

A geometric approach to dense Cayley digraphs of finite Abelian groups ¹

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Abstract

We give a method for constructing infinite families of dense (or eventually likely dense) Cayley digraphs of finite Abelian groups. The diameter of the digraphs is obtained by means of the related *minimum distance diagrams*. A *dilating* technique for these diagrams, which can be used for any degree of the digraph, is applied to generate the digraphs of the family. Moreover, two infinite families of digraphs with distinguished metric properties will be given using these methods. The first family contains digraphs with asymptotically large ratio between the order and the diameter as the degree increases (moreover it is the first known asymptotically dense family). The second family, for fixed degree $d = 3$, contains digraphs with the current best known density.

Keywords: Cayley digraph, minimum distance diagram, dilation, diameter, density.

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1 Introduction

Given a finite Abelian group Γ of order $N = |\Gamma|$, consider a *generating set* $B = \{\gamma_1, \dots, \gamma_d\}$ of Γ . The *Cayley digraph of Γ with respect to B* is denoted by $G = \text{Cay}(\Gamma, B)$. The set of vertices of G is $V(G) = \Gamma$ and the set of arcs is $A(G) = \{(\alpha, \beta) : \beta - \alpha \in B\}$. These digraphs are *regular* and *vertex transitive*. The (out and in) *degree* and *diameter* of G are denoted by $d(G)$ and $k(G)$, respectively. We write $G \cong G'$ when G and G' are isomorphic digraphs.

Let $\text{NA}_{d,k}$ (respectively, $\text{NC}_{d,k}$) be the maximum number of vertices that a Cayley digraph of an Abelian group (respectively, of a cyclic group), with degree d and diameter k , can have. Let us denote $\text{lb}(d, k)$ the *lower bound* for $\text{NA}_{d,k}$. Then, from [2, Theorem 9.1] of Dougherty and Faber in 2004, it follows that

$$(1) \quad \text{lb}(d, k) = \frac{c}{d(\ln d)^{1+\log_2 e}} \frac{k^d}{d!} + O(k^{d-1}) \leq \text{NA}_{d,k} < \binom{k+d}{k},$$

for some constant c . As far as we know, no constructions of Cayley digraphs G of order $N(G) \sim \text{lb}(b, k)$ are known.

The *density* $\delta(G)$ of a digraph G is defined by $\delta(G) = N(G)/(k(G) + d)^d$. Let us denote by $\Delta_{d,k} = \max\{\delta(G) : d(G) = d, k(G) = k\}$ and $\Delta_d = \max\{\Delta_{d,k} : d(G) = d\}$. The only value of d for which Δ_d is known is 2, Forcade and Lamoreaux in 2000 proved that $\Delta_2 = \frac{1}{3}$ in [5, Section 4]. That density is attained by $G_t = \text{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}_{3t}, \{(1, -1), (0, 1)\})$, with $N(G_t) = 3t^2$ and $k(G_t) = 3t - 2$. No Cayley digraph of a cyclic group attains this density for $d = 2$. Although the value of Δ_3 is not known, Fiduccia, Forcade and Zito in 1998 [3, Corollary 3.6] proved that $\Delta_3 \leq \frac{3}{25} = 0.12$. Numerical evidences seem to label this value as too optimistic. The maximum density attained by known Cayley digraphs is $\delta_0 = 0.084$ and they have been found by computer search. These digraphs are $G_0 = \text{Cay}(\mathbb{Z}_{84}, \{2, 9, 35\})$ and $G_1 = \text{Cay}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{168}, \{(1, 0, 2), (0, 0, 9), (0, 1, 35)\})$ in [3, Table 1] and $G'_1 \cong G_1$, $G_2 = \text{Cay}(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{252}, \{(0, 0, 2), (0, 1, 9), (1, 0, 35)\})$ and $G_3 = \text{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{336}, \{(0, 1, 2), (0, 0, 9), (1, 0, 35)\})$ in [2, Table 8.2].

Remark 1.1 A large value of the ratio $N(G)/k(G)$ does not guarantee a large density of G .

In this work we study some metric properties of Cayley digraphs which allow us to provide two outstanding infinite families of dense or eventually likely dense digraphs. The first one has a large ratio N/k as d increases. Finally, each member of the second family has density $\delta = \delta_0$.

2 Minimum distance diagrams

Each Cayley digraph has (at least one) related *minimum distance diagram* (MDD for short) which encodes its metrical properties. Given $\mathbf{a} \in \mathbb{N}^d$, we denote the *unitary cube* $\llbracket \mathbf{a} \rrbracket = [a_1, a_1 + 1) \times [a_2, a_2 + 1) \times \dots \times [a_d, a_d + 1) \in \mathbb{R}^d$. We also denote the *cone* $\nabla(\mathbf{a})$ as the set of cubes $\nabla(\mathbf{a}) = \{\llbracket \mathbf{b} \rrbracket : \mathbf{0} \leq \mathbf{b} \leq \mathbf{a}\}$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^d$ and the inequality acts over all the coordinates $0 \leq a_i \leq b_i$. From now on, we use the ℓ_1 norm $\|\mathbf{a}\|_1 = |a_1| + \dots + |a_d|$.

Definition 2.1 [MDD] Given a finite Abelian group $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$ of order $N = |\Gamma|$, let us consider the map $\phi : \mathbb{N}^n \longrightarrow \Gamma$ given by $\phi(\mathbf{a}) = a_1\gamma_1 + \dots + a_n\gamma_n$. A *minimum distance diagram* related to the Cayley digraph $\text{Cay}(\Gamma, \{\gamma_1, \dots, \gamma_n\})$, denoted by \mathcal{L} , is a set of N unitary cubes $\mathcal{L} = \{\llbracket \mathbf{a}_0 \rrbracket, \dots, \llbracket \mathbf{a}_{N-1} \rrbracket\}$ such that

- (i) $\{\phi(\mathbf{a}) : \llbracket \mathbf{a} \rrbracket \in \mathcal{L}\} = \Gamma$,
- (ii) $\llbracket \mathbf{a} \rrbracket \in \mathcal{L} \Rightarrow \nabla(\mathbf{a}) \subset \mathcal{L}$.
- (iii) $\|\mathbf{a}\|_1 = \min\{\|\mathbf{x}\|_1 : \mathbf{x} \in \phi^{-1}(\phi(\mathbf{a}))\}$, for all $\llbracket \mathbf{a} \rrbracket \in \mathcal{L}$.

Given a MDD \mathcal{L} related to G , we denote the *diameter of \mathcal{L}* as $k(\mathcal{L}) = \max\{\|\mathbf{a}\|_1 : \llbracket \mathbf{a} \rrbracket \in \mathcal{L}\}$. It is not difficult to see that $k(\mathcal{L}) = k(G)$.

Given a unitary cube $\llbracket \mathbf{a} \rrbracket$, we define the set of cubes $t\llbracket \mathbf{a} \rrbracket$, with $t \in \mathbb{N}$ and $t \geq 1$, by $t\llbracket \mathbf{a} \rrbracket = \{\llbracket t\mathbf{a} + (\alpha_1, \dots, \alpha_n) \rrbracket : 0 \leq \alpha_1, \dots, \alpha_n \leq t - 1\}$.

Definition 2.2 [MDD's Dilation] Given a MDD \mathcal{L} related to some Cayley digraph, the *t-dilation* $t\mathcal{L}$ of \mathcal{L} is $t\mathcal{L} = \{t\llbracket \mathbf{a} \rrbracket : \llbracket \mathbf{a} \rrbracket \in \mathcal{L}\}$.

It has been shown that MDDs tessellate \mathbb{R}^d by translation through d integral vectors. Given an MDD \mathcal{L} related to the Cayley digraph $G = \text{Cay}(\Gamma, B)$, assume that \mathcal{L} tessellates through the vectors $A = \{\mathbf{u}_1, \dots, \mathbf{u}_d\} \subset \mathbb{Z}^d$. Consider the matrix $M \in \mathbb{Z}^{d \times d}$ defined by the column vectors of A . Then, $M\mathbb{Z}^d = \{M\mathbf{x} : \mathbf{x} \in \mathbb{Z}^d\}$ is a normal subgroup of \mathbb{Z}^d and the group $\mathbb{Z}^d/M\mathbb{Z}^d$ is an Abelian group of order $N(G) = |\det M|$. Consider the Cayley digraph $G_M = \text{Cay}(\mathbb{Z}^d/M\mathbb{Z}^d, \{\mathbf{e}_1, \dots, \mathbf{e}_d\})$, where \mathbf{e}_i are the canonical unitary vectors. Then, $G_M \cong G$.

Assume that the matrix $S = \text{diag}(s_1, \dots, s_d) \in \mathbb{Z}^{d \times d}$ is the *Smith normal form* of M , and that $U, V \in \mathbb{Z}^{d \times d}$ are the unimodular matrices such that $S = UMV$. Then, the isomorphism of digraphs $G_M \cong G_S \cong \text{Cay}(\mathbb{Z}_{s_1} \oplus \dots \oplus \mathbb{Z}_{s_d}, \{U\mathbf{e}_1, \dots, U\mathbf{e}_d\})$ holds ([4]). This property allows us to find the two families of the next section.

Theorem 2.3 For an integer $t \geq 1$, the following holds:

(a) \mathcal{L} is an MDD related to $G_M \Leftrightarrow t\mathcal{L}$ is an MDD related to G_{tM} .

(b) $k(t\mathcal{L}) = t(k(\mathcal{L}) + d) - d$.

Corollary 2.4 Consider the Cayley digraph $G = \text{Cay}(\mathbb{Z}_{s_1} \oplus \dots \oplus \mathbb{Z}_{s_d}, \{U\mathbf{e}_1, \dots, U\mathbf{e}_d\})$. Denote $tG = \text{Cay}(\mathbb{Z}_{ts_1} \oplus \dots \oplus \mathbb{Z}_{ts_d}, \{U\mathbf{e}_1, \dots, U\mathbf{e}_d\})$. Then, $k(tG) = t(k(G) + d) - d$.

3 Applications

In this section we apply Corollary 2.4 to obtain two infinite families of Cayley digraphs with good metrical properties. More precisely, one family with large ratio N/k for any degree d and another family with the largest known density $\delta_0 = 0.084$ for fixed degree $d = 3$.

Lemma 3.1 ([1]) Consider the integral vectors $B'_d = \{(1, 1, \dots, 1), (2, 1, \dots, 1), \dots, (1, 1, \dots, 2)\} \subset \mathbb{Z}^{d-1}$, $d \geq 2$. Then, $G'_d = \text{Cay}(\mathbb{Z}_{d+1} \oplus \binom{d-1}{\cdot \cdot \cdot} \oplus \mathbb{Z}_{d+1}, B'_d)$ has diameter $k(G'_d) = \binom{d}{2}$.

Proposition 3.2 Consider $B_d = \{(1, 1, 1, \dots, 1), (1, 2, 1, \dots, 1), \dots, (1, 1, \dots, 1, 2)\} \subset \mathbb{Z}^d$. Then, the Cayley digraph $G_{d,t} = \text{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}_{t(d+1)} \oplus \binom{d-1}{\cdot \cdot \cdot} \oplus \mathbb{Z}_{d+1}, B_d)$ has diameter $k(G_{d,t}) = t \binom{d+1}{2} - d$, for $d \geq 2$ and $t \geq 1$.

This proposition comes from the digraph isomorphism $G_{d,1} \cong G'$, where G' is the Cayley digraph of Lemma 3.1, and the application of Corollary 2.4. Notice that, by using the Stirling's formula, the lower bound expression $\text{lb}(d, k)$ in (1), gives

$$\text{lb}(d, k) \sim \frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d - (\ln \ln d)(1 + \log_2 e)} \left(\frac{k}{d}\right)^d + O(k^{d-1}),$$

with the multiplicative factor of $\left(\frac{k}{d}\right)^d$ being

$$\frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d - (\ln \ln d)(1 + \log_2 e)} \sim \frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d}.$$

Notice that $N(G_{d,t}) = t^d (d+1)^{d-1}$ and $k_{d,t} = k(G_{d,t}) = t \binom{d+1}{2} - d$ hold for all t . Thus, as $t = \frac{k_{d,t} + d}{\binom{d+1}{2}}$, we get

$$N(G_{d,t}) = \frac{2^d}{d+1} \left(\frac{k_{d,t}}{d} + 1\right)^d = \frac{2^d}{d+1} \left(\frac{k_{d,t}}{d}\right)^d + O(k_{d,t}^{d-1}),$$

with the multiplicative factor of $\left(\frac{k_{d,t}}{d}\right)^d$ being $e^{d \ln 2 - \ln(d+1)}$.

Now, let us focus our attention on the density. For fixed degree $d = 3$, a computer search gives $k(G_0) = 7$ with $G_0 = \text{Cay}(\mathbb{Z}_{84}, \{2, 9, 35\}) = 7$. Then, $\delta(G_0) = \delta_0$ holds. This digraph has one related MDD given in Figure 1. This result is well known and it has been remarked by several authors ([2,3,5]). We refer to this MDD as \mathcal{L}_0 .

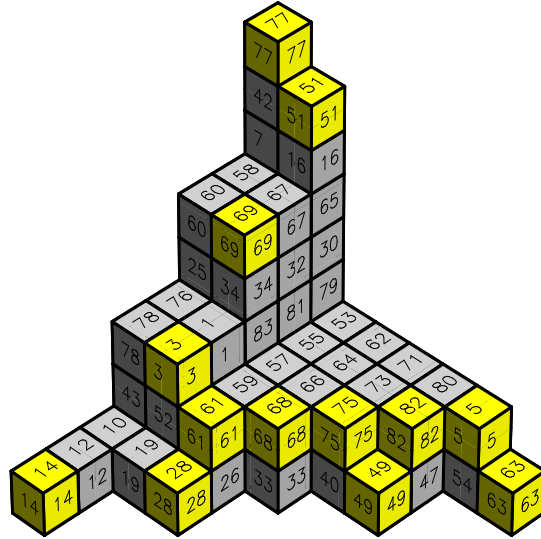


Fig. 1. MDD related to the digraph G_0

The MDD tessellates \mathbb{R}^3 through the vectors $B = \{(1, 5, 2), (2, 2, 2), (-6, 4, 3)\}$. Set the matrix of column vectors B as M . The Smith normal form is

$$S = \text{diag}(1, 1, 84) = UMV = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 10 \\ 7 & -3 & -38 \end{pmatrix} \begin{pmatrix} 1 & 2 & -6 \\ 5 & 2 & 4 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 26 \\ 0 & 1 & 23 \\ -1 & 0 & -2 \end{pmatrix}.$$

Then, by Corollary 2.4, we get the following result.

Proposition 3.3 *The family of Abelian Cayley digraphs*

$$G_t = \text{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}_t \oplus \mathbb{Z}_{84t}, \{(1, 10, -38), (0, 1, -3), (0, -2, 7)\})$$

has order $N_t = 84t^3$ and diameter $k_t = 10t - 3$, for $t \geq 1$.

Notice that

$$\delta(G_t) = \frac{N_t}{(k_t + 3)^3} = \frac{84t^3}{10^3 t^3} = \delta_0, \text{ for all } t \geq 1.$$

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