mu\hat{u}_x\overline{\hat{u}_x^*} + h_1(\theta + d_1\psi)\overline{(\theta^* + d_1\psi^*)} + \right. \\ & \left. d_1k\Phi(\theta)_x\overline{\Phi(\theta^*)_x} + md_1k^2\Phi(\theta)_{xx}\overline{\Phi(\theta^*)_{xx}} \right] dx. \end{aligned} \quad (3.3)$$

We define

$$\mathcal{B} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu}{\rho}D^2 & 0 & \frac{a}{\rho}D & \frac{a}{d_1}\rho D \\ 0 & 0 & 0 & I \\ 0 & \frac{a}{h_1d_1}D & \frac{k}{h_1d_1}D^2\Phi & -\frac{1}{d_1} \end{pmatrix}. \quad (3.4)$$

Our problem can be transformed in the following Cauchy problem in the Hilbert space  $\mathcal{H}_1$ :

$$\frac{d\omega}{dt} = \mathcal{B}\omega, \quad \omega_0 = (\hat{u}_0, \hat{v}_0, \theta_0, \psi_0), \quad (3.5)$$

where  $\omega = (\hat{u}, \hat{v}, \theta, \psi)$ . The domain  $\mathcal{D}_1$  of  $\mathcal{B}$  is the set of  $\omega \in \mathcal{H}_1$  such that  $\mathcal{B}\omega \in \mathcal{H}_1$ . It is a dense subspace of  $\mathcal{H}$ .

The existence and the uniqueness of solutions follows as in Section 2.1, we have the dissipativity of  $\mathcal{B}$  and  $\text{Range}(\mathcal{B}) = \mathcal{H}_1$ , implying

**Theorem 3.1** *The operator  $\mathcal{B}$  generates a contraction semigroup  $\{e^{t\mathcal{B}}\}_{t \geq 0}$ , and for  $\omega_0 \in \mathcal{D}_1$  there exists a unique solution  $\omega \in C^0([0, \infty), \mathcal{D}_1) \cap C^1([0, \infty), \mathcal{H}_1)$ .*

Now we prove the interesting fact that the system, for the boundary conditions (1.6), is *not* exponentially stable. The existence and uniqueness is obtained for these boundary conditions in a similar way, but it is easier accessible with the method used below. To exclude trivial non-decaying solutions we assume that the initial conditions belong to the domain.

Taken into account (1.7)<sub>3</sub>,  $\theta = \phi - mk\phi_{xx}$ , the system (1.7) can be written as follows:

$$\begin{cases} \rho \ddot{u} = \mu u_{xx} + a(\phi_x - mk\phi_{xxx}) \\ h_1 \left[ \dot{\phi} - mk\dot{\phi}_{xx} + d_1 \left( \ddot{\phi} - mk\ddot{\phi}_{xx} \right) \right] - a(\dot{u}_x + d_1 \ddot{u}_x) = k\phi_{xx}. \end{cases} \quad (3.6)$$

**Theorem 3.2** *The corresponding semigroup is not exponentially stable.*

*Proof.* We will see that, for all sufficiently small  $\epsilon > 0$ , there exist solutions of the form

$$u(x, t) = K_1 \exp(\omega t) \sin(nx), \quad \phi(x, t) = K_2 \exp(\omega t) \cos(nx), \quad (3.7)$$

such that  $\text{Re}(\omega) > -\epsilon$ . This will prove that we do not have uniform exponential decay of the system. Suppose that  $u$  and  $\phi$  are as in (3.7). Then, replacing them in (3.6), the following linear and homogeneous system in the unknowns  $K_1$  and  $K_2$  is obtained:

$$\begin{pmatrix} n^2\mu + \rho\omega^2 & an(1 + kmn^2) \\ -an\omega(1 + d_1\omega) & kn^2 + \omega h_1(1 + kmn^2)(1 + d_1\omega) \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.8)$$

This linear system will have nontrivial solution if, and only if, the determinant of the coefficients matrix is null. Let  $p(x)$  be this determinant once  $\omega$  is replaced by  $x$ .

$$\begin{aligned} p(x) = & x^4 (d_1 h_1 k m \rho n^2 + d_1 h_1 \rho) + x^3 (h_1 k m \rho n^2 + h_1 \rho) \\ & + x^2 (a^2 d_1 k m n^4 + d_1 h_1 k m \mu n^4 + a^2 d_1 n^2 + d_1 h_1 \mu n^2 + k \rho n^2) \\ & + x (a^2 k m n^4 + h_1 k m \mu n^4 + a^2 n^2 + h_1 \mu n^2) + k \mu n^4. \end{aligned} \quad (3.9)$$

It is a polynomial of degree four. To prove that  $p(x)$  has roots as near as desired to the imaginary axis, we will show that for any  $\epsilon > 0$  there are roots of  $p(x)$  located at the right of the vertical line  $\text{Re}(z) = -\epsilon$ , or, equivalently, that the polynomial  $p(x - \epsilon)$  has a root with positive real part. To prove that, we use the Routh-Hurwitz theorem (see [17]). It assesses that, if  $a_0 > 0$ , then all the roots of a polynomial  $a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$  have negative real part if, and only if, all the leading minors of the matrix

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

are positive. We denote by  $L_i$ , for  $i = 1, 2, 3, 4$ , the leading minors of this matrix.

In our case,

$$\begin{aligned}
a_0 &= n^2 d_1 h_1 k m \rho + d_1 h_1 \rho, \\
a_1 &= n^2 (h_1 k m \rho - 4 d_1 h_1 k m \epsilon \rho) + h_1 \rho - 4 d_1 h_1 \epsilon \rho, \\
a_2 &= n^4 (d_1 k m a^2 + d_1 h_1 k m \mu) + n^2 (d_1 a^2 + d_1 h_1 \mu + 6 d_1 h_1 k m \epsilon^2 \rho + k \rho - 3 h_1 k m \epsilon \rho) \\
&\quad + 6 d_1 h_1 \epsilon^2 \rho - 3 h_1 \epsilon \rho, \\
a_3 &= n^4 (k m a^2 - 2 d_1 k m \epsilon a^2 + h_1 k m \mu - 2 d_1 h_1 k m \epsilon \mu) \\
&\quad + n^2 (-4 d_1 h_1 k m \rho \epsilon^3 + 3 h_1 k m \rho \epsilon^2 - 2 a^2 d_1 \epsilon - 2 d_1 h_1 \mu \epsilon - 2 k \rho \epsilon + a^2 + h_1 \mu) \\
&\quad - 4 d_1 h_1 \epsilon^3 \rho + 3 h_1 \epsilon^2 \rho, \\
a_4 &= n^4 (d_1 k m \epsilon^2 a^2 - k m \epsilon a^2 + d_1 h_1 k m \epsilon^2 \mu + k \mu - h_1 k m \epsilon \mu) \\
&\quad + n^2 (d_1 h_1 k m \rho \epsilon^4 - h_1 k m \rho \epsilon^3 + a^2 d_1 \epsilon^2 + d_1 h_1 \mu \epsilon^2 + k \rho \epsilon^2 - a^2 \epsilon - h_1 \mu \epsilon) \\
&\quad + d_1 h_1 \epsilon^4 \rho - h_1 \epsilon^3 \rho.
\end{aligned} \tag{3.10}$$

Direct computations give that  $L_2$  is a polynomial of degree six in  $n$ :

$$L_2 = -2d_1^2 h_1 k^2 m^2 \epsilon \rho (a^2 + h_1 \mu) n^6 + R_4(n), \tag{3.11}$$

where  $R_4(n)$  is a polynomial on  $n$  of degree 4. Then, for  $n$  sufficiently large, the sign of  $L_2$  is determined by the coefficient of  $n^6$ :  $-2d_1^2 h_1 k^2 m^2 \epsilon \rho (a^2 + h_1 \mu) < 0$ . For  $n$  large enough,  $L_2$  is negative and  $p(x - \epsilon)$  has at least one root with positive real part. Then, a uniform rate of decay of exponential type for all the solutions of system (3.6) cannot be obtained and so, the decay of the solutions is slow.  $\square$

We recall that  $d_1 = 0$  corresponds to the classical law with two temperatures, where the exponential stability is known, cp. [12]. Also, the case  $m = 0$  corresponds to the Lord-Shulman theory where the exponential stability is known [14].

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